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PAIRS TRADING UNDER GEOMETRIC BROWNIAN MOTIONS WITH REGIME SWITCHING

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Abstract. This paper is about an optimal pairs trading rule. A pairs position consists of a long position in one stock and a short position in the other. The problem is to find stopping times to open and then close the pairs position to maximize expected reward functions. In this paper, we consider the optimal pairs trading rule with one round trip. The underlying stock prices follow a general geometric Brownian motion with regime switching. The optimal policy is characterized by threshold curves obtained by solving the associated HJB equations (quasi-variational inequalities). Moreover, numerical examples are provided to illustrate optimal policies.

1. Introduction

This paper is concerned with pairs trading of stocks. The idea behind pairs trading is to track the price movements of a pairs of stocks over time and compare their relative price strengths. A pairs position consists of a short position in the stronger stock and a long position in the weaker one. A pairs trade is about buying and then selling such pairs positions. The strategy bets on the reversal of their price strength. What makes the strategy attractive is its 'market neutral' nature in the sense that it can be profitable under any market conditions. Pairs trading was initially introduced by Bamberger and followed by Tartaglia's quantitative group at Morgan Stanley in the 1980s; see Gatev et al. [4] for related history and background details. There are many in-depth discussions in connection with the cause of the divergence and subsequent convergence; see the book by Vidyamurthy [15] and references therein.

Mathematical trading rules have been studied for many years. For example, Zhang [17] considered a selling rule determined by two threshold levels, a target price and a stop-loss limit. In [17], such optimal threshold levels are obtained by solving a set of two-point boundary value problems. Guo and Zhang [5] studied the optimal selling rule under a model with switching geometric Brownian motion. Using a smooth-fit technique, they obtained the optimal threshold levels by solving a set of algebraic equations. These papers are concerned with the selling side of trading in which the underlying price models are of GBM type. Dai et al. [1] developed a trend following rule based on a conditional probability indicator. They

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showed that the optimal trading rule can be determined by two threshold curves which can be obtained by solving the associated Hamilton-Jacobi-Bellman (HJB) equations. A similar idea was developed following a confidence interval approach by Iwarere and Barmish [7]. Besides, Merhi and Zervos [9] studied an investment capacity expansion/reduction problem following a dynamic programming approach under a geometric Brownian motion market model. In connection with mean reversion trading, Zhang and Zhang [18] obtained a buy-low and sell-high policy by characterizing the 'low' and 'high' levels in terms of the mean reversion parameters. Song and Zhang [11] studied pairs trading under a mean reversion model. It is shown that the optimal trading rule can be determined by threshold levels that can be obtained by solving a set of algebraic equations. A set of sufficient conditions are also provided to establish the desired optimality. Deshpande and Barmish [2] introduced a control-theoretic approach. In particular, they were able to relax the requirement for spread functions and showed that their trading algorithm produces positive expected returns. Other related pairs technologies can be found in Elliott et al. [3] and Whistler [16]. Recently, Tie et al.[14] studied an optimal pairs trading rule under geometric Brownian motions.. The objective is to initiate and close the positions of the pair sequentially to maximize a discounted payoff function. Using a dynamic programming approach, they studied the problem under a geometric Brownian motion model and proved that the buying and selling can be determined by two threshold curves in closed form. They also demonstrate the optimality of their trading strategy.

Market models with regime switching are important in market analysis. In this paper, we consider a geometric Brownian motion with regime switching. The market mode is represented by a two-state Markov chain. In a recent paper, Tie and Zhang [13] treated the selling part of pairs trading that generalizes the results of Hu and Oksendal [6] by incorporating models with regime switching. They showed that the optimal selling rule can be determined by two threshold curves and established a set of sufficient conditions that guarantee the optimality of the policy. To complete the circle of pairs trading, one has to come up with the buying part of the trading rule to determine how much divergence is needed that triggers the entry of the position. It is the focus of this paper. In particular, we study pairs trading under geometric Brownian motions with regime switching. The objective is to buy and then sell a pairs position to maximize the expected return. Using a smooth-fit method, we characterize the trading policies in terms of threshold curves which can be determined by a set of algebraic equations, We also provide a set of sufficient conditions for the optimality of the trading policy. Finally, we present numerical examples to illustrate the results.

This paper is organized as follows. In §2, we formulate the pairs trading problem under consideration. In §3, we study the associated HJB equations and their solutions. In §4, we provide a set of sufficient conditions that guarantee the optimality of our trading rule. Numerical examples are given in §5. Some concluding remarks are given in §6. Finally, key steps for pairs selling rules are given in Appendix.

2. Problem formulation

Our pairs trading strategy involves two stocks $\mathbf{S}^1$ and $\mathbf{S}^2$. Let $\{X_t^1, t \geq 0\}$ denote the prices of stock $\mathbf{S}^1$ and $\{X_t^2, t \geq 0\}$ that of stock $\mathbf{S}^2$. They satisfy the following stochastic differential equation:

$$d\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} X_t^1 & \\ & X_t^2 \end{pmatrix} \left[\begin{pmatrix} \mu_1(\alpha_t) \\ \mu_2(\alpha_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_{11}(\alpha_t) & \sigma_{12}(\alpha_t) \\ \sigma_{21}(\alpha_t) & \sigma_{22}(\alpha_t) \end{pmatrix} d\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} \right], \tag{2.1}$$

where $\alpha_t \in \mathcal{M} = \{1, 2\}$ is a two-state Markov chain and (W_t^1, W_t^2) a two-dimensional standard Brownian motion. Here, for $\alpha = 1, 2$, $\mu_i(\alpha)$, $i = 1, 2$, are the return rates and $\sigma_{ij}(\alpha)$, $i, j = 1, 2$, the volatility constants.

Let Q be the generator of α_t given by $Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$, with $\lambda_1 > 0$ and $\lambda_2 > 0$. We assume α_t and (W_t^1, W_t^2) are independent.

In this paper, we assume, for simplicity, a pairs position consists of one-share long position in stock $\mathbf{S}^1$ and one-share short position in stock $\mathbf{S}^2$. Let $\mathbf{Z}$ denote the corresponding pairs position. One share in $\mathbf{Z}$ represents the combination of one share long position in $\mathbf{S}^1$ and one share short position in $\mathbf{S}^2$.

Remark 2.1. Intuitively, if stock $\mathbf{S}^1$ is cheap (i.e., X_t^1 is small) and stock $\mathbf{S}^2$ is dear (i.e., X_t^2 is large), then one should buy $\mathbf{S}^1$ and sell (short) $\mathbf{S}^2$. This amounts to open a pairs position $\mathbf{Z}$. The idea of pairs trading is to bet on the eventual price reversal. Therefore, one should close the pairs position $\mathbf{Z}$ by selling $\mathbf{S}^1$ and buying back $\mathbf{S}^2$ after, relatively speaking, substantial rises of $\mathbf{S}^1$ and/or adequate falls of $\mathbf{S}^2$ in their prices.

We consider one round trip pairs trading. The net position at any time can be either long (with one share of $\mathbf{Z}$) or flat (no stock position of either $\mathbf{S}^1$ or $\mathbf{S}^2$). Let $i = 0, 1$ denote the initial net position and let τ_0, τ_1, τ_2 denote stopping times with $\tau_1 \leq \tau_2$.. If initially the net position is flat ($i = 0$), then one should start to buy a share of $\mathbf{Z}$. That is, to first buy at τ_1 and then sell at τ_2. The decision is denoted by $\Lambda_0 = \{\tau_1, \tau_2\}$. If initially the net position is long ($i = 1$), then one should sell $\mathbf{Z}$. The corresponding decision is denoted by $\Lambda_1 = \{\tau_0\}$.

Let K denote the fixed percentage of transaction costs associated with buying or selling of stocks $\mathbf{S}^i$, $i = 1, 2$. For example, the cost to establish the pairs position $\mathbf{Z}$ at $t = t_1$ is $(1 + K)X_{t_1}^1 - (1 - K)X_{t_2}^2$ and the proceeds to close it at a later time $t = t_2$ is $(1 - K)X_{t_2}^1 - (1 + K)X_{t_2}^2$. For ease of notation, let $\beta_{\mathrm{b}} = 1 + K$ and $\beta_{\mathrm{s}} = 1 - K$.

Given the initial state (x_1, x_2, α), the initial net position $i = 0, 1$, and the decision variables Λ_0 and Λ_1, the corresponding reward functions

$$\begin{aligned} J_0(x_1, x_2, \alpha, \Lambda_0) &= E\big\{[e^{-\rho\tau_2}(\beta_{\mathrm{s}} X_{\tau_2}^1 - \beta_{\mathrm{b}} X_{\tau_2}^2) I_{\{\tau_2 < \infty\}} \\ &\qquad - e^{-\rho\tau_1}(\beta_{\mathrm{b}} X_{\tau_1}^1 - \beta_{\mathrm{s}} X_{\tau_1}^2) I_{\{\tau_1 < \infty\}}]\big\}, \\ J_1(x_1, x_2, \alpha, \Lambda_1) &= E\big\{e^{-\rho\tau_0}(\beta_{\mathrm{s}} X_{\tau_0}^1 - \beta_{\mathrm{b}} X_{\tau_0}^2) I_{\{\tau_0 < \infty\}}\big\}, \end{aligned} \tag{2.2}$$

where $\rho > 0$ is a given discount factor and I_A is the indicator function of an event A.

Let $\mathcal{F}_t = \sigma\{(X_r^1, X_r^2, \alpha_r) : r \leq t\}$. The problem is to find $\{\mathcal{F}_t\}$ stopping times τ_0, τ_1, and τ_2, to maximize J_i. For $i = 0, 1$, let $V_i(x_1, x_2, \alpha)$ denote the value functions with the initial state $(X_0^1, X_0^2, \alpha_0) = (x_1, x_2, \alpha)$ and initial net positions $i = 0, 1$. That is, $V_i(x_1, x_2, \alpha) = \sup_{\Lambda_i} J_i(x_1, x_2, \alpha, \Lambda_i)$, $i = 0, 1$.

Remark 2.2. We would like to point out that our 'one-share' pair position is not as restrictive as it appears. For example, one can consider any pairs with n_1 shares of long position in $\mathbf{S}^1$ and n_2 shares of short position in $\mathbf{S}^2$. To treat this case, one only has to make change of the state variables $(X_t^1, X_t^2) \to (n_1 X_t^1, n_2 X_t^2)$. Due to the nature of GBMs, the corresponding system equation in (2.1) will remain the same. The modification only affects the reward function in (2.2) implicitly.

Throughout this paper, we impose the following conditions:

(A1) $\rho > \mu_j(\alpha)$, for $\alpha = 1, 2$ and $j = 1, 2$.

Under these conditions, we can establish the lower and upper bounds for the value functions as follows.

Lemma 2.3. *For some constant C, the inequalities hold*

$$0 \le V_0(x_1, x_2, \alpha) \le C x_2. \tag{2.3}$$

In addition, we have

$$\beta_{\mathrm{s}} x_1 - \beta_{\mathrm{b}} x_2 \le V_1(x_1, x_2, \alpha) \le \beta_{\mathrm{s}} x_1. \tag{2.4}$$

Proof. We first consider the inequalities in (2.3). Clearly, $V_0 \ge 0$. To see $V_0 \le C x_2$, note that

$$\begin{aligned} J_0(x_1, x_2, \alpha, \Lambda_0) &\le E\big\{[e^{-\rho\tau_2}(X_{\tau_2}^1 - X_{\tau_2}^2) I_{\{\tau_2 < \infty\}} - e^{-\rho\tau_1}(X_{\tau_1}^1 - X_{\tau_1}^2) I_{\{\tau_1 < \infty\}}]\big\} \\ &= E[e^{-\rho\tau_2} X_{\tau_2}^1 I_{\{\tau_2 < \infty\}} - e^{-\rho\tau_1} X_{\tau_1}^1 I_{\{\tau_1 < \infty\}}] \\ &\quad - E[e^{-\rho\tau_2} X_{\tau_2}^2 I_{\{\tau_2 < \infty\}} - e^{-\rho\tau_1} X_{\tau_1}^2 I_{\{\tau_1 < \infty\}}]. \end{aligned}$$

Following from the proof of Lemma 3.1 of Tie et al. [14], we can show the first term above is less than or equal to 0. To find an upper bound for the second term, it suffices to show

$$E\int_{\tau_1}^{\tau_2} e^{-\rho t} X_t^2 (\rho - \mu_2(\alpha_t)) dt \le C x_2.$$

To this end, let $\mu_{\min} = \min\{\mu_2(1), \mu_2(2)\}$ and $\mu_{\max} = \max\{\mu_2(1), \mu_2(2)\}$. Then, we have

$$E\int_{\tau_1}^{\tau_2} e^{-\rho t} X_t^2 (\rho - \mu_2(\alpha_t)) dt \le (\rho - \mu_{\min}) \int_0^\infty e^{-\rho t} E X_t^2 dt.$$

Note that

$$E X_t^2 = x_2 + E\int_0^t X_s^2 \mu_2(\alpha_s) ds \le x_2 + \mu_{\max} \int_0^t E X_s^2 ds.$$

Use Gronwall's inequality to obtain $E X_t^2 \le x_2 e^{\mu_{\max} t}$. It follows that

$$\int_0^\infty e^{-\rho t} E X_t^2 dt = \frac{x_2}{\rho - \mu_{\max}}.$$

Therefore, we have

$$E\int_{\tau_1}^{\tau_2} e^{-\rho t} X_t^2 (\rho - \mu_2(\alpha_t)) dt \le \frac{(\rho - \mu_{\min}) x_2}{\rho - \mu_{\max}} =: C x_2.$$

Similarly, the inequalities in (2.4) can be obtained. □

3. HJB EQUATIONS

In this paper, we follow the dynamic programming approach and focus on the associated HJB equations. For $i = 1, 2$, let

$$\begin{aligned} \mathcal{A}_i = & \frac{1}{2}\left[a_{11}(i)x_1^2\frac{\partial^2}{\partial x_1^2} + 2a_{12}(i)x_1x_2\frac{\partial^2}{\partial x_1\partial x_2} + a_{22}(i)x_2^2\frac{\partial^2}{\partial x_2^2}\right] \\ & + \mu_1(i)x_1\frac{\partial}{\partial x_1} + \mu_2(i)x_2\frac{\partial}{\partial x_2} \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} a_{11}(i) &= \sigma_{11}^2(i) + \sigma_{12}^2(i), \\ a_{12}(i) &= \sigma_{11}(i)\sigma_{21}(i) + \sigma_{12}(i)\sigma_{22}(i), \text{ and} \\ a_{22}(i) &= \sigma_{21}^2(i) + \sigma_{22}^2(i). \end{aligned}$$

Formally, the associated HJB equations have the form:

$$\begin{cases} \min\{(\rho - \mathcal{A}_1)v_0(x_1,x_2,1) - \lambda_1(v_0(x_1,x_2,2) - v_0(x_1,x_2,1)), \\ \qquad\qquad v_0(x_1,x_2,1) - v_1(x_1,x_2,1) + \beta_b x_1 - \beta_s x_2\} = 0, \\ \min\{(\rho - \mathcal{A}_2)v_0(x_1,x_2,2) - \lambda_2(v_0(x_1,x_2,1) - v_0(x_1,x_2,2)), \\ \qquad\qquad v_0(x_1,x_2,2) - v_1(x_1,x_2,2) + \beta_b x_1 - \beta_s x_2\} = 0, \end{cases} \tag{3.2}$$

$$\begin{cases} \min\{(\rho - \mathcal{A}_1)v_1(x_1,x_2,1) - \lambda_1(v_1(x_1,x_2,2) \\ \qquad\qquad -v_1(x_1,x_2,1)),\ v_1(x_1,x_2,1) - \beta_{\mathrm{s}} x_1 + \beta_{\mathrm{b}} x_2\} = 0, \\ \min\{(\rho - \mathcal{A}_2)v_1(x_1,x_2,2) - \lambda_2(v_1(x_1,x_2,1) \\ \qquad\qquad -v_1(x_1,x_2,2)),\ v_1(x_1,x_2,2) - \beta_{\mathrm{s}} x_1 + \beta_{\mathrm{b}} x_2\} = 0. \end{cases} \tag{3.3}$$

For ease of notation, let $u_1 = v_0(x_1, x_2, 1)$, $u_2 = v_0(x_1, x_2, 2)$, $u_3 = v_1(x_1, x_2, 1)$, and $u_4 = v_1(x_1, x_2, 2)$.

To solve the above HJB equations, we first convert them into single variable equations. Let $y = x_2/x_1$ and $u_i(x_1, x_2) = x_1 w_i(x_2/x_1)$, for some function $w_i(y)$ and $i = 1, 2, 3, 4$. Then we have by direct calculation that

$$\begin{aligned} &\frac{\partial u_i}{\partial x_1} = w_i(y) - yw_i'(y),\ \frac{\partial u_i}{\partial x_2} = w_i'(y), \\ &\frac{\partial^2 u_i}{\partial x_1^2} = \frac{y^2 w_i''(y)}{x_1},\ \frac{\partial^2 u_i}{\partial x_2^2} = \frac{w_i''(y)}{x_1},\ \text{and}\ \frac{\partial^2 u_1}{\partial x_1 \partial x_2} = -\frac{yw_i''(y)}{x_1}. \end{aligned}$$

Write $\mathcal{A}_j u_i$ in terms of w_i to obtain

$$\mathcal{A}_j u_i = x_1\left\{\sigma_j y^2 w_i''(y) + [\mu_2(j) - \mu_1(j)]yw_i'(y) + \mu_1(j)w_i(y)\right\},$$

where $\sigma_j = (a_{11}(j) - 2a_{12}(j) + a_{22}(j))/2$.

Then, the HJB equations can be given in terms of y and w_i as follows:

$$\begin{aligned}
&\min\left\{(\rho+\lambda_1-\mathcal{L}_1)w_1(y)-\lambda_1 w_2(y),\ w_1(y)-w_3(y)+\beta_{\rm b}-\beta_{\rm s}y\right\}=0,\\
&\min\left\{(\rho+\lambda_2-\mathcal{L}_2)w_2(y)-\lambda_2 w_1(y),\ w_2(y)-w_4(y)+\beta_{\rm b}-\beta_{\rm s}y\right\}=0,\\
&\min\left\{(\rho+\lambda_1-\mathcal{L}_1)w_3(y)-\lambda_1 w_4(y),\ w_3(y)+\beta_{\rm b}y-\beta_{\rm s}\right\}=0,\\
&\min\left\{(\rho+\lambda_2-\mathcal{L}_2)w_4(y)-\lambda_2 w_3(y),\ w_4(y)+\beta_{\rm b}y-\beta_{\rm s}\right\}=0,
\end{aligned}\tag{3.4}$$

where

$$\mathcal{L}_j[w_i(y)]=\sigma_j y^2 w_i''(y)+[\mu_2(j)-\mu_1(j)]yw_i'(y)+\mu_1(j)w_i(y).\tag{3.5}$$

In this paper, we only consider the case when $\sigma_j\neq 0$, $j=1,2$. If either $\sigma_1=0$ and/or $\sigma_2=0$, the problem reduces to a (partial) first order case and can be treated in a similar and simpler way. Next, we consider the joint equations $(\rho+\lambda_1-\mathcal{L}_1)w_1=\lambda_1 w_2$ and $(\rho+\lambda_2-\mathcal{L}_2)w_2=\lambda_2 w_1$. Combine them to obtain

$$(\rho+\lambda_1-\mathcal{L}_1)(\rho+\lambda_2-\mathcal{L}_2)w_2=\lambda_1\lambda_2 w_2 \quad\text{and}\quad (\rho+\lambda_2-\mathcal{L}_2)(\rho+\lambda_1-\mathcal{L}_1)w_1=\lambda_1\lambda_2 w_1.$$

Both w_1 and w_2 must satisfy

$$[(\rho+\lambda_1-\mathcal{L}_1)(\rho+\lambda_2-\mathcal{L}_2)-\lambda_1\lambda_2]w=0.$$

Note that the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are the Euler type and the solutions to the above equation are of the form $w_i=y^\delta$. Thus, δ must satisfy the equation

$$P(\delta):=[\rho+\lambda_1-A_1(\delta)][\rho+\lambda_2-A_2(\delta)]-\lambda_1\lambda_2=0,\tag{3.6}$$

where

$$A_j(\delta)=\sigma_j\delta(\delta-1)+[(\mu_2(j)-\mu_1(j)]\delta+\mu_1(j)=\sigma_j\delta^2-[\sigma_j+\mu_1(j)-\mu_2(j)]\delta+\mu_1(j).\tag{3.7}$$

Note that $\rho+\lambda_1-A_1(\zeta)=0$ and $\rho+\lambda_2-A_2(\nu)=0$ have roots, respectively,

$$\begin{aligned}
\zeta_1&=\frac{1}{2}+\frac{\mu_1(1)-\mu_2(1)}{2\sigma_1}+\sqrt{\left(\frac{1}{2}+\frac{\mu_1(1)-\mu_2(1)}{2\sigma_1}\right)^2+\frac{\rho+\lambda_1-\mu_1(1)}{\sigma_1}},\\
\zeta_2&=\frac{1}{2}+\frac{\mu_1(1)-\mu_2(1)}{2\sigma_1}-\sqrt{\left(\frac{1}{2}+\frac{\mu_1(1)-\mu_2(1)}{2\sigma_1}\right)^2+\frac{\rho+\lambda_1-\mu_1(1)}{\sigma_1}},
\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}
\nu_1&=\frac{1}{2}+\frac{\mu_1(2)-\mu_2(2)}{2\sigma_2}+\sqrt{\left(\frac{1}{2}+\frac{\mu_1(2)-\mu_2(2)}{2\sigma_2}\right)^2+\frac{\rho+\lambda_2-\mu_1(1)}{\sigma_2}},\\
\nu_2&=\frac{1}{2}+\frac{\mu_1(2)-\mu_2(2)}{2\sigma_2}-\sqrt{\left(\frac{1}{2}+\frac{\mu_1(2)-\mu_2(2)}{2\sigma_2}\right)^2+\frac{\rho+\lambda_2-\mu_1(1)}{\sigma_2}}.
\end{aligned}\tag{3.9}$$

Note also that $\zeta_1>1$ and $\nu_1>1$, and $\zeta_2<0$ and $\nu_2<0$.

It is elementary to show that the equation $P(\delta)=0$ has four distinct roots δ_j, $1\le j\le 4$ with $\delta_4<\delta_3<0<1<\delta_2<\delta_1$. The δ_j, ζ_j and ν_j should have relation

$$\delta_4<\min\{\zeta_2,\nu_2\},\quad 0>\delta_3>\max\{\zeta_2,\nu_2\},\ 0<\delta_2<\min\{\zeta_1,\nu_1\},\quad\text{and}\quad \delta_1>\max\{\zeta_1,\nu_1\}.$$

The general solutions of the equations

$$(\rho + \lambda_1 - \mathcal{L}_1)w_1 = \lambda_1 w_2 \quad \text{and} \quad (\rho + \lambda_2 - \mathcal{L}_2)w_2 = \lambda_2 w_1$$

can be given as

$$w_1 = \sum_{j=1}^{4} c_{1j} y^{\delta_j} \quad \text{and} \quad w_2 = \sum_{j=1}^{4} c_{2j} y^{\delta_j},$$

for constants c_{ij}. Substituting them into the original equations leads to

$$\sum_{j=1}^{4} c_{1j}(\rho + \lambda_1 - A_1(\delta_j))y^{\delta_j} = \lambda_1 \sum_{j=1}^{4} c_{2j} y^{\delta_j}$$

and

$$\sum_{j=1}^{4} c_{2j}(\rho + \lambda_2 - A_2(\delta_j))y^{\delta_j} = \lambda_2 \sum_{j=1}^{4} c_{1j} y^{\delta_j}.$$

Hence, we have

$$c_{1,j}(\rho + \lambda_1 - A_1(\delta_j)) = \lambda_1 c_{2j} \quad \text{and} \quad c_{2j}(\rho + \lambda_2 - A_2(\delta_j)) = \lambda_2 c_{1j}.$$

Let $\eta_j = (\rho + \lambda_1 - A_1(\delta_j))/\lambda_1$. Then, we have

$$\eta_j = \frac{\rho + \lambda_1 - A_1(\delta_j)}{\lambda_1} = \frac{\lambda_2}{\rho + \lambda_2 - A_2(\delta_j)} \tag{3.10}$$

Necessarily, $c_{2j} = \eta_j c_{1j}$. Hence,

$$w_1 = \sum_{j=1}^{4} c_{1j} y^{\delta_j} \quad \text{and} \quad w_2 = \sum_{j=1}^{4} \eta_j c_{1j} y^{\delta_j}. \tag{3.11}$$

Similarly we can show the general solutions of $(\rho+\lambda_1-\mathcal{L}_1)w_3 = \lambda_1 w_4$ and $(\rho+\lambda_2 - \mathcal{L}_2)w_4 = \lambda_2 w_3$ are given by

$$w_3 = \sum_{j=1}^{4} d_{1j} y^{\delta_j} \quad \text{and} \quad w_4 = \sum_{j=1}^{4} \eta_j d_{1j} y^{\delta_j}, \tag{3.12}$$

for constants d_{ij}.

By direct computation, we can show

$$\{y > 0 : \ w_1 - w_3 + \beta_b - \beta_s y = 0\} \cap \{y > 0 : \ w_3 + \beta_b y - \beta_s = 0\} = \emptyset,$$

$$\{y > 0 : \ w_2 - w_4 + \beta_b - \beta_s y = 0\} \cap \{y > 0 : \ w_4 + \beta_b y - \beta_s = 0\} = \emptyset.$$

In view of Remark 2.1, if $\alpha = 1$, we divide the first quadrant into three regions $\{(x_1, x_2) > 0 : \ x_2 \geq k_1 x_1\}$ (open position region), $\{(x_1, x_2) > 0 : \ k_3 x_1 < x_2 < k_1 x_1\}$, (hold region) and $\{(x_1, x_2) > 0 : \ x_2 \leq k_3 x_1\}$ (close position region), for some positive constants k_1 and k_3. If $\alpha = 2$, we can do so similarly with regions $\{(x_1, x_2) > 0 : \ x_2 \geq k_2 x_1\}$ (open position region), $\{(x_1, x_2) > 0 : \ k_4 x_1 < x_2 < k_2 x_1\}$ (hold region), and $\{(x_1, x_2) > 0 : \ x_2 \leq k_4 x_1\}$ (close position region), for some positive k_2 and k_4. These regions are illustrated in Figure 1. A main objective is to determine these key thresholds (k_1, k_2, k_3, and k_4).

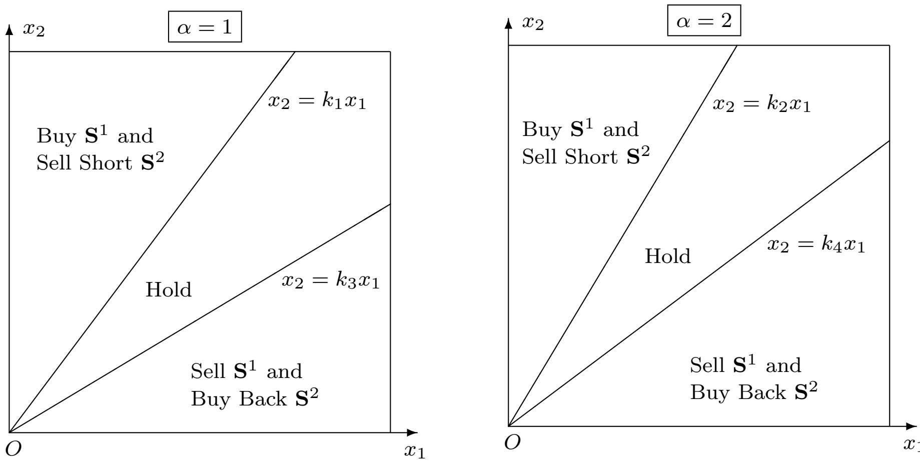

FIGURE 1. Switching Regions $\alpha = 1$ (left) and $\alpha = 2$ (right)

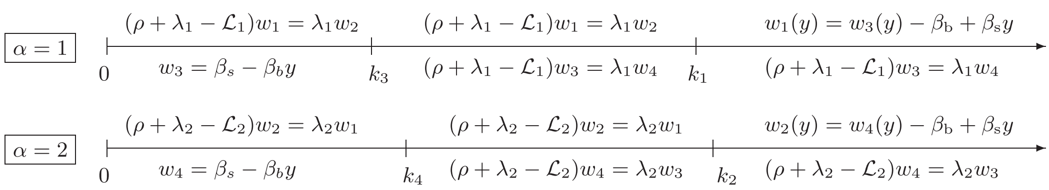

FIGURE 2. Equalities of HJB equations

Remark 3.1. In this paper, Theorems 1 and 2 (to follow) provide formulas for the computation of these key levels. In particular, one can start with (7.6) and (7.7) for k_3 and k_4. Then, solve the equations (3.17) in Case I ($k_3 < k_1 < k_4 < k_2$); (3.21) in Case II ($k_3 < k_4 < k_1 < k_2$); and (3.24) in Case III ($k_3 < k_4 < k_2 < k_1$) for k_1 and k_2.

Note here $k_3 < k_1$ and $k_4 < k_2$. As a result, recall the change of variables ($y = x_2/x_1$), the equations in (3.4) can be specified as follows:

$$\begin{cases} w_3 = \beta_s - \beta_b y \quad \text{and} \quad (\rho + \lambda_1 - \mathcal{L}_1)w_1 = \lambda_1 w_2 & \text{when } y < k_3, \\ (\rho + \lambda_1 - \mathcal{L}_1)w_1 = \lambda_1 w_2 \quad \text{and} \quad (\rho + \lambda_1 - \mathcal{L}_1)w_3 = \lambda_1 w_4 & \text{when } k_3 < y < k_1, \\ w_1 = w_3 + \beta_s y - \beta_b \quad \text{and} \quad (\rho + \lambda_1 - \mathcal{L}_1)w_3 = \lambda_1 w_4 & \text{when } y > k_1, \end{cases}$$

$$\begin{cases} w_4 = \beta_s - \beta_b y \quad \text{and} \quad (\rho + \lambda_2 - \mathcal{L}_2)w_2 = \lambda_2 w_1 & \text{when } y < k_4, \\ (\rho + \lambda_2 - \mathcal{L}_2)w_2 = \lambda_2 w_1 \quad \text{and} \quad (\rho + \lambda_2 - \mathcal{L}_2)w_4 = \lambda_2 w_3 & \text{when } k_4 < y < k_2, \\ w_2 = w_4 + \beta_s y - \beta_b \quad \text{and} \quad (\rho + \lambda_2 - \mathcal{L}_2)w_4 = \lambda_2 w_3 & \text{when } y > k_2. \end{cases}$$

Each of these intervals and the corresponding equalities are given in Figure 2.

We have four threshold parameters k_1, k_2, k_3 and k_4 to be determined. There are a number of ways to order them. Recall that $k_3 < k_1$ and $k_4 < k_2$. The largest is either k_1 or k_2 and the smallest is either k_3 or k_4. If k_3 is the smallest, then we can place k_1 at three different places. So this will lead to the following three cases.

$$k_3 \leq k_1 \leq k_4 \leq k_2, \quad k_3 \leq k_4 \leq k_1 \leq k_2, \quad k_3 \leq k_4 \leq k_2 \leq k_1.$$

Similarly if k_4 is the smallest, then we can place k_2 at three different places. Hence the next three possibilities:

$$k_4 \leq k_2 \leq k_3 \leq k_1, \quad k_4 \leq k_3 \leq k_2 \leq k_1, \quad k_4 \leq k_3 \leq k_1 \leq k_2.$$

In this paper, we only consider these cases with $k_3 < k_4$. The rest cases can be treated in a similar way.

On the region $(0, k_1 \wedge k_2]$ with $k_1 \wedge k_2 = \min\{k_1, k_2\}$, we have

$$(\rho + \lambda_1 - \mathcal{L}_1)w_1 = \lambda_1 w_2 \quad \text{and} \quad (\rho + \lambda_2 - \mathcal{L}_2)w_2 = \lambda_2 w_1,$$

this implies

$$w_1 = \sum_{j=1}^{4} c_{1j} y^{\delta_j} \quad \text{and} \quad w_2 = \sum_{j=1}^{4} \eta_j c_{1j} y^{\delta_j}.$$

in this region. Recall Lemma 2.3 and $\delta_3 < 0$, $\delta_4 < 0$. It follows that the coefficients for y^{δ_3} and y^{δ_4} have to be zero. Thus, we have

$$w_1 = C_1 y^{\delta_1} + C_2 y^{\delta_2} \quad \text{and} \quad w_2 = C_1 \eta_1 y^{\delta_1} + C_2 \eta_2 y^{\delta_2}.$$

Similarly, in the region $[k_3 \vee k_4, \infty)$ with $k_3 \vee k_4 = \max\{k_3, k_4\}$,

$$(\rho + \lambda_1 - \mathcal{L}_1)w_3 = \lambda_1 w_4 \quad \text{and} \quad (\rho + \lambda_2 - \mathcal{L}_2)w_4 = \lambda_2 w_3,$$

the linear growth conditions (recall $\delta_1, \delta_2 > 1$) yield

$$w_3 = C_3 y^{\delta_3} + C_4 y^{\delta_4} \quad \text{and} \quad w_4 = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4}.$$

To solve the HJB equations, we first note that w_3 and w_4 are not coupled with w_1 and w_2 and can be found separately. This is treated as a pure selling problem in Tie and Zhang [13]. In this paper, we only consider the case ($k_3 < k_4$) and provide key steps for this case in Appendix for the sake of completeness.

Solving for w_1 and w_2. In this section, we solve for w_1 and w_2 using the solution w_3 and w_4. Recall that w_1 and w_2 satisfy the HJB equations

$$\begin{aligned} &\min\Big\{(\rho + \lambda_1 - \mathcal{L}_1)w_1(y) - \lambda_1 w_2(y),\ w_1(y) - w_3(y) + \beta_{\mathrm{b}} - \beta_{\mathrm{s}} y\Big\} = 0, \\ &\min\Big\{(\rho + \lambda_2 - \mathcal{L}_2)w_2(y) - \lambda_2 w_1(y),\ w_2(y) - w_4(y) + \beta_{\mathrm{b}} - \beta_{\mathrm{s}} y\Big\} = 0. \end{aligned} \tag{3.13}$$

To find threshold type solutions, we are to determine k_1 and k_2 so that on $(0, k_1)$: $(\rho + \lambda_1 - \mathcal{L}_1)w_1(y) - \lambda_1 w_2(y) = 0$ and $w_1(y) - w_3(y) + \beta_{\mathrm{b}} - \beta_{\mathrm{s}} y \geq 0$; on $[k_1, \infty)$: $(\rho + \lambda_1 - \mathcal{L}_1)w_1(y) - \lambda_1 w_2(y) \geq 0$ and $w_1(y) - w_3(y) + \beta_{\mathrm{b}} - \beta_{\mathrm{s}} y = 0$; on $(0, k_2)$: $(\rho + \lambda_2 - \mathcal{L}_2)w_2(y) - \lambda_2 w_1(y) = 0$ and $w_2(y) - w_4(y) + \beta_{\mathrm{b}} - \beta_{\mathrm{s}} y \geq 0$; and on $[k_2, \infty)$: $(\rho + \lambda_2 - \mathcal{L}_2)w_2(y) - \lambda_2 w_1(y) \geq 0$ and $w_2(y) - w_4(y) + \beta_{\mathrm{b}} - \beta_{\mathrm{s}} y = 0$.

Recall that $k_4 < k_2$ and $k_3 < k_1$. Recall also the condition $k_3 < k_4$. We need consider the three cases $k_3 \leq k_1 \leq k_4 \leq k_2$, $k_3 \leq k_4 \leq k_1 \leq k_2$, $k_3 \leq k_4 \leq k_2 \leq k_1$. To focus on key ideas, we only treat each of these cases with strict inequalities. Cases with equalities can be dealt with in a similar way.

Case I: $k_3 < k_1 < k_4 < k_2$**.** First, we consider the case when $k_3 < k_1 < k_4 < k_2$. For $0 < y < k_1$, we have $(\rho + \lambda_1 - \mathcal{L}_1)w_1(y) - \lambda_1 w_2(y) = 0$ and $(\rho + \lambda_2 - \mathcal{L}_2)w_2(y) - \lambda_2 w_1(y) = 0$. Their general solutions have the form:

$$w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2} \quad \text{and} \quad w_2(y) = C_1\eta_1 y^{\delta_1} + C_2\eta_2 y^{\delta_2}.$$

For $k_1 \le y \le k_2$, we have $w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y$ and $(\rho + \lambda_2 - \mathcal{L}_2)w_2(y) - \lambda_2 w_1(y) = 0$. For $k_2 \le y < \infty$, we have $w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y$ and $w_2(y) = w_4(y) - \beta_{\rm b} + \beta_{\rm s} y$. Recall that the solution $w_3(y)$ and $w_4(y)$ in (7.9) (Appendix). This leads to, on $[k_1, k_4]$, $w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y = E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - \beta_{\rm b} + (\beta_{\rm s} - a_2)y$ and $w_2(y)$ satisfies

$$(\rho + \lambda_2 - \mathcal{L}_2)w_2(y) = \lambda_2 w_1(y) = \lambda_2[E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - \beta_{\rm b} + (\beta_{\rm s} - a_2)y].$$

Then the solution $w_2(y) = B_1 y^{\nu_1} + B_2 y^{\nu_2} + w_{2,p_1}(y)$, where $B_1 y^{\nu_1} + B_2 y^{\nu_2}$ is the general solution of the homogeneous differential equation $(\rho + \lambda_2 - \mathcal{L}_2)w_2(y) = 0$ with ν_1 and ν_2 given in (3.9). A particular solution of

$$(\rho + \lambda_2 - \mathcal{L}_2)w_2(y) = \lambda_2 w_1(y) = \lambda_2[E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - \beta_{\rm b} + (\beta_{\rm s} - a_2)y]$$

can be given by

$$\begin{aligned} w_{2,p_1}(y) = {} & \frac{\lambda_2 E_1}{\rho + \lambda_2 - A_2(\zeta_1)} y^{\zeta_1} + \frac{\lambda_2 E_2}{\rho + \lambda_2 - A_2(\zeta_2)} y^{\zeta_2} \\ & + \frac{\lambda_2(a_1 - \beta_{\rm b})}{\rho + \lambda_2 - \mu_1(2)} + \frac{\lambda_2(\beta_{\rm s} - a_2)}{\rho + \lambda_2 - \mu_2(2)} y. \end{aligned} \tag{3.14}$$

Next, on the interval $[k_4, k_2]$, $w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y$ and $w_2(y)$ satisfies the inhomogenous equation $(\rho + \lambda_2 - \mathcal{L}_2)w_2(y) = \lambda_2 w_1(y) = \lambda_2(C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y)$. Similarly, a general solution $w_2(y) = D_1 y^{\nu_1} + D_2 y^{\nu_2} + w_{2,p_2}(y)$, where $w_{2,p_2}(y)$ is the particular solution given by

$$\begin{aligned} w_{2,p_2}(y) = {} & \frac{\lambda_2 C_3}{\rho + \lambda_2 - A_2(\delta_3)} y^{\delta_3} + \frac{\lambda_2 C_4}{\rho + \lambda_2 - A_2(\delta_4)} y^{\delta_4} \\ & - \frac{\lambda_2 \beta_{\rm b}}{\rho + \lambda_2 - \mu_1(2)} + \frac{\lambda_2 \beta_{\rm s} y}{\rho + \lambda_2 - \mu_2(2)}. \end{aligned} \tag{3.15}$$

Recall that $\eta_3 = \lambda_2/(\rho + \lambda_2 - A_2(\delta_3))$ and $\eta_4 = \lambda_2//(\rho + \lambda_2 - A_2(\delta_4))$. It follows that

$$w_{2,p_2}(y) = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \frac{\lambda_2 \beta_{\rm b}}{\rho + \lambda_2 - \mu_1(2)} + \frac{\lambda_2 \beta_{\rm s} y}{\rho + \lambda_2 - \mu_2(2)}.$$

Finally, on the interval $[k_2, \infty)$, $w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y$ and $w_2(y) = w_4(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y$. These computations can be summarized as follows:

(3.16)

$$\begin{array}{ll}
\text{On } (0,k_1): & w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2},\\
& w_2(y) = C_1\eta_1 y^{\delta_1} + C_2\eta_2 y^{\delta_2},\\
\text{On } [k_1,k_4): & w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y = E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - \beta_{\rm b} + (\beta_{\rm s} - a_2)y,\\
& w_2(y) = B_1 y^{\nu_1} + B_2 y^{\nu_2} + w_{2,p_1}(y),\\
\text{On } [k_4,k_2]: & w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y,\\
& w_2(y) = D_1 y^{\nu_1} + D_2 y^{\nu_2} + w_{2,p_2}(y),\\
\text{On } (k_2,\infty): & w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y,\\
& w_2(y) = w_4(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y,
\end{array}$$

where

$$\begin{aligned}
w_{2,p_1}(y) &= \frac{\lambda_2 E_1}{\rho+\lambda_2 - A_2(\zeta_1)} y^{\zeta_1} + \frac{\lambda_2 E_2}{\rho+\lambda_2 - A_2(\zeta_2)} y^{\zeta_2}\\
&\quad + \frac{\lambda_2(a_1-\beta_{\rm b})}{\rho+\lambda_2-\mu_1(2)} + \frac{\lambda_2(\beta_{\rm s}-a_2)}{\rho+\lambda_2-\mu_2(2)} y,\\
w_{2,p_2}(y) &= C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \frac{\lambda_2\beta_{\rm b}}{\rho+\lambda_2-\mu_1(2)} + \frac{\lambda_2\beta_{\rm s} y}{\rho+\lambda_2-\mu_2(2)}.
\end{aligned}$$

We follow the smooth-fit method to determine parameters C_1, C_2, B_1, B_2, D_1, D_2, k_1 and k_2. The continuity of $w_1(y)$, $w_2(y)$, $w_1'(y)$ and $w_2'(y)$ at k_1 yields

$$\begin{aligned}
C_1 k_1^{\delta_1} + C_2 k_1^{\delta_2} &= w_3(k_1) + \beta_{\rm s} k_1 - \beta_{\rm b},\\
C_1\delta_1 k_1^{\delta_1-1} + C_2\delta_2 k_1^{\delta_2-1} &= w_3'(k_1) + \beta_{\rm s},\\
C_1\eta_1 k_1^{\delta_1} + C_2\eta_2 k_1^{\delta_2} &= B_1 k_1^{\nu_1} + B_2 k_1^{\nu_2} + w_{2,p_1}(k_1),\\
C_1\eta_1\delta_1 k_1^{\delta_1-1} + C_2\eta_2\delta_2 k_1^{\delta_2-1} &= B_1\nu_1 k_1^{\nu_1-1} + B_2\nu_2 k_1^{\nu_2-1} + w_{2,p_1}'(k_1).
\end{aligned}$$

The continuity of $w_2(y)$ and $w_2'(y)$ at k_4 yields

$$\begin{aligned}
B_1 k_4^{\nu_1} + B_2 k_4^{\nu_2} + w_{2,p_1}(k_4) &= D_1 k_4^{\nu_1} + D_2 k_4^{\nu_2} + w_{2,p_2}(k_4),\\
B_1\nu_1 k_4^{\nu_1-1} + B_2\nu_2 k_4^{\nu_2-1} + w_{2,p_1}'(k_4) &= D_1\nu_1 k_4^{\nu_1-1} + D_2\nu_2 k_4^{\nu_2-1} + w_{2,p_2}'(k_4).
\end{aligned}$$

The continuity of $w_2(y)$ and $w_2'(y)$ at k_2 yields

$$\begin{aligned}
D_1 k_2^{\nu_1} + D_2 k_2^{\nu_2} + w_{2,p_2}(k_2) &= w_4(k_2) - \beta_{\rm b} + \beta_{\rm s} k_2,\\
D_1\nu_1 k_2^{\nu_1-1} + D_2\nu_2 k_2^{\nu_2-1} + w_{2,p_2}'(k_2) &= w_4'(k_2) + \beta_{\rm s}.
\end{aligned}$$

Let

$$\Lambda = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \text{ and } \Phi(t,s_1,s_2) = \begin{pmatrix} t^{s_1} & t^{s_2} \\ s_1 t^{s_1} & s_2 t^{s_2} \end{pmatrix}.$$

Then, we have

$$\Phi^{-1}(t,s_1,s_2) = \frac{1}{s_2-s_1}\begin{pmatrix} s_2 t^{-s_1} & -t^{-s_1} \\ -s_1 t^{-s_2} & t^{-s_2}. \end{pmatrix}.$$

Using these matrices, we can write the first four equations at k_1 as

$$\Phi(k_1,\delta_1,\delta_2)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} w_3(k_1)+\beta_{\mathrm{s}}k_1-\beta_{\mathrm{b}} \\ k_1[w_3'(k_1)+\beta_{\mathrm{s}}] \end{pmatrix},$$
$$\Phi(k_1,\delta_1,\delta_2)\Lambda\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \Phi(k_1,\nu_1,\nu_2)\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} + \begin{pmatrix} w_{2,p_1}(k_1) \\ k_1 w_{2,p_1}'(k_1) \end{pmatrix}.$$

It follows, by solving for C_1, C_2, B_1 and B_2, that

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \Phi^{-1}(k_1,\delta_1,\delta_2)\begin{pmatrix} w_3(k_1)+\beta_{\mathrm{s}}k_1-\beta_{\mathrm{b}} \\ k_1[w_3'(k_1)+\beta_{\mathrm{s}}] \end{pmatrix},$$
$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \Phi^{-1}(k_1,\nu_1,\nu_2)\Bigg[\Phi(k_1,\delta_1,\delta_2)\Lambda\Phi^{-1}(k_1,\delta_1,\delta_2) \begin{pmatrix} w_3(k_1)+\beta_{\mathrm{s}}k_1-\beta_{\mathrm{b}} \\ k_1[w_3'(k_1)+\beta_{\mathrm{s}}] \end{pmatrix} - \begin{pmatrix} w_{2,p_1}(k_1) \\ k_1 w_{2,p_1}'(k_1) \end{pmatrix}\Bigg].$$

In addition, simple calculation yields

$$\Phi(k_1,\delta_1,\delta_2)\Lambda\Phi^{-1}(k_1,\delta_1,\delta_2) = \frac{1}{\delta_2-\delta_1}\begin{pmatrix} \eta_1\delta_2-\eta_2\delta_1 & \eta_2-\eta_1 \\ \delta_1\delta_2(\eta_1-\eta_2) & \eta_2\delta_2-\eta_1\delta_1 \end{pmatrix}.$$

Note that this matrix is independent of k_1. Moreover, we can write (from the continuity of w_2 and w_2' at k_4)

$$\Phi(k_4,\nu_1,\nu_2)\begin{pmatrix} B_1-D_1 \\ B_2-D_2 \end{pmatrix} = \begin{pmatrix} w_{2,p_2}(k_4)-w_{2,p_1}(k_4) \\ k_4[w_{2,p_2}'(k_4)-w_{2,p_1}'(k_4)] \end{pmatrix}.$$

This yields

$$\begin{pmatrix} B_1-D_1 \\ B_2-D_2 \end{pmatrix} = \Phi^{-1}(k_4,\nu_1,\nu_2)\begin{pmatrix} w_{2,p_2}(k_4)-w_{2,p_1}(k_4) \\ k_4[w_{2,p_2}'(k_4)-w_{2,p_1}'(k_4)] \end{pmatrix}.$$

Finally, follow from the continuity of w_2 and w_2' at k_2, we write

$$\Phi(k_2,\nu_1,\nu_2)\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} w_4(k_2)-w_{2,p_2}(k_2)-\beta_{\mathrm{b}}+\beta_{\mathrm{s}}k_2 \\ k_2[w_4'(k_2)-w_{2,p_2}'(k_2)+\beta_{\mathrm{s}}] \end{pmatrix}.$$

This gives

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \Phi^{-1}(k_2,\nu_1,\nu_2)\begin{pmatrix} w_4(k_2)-w_{2,p_2}(k_2)-\beta_{\mathrm{b}}+\beta_{\mathrm{s}}k_2 \\ k_2[w_4'(k_2)-w_{2,p_2}'(k_2)+\beta_{\mathrm{s}}] \end{pmatrix}.$$

Combine this with the previous formula to obtain the second expression or B_1 and B_2:

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \Phi^{-1}(k_4,\nu_1,\nu_2)\begin{pmatrix} w_{2,p_2}(k_4)-w_{2,p_1}(k_4) \\ k_4[w_{2,p_2}'(k_4)-w_{2,p_1}'(k_4)] \end{pmatrix} + \Phi^{-1}(k_2,\nu_1,\nu_2)\begin{pmatrix} w_4(k_2)-w_{2,p_2}(k_2)-\beta_{\mathrm{b}}+\beta_{\mathrm{s}}k_2 \\ k_2[w_4'(k_2)-w_{2,p_2}'(k_2)+\beta_{\mathrm{s}}] \end{pmatrix}.$$

Eliminate (B_1, B_2) to obtain the following equations for k_1 and k_2:
(3.17)

$$\Phi^{-1}(k_1,\nu_1,\nu_2)\left[\Phi(k_1,\delta_1,\delta_2)\Lambda\Phi^{-1}(k_1,\delta_1,\delta_2)\begin{pmatrix} w_3(k_1)+\beta_s k_1-\beta_b \\ k_1 w_3'(k_1)+\beta_s k_1\end{pmatrix} - \begin{pmatrix} w_{2,p_1}(k_1) \\ k_1 w_{2,p_1}'(k_1)\end{pmatrix}\right]$$
$$= \Phi^{-1}(k_4,\nu_1,\nu_2)\begin{pmatrix} w_{2,p_2}(k_4)-w_{2,p_1}(k_4) \\ k_4[w_{2,p_2}'(k_4)-w_{2,p_1}'(k_4)]\end{pmatrix} + \Phi^{-1}(k_2,\nu_1,\nu_2)\begin{pmatrix} w_4(k_2)-w_{2,p_2}(k_2)-\beta_b+\beta_s k_2 \\ k_2[w_4'(k_2)-w_{2,p_2}'(k_2)+\beta_s]\end{pmatrix}.$$

Formula (3.17) yields two equations of k_1 and k_2. The existence of k_1 and k_2 can proved by following the method in Lemma 4.2 of [12]. Once we find k_1 and k_2 and note that the constants B_1, B_2, C_1, C_2, D_1, and D_2 can be written as functions of k_1 and k_2. So are functions $w_1(y)$ and $w_2(y)$. In view of this, k_1 and k_2 have to be determined so that the following variational inequalities are satisfied:

$$\begin{array}{ll} \text{On } (0,k_1): & w_1(y)-w_3(y)+\beta_b-\beta_s y\ge 0,\\ & w_2(y)-w_4(y)+\beta_b-\beta_s y\ge 0,\\ \text{On } [k_1,k_2]: & (\rho+\lambda_1-\mathcal{L}_1)w_1(y)-\lambda_1 w_2(y)\ge 0,\\ & w_2(y)-w_4(y)+\beta_b-\beta_s y\ge 0,\\ \text{On } (k_2,\infty): & (\rho+\lambda_1-\mathcal{L}_1)w_1(y)-\lambda_1 w_2(y)\ge 0,\\ & (\rho+\lambda_2-\mathcal{L}_2)w_2(y)-\lambda_2 w_1(y)\ge 0.\end{array} \tag{3.18}$$

To facilitate numerical computations, we provide equivalent inequalities for those involving the differential operators $\mathcal{L}_j$. First, we consider the two inequalities on the interval $[k_2,\infty)$:

$$(\rho+\lambda_1-\mathcal{L}_1)w_1(y)-\lambda_1 w_2(y)\ge 0 \quad\text{and}\quad (\rho+\lambda_2-\mathcal{L}_2)w_2(y)-\lambda_2 w_1(y)\ge 0. \tag{3.19}$$

Recall that $w_1(y)=w_3(y)-\beta_b+\beta_s y$ and $w_2(y)=w_4(y)-\beta_b+\beta_s y$, and we apply $\mathcal{L}_1$ to $w_1(y)$ and $\mathcal{L}_2$ to $w_2(y)$ to get

$$(\rho+\lambda_1-\mathcal{L}_1)w_1(y)=\lambda_1 w_4(y)+(\rho+\lambda_1-\mu_2(1))\beta_s y-(\rho+\lambda_1-\mu_1(1))\beta_b,$$
$$(\rho+\lambda_2-\mathcal{L}_2)w_2(y)=\lambda_2 w_3(y)+(\rho+\lambda_2-\mu_2(2))\beta_s y-(\rho+\lambda_2-\mu_1(2))\beta_b.$$

Then (3.19) is equivalent to

$$(\rho+\lambda_1-\mu_2(1))\beta_s y-(\rho+\lambda_1-\mu_1(1))\beta_b\ge\lambda_1(\beta_s y-\beta_b),$$
$$(\rho+\lambda_2-\mu_2(2))\beta_s y-(\rho+\lambda_2-\mu_1(2))\beta_b\ge\lambda_2(\beta_s y-\beta_b).$$

Simplify to obtain

$$(\rho-\mu_2(1))\beta_s y-(\rho-\mu_1(1))\beta_b\ge 0 \quad\text{and}\quad (\rho-\mu_2(2))\beta_s y-(\rho-\mu_1(2))\beta_b\ge 0.$$

These inequalities hold as long as

$$k_2\ge\frac{(\rho-\mu_1(j))\beta_b}{(\rho-\mu_2(j))\beta_s}\quad\text{for } j=1,2.$$

Next, we consider the inequality involving $\mathcal{L}_1$, i.e, $(\rho+\lambda_1-\mathcal{L}_1)w_1(y)-\lambda_1 w_2(y)\ge 0$ on $[k_1,k_2]$. Recall that $w_1=w_3-\beta_b+\beta_s y$ and w_2 satisfies $(\rho+\lambda_2-\mathcal{L}_2)w_2(y)=$

$\lambda_2 w_1(y)$ on this interval. Applying $(\rho+\lambda_1-\mathcal{L}_1)$ to w_1 yield

$$(\rho+\lambda_1-\mathcal{L}_1)w_1=(\rho+\lambda_1-\mathcal{L}_1)w_3+(\rho+\lambda_1-\mu_2(1))\beta_{\rm s}y-(\rho+\lambda_1-\mu_1(1))\beta_{\rm b}.$$

Recall that $k_3<k_1<k_4<k_2$ and $(\rho+\lambda_1-\mathcal{L}_1)w_3=\lambda_1 w_4$. It follows that

$$(\rho+\lambda_1-\mathcal{L}_1)w_1=\lambda_1 w_4+(\rho+\lambda_1-\mu_2(1))\beta_{\rm s}y-(\rho+\lambda_1-\mu_1(1))\beta_{\rm b}.$$

Recall also that $w_4=\beta_{\rm s}-\beta_{\rm b}y$ on the interval $[0,k_4]$. Hence on interval $[k_1,k_4]\subset[0,k_4]$, $(\rho+\lambda_1-\mathcal{L}_1)w_1(y)-\lambda_1 w_2(y)\geq 0$ is equivalent to

$$\lambda_1(\beta_{\rm s}-\beta_{\rm b}y)+(\rho+\lambda_1-\mu_2(1))\beta_{\rm s}y-(\rho+\lambda_1-\mu_1(1))\beta_{\rm b}\geq\lambda_1 w_2.$$

Since $w_2(y)=B_1y^{\nu_1}+B_2y^{\nu_2}+w_{2,p_1}(y)$ on the interval $[k_1,k_4]$, the above inequality is equivalent to

$$B_1y^{\nu_1}+B_2y^{\nu_2}+w_{2,p_1}(y)\leq\left[\frac{\rho-\mu_2(1)}{\lambda_1}\beta_{\rm s}+\beta_{\rm s}-\beta_{\rm b}\right]y-\left[\frac{\rho-\mu_1(1)}{\lambda_1}\beta_{\rm b}+\beta_{\rm b}-\beta_{\rm s}\right].$$

Similarly on the interval $[k_4,k_2]$, $w_2(y)=D_1y^{\nu_1}+D_2y^{\nu_2}+w_{2,p_2}(y)$, and the inequality is equivalent to

$$D_1y^{\nu_1}+D_2y^{\nu_2}+w_{2,p_2}(y)\leq\left[\frac{\rho-\mu_2(1)}{\lambda_1}\beta_{\rm s}+\beta_{\rm s}-\beta_{\rm b}\right]y-\left[\frac{\rho-\mu_1(1)}{\lambda_1}\beta_{\rm b}+\beta_{\rm b}-\beta_{\rm s}\right].$$

Case II: $k_3<k_4<k_1<k_2$. Next, we treat the case $(k_3<k_4<k_1<k_2)$. Note that, for $0<y<k_1$, we have $(\rho+\lambda_1-\mathcal{L}_1)w_1(y)-\lambda_1 w_2(y)=0$ and $(\rho+\lambda_2-\mathcal{L}_2)w_2(y)-\lambda_2 w_1(y)=0$. Their general solutions are of the forms

$$w_1(y)=C_1y^{\delta_1}+C_2y^{\delta_2}\quad\text{and}\quad w_2(y)=C_1\eta_1y^{\delta_1}+C_2\eta_2y^{\delta_2}.$$

For $k_1\leq y\leq k_2$, we have $w_1(y)=w_3(y)-\beta_{\rm b}+\beta_{\rm s}y$ and $(\rho+\lambda_2-\mathcal{L}_2)w_2(y)-\lambda_2 w_1(y)=0$. For $k_2<y<\infty$, we have $w_1(y)=w_3(y)-\beta_{\rm b}+\beta_{\rm s}y$ and $w_2(y)=w_4(y)-\beta_{\rm b}+\beta_{\rm s}y$. Recall also the solutions $w_3(y)$ and $w_4(y)$ in (7.9) (Appendix): It follows that, on the interval $[k_1,k_2]$, $w_1(y)=w_3(y)-\beta_{\rm b}+\beta_{\rm s}y=E_1y^{\zeta_1}+E_2y^{\zeta_2}+a_1-\beta_{\rm b}+(\beta_{\rm s}-a_2)y$; and $w_2(y)$ satisfies the equation $(\rho+\lambda_2-\mathcal{L}_2)w_2(y)=\lambda_2 w_1(y)=\lambda_2[E_1y^{\zeta_1}+E_2y^{\zeta_2}+a_1-\beta_{\rm b}+(\beta_{\rm s}-a_2)y]$. Then the general solution $w_2(y)=B_1y^{\nu_1}+B_2y^{\nu_2}+w_{2,p}(y)$ where the particular solution

$$\begin{aligned}w_{2,p}(y)=&\frac{\lambda_2E_1}{\rho+\lambda_2-A_2(\zeta_1)}y^{\zeta_1}+\frac{\lambda_2E_2}{\rho+\lambda_2-A_2(\zeta_2)}y^{\zeta_2}\\&+\frac{\lambda_2(a_1-\beta_{\rm b})}{\rho+\lambda_2-\mu_1(2)}+\frac{\lambda_2(\beta_{\rm s}-a_2)}{\rho+\lambda_2-\mu_2(2)}y.\end{aligned}$$

In this paper, the use of parameters A_i, B_i, C_i, etc is limited to the particular section. They may be different across sections if no confusion arises.

Finally, on the interval (k_2,∞), we have

$$\begin{aligned}w_1(y)&=w_3(y)-\beta_{\rm b}+\beta_{\rm s}y=C_3y^{\delta_3}+C_4y^{\delta_4}-\beta_{\rm b}+\beta_{\rm s}y,\\w_2(y)&=w_4(y)-\beta_{\rm b}+\beta_{\rm s}y=C_3\eta_3y^{\delta_3}+C_4\eta_4y^{\delta_4}-\beta_{\rm b}+\beta_{\rm s}y.\end{aligned}$$

Summarize the above computation to obtain

(3.20)
$$\begin{aligned}
\text{On } (0,k_1): \quad & w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2},\\
& w_2(y) = C_1\eta_1 y^{\delta_1} + C_2\eta_2 y^{\delta_2},\\
\text{On } [k_1,k_2]: \quad & w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y = E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - \beta_{\rm b} + (\beta_{\rm s} - a_2)y,\\
& w_2(y) = B_1 y^{\nu_1} + B_2 y^{\nu_2} + w_{2,p}(y),\\
\text{On } (k_2,\infty): \quad & w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y,\\
& w_2(y) = w_4(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y.
\end{aligned}$$

Next, we use the smooth-fit conditions to determine the parameters C_1, C_2, B_1, B_2, k_1 and k_2. First, the continuity of $w_1(y)$, $w_2(y)$, $w_1'(y)$ and $w_2'(y)$ at k_1 yields

$$\begin{aligned}
C_1 k_1^{\delta_1} + C_2 k_1^{\delta_2} &= w_3(k_1) + \beta_{\rm s} k_1 - \beta_{\rm b},\\
C_1\delta_1 k_1^{\delta_1-1} + C_2\delta_2 k_1^{\delta_2-1} &= w_3'(k_1) + \beta_{\rm s},\\
C_1\eta_1 k_1^{\delta_1} + C_2\eta_2 k_1^{\delta_2} &= B_1 k_1^{\nu_1} + B_2 k_1^{\nu_2} + w_{2,p}(k_1),\\
C_1\eta_1\delta_1 k_1^{\delta_1-1} + C_2\eta_2\delta_2 k_1^{\delta_2-1} &= B_1\nu_1 k_1^{\nu_1-1} + B_2\nu_2 k_1^{\nu_2-1} + w_{2,p}'(k_1).
\end{aligned}$$

Similarly, the continuity of $w_2(y)$ and $w_2'(y)$ at k_2 yields

$$\begin{aligned}
B_1 k_2^{\nu_1} + B_2 k_2^{\nu_2} + w_{2,p}(k_2) &= w_4(k_2) - \beta_{\rm b} + \beta_{\rm s} k_2,\\
B_1\nu_1 k_2^{\nu_1-1} + B_2\nu_2 k_2^{\nu_2-1} + w_{2,p}'(k_2) &= w_4'(k_2) + \beta_{\rm s}.
\end{aligned}$$

We can write them in matrix form:

$$\begin{aligned}
\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= \Phi^{-1}(k_1,\delta_1,\delta_2)\begin{pmatrix} w_3(k_1) + \beta_{\rm s} k_1 - \beta_{\rm b} \\ k_1[w_3'(k_1) + \beta_{\rm s}] \end{pmatrix},\\
\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} &= \Phi^{-1}(k_1,\nu_1,\nu_2)\Bigg[\Phi(k_1,\delta_1,\delta_2)\Lambda\Phi^{-1}(k_1,\delta_1,\delta_2)\\
&\qquad \begin{pmatrix} w_3(k_1) + \beta_{\rm s} k_1 - \beta_{\rm b} \\ k_1[w_3'(k_1) + \beta_{\rm s}] \end{pmatrix} - \begin{pmatrix} w_{2,p}(k_1) \\ k_1 w_{2,p}'(k_1) \end{pmatrix}\Bigg].
\end{aligned}$$

The continuity of w_2 and w_2' at k_2 leads to the equations

$$\Phi(k_2,\nu_1,\nu_2)\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} w_4(k_2) - w_{2,p}(k_2) - \beta_{\rm b} + \beta_{\rm s} k_2 \\ k_2[w_4'(k_2) - w_{2,p}'(k_2) + \beta_{\rm s}] \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \Phi^{-1}(k_2,\nu_1,\nu_2)\begin{pmatrix} w_4(k_2) - w_{2,p}(k_2) - \beta_{\rm b} + \beta_{\rm s} k_2 \\ k_2[w_4'(k_2) - w_{2,p}'(k_2) + \beta_{\rm s}] \end{pmatrix}.$$

Eliminate B_1 and B_2 to obtain the equations for k_1 and k_2:

(3.21)
$$\begin{aligned}
&\Phi^{-1}(k_1,\nu_1,\nu_2)\Bigg[\Phi(k_1,\delta_1,\delta_2)\Lambda\Phi^{-1}(k_1,\delta_1,\delta_2)\begin{pmatrix} w_3(k_1) + \beta_{\rm s} k_1 - \beta_{\rm b} \\ k_1[w_3'(k_1) + \beta_{\rm s}] \end{pmatrix}\\
&\qquad - \begin{pmatrix} w_{2,p}(k_1) \\ k_1 w_{2,p}'(k_1) \end{pmatrix}\Bigg]\\
= \; &\Phi^{-1}(k_2,\nu_1,\nu_2)\begin{pmatrix} w_4(k_2) - w_{2,p}(k_2) - \beta_{\rm b} + \beta_{\rm s} k_2 \\ k_2[w_4'(k_2) - w_{2,p}'(k_2) + \beta_{\rm s}] \end{pmatrix}.
\end{aligned}$$

Recall that the constants B_1, B_2, C_1, and C_2 can be represented as functions of k_1 and k_2. So are functions $w_1(y)$ and $w_2(y)$. Therefore, k_1 and k_2 need to be determined so that the following variational inequalities are satisfied:

$$\begin{array}{ll} \text{On } (0,k_1): & w_1(y) - w_3(y) + \beta_{\mathrm{b}} - \beta_{\mathrm{s}} y \geq 0, \\ & w_2(y) - w_4(y) + \beta_{\mathrm{b}} - \beta_{\mathrm{s}} y \geq 0, \\ \text{On } [k_1,k_2]: & (\rho + \lambda_1 - \mathcal{L}_1) w_1(y) - \lambda_1 w_2(y) \geq 0, \\ & w_2(y) - w_4(y) + \beta_{\mathrm{b}} - \beta_{\mathrm{s}} y \geq 0, \\ \text{On } (k_2,\infty): & (\rho + \lambda_1 - \mathcal{L}_1) w_1(y) - \lambda_1 w_2(y) \geq 0, \\ & (\rho + \lambda_2 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y) \geq 0. \end{array} \tag{3.22}$$

Next, we consider equivalent inequalities for those involving the differential operators $\mathcal{L}_j$. First, on the interval $[k_2,\infty)$, the variational inequalities are equivalent to

$$\begin{aligned} &(\rho + \lambda_1 - \mu_2(1))\beta_{\mathrm{s}} y - (\rho + \lambda_1 - \mu_1(1))\beta_{\mathrm{b}} \geq \lambda_1(\beta_{\mathrm{s}} y - \beta_{\mathrm{b}}), \\ &(\rho + \lambda_2 - \mu_2(2))\beta_{\mathrm{s}} y - (\rho + \lambda_2 - \mu_1(2))\beta_{\mathrm{b}} \geq \lambda_2(\beta_{\mathrm{s}} y - \beta_{\mathrm{b}}). \end{aligned}$$

as in Case I. The equivalent conditions for these inequalities to hold are

$$k_2 \geq \frac{(\rho - \mu_1(j))\beta_{\mathrm{b}}}{(\rho - \mu_2(j))\beta_{\mathrm{s}}} \quad \text{for } j = 1, 2.$$

Move on to the interval $[k_1,k_2]$ and recall $w_1 = w_3 - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y$. Apply $(\rho + \lambda_1 - \mathcal{L}_1)$ to w_1 to obtain

$$(\rho + \lambda_1 - \mathcal{L}_1) w_1 = (\rho + \lambda_1 - \mathcal{L}_1) w_3 + (\rho + \lambda_1 - \mu_2(1))\beta_{\mathrm{s}} y - (\rho + \lambda_1 - \mu_1(1))\beta_{\mathrm{b}}.$$

In addition, recall that $k_3 < k_4 < k_1 < k_2$ and $(\rho + \lambda_1 - \mathcal{L}_1) w_3 = \lambda_1 w_4$. It follows that

$$(\rho + \lambda_1 - \mathcal{L}_1) w_1 = \lambda_1 w_4 + (\rho + \lambda_1 - \mu_2(1))\beta_{\mathrm{s}} y - (\rho + \lambda_1 - \mu_1(1))\beta_{\mathrm{b}}.$$

Recall also that $w_4 = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4}$ for $y \geq k_4$ and $w_2(y) = B_1 y^{\nu_1} + B_2 y^{\nu_2} + w_{2,p}(y)$. Hence the inequality $(\rho + \lambda_1 - \mathcal{L}_1) w_1(y) - \lambda_1 w_2(y) \geq 0$ is equivalent to

$$\begin{aligned} &B_1 y^{\nu_1} + B_2 y^{\nu_2} + w_{2,p}(y) \\ &\qquad \leq C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} + \left[\frac{\rho + \lambda_1 - \mu_2(1)}{\lambda_1}\right]\beta_{\mathrm{s}} y - \left[\frac{\rho + \lambda_1 - \mu_1(1)}{\lambda_1}\right]\beta_{\mathrm{b}}. \end{aligned}$$

Case III: $k_3 < k_4 < k_2 < k_1$. Finally, we consider the last case ($k_3 < k_4 < k_2 < k_1$). For $0 < y < k_2$, we have the equations

$$(\rho + \lambda_1 - \mathcal{L}_1) w_1(y) - \lambda_1 w_2(y) = 0 \quad \text{and} \quad (\rho + \lambda_2 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y) = 0.$$

Their general solutions can be given by

$$w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2} \quad \text{and} \quad w_2(y) = C_1\eta_1 y^{\delta_1} + C_2\eta_2 y^{\delta_2}.$$

For $k_1 \leq y \leq k_2$, we have

$$w_1(y) = w_3(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y \quad \text{and} \quad (\rho + \lambda_2 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y) = 0.$$

For $k_2 < y < \infty$, we have

$$w_1(y) = w_3(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y \quad \text{and} \quad w_2(y) = w_4(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y.$$

Recall the solutions w_3 and w_4 given in (7.9) (Appendix). It follows that, on the interval $[k_2, k_1]$,

$$w_2(y) = w_4(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y;$$

and $w_1(y)$ satisfies

$$(\rho + \lambda_1 - \mathcal{L}_1)w_1(y) = \lambda_1 w_2(y) = \lambda_1[C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y].$$

Then the general solution $w_1(y) = B_1 y^{\zeta_1} + B_2 y^{\zeta_2} + w_{1,p}(y)$ where the particular solution

$$\begin{aligned} w_{1,p}(y) = & \frac{\lambda_1 C_3\eta_3}{\rho + \lambda_1 - A_1(\delta_3)} y^{\delta_3} + \frac{\lambda_1\eta_4 C_4}{\rho + \lambda_1 - A_1(\delta_4)} y^{\delta_4} \\ & - \frac{\lambda_1\beta_{\rm b}}{\rho + \lambda_1 - \mu_1(1)} + \frac{\lambda_1\beta_{\rm s}}{\rho + \lambda_1 - \mu_2(1)} y. \end{aligned}$$

Note that $\lambda_1/(\rho + \lambda_1 - A_1(\delta_3)) = 1/\eta_3$ and $\lambda_1/(\rho + \lambda_1 - A_1(\delta_4)) = 1/\eta_4$. These imply

$$\begin{aligned} w_{1,p}(y) &= C_3 y^{\delta_3} + C_4 y^{\delta_4} - \frac{\lambda_1\beta_{\rm b}}{\rho + \lambda_1 - \mu_1(1)} + \frac{\lambda_1\beta_{\rm s} y}{\rho + \lambda_1 - \mu_2(1)} \\ &= w_3(y) - \frac{\lambda_1\beta_{\rm b}}{\rho + \lambda_1 - \mu_1(1)} + \frac{\lambda_1\beta_{\rm s} y}{\rho + \lambda_1 - \mu_2(1)}. \end{aligned}$$

Finally, on the interval $[k_1, \infty)$, we have

$$\begin{aligned} w_1(y) &= w_3(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y, \\ w_2(y) &= w_4(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y. \end{aligned}$$

Summarize the above computation to obtain

(3.23)

$$\begin{aligned} \text{On } (0, k_2): \quad & w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2}, \\ & w_2(y) = C_1\eta_1 y^{\delta_1} + C_2\eta_2 y^{\delta_2}, \\ \text{On } [k_2, k_1]: \quad & w_1(y) = B_1 y^{\zeta_1} + B_2 y^{\zeta_2} + w_{1,p}(y), \\ & w_2(y) = w_4(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y, \\ \text{On } (k_1, \infty): \quad & w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y, \\ & w_2(y) = w_4(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y. \end{aligned}$$

Next, we apply the smooth-fit method to determine the parameters C_1, C_2, B_1, B_2, k_1 and k_2. First, the continuity of $w_1(y)$, $w_2(y)$, $w_1'(y)$ and $w_2'(y)$ at k_2 yields

$$\begin{aligned} C_1 k_2^{\delta_1} + C_2 k_2^{\delta_2} &= B_1 k_2^{\zeta_1} + B_2 k_2^{\zeta_2} + w_{1,p}(k_2), \\ C_1\delta_1 k_2^{\delta_1 - 1} + C_2\delta_2 k_2^{\delta_2 - 1} &= B_1\zeta_1 k_2^{\zeta_1 - 1} + B_2\zeta_2 k_2^{\zeta_2 - 1} + w_{1,p}'(k_2), \\ C_1\eta_1 k_2^{\delta_1} + C_2\eta_2 k_2^{\delta_2} &= w_4(k_2) + \beta_{\rm s} k_2 - \beta_{\rm b}, \\ C_1\eta_1\delta_1 k_2^{\delta_1 - 1} + C_2\eta_2\delta_2 k_2^{\delta_2 - 1} &= w_4'(k_2) + \beta_{\rm s}. \end{aligned}$$

The continuity of $w_1(y)$ and $w_1'(y)$ at k_1 yields

$$\begin{aligned} B_1 k_1^{\zeta_1} + B_2 k_1^{\zeta_2} + w_{1,p}(k_1) &= w_3(k_1) - \beta_{\rm b} + \beta_{\rm s} k_1, \\ B_1\zeta_1 k_1^{\zeta_1 - 1} + B_2\zeta_2 k_1^{\zeta_2 - 1} + w_{1,p}'(k_1) &= w_3'(k_1) + \beta_{\rm s}. \end{aligned}$$

Solve for C_1, C_2, B_1 and B_2 to obtain

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \Lambda^{-1}\Phi^{-1}(k_2,\delta_1,\delta_2)\begin{pmatrix} w_4(k_2)+\beta_s k_2-\beta_b \\ k_2[w_4'(k_2)+\beta_s] \end{pmatrix},$$

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \Phi^{-1}(k_2,\zeta_1,\zeta_2)\left[\Phi(k_2,\delta_1,\delta_2)\Lambda^{-1}\Phi^{-1}(k_2,\delta_1,\delta_2)\begin{pmatrix} w_4(k_2)+\beta_s k_2-\beta_b \\ k_2[w_4'(k_2)+\beta_s] \end{pmatrix} - \begin{pmatrix} w_{1,p}(k_2) \\ k_2 w_{1,p}'(k_2) \end{pmatrix}\right].$$

The continuity of w_1 and w_1' at k_1 yields the system

$$\Phi(k_1,\zeta_1,\zeta_2)\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} w_3(k_1)-w_{1,p}(k_1)-\beta_b+\beta_s k_1 \\ k_1[w_3'(k_1)-w_{1,p}'(k_1)+\beta_s]. \end{pmatrix}$$

This gives

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \Phi^{-1}(k_1,\zeta_1,\zeta_2)\begin{pmatrix} w_3(k_1)-w_{1,p}(k_1)-\beta_b+\beta_s k_1 \\ k_1[w_3'(k_1)-w_{1,p}'(k_1)+\beta_s] \end{pmatrix}.$$

Eliminate B_1 and B_2 to obtain the following equations for k_1 and k_2:

$$\begin{aligned}\Phi^{-1}(k_2,\zeta_1,\zeta_2)\Bigg[&\Phi(k_2,\delta_1,\delta_2)\Lambda^{-1}\Phi^{-1}(k_2,\delta_1,\delta_2) \\ &\begin{pmatrix} w_4(k_2)+\beta_s k_2-\beta_b \\ k_2[w_4'(k_2)+\beta_s] \end{pmatrix} - \begin{pmatrix} w_{1,p}(k_2) \\ k_2 w_{1,p}'(k_2) \end{pmatrix}\Bigg] \\ &= \Phi^{-1}(k_1,\zeta_1,\zeta_2)\begin{pmatrix} w_3(k_1)-w_{1,p}(k_1)-\beta_b+\beta_s k_1 \\ k_1[w_3'(k_1)-w_{1,p}'(k_1)+\beta_s] \end{pmatrix}.\end{aligned} \tag{3.24}$$

Again, note that the constants B_1, B_2, C_1, and C_2 can be given as functions of k_1 and k_2. So are functions $w_1(y)$ and $w_2(y)$. Therefore, k_1 and k_2 need to be determined so that the following variational inequalities are satisfied:

$$\begin{array}{ll} \text{On } (0,k_2): & w_1(y)-w_3(y)+\beta_b-\beta_s y\geq 0, \\ & w_2(y)-w_4(y)+\beta_b-\beta_s y\geq 0, \\ \text{On } [k_2,k_1]: & w_1(y)-w_3(y)+\beta_b-\beta_s y\geq 0, \\ & (\rho+\lambda_2-\mathcal{L}_2)w_2(y)-\lambda_2 w_1(y)\geq 0, \\ \text{On } (k_1,\infty): & (\rho+\lambda_1-\mathcal{L}_1)w_1(y)-\lambda_1 w_2(y)\geq 0, \\ & (\rho+\lambda_2-\mathcal{L}_2)w_2(y)-\lambda_2 w_1(y)\geq 0. \end{array} \tag{3.25}$$

Finally, to see equivalent conditions for the above inequalities involving $\mathcal{L}_j$, we first note that, on the interval (k_1,∞), the variational inequalities are equivalent to (as in Case II by switching the roles of k_1 and k_2, (and w_1 and w_2),

$$k_1 \geq \frac{(\rho-\mu_1(j))\beta_b}{(\rho-\mu_2(j))\beta_s} \quad \text{for } j=1,2.$$

Next, on the interval $[k_2,k_1]$, to relate $(\rho+\lambda_2-\mathcal{L}_2)w_2(y)-\lambda_2 w_1(y)\geq 0$, recall that $w_1(y)=B_1y^{\zeta_1}+B_2y^{\zeta_2}+w_{1,p}(y)$ and $w_2(y)=w_4(y)-\beta_b+\beta_s y$ on $[k_2,k_1]$. Apply $(\rho+\lambda_2-\mathcal{L}_2$ to $w_2)$ to obtain

$$(\rho+\lambda_2-\mathcal{L}_2)w_2(y)=\lambda_2 w_3-(\rho+\lambda_2-\mu_1(2))\beta_b+(\rho+\lambda_2-\mu_2(2))\beta_s y.$$

Hence, $(\rho+\lambda_2-\mathcal{L}_2)w_2(y)-\lambda_2 w_1(y)\ge 0$ is equivalent to

$$B_1y^{\zeta_1}+B_2y^{\zeta_2}+w_{1,p}(y)\le C_3y^{\delta_3}+C_4y^{\delta_4}-\left[\frac{\rho+\lambda_2-\mu_1(2)}{\lambda_2}\right]\beta_{\rm b}+\left[\frac{\rho+\lambda_2-\mu_2(2)}{\lambda_2}\right]\beta_{\rm s}y.$$

4. Verification theorems

In this section, we provide verification theorems for Cases I, II, and III. First, we recall the optimal selling rule given in Tie and Zhang [13].

Theorem 4.1. (Selling Rule $k_3<k_4$). *Assume* (A1). *Let* k_3 *and* k_4 *be given in* (7.6) *and* (7.7)*, resp. Let* $w_3(y)$ *and* $w_4(y)$ *be given as in* (7.9) *such that the variational inequalities in* (7.10) *are satisfied. Then,* $v_1(x_1,x_2,1)=x_1w_3(x_2/x_1)=V_1(x_1,x_2,1)$ *and* $v_1(x_1,x_2,2)=x_1w_4(x_2/x_1)=V_1(x_1,x_2,2)$*. Let* $D_{\rm S}=\{(x_1,x_2,1):\ x_2>k_3x_1\}\cup\{(x_1,x_2,2):\ x_2>k_4x_1\}$*. Let* $\tau_0^*=\inf\{t:\ (X_t^1,X_t^2,\alpha_t)\notin D_{\rm S}\}$*. Then* τ_0^* *is optimal.*

Theorem 4.2. (Buying Rule). *Assume* (A1)*. Let* k_1 *and* k_2 *be given by* (3.17) *in Case I* (*by* (3.21) *in Case II and* (3.24) *in Case III, resp.*)*. Let also* $w_1(y)$ *and* $w_2(y)$ *be given by* (3.16) *in Case I* (*by* (3.20) *in Case II and* (3.23) *in Case III, resp.*) *Suppose the variational inequalities in* (3.18) *hold* (*Case I*) (*in* (3.22) (*Case II*) *and* (3.25) (*Case III*)*, resp.*)*. Then,* $v_0(x_1,x_2,1)=x_1w_1(x_2/x_1)=V_0(x_1,x_2,1)$ *and* $v_0(x_1,x_2,2)=x_1w_2(x_2/x_1)=V_0(x_1,x_2,2)$*. Let* $D_{\rm B}=\{(x_1,x_2,1):\ x_2<k_1x_1\}\cup\{(x_1,x_2,2):\ x_2<k_2x_1\}$*. Define* $\tau_1^*=\inf\{t:\ (X_t^1,X_t^2,\alpha_t)\notin D_{\rm B}\}$ *and* $\tau_2^*=\inf\{t\ge\tau_1^*:\ (X_t^1,X_t^2,\alpha_t)\notin D_{\rm S}\}$*. Then* $\Lambda_0=(\tau_1^*,\tau_2^*)$ *is optimal.*

Proof. The proof is similar to that of [14, Theorem 5.1]. We sketch key steps for the sake of completeness. First, we show $v_i(x_1,x_2,\alpha)\ge J_i(x_1,x_2,\alpha,\Lambda_i)$. To this end, note that, in view of the variational inequalities in the HJB equations, for any stopping times $0\le\theta_1\le\theta_2$, a.s.,

$$E\left(e^{-\rho\theta_1}v_i(X_{\theta_1}^1,X_{\theta_1}^2,\alpha_{\theta_1})I_{\{\theta_1<\infty\}}\right)\ge E\left(e^{-\rho\theta_2}v_i(X_{\theta_2}^1,X_{\theta_2}^2,\alpha_{\theta_2})I_{\{\theta_2<\infty\}}\right),\quad fori=0,1. \tag{4.1}$$

Given $\Lambda_0=(\tau_1,\tau_2)$, it follows that

$$\begin{aligned}
v_0(x_1,x_2,\alpha)\ge\ & E\left(e^{-\rho\tau_1}v_0(X_{\tau_1}^1,X_{\tau_1}^2,\alpha_{\tau_1})I_{\{\tau_1<\infty\}}\right)\\
\ge\ & E\left(e^{-\rho\tau_1}\left(v_1(X_{\tau_1}^1,X_{\tau_1}^2,\alpha_{\tau_1})-\beta_{\rm b}X_{\tau_1}^1+\beta_{\rm s}X_{\tau_1}^2\right)I_{\{\tau_1<\infty\}}\right)\\
=\ & E\left(e^{-\rho\tau_1}v_1(X_{\tau_1}^1,X_{\tau_1}^2,\alpha_{\tau_1})I_{\{\tau_1<\infty\}}-e^{-\rho\tau_1}(\beta_{\rm b}X_{\tau_1}^1-\beta_{\rm s}X_{\tau_1}^2)I_{\{\tau_1<\infty\}}\right)\\
\ge\ & E\left(e^{-\rho\tau_2}v_1(X_{\tau_2}^1,X_{\tau_2}^2,\alpha_{\tau_2})I_{\{\tau_2<\infty\}}-e^{-\rho\tau_1}(\beta_{\rm b}X_{\tau_1}^1-\beta_{\rm s}X_{\tau_1}^2)I_{\{\tau_1<\infty\}}\right)\\
\ge\ & E\left(e^{-\rho\tau_2}(\beta_{\rm s}X_{\tau_2}^1-\beta_{\rm b}X_{\tau_2}^2)I_{\{\tau_2<\infty\}}-e^{-\rho\tau_1}(\beta_{\rm b}X_{\tau_1}^1-\beta_{\rm s}X_{\tau_1}^2)I_{\{\tau_1<\infty\}}\right)\\
=\ & J_0(x_1,x_2,\alpha,\Lambda_0).
\end{aligned}$$

Next, we establish the equality $v_i(x_1,x_2)=J_i(x_1,x_2,\Lambda_i^*)$. Recall that $\tau_1^*=\inf\{t\ge 0:\ (X_t^1,X_t^2,\alpha_t)\in D_{\rm B}\}$ and $\tau_2^*=\inf\{t\ge\tau_1^*:\ (X_t^1,X_t^2,\alpha_t)\in D_{\rm S}\}$. Using Dynkin's formula over the intervals $(0,\tau_1^*)$ and (τ_1^*,τ_2^*) to obtain

$$\begin{aligned}
v_0(x_1,x_2,\alpha)=\ & E[e^{-\rho\tau_1^*}v_0(X_{\tau_1^*}^1,X_{\tau_1^*}^1,\alpha_{\tau_1^*})I_{\{\tau_1^*<\infty\}}]\\
=\ & E[e^{-\rho\tau_1^*}(v_1(X_{\tau_1^*}^1,X_{\tau_1^*}^1,\alpha_{\tau_1^*})-\beta_{\rm b}X_{\tau_1^*}^1+\beta_{\rm s}X_{\tau_2^*}^2)I_{\{\tau_1^*<\infty\}}].
\end{aligned}$$

We have also

$$\begin{aligned} E\left(e^{-\rho\tau_1^*}v_1(X^1_{\tau_1^*},X^2_{\tau_1^*},\alpha_{\tau_1^*})I_{\{\tau_1^*<\infty\}}\right) &= E\left(e^{-\rho\tau_2^*}v_1(X^1_{\tau_2^*},X^2_{\tau_2^*},\alpha_{\tau_2^*})I_{\{\tau_2^*<\infty\}}\right) \\ &= E\left(e^{-\rho\tau_2^*}\left(\beta_{\mathrm{s}}X^1_{\tau_2^*}-\beta_{\mathrm{b}}X^2_{\tau_2^*}\right)I_{\{\tau_2^*<\infty\}}\right). \end{aligned}$$

Combine these two equalities to obtain $v_0(x_1,x_2,\alpha)=J_0(x_1,x_2,\alpha,\Lambda_0^*)$. □

5. Numerical examples

In this section, we give three examples, one for each of the three cases.

Example 5.1. (Case I: $k_3<k_1<k_4<k_2$**).** In this example, we take

$$\begin{array}{llll} \mu_1(1)=0.30, & \mu_2(1)=0.27, & \mu_1(2)=-0.43, & \mu_2(2)=-0.66, \\ \sigma_{11}(1)=0.44, & \sigma_{12}(1)=0.27, & \sigma_{21}(1)=0.31, & \sigma_{22}(1)=0.60, \\ \sigma_{11}(2)=0.19, & \sigma_{12}(2)=0.65, & \sigma_{21}(2)=0.28, & \sigma_{22}(2)=0.15, \\ \lambda_1=6.0, & \lambda_2=10.0, & K=0.001, & \rho=0.50. \end{array}$$

First, we solve (7.6) and (7.7) for k_3 and k_4 followed by (3.17) for k_1 and k_2. We obtain $k_1=0.597020$, $k_2=0.690976$, $k_3=0.578407$, and $k_4=0.601707$. Using these to calculate the rest parameters to get $B_1=-1082.994378$, $B_2=0.002139$, $C_1=6.721641$, $C_2=-0.043221$, $C_3=0.189389$, $C_4=-0.000004$, $D_1=-0.078520$, $D_2=-0.0007050$, $E_1=1.377957$, and $E_2=4.440166$. Plugging these numbers into (3.16) and (7.9) to obtain the corresponding value functions. We verify that all the variational inequalities in (3.18) and (7.10) are satisfied. Finally, the graphs of these value functions are given in Figure 3.

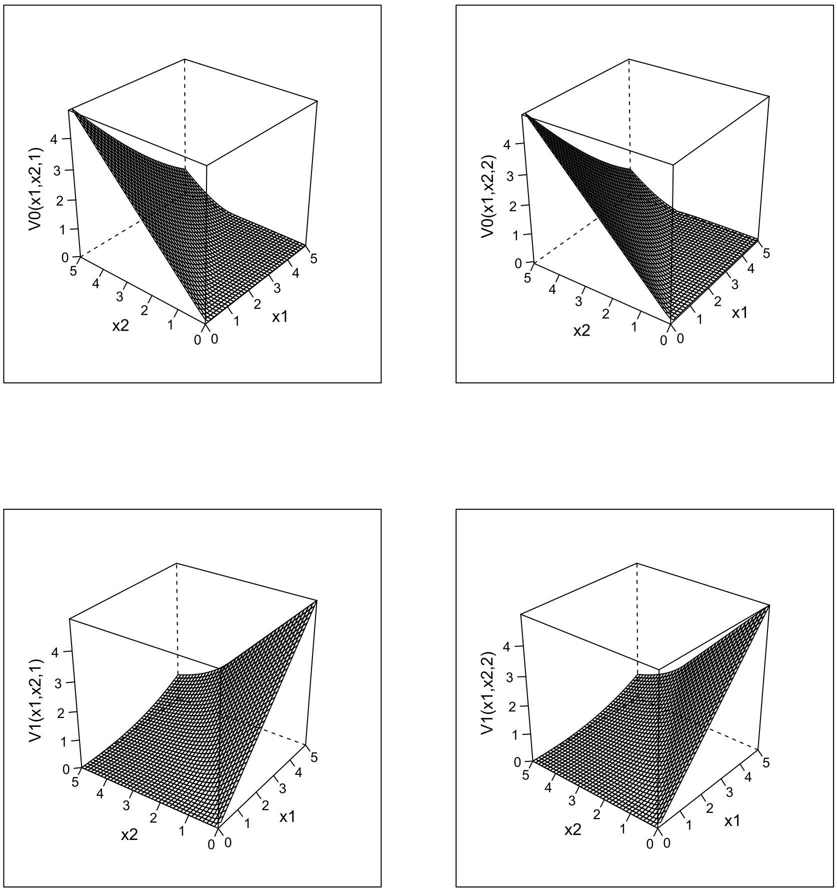

FIGURE 3. Value Functions $V_0(x_1, x_2, 1)$, $V_0(x_1, x_2, 2)$, $V_1(x_1, x_2, 1)$, and $V_1(x_1, x_2, 2)$

Example 5.2. (Case II: $k_3 < k_4 < k_1 < k_2$**).** In this example, we take

$$\begin{array}{llll} \mu_1(1) = -0.26, & \mu_2(1) = -0.56, & \mu_1(2) = -0.4, & \mu_2(2) = 0.22, \\ \sigma_{11}(1) = 0.37, & \sigma_{12}(1) = 0.46, & \sigma_{21}(1) = 0.59, & \sigma_{22}(1) = 0.59, \\ \sigma_{11}(2) = 0.47, & \sigma_{12}(2) = 0.31, & \sigma_{21}(2) = 0.28, & \sigma_{22}(2) = 0.68, \\ \lambda_1 = 6.0, & \lambda_2 = 10.0, & K = 0.001, & \rho = 0.50. \end{array}$$

Similarly as in Example 1, we solve (7.6) and (7.7) and then (3.21) to obtain $k_1 = 0.929500$, $k_2 = 0.962000$, $k_3 = 0.678861$, and $k_4 = 0.810852$. Then, we calculate and get $B_1 = 0.295000$, $B_2 = 0.021266$, $C_1 = 0.078164$, $C_2 = 0.048996$, $C_3 = 0.097388$, $C_4 = -0.000156$, $E_1 = 0.225207$, and $E_2 = 0.000199$. Feeding these numbers into (3.20) and (7.9) to obtain the corresponding value functions. It can be shown that all the variational inequalities in (3.22) and (7.10) are satisfied. The graphs of the value functions are given in Figure 4.

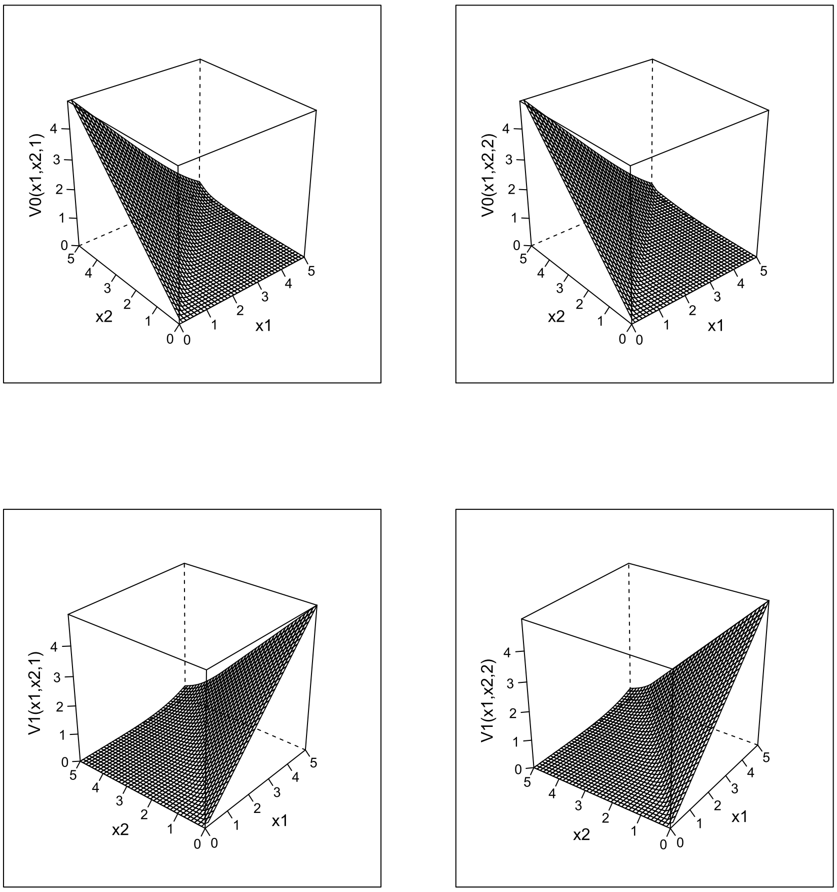

FIGURE 4. Value Functions $V_0(x_1, x_2, 1)$, $V_0(x_1, x_2, 2)$, $V_1(x_1, x_2, 1)$, and $V_1(x_1, x_2, 2)$

Example 5.3. (Case III: $k_3 < k_4 < k_2 < k_1$). Finally, in this example, we take

$$\begin{array}{llll} \mu_1(1) = 0.20, & \mu_2(1) = 0.25, & \mu_1(2) = -0.30, & \mu_2(2) = -0.35, \\ \sigma_{11}(1) = 0.30, & \sigma_{12}(1) = 0.10, & \sigma_{21}(1) = 0.10, & \sigma_{22}(1) = 0.35, \\ \sigma_{11}(2) = 0.40, & \sigma_{12}(2) = 0.20, & \sigma_{21}(2) = 0.20, & \sigma_{22}(2) = 0.45, \\ \lambda_1 = 6.0, & \lambda_2 = 10.0, & K = 0.001, & \rho = 0.50. \end{array}$$

Similarly as in previous examples, we solve (7.6) and (7.7) and then (3.24) to obtain $k_1 = 1.379000$, $k_2 = 1.212000$, $k_3 = 0.723277$, and $k_4 = 0.737941$. Then, we calculate and get $B_1 = 0.000175$, $B_2 = 0.043496$, $C_1 = -0.000069$, $C_2 = 0.147433$, $C_3 = 0.114418$, $C_4 = -0.000006$, $E_1 = 0.291176$, and $E_2 = 0.000294$. Using these numbers in (3.23) and (7.9) to obtain the corresponding value functions. It can be shown that all the variational inequalities in (3.25) and (7.10) are satisfied. The graphs of the value functions are given in Figure 5.

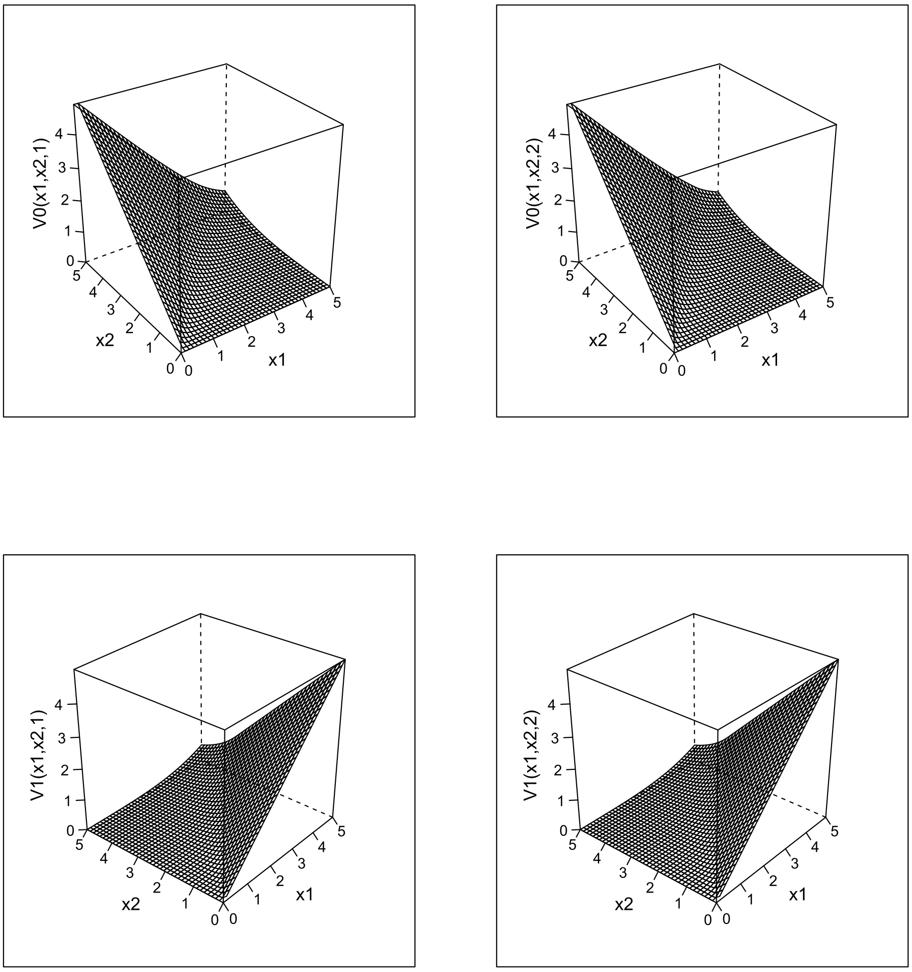

FIGURE 5. Value Functions $V_0(x_1, x_2, 1)$, $V_0(x_1, x_2, 2)$, $V_1(x_1, x_2, 1)$, and $V_1(x_1, x_2, 2)$

6. CONCLUSIONS

This paper is about an optimal pairs trading rule. The main results include threshold type trading rules and sufficient optimality conditions in terms of verification theorems. It would be interesting to consider models in which the market mode α_t is not directly observable. In this case, the Wonham filter can be used for calculation of the conditional probabilities of $\alpha = 1$ given the stock prices up to time t. Some ideas along this line have been used in Dai et al. [1] in connection with trend following trading.

7. Appendix: The solutions w_3 and w_4

In this appendix, we sketch the key steps in derivation of solutions w_3 and w_4. Recall the corresponding HJB equations:

$$\begin{aligned} &\min\Big\{(\rho+\lambda_1-\mathcal{L}_1)w_3(y)-\lambda_1 w_4(y),\ w_3(y)+\beta_{\rm b}y-\beta_{\rm s}\Big\}=0,\\ &\min\Big\{(\rho+\lambda_2-\mathcal{L}_2)w_4(y)-\lambda_2 w_3(y),\ w_4(y)+\beta_{\rm b}y-\beta_{\rm s}\Big\}=0. \end{aligned} \tag{7.1}$$

In this appendix, we only consider the case $k_3<k_4$. Details on other cases can be found in [13]. First, we divide the interval $(0,\infty)$ into three subintervals:

$$\Gamma_1=(0,k_3),\quad \Gamma_2=(k_3,k_4),\quad \text{and}\quad \Gamma_3=[k_4,\infty).$$

Note that $w_3=w_4=\beta_s-\beta_b y$ on Γ_1;

$$w_3=C_3y^{\delta_3}+C_4y^{\delta_4}\quad\text{and}\quad w_4=\eta_3C_3y^{\delta_3}+\eta_4C_4y^{\delta_4}\ \text{on}\ \Gamma_3;$$

and $w_4=\beta_s-\beta_b y$ and $(\rho+\lambda_1-\mathcal{L}_1)w_3(y)=\lambda_1 w_4(y)$ on Γ_2. To solve the non-homogeneous linear equation of Euler type:

$$(\rho+\lambda_1-\mathcal{L}_1)w_3(y)=\lambda_1 w_4(y)=\lambda_1(\beta_s-\beta_b y),$$

let

$$a_1=\frac{\lambda_1\beta_s}{\rho+\lambda_1-\mu_1(1)}\quad\text{and}\quad a_2=\frac{\lambda_1\beta_b}{\rho+\lambda_1-\mu_2(1)}. \tag{7.2}$$

Then a particular solution can be given as $w_{3,p}(y)=a_1-a_2y$. The general solution is given by

$$w_3=E_1y^{\zeta_1}+E_2y^{\zeta_2}+a_1-a_2y, \tag{7.3}$$

where ζ_1 and ζ_2 are given by (3.8).

Next we apply smooth-fit conditions to find the parameters C_1, C_2, E_1, E_2, k_3 and k_4. First the continuity of w_4 and its derivative at k_4 yield

$$\begin{aligned} \beta_s-\beta_b k_4&=\eta_3C_3k_4^{\delta_3}+\eta_4C_4k_4^{\delta_4},\\ -\beta_b&=\eta_3\delta_3C_3k_4^{\delta_3-1}+\eta_4\delta_4C_4k_4^{\delta_4-1}. \end{aligned}$$

The continuity of w_3 and its derivative at k_3 and k_4 yield

$$\begin{aligned} \beta_s-\beta_b k_3&=E_1k_3^{\zeta_1}+E_2k_3^{\zeta_2}+a_1-a_2k_3,\\ -\beta_b&=E_1\zeta_1k_3^{\zeta_1-1}+E_2\zeta_2k_3^{\zeta_2-1}-a_2,\\ E_1k_4^{\zeta_1}+E_2k_4^{\zeta_2}+a_1-a_2k_4&=C_3k_4^{\delta_3}+C_4k_4^{\delta_4},\\ E_1\zeta_1k_4^{\zeta_1-1}+E_2\zeta_2k_4^{\zeta_2-1}-a_2&=\delta_3C_3k_4^{\delta_3-1}+\delta_4C_4k_4^{\delta_4-1}. \end{aligned}$$

Let

$$\Phi(t,s_1,s_2)=\begin{pmatrix} t^{s_1} & t^{s_2}\\ s_1t^{s_1} & s_2t^{s_2}\end{pmatrix}\ \text{and}\ \Lambda=\begin{pmatrix}\eta_1 & 0\\ 0 & \eta_2\end{pmatrix} \tag{7.4}$$

Then the above system can be rewritten as

$$\Phi(k_4,\delta_3,\delta_4)\Lambda\begin{pmatrix}C_3\\C_4\end{pmatrix}=\begin{pmatrix}\beta_\mathrm{s}-\beta_\mathrm{b}k_4\\-\beta_\mathrm{b}k_4\end{pmatrix},$$
$$\Phi(k_3,\zeta_1,\zeta_2)\begin{pmatrix}E_1\\E_2\end{pmatrix}=\begin{pmatrix}(\beta_\mathrm{s}-a_1)-(\beta_\mathrm{b}-a_2)k_3\\-(\beta_\mathrm{b}-a_2)k_3\end{pmatrix}$$
$$\Phi(k_4,\zeta_1,\zeta_2)\begin{pmatrix}E_1\\E_2\end{pmatrix}+\begin{pmatrix}a_1-a_2k_4\\-a_2k_4\end{pmatrix}=\Phi(k_4;\delta_3,\delta_4)\begin{pmatrix}C_3\\C_4\end{pmatrix}.$$

Eliminate the parameters C_1, C_2, E_1, and E_2 to obtain the equations for k_3 and k_4:

$$\begin{aligned}&\Phi(k_4,\zeta_1,\zeta_2)\Phi^{-1}(k_3,\zeta_1,\zeta_2)\begin{pmatrix}(\beta_\mathrm{s}-a_1)-(\beta_\mathrm{b}-a_2)k_3\\-(\beta_\mathrm{b}-a_2)k_3\end{pmatrix}+\begin{pmatrix}a_1-a_2k_4\\-a_2k_4\end{pmatrix}\\&=\Phi(k_4,\delta_3,\delta_4)\Lambda^{-1}\Phi^{-1}(k_4,\delta_3,\delta_4)\begin{pmatrix}\beta_\mathrm{s}-\beta_\mathrm{b}k_4\\-\beta_\mathrm{b}k_4\end{pmatrix}.\end{aligned}\tag{7.5}$$

Let $r=k_4/k_3$. Some simple calculations yield

$$\Phi(k_4,\zeta_1,\zeta_2)\Phi^{-1}(k_3,\zeta_1,\zeta_2)=\frac{1}{\zeta_1-\zeta_2}\begin{pmatrix}\zeta_1r^{\zeta_2}-\zeta_2r^{\zeta_1}&r^{\zeta_1}-r^{\zeta_2}\\\zeta_1\zeta_2(r^{\zeta_2}-r^{\zeta_1})&\zeta_1r^{\zeta_1}-\zeta_2r^{\zeta_2}\end{pmatrix}$$
$$\Phi(k_4,\delta_3,\delta_4)\Lambda^{-1}\Phi^{-1}(k_4,\delta_3,\delta_4)=\frac{1}{\eta_1\eta_2(\delta_3-\delta_4)}\begin{pmatrix}\eta_1\delta_3-\eta_2\delta_4&\eta_2-\eta_1\\\delta_3\delta_4(\eta_1-\eta_2)&\eta_2\delta_3-\eta_1\delta_4\end{pmatrix}.$$

We can rewrite these (7.5) as follows

$$\frac{1}{\zeta_1-\zeta_2}\begin{pmatrix}(\zeta_2-1)(\beta_\mathrm{b}-a_2)k_3-\zeta_2(\beta_\mathrm{s}-a_1)&\zeta_1(\beta_\mathrm{s}-a_1)+(1-\zeta_1)(\beta_\mathrm{b}-a_2)k_3\\\zeta_1[(\zeta_2-1)(\beta_\mathrm{b}-a_2)k_3-\zeta_2(\beta_\mathrm{s}-a_1)]&\zeta_2[\zeta_1(\beta_\mathrm{s}-a_1)+(1-\zeta_1)(\beta_\mathrm{b}-a_2)k_3]\end{pmatrix}\begin{pmatrix}r^{\zeta_1}\\r^{\zeta_2}\end{pmatrix}$$
$$=\frac{1}{\eta_1\eta_2(\delta_3-\delta_4)}\begin{pmatrix}\eta_1\delta_3-\eta_2\delta_4&\eta_2-\eta_1\\\delta_3\delta_4(\eta_1-\eta_2)&\eta_2\delta_3-\eta_1\delta_4\end{pmatrix}\begin{pmatrix}\beta_\mathrm{s}-\beta_\mathrm{b}k_4\\-\beta_\mathrm{b}k_4\end{pmatrix}-\begin{pmatrix}a_1-a_2k_4\\-a_2k_4\end{pmatrix}.$$

Let

$$\alpha_1=(\zeta_2-1)(\beta_\mathrm{b}-a_2)k_3-\zeta_2(\beta_\mathrm{s}-a_1)\quad\text{and}\quad\alpha_2=(\zeta_2-1)(\beta_\mathrm{b}-a_2)k_3-\zeta_2(\beta_\mathrm{s}-a_1).$$

The matrix on the lefthand side is

$$\frac{1}{\zeta_1-\zeta_2}\begin{pmatrix}\alpha_1&\alpha_2\\\zeta_1\alpha_1&\zeta_2\alpha_2\end{pmatrix}\quad\text{with inverse}\quad\begin{pmatrix}-\frac{\zeta_2}{\alpha_1}&\frac{1}{\alpha_1}\\\frac{\zeta_1}{\alpha_2}&-\frac{1}{\alpha_2}\end{pmatrix}.$$

This yields

$$\begin{pmatrix}r^{\zeta_1}\\r^{\zeta_2}\end{pmatrix}=\begin{pmatrix}-\frac{\zeta_2}{\alpha_1}&\frac{1}{\alpha_1}\\\frac{\zeta_1}{\alpha_2}&-\frac{1}{\alpha_2}\end{pmatrix}$$
$$\left[\frac{1}{\eta_1\eta_2(\delta_3-\delta_4)}\begin{pmatrix}\eta_1\delta_3-\eta_2\delta_4&\eta_2-\eta_1\\\delta_3\delta_4(\eta_1-\eta_2)&\eta_2\delta_3-\eta_1\delta_4\end{pmatrix}\begin{pmatrix}\beta_\mathrm{s}-\beta_\mathrm{b}k_4\\-\beta_\mathrm{b}k_4\end{pmatrix}-\begin{pmatrix}a_1-a_2k_4\\-a_2k_4\end{pmatrix}\right]$$

Simplify them to obtain

$$\begin{aligned}
&[\zeta_2(\beta_s-a_1)+(1-\zeta_2)(\beta_b-a_2)k_3]r^{\zeta_1}+\zeta_2 a_1+(1-\zeta_2)a_2k_4\\
&\quad=\frac{-\delta_4\beta_s+(\delta_4-1)\beta_bk_4}{\eta_3(\delta_3-\delta_4)}(\zeta_2-\delta_3)+\frac{\delta_3\beta_s+(1-\delta_3)\beta_bk_4}{\eta_4(\delta_3-\delta_4)}(\zeta_2-\delta_4),\\
&[-\zeta_1(\beta_s-a_1)+(\zeta_1-1)(\beta_b-a_2)k_3]r^{\zeta_2}+(\zeta_1-1)a_2k_4-\zeta_1 a_1\\
&\quad=\frac{-\delta_4\beta_s+(\delta_4-1)\beta_bk_4}{\eta_3(\delta_3-\delta_4)}(\delta_3-\zeta_1)+\frac{\delta_3\beta_s+(1-\delta_3)\beta_bk_4}{\eta_4(\delta_3-\delta_4)}(\delta_4-\zeta_1).
\end{aligned}$$

Let

$$\begin{aligned}
A_1&=\frac{-\delta_4\beta_s(\zeta_2-\delta_3)}{\eta_3(\delta_3-\delta_4)}+\frac{\delta_3\beta_s(\zeta_2-\delta_4)}{\eta_4(\delta_3-\delta_4)}-\zeta_2a_1,\\
B_1&=\frac{(\delta_4-1)(\zeta_2-\delta_3)\beta_b}{\eta_3(\delta_3-\delta_4)}+\frac{(1-\delta_3)\beta_b(\zeta_2-\delta_4)}{\eta_4(\delta_3-\delta_4)}-(1-\zeta_2)a_2;\\
A_2&=\frac{-\delta_4\beta_s(\delta_3-\zeta_1)}{\eta_3(\delta_3-\delta_4)}+\frac{\delta_3\beta_s(\delta_4-\zeta_1)}{\eta_4(\delta_3-\delta_4)}+\zeta_1a_1,\\
B_2&=\frac{(\delta_4-1)(\delta_3-\zeta_1)\beta_b}{\eta_3(\delta_3-\delta_4)}+\frac{(1-\delta_3)\beta_b(\delta_4-\zeta_1)}{\eta_4(\delta_3-\delta_4)}-(\zeta_1-1)a_2
\end{aligned}$$

Then we can rewrite the above system as

$$\begin{aligned}
[\zeta_2(\beta_s-a_1)+(1-\zeta_2)(\beta_b-a_2)k_3]r^{\zeta_1}&=A_1+B_1k_4\\
[-\zeta_1(\beta_s-a_1)+(\zeta_1-1)(\beta_b-a_2)k_3]r^{\zeta_2}&=A_2+B_2k_4
\end{aligned}$$

Since $k_4=rk_3$, we can obtain

$$k_3=\frac{A_1-\zeta_2(\beta_s-a_1)r^{\zeta_1}}{(1-\zeta_2)(\beta_b-a_2)r^{\zeta_1}-B_1r}=\frac{A_2+\zeta_1(\beta_s-a_1)r^{\zeta_2}}{(\zeta_1-1)(\beta_b-a_2)r^{\zeta_2}-B_2r},\tag{7.6}$$

and

$$k_4=rk_3=\frac{A_1r-\zeta_2(\beta_s-a_1)r^{\zeta_1+1}}{(1-\zeta_2)(\beta_b-a_2)r^{\zeta_1}-B_1r}=\frac{A_2r+\zeta_1(\beta_s-a_1)r^{\zeta_2+1}}{(\zeta_1-1)(\beta_b-a_2)r^{\zeta_2}-B_2r}.\tag{7.7}$$

The second equality in (7.6) yields an equation for r:

$$\frac{A_1-\zeta_2(\beta_s-a_1)r^{\zeta_1}}{(1-\zeta_2)(\beta_b-a_2)r^{\zeta_1}-B_1r}=\frac{A_2+\zeta_1(\beta_s-a_1)r^{\zeta_2}}{(\zeta_1-1)(\beta_b-a_2)r^{\zeta_2}-B_2r}.$$

Since we assume that $k_3<k_4$, we need to show that the above equation has a unique solution $r>1$. Once we find r, we can find k_3 and k_4 from (7.6) and (7.7). Then C_1, C_2, E_1 and E_2 can be given as follows:

$$\begin{aligned}
C_3&=\tfrac{-\delta_4\beta_s+(\delta_4-1)\beta_bk_4}{\eta_3(\delta_3-\delta_4)k_4^{\delta_3}}, & C_4&=\tfrac{\delta_3\beta_s+(1-\delta_3)\beta_bk_4}{\eta_4(\delta_3-\delta_4)k_4^{\delta_4}},\\
E_1&=\tfrac{-\zeta_2(\beta_s-a_1)-(1-\zeta_2)(\beta_b-a_2)k_3}{(\zeta_1-\zeta_2)k_3^{\zeta_1}}, & E_2&=\tfrac{\zeta_1(\beta_s-a_1)-(\zeta_1-1)(\beta_b-a_2)k_3}{(\zeta_1-\zeta_2)k_3^{\zeta_2}}.
\end{aligned}\tag{7.8}$$

We summarize the solutions w_3 and w_4 as follows:

$$\begin{aligned}
(0,k_3):&\quad w_3=\beta_s-\beta_by,\\
[k_3,k_4]:&\quad w_3=E_1y^{\zeta_1}+E_2y^{\zeta_2}+a_1-a_2y,\\
(k_4,\infty):&\quad w_3=C_3y^{\delta_3}+C_4y^{\delta_4},\\
[0,k_4]:&\quad w_4=\beta_s-\beta_by,\\
(k_4,\infty):&\quad w_4=C_3\eta_3y^{\delta_3}+C_4\eta_4y^{\delta_4},
\end{aligned}\tag{7.9}$$

where a_1 and a_2 are given by (7.2), C_1, C_2, E_1 and E_2 are given by (7.8); and η_1 and η_2 are given by (3.10). In addition, we assume the inequalities to hold:

$$\begin{array}{ll} (0, k_3): & (\rho + \lambda_1 - \mathcal{L}_1)w_3(y) - \lambda_1 w_4(y) \geq 0, \\ [k_3, k_4]: & w_3 = E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - a_2 y \geq \beta_s - \beta_b y, \\ [k_4, \infty): & w_3 = C_3 y^{\delta_3} + C_4 y^{\delta_4} \geq \beta_s - \beta_b y, \\ (0, k_4): & (\rho + \lambda_2 - \mathcal{L}_2)w_4(y) - \lambda_2 w_3(y) \geq 0, \\ [k_4, \infty): & w_4 = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4} \geq \beta_s - \beta_b y. \end{array} \tag{7.10}$$

Some sufficient conditions for these inequalities can be found in Tie and Zhang [13].

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