Pure and Applied Functional Analysis
Volume 6, Number 3, 2021, i–iii

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

PREFACE: 30 YEARS OF EXTRAPOLATION

SERGEY ASTASHKIN AND MARIO MILMAN

Parameterized families of spaces (scales) play an important role in modern Analysis. Indeed, parameters often are associated with the description of the qualitative properties (e.g. integrability, smoothness, etc) of the elements of function spaces and are used in the construction of the corresponding norms. Over the last 60+ years or so, Interpolation Theory has evolved as a very powerful framework to study operators acting on classical scales of spaces and, in particular, for developing methods to construct new scales of function spaces with special properties. Classically, starting with a pair of spaces, interpolation allows one to create parameterized scales of "intermediate" spaces and often the values of the parameters involved correspond to the relative "strength" of the underlying properties one is studying[1]. In this fashion, Interpolation theory not only facilitates the interactions and applications of the general methods of Functional Analysis to other parts of Analysis (e.g. Function Spaces (cf. [6], [19]), PDEs (cf. [1], [13], [20], [22], [23], [28]), Embedding Theory (cf. [11]), Harmonic Analysis ([11], [12], [25]), Approximation Theory ([10], [29]), etc.), but also leads to the creation of new "scale specific"methods of its own (cf. [24], [25]). Indeed, many results in Approximation Theory, Harmonic Analysis, PDE's, etc, crucially rely on interpolation methods, even in their very formulation. Moreover, the platform provided by the theory can be also used to extend the fundamental results of Functional Analysis itself. For example, the extensions of the Open Mapping theorem to scales of spaces (cf. [27]), development of tools to study Wavelets, (cf. [12]) applications to the Geometry of Banach spaces (cf. [3], [8], [17], [18], [21]), connections to differential geometry via reinterpreting scales of spaces as "curves" (cf. [26], [2]), commutators and Non Linear PDEs (cf. [13], [25], and references therein), Operator inequalities (cf. [2]).

Many well-known mathematicians contributed to lay down the foundations of the theory, among others we mention, N. Aronszajn, A. P. Calderón, M. Cotlar, E. Gagliardo, M. Krein, J. L. Lions, J. Marcinkiewicz, B. S. Mityagin, J. Peetre, M. Riesz, R. Rochberg, R. Thorin, E. A. Stein, A. Zygmund, to name but just a few. These, and more recent developments, are recorded in many books devoted to Interpolation theory (cf. [6] , [7], [9], [10], [19], [23], [28], [29]) to which we refer for a full development and detailed historical perspective.

As it turns out, it is important to determine the optimal range of the parameters that guarantee the validity of the results (e.g. properties, inequalities, methods, etc) under consideration. In fact, determining the limiting admissible values of the

2020 *Mathematics Subject Classification.* 46B70, 46E30, 46E35, 46M35.
Key words and phrases. Interpolation, extrapolation, function spaces.

[1]This can be also useful in formulating concepts like "1/2" derivative, etc.

above parameters often helps to advance our understanding of the problems at hand. The usual methods of interpolation theory which, generally speaking, generate new "intermediate" inequalities from two given ones, do not help per se, in this endeavor. It is precisely to address these issues that Extrapolation theory was created as a complement to Interpolation. It was observed that if qualitative information is available on how the parameters of the intervening inequalities deteriorated as they approach the end of their range of validity, it is possible to construct end point spaces (usually different from the ones of the original scale), the so-called "extrapolation spaces", for which a valid end point estimate ("extrapolation estimate") would hold.

By the mid eighties a new theory of Extrapolation started to emerge with the objective to understand the limiting cases of interpolation inequalities and the construction of extrapolation spaces needed to formulate the corresponding end point results. One could trace the origins of extrapolation theory to a rather elementary, but yet powerful, theorem by Yano [30] concerning decaying L^p estimates, which can be considered as the first "extrapolation theorem". Taking Yano's result as a starting point, Jawerth-Milman [16] initiated the development of a theory of Extrapolation which by now has received contributions from several mathematicians worldwide (for recent references cf. [4], [5], and the references therein).

The first formal publications on extrapolation theory appeared in 1988-89 (cf. [14], [15]) and the number of contributions devoted to extrapolation has increased steadily since. To celebrate the first thirty years, and with the aim to promote further developments of the theory, we decided to organize this special issue of Pure and Applied Functional Analysis devoted to Extrapolation. The volume collects contributions from authors from Argentina, Australia, Czech Republic, Greece, Italy, Kazakhstan, Pakistan, Russia, Sweden, US, and gives some perspective on recent work on Extrapolation in different directions. A specially prepared paper by the editors, that includes a brief Introduction to the theory as well as a fresh list of Open Problems, closes the volume.

References

[1] H. Amann, *Linear and Quasilinear Parabolic Problems, Vol. 1*, Monographs in Mathematics, Birkhäuser, Basel, 1995.

[2] E. Andruchow, G. Corach, M. Milman and D. Stojanoff, *Geodesics and interpolation*, Rev. Un. Mat. Argentina **40** (1997), 83–91.

[3] S.V. Astashkin, *Rademacher System in Function Spaces*, Birkhäusser, Switzerland 2017.

[4] S. V. Astashkin, K. V. Lykov,and M. Milman, *Limiting interpolation spaces via extrapolation*, J. Approx. Theory **240** (2019), 16–70.

[5] S. V. Astashkin and K. V. Lykov, *Jawerth-Milman extrapolation theory: some recent developments with applications*, in Functional Analysis, Harmonic Analysis and Image Processing: A collection of papers in honor of Björn Jawerth, Amer. Math. Soc., Contemporary Mathematics, vol 693, Providence, 2017, pp. 7–53.

[6] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, New York, 1988.

[7] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin, 1976.

[8] M. Sh. Braverman, *Independent Random Variables and Rearrangement Invariant Spaces*, London Math. Soc. Lecture Note Ser., vol. 194, Cambridge University Press, Cambridge, 1994.

[9] Yu. A. Brudnyi and N. Ya. Krugljak, *Interpolation Functors and Interpolation Spaces*, North-Holland, Amsterdam, 1991.

[10] P. L Butzer and H. Berens, *Semi-groups of Operators and Approximation*, Springer-Verlag, Berlin, 1967.

[11] O. Domínguez and S. Tikhonov, *Embeddings of smooth function spaces, extrapolations, and related inequalities*, preprint, arXiv:1909.12818.
[12] M. Frazier, B. Jawerth, and G. Weiss, *Littlewood-Paley theory and the study of function spaces*, CBMS Regional Conference Series in Mathematics, vol. 79, American Mathematical Society, Providence, RI, 1991.
[13] T. Iwaniec and G. Martin, *Geometric Function Theory and Non-linear Analysis*, Oxford University Press, London, 2001.
[14] B. Jawerth and M. Milman, *A Theory of Extrapolation Spaces. First Applications*, Compt. Rend. Acad. Sc. Paris **309** (1989), 175–179.
[15] B. Jawerth and M. Milman, *A theory of Extrapolation Spaces. Further Applications*, Compt. Rend. Acad. Sc. Paris **309** (1989), 225-229.
[16] B. Jawerth and M. Milman, *Extrapolation Spaces with applications*, Mem. of the Amer. Math. Soc., vol. 89, No. 440, 1991.
[17] W. B. Johnson and J. Lindenstrauss (editors), *Handbook of the Geometry of Banach Spaces*, Elsevier, Amsterdam, 2003.
[18] W. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri, *Symmetric structures in Banach spaces.* Mem. Amer. Math. Soc., vol. 19, No. 217, 1979.
[19] S. G. Krein, Ju. I. Petunin and E. M. Semenov, *Interpolation of Linear Operators*, Trans. Amer. Math. Soc., vol. 54, Providence, RI, 1982.
[20] G. Leoni, *A First Course in Sobolev Spaces*, Graduate Studies in Mathematics **105**, Amer. Math. Soc., Providence, RI, 2009.
[21] J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces II. Function spaces*, Springer-Verlag, Berlin, 1979.
[22] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications Vol. 1*, Springer, Berlin, 1972.
[23] A. Lunardi, *Interpolation Theory. Second Edition*, Scuola Normale Superiore di Pisa (Nuova Serie), Edizioni della Normale, Pisa, 2009.
[24] V. I. Ovchinnikov, *The method of orbits in interpolation theory*, Mathematical Reports, Vol. 1, Part 2, Harwood Academic Publishers, 1984, pp. 349–516.
[25] R. Rochberg, *Uses of commutator theorems in analysis*, Contemp. Math. **445** (2007), 277–295.
[26] S. Semmes, *Interpolation of Banach spaces, differential geometry and differential equations*, Rev. Mat. Iberoamericana **4** (1988), 155–176.
[27] I. Ya. Shneiberg, *Spectral properties of linear operators in interpolation families of Banach spaces*, Mat. Issled. **9** (1974), 214–229.
[28] L. Tartar, *An Introduction to Sobolev Spaces and Interpolation Spaces*, Lecture Notes of the Unione Matematica 3 Italiana, Springer, Berlin, 2007.
[29] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, VEB, Berlin, 1978.
[30] S. Yano, *Notes on Fourier analysis. XXIX. An extrapolation theorem*, J. Math. Soc. Japan **3** (1951), 296–305.

Sergey Astashkin
Samara National Research University, Moskovskoye shosse 34, 443086, Samara, Russia
E-mail address: `astash56@mail.ru`

Mario Milman
Instituto Argentino de Matematica, Buenos Aires, Argentina
E-mail address: `Mario.Milman@icloud.com`