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PREFACE: CALCULUS OF VARIATIONS, PDE AND THEIR APPLICATIONS

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Special Issue on Calculus of Variations, PDE and Their Applications dedicated to Professor Luc Tartar on the occasion of his 75th birthday.

Preface by Roger Lewandowski, Simeon Reich and Alexander Zaslavski

This special issue on Calculus of Variations, PDE and Their Applications is dedicated to Professor Luc Tartar on the occasion of his 75th birthday.

Luc Tartar was born in 1946 in the 12th arrondissement of Paris and completed his secondary education in Algeria. He returned to France in 1962 and attended preparatory classes at Lycée Charlemagne in Paris before joining École Polytechnique in 1965 with the intention of becoming an engineer. However, at the start of his second year, a presentation by Laurent Schwartz inspired him to pursue mathematical research. During that same year, he attended Jacques-Louis Lions' course on numerical analysis, a formative experience for his future work.

In 1971, Luc defended his doctoral dissertation under the supervision of Jacques-Louis Lions. His thesis focused on nonlinear interpolation and its applications to the regularity of solutions to partial differential equations (PDEs) with coefficients that are not necessarily of class C^∞. He quickly obtained a position as a Maître de conférences at Paris Dauphine University. During the 1974-1975 academic year, he was on sabbatical at the University of Wisconsin in Madison, where he served as a visiting associate professor, followed by a professorship at Université d'Orsay until 1982. From 1982 to 1988, he was on secondment at the "Commissariat à l'énergie Atomique" (CEA) under the direction of Robert Dautray. In 1987, he was elected a corresponding member of the French Academy of Sciences, nominated by Jean Leray. Later that year, he joined Carnegie Mellon University in Pittsburgh (USA), where he became a University Professor in 1994. In 2006, he was elected a foreign member of the Istituto Lombardo Accademia di Scienze e Lettere upon the nomination of Enrico Magenes. He retired in 2012 and returned to France.

Luc's early research on interpolation in the 1970s demonstrated his remarkable mathematical creativity, scientific originality, and deep understanding of PDEs. Notably, he authored a series of lecture notes from courses he gave at the University of Wisconsin-Madison, which became known as his *Madison Lectures* [4]. These lectures were highly influential for young researchers of the next generation. Among the results presented in these notes is a beautiful theorem in functional analysis, now known as the *Petree-Tartar Lemma*, which provides an elegant proof of the Aubin-Lions compactness lemma.

In 1969, Luc met François Murat, with whom he shared an office when the Jacques-Louis Lions laboratory was established. Starting in the 1970s, Luc and François held regular meetings every Friday at the Jussieu campus (Paris, France)

to discuss problems now known as *homogenization*, which initially aimed to analyze composite materials by linking their microstructure to macroscopic properties. A significant milestone in their collaboration came when Luc formulated and proved what is now known as the *div-curl lemma*, a cornerstone of the theory of compensated compactness [3, 5].

Luc then introduced oscillating test functions, which made it possible to extract convergent subsequences in elliptic problems with uniformly bounded and coercive matrices, without any periodicity assumption. This led to a general theory of homogenization. Luc presented these ideas, including compensated compactness and a corrector result, during the prestigious Peccot Lectures at the Collège de France in the spring of 1977. His fruitful method paved the way for numerous developments in homogenization (see, for example, Bensoussan-Lions-Papanicolaou [1]). Luc rigorously proved results postulated by engineers, such as the celebrated Hashin-Shtrikman bounds [6]. In parallel, he continued to explore a wide range of PDE problems, including hyperbolic systems particularly Boltzmann equations with a finite number of velocities and Navier-Stokes equations. He introduced a new approach based on compensated compactness to handle limits in the viscous regularization of the 1D Burgers equation [7].

In the 1980s, Luc pursued a deep investigation of oscillatory phenomena in PDEs with the goal of providing a rigorous framework for mechanics and physics, often challenging established scientific dogmas. This research led to his invention of *H-measures* [8], a powerful tool for studying the relationship between small and large-scale behavior in physical phenomena and the propagation of oscillations in PDEs. This groundbreaking discovery found numerous applications, such as the existence of singular solutions for the Euler equations at arbitrary energy levels [14].

Luc was an invited speaker at the International Congress of Mathematicians (ICM) in Kyoto, where he presented a lecture titled *"H-measures and Applications"*. In this talk, he explained how H-measures could track the direction in which energy propagates, unlike classical thermodynamics, which simply states that energy dissipates locally. Starting from the linear wave equation, Luc derived a kinetic model a result that was rigorously proved rather than postulated, unlike the classical Maxwell-Boltzmann model. An open question remains whether this analysis can be extended to certain semilinear models.

Luc has authored four books, three of which are based on lecture notes. His book *An Introduction to Navier-Stokes Equations and Oceanography* [9], starting from the book of Roger Lewandowski *Analyse Mathématique et Océanographie* [2], provides an in-depth development of Navier-Stokes theory, making it an essential reference in the field.

His book *An Introduction to Sobolev Spaces and Interpolation Spaces* [10] is one of the few comprehensive texts offering a profound understanding of Sobolev spaces. It stands out as the only text where fractional Sobolev spaces are introduced in full generality, along with the theory of traces for Lipschitz domains.

In *From Hyperbolic Systems to Kinetic Theory: A Personalized Quest* [11], Luc examines the connections between hyperbolic differential equations and kinetic theories, offering a deep and personalized analysis of their applications in fluid dynamics and continuum mechanics.

His book *The General Theory of Homogenization: A Personalized Introduction* [12] presents advanced mathematical methods for homogenization, with applications to composite materials and the interaction between microstructures and macroscopic properties. The book also introduces H-measures and rigorously demonstrates how complex problems in continuum mechanics can be simplified using this approach.

Luc's formative years were shaped by a Reformed Protestant background. His father was a pastor, and he is married to a pastor. This upbringing likely instilled in him an unwavering sense of ethics and integrity, both scientifically and personally. He has consistently stood by his principles of justice, rigor, and honesty, even when it risked his academic career, making him a role model for many younger researchers. Beyond his scientific work, he has maintained a keen interest in religious texts and authored a non-scientific book titled *Promenade Critique dans des Textes Monothéistes* [13]. In this work, he exercises logical reasoning (without mathematics) to critically examine biblical stories and uncover contradictions, sometimes using archaeological findings. His aim is not to promote conversion but to encourage believers to practice their faith more thoughtfully by questioning what deserves to be questioned.

Without a doubt, Luc Tartar is one of the most brilliant mathematicians of his generation. He has opened numerous avenues of research and inspired countless researchers in the fields of PDEs and mathematical physics. His major results and the program he developed constitute a lasting legacy for mathematics.

In this special issue we present papers authored by a selected group of experts in the areas of Calculus of Variations, PDE and their applications. The papers collected here have been contributed by collaborators, friends and colleagues of Professor Tartar, who were influenced by his scientific work. The special issue contains eleven papers contributed by researchers from Canada, Czech Republic, Croatia, France, Germany, Israel, Italy, United Kingdom, and the USA.

These papers cover a wide spectrum of important problems and topics of current research interest, including the ideal magnetohydrodynamics and field dislocation mechanics, singular boundary condition for a degenerated turbulent toy model, a variational approach to strain-limiting viscoelasticity in one space dimension, strict positivity of the temperature in the Navier-Stokes-Fourier system, a continued fraction approximation for the effective elasticity tensor of two-dimensional polycrystals as a function of the crystal elasticity tensor, anisotropic distributions, numerical simulation of polarized light, Sobolev-Slobodeckii embeddings and space diagrams, solvability in the sense of sequences for some non-Fredholm operators with a drift, polyconvex gradient flows, and generalized nonexpansive mappings.

Therefore, we feel that this special issue will be highly important for many mathematicians, who are interested in recent developments in Analysis, Calculus of Variations and PDE as well as in their diverse applications.

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