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COMPARISON THEOREMS OF VISCOSITY SOLUTIONS FOR HAMILTON-JACOBI EQUATIONS WITH CO-INVARIANT DERIVATIVES OF FRACTIONAL ORDERS

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Abstract. We consider Hamilton-Jacobi equations with co-invariant derivatives of fractional orders which typically include Hamilton-Jacobi-Bellman equations of optimal control and Isaacs equations of differential games in fractional order systems governed by Caputo differential equations with orders less than one. In the frame of a viscosity solution notion of Gomoyunov (2021) and also Masuda (2021) with co-invariant derivatives of Gomoyunov (2020), we discuss comparison results on viscosity solutions for the Hamilton-Jacobi equations. Finding an appropriate smooth functional to measure a penalized distance of state trajectories, we obtain a comparison theorem by doubling variables arguments under no specific types of continuity assumptions for solutions. We also remark that our choice of the smooth distance functional is valid in comparison arguments for functional Hamilton-Jacobi equations.

1. Introduction

Hamilton-Jacobi equations are nonlinear partial differential equations (abbreviated as PDEs) of first order which originally have been studied in contexts of calculus of variations and analytical mechanics. Solutions of Hamilton-Jacobi equations play crucial roles to integrate Hamiltonian systems in classical mechanics (cf.[1]). As a different perspective of Lagrangian mechanics, if velocity of a mass point is regarded as a control parameter, Hamilton-Jacobi equations in classical mechanics are particular cases of Hamilton-Jacobi-Bellman (HJB) equations in optimal control theory. HJB equations are nonlinear PDEs for value functions which are formally derived by using dynamic programming principles (DPPs). It is fundamental to find solutions of HJB equations to construct optimal feedback controls of optimal control problems (cf. [10]). To apply Hamilton-Jacobi and HJB equations for classical mechanics and optimal control problems, solutions have to be smooth. However it is known that they are not necessarily differentiable everywhere. Hence weak solution notions are required for general nonlinear PDEs of first order. In the rest of the present paper, we use terminology *Hamilton-Jacobi equations* for general nonlinear PDEs of first order of any types.

There are two notions of weak solutions for Hamilton-Jacobi equations which widely succeed in theories and applications. The first is the notion of viscosity

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solutions introduced by Crandall and Lions ([3]). A key to the definition is to use smooth test functions touching solutions from above or below and this idea fits well to dynamic programming principles. There are a lot of studies of viscosity solutions for control problems, but also for various types of PDEs, not limited to control theory (cf. [2], [11]). The second is minimax solutions originated from u- and v-stable functions in the theory of positional differential games by Krasovskii. Minimax solutions are functions whose graphs are weakly invariant with respect to flows of generalized characteristic inclusions, thus the idea of minimax solutions is closely related to methods of characteristics for nonlinear PDEs of first order (cf.[28]). Although it seems that viscosity and minimax solutions are different notions, it turns out they are equivalent weak solution notions (cf. [28, Theorem 4.3]).

Viscosity and minimax solution theories mentioned in the above are designed for Hamilton-Jacobi equations related to Markovian systems, that is, systems where states of the future are determined by the current state. Recently systems with path-dependent properties attract attention of a lot of researchers from views of theories and applications. Since Markovian properties are not expected any more in natural forms, states of the future should be described by functionals of past state trajectories, hence theory of Hamilton-Jacobi equations of functional types is needed to be developed. Here note that appropriate derivative notions on spaces of state trajectories are needed for defining Hamilton-Jacobi equations of functional types. For deterministic systems governed by path-dependent ordinary differential equations, theories of viscosity solutions and minimax solutions are developed in [23] and [22], respectively, using co-invariant derivatives of [20] (cf. [16], [17], [18] for related works of viscosity solutions). Motivated by Dupire's vertical and horizontal derivatives ([5]), viscosity solutions for path-dependent stochastic systems have been intensively studied, for instance, in [6], [7], [8], [25], [26] and [29] (see [32] for literature of these directions).

Fractional order systems are also systems with path-dependent properties. In the path-dependent systems mentioned in the above paragraph, it is supposed that coefficients of differential equations depend not only on current state but also on past state trajectories which bring path-dependent nature to the systems. For fractional order systems, definitions of derivatives of fractional orders depend on past trajectories, thus the path-dependency in fractional order systems are different from that in path-dependent systems. To derive HJB and Isaacs equations for fractional order systems, DPPs (flow properties) in some forms and derivative notions over path-spaces are required. [13] proposes initial conditions on Caputo differential equations to have DPPs, and derives HJB equations of functional types with co-invariant derivatives with fractional order. Then [14] develops a viscosity theory for Hamilton-Jacobi equations of fractional order systems (see also [24]). A minimax solution notion for such fractional order systems is also studied in [15]. For stochastic systems governed by stochastic Volterra integral equations, [30] finds flow properties by adding extra path components and obtains path-dependent PDEs under the spirit of derivatives of Dupire. A viscosity theory for such path-dependent PDEs is also sought in [31].

In this paper, we are concerned with comparison results of viscosity solutions for Hamilton-Jacobi equations related to fractional order systems governed by Caputo differential equations with orders less than one. As mentioned above, a viscosity solution notion is proposed by [14] (also [24]) using co-invariant derivatives with fractional order. For the viscosity solution theory, comparison theorems are important, for instance, to characterize value functions of optimal control and differential games as unique viscosity solutions of HJB and Isaacs equations. In proofs of comparison theorems for usual viscosity theory, doubling variables arguments are often used to measure a penalized distance of two states which is naturally taken by smooth Euclidean distance. For Hamilton-Jacobi equations of functional types, we need to measure a distance of two past trajectories. However there is not an obvious way to find smooth distance functionals which are useful for comparison arguments. Moreover, for Hamilton-Jacobi equations of fractional order systems, it seems that there are few examples of smooth functionals in the sense of co-invariant differentiability with fractional orders. [14] finds a smooth distance functional and proves a comparison theorem holds by assuming certain L^1-type Lipschitz continuity condition on solutions. In this paper, we propose a smooth distance functional different from [14] and obtain a comparison theorem for viscosity solutions of Hamilton-Jacobi equations. Our conditions on Hamiltonians are restrictive a little compared to [14], but we do not require any types of continuity conditions on solutions like [14]. Thanks to the form of our distance functional, the proof is much simpler than [14].

The paper is organized as follows. In section 2, we review notions and results on fractional calculus such as Riemann-Liouville integrals and Caputo derivatives required in the present paper. We also recall co-invariant derivatives of fractional orders on spaces of state trajectories given by [13]. In section 3, we introduce Hamilton-Jacobi equations with co-invariant derivatives of fractional orders which cover HJB and Isaacs equations for systems governed by Caputo differential equations. Following [14] (also [24]), we give a definition of viscosity solutions under co-invariant derivative of fractional orders. We also mention that value functionals of differential games are viscosity solutions of the Isaacs equations. In section 4, we give a comparison theorem of viscosity solutions. With a choice of a new smooth distance functional, we obtain a comparison result on viscosity sub/super solutions using a doubling variables argument. In section 5, we give a remark on the relation of the comparison argument of section 4 to the case of order one which is studied in [23]. We mention that our choice of the distance functional still works for the argument of [23], thus our distance functional can be regarded as an extension to fractional order cases. We collect auxiliary results and proofs in appendices.

2. Riemann-Louville integrals, Caputo derivatives and co-invariant derivatives

We review notions and results on fractional calculus which are needed for the discussion of the present paper (see [4], [19], [27] for more information). Let $p > 0$. For $\phi \in L^\infty[0,t] = L^\infty([0,t];\mathbb{R}^n)$, define $I_0^p\phi : [0,t] \to \mathbb{R}^n$ by

$$(I_0^p\phi)(s) = \frac{1}{\Gamma(p)} \int_0^s \phi(r)(s-r)^{p-1}dr,\ s \in [0,t],$$

where $\Gamma(p)$ is Gamma function with order p:

$$\Gamma(p) = \int_0^\infty s^{p-1}e^{-s}ds.$$

$I_0^p\phi$ is called *Riemann-Louville integral of ϕ with order p*. We set $(I_0^0\phi)(r) = \phi(r)$, that is, I_0^0 is an identity operator. Note that Riemann-Louville integrals are extensions of multiple integrals of positive integer orders.

Let $0 < \alpha \le 1$. Let $\xi \in L^\infty[0,t]$. The *Caputo derivative* of ξ with order α at $s \in [0,t]$ is defined by the following if the derivative exists:

$$({}^CD_0^\alpha\xi)(s) = \frac{d}{ds}I_0^{1-\alpha}(\xi(\cdot) - \xi(0))(s).$$

More precisely, ${}^CD_0^\alpha\xi$ is given by

$$({}^CD_0^\alpha\xi)(s) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{ds}\displaystyle\int_0^s \frac{\xi(r)-\xi(0)}{(s-r)^\alpha}dr & (0<\alpha<1) \\ \dfrac{d\xi}{ds}(s) \quad (\alpha = 1). \end{cases}$$

We denote $AC^\alpha[0,t]$ by

$$\begin{aligned} AC^\alpha[0,t] = \{\xi : [0,t] \to \mathbb{R}^n;\ \exists\phi \in L^\infty[0,t] \\ s.t.\ \xi(s) = \xi(0) + (I_0^\alpha\phi)(s),\ s \in [0,t]\}. \end{aligned} \tag{2.1}$$

If $\xi \in AC^\alpha[0,t]$, ξ is α-Hölder continuous on $[0,t]$ (see Proposition 2.1) and Caputo differentiable at almost every $s \in [0,t]$:

$$({}^CD_0^\alpha\xi)(s) = \phi(s),\ \ a.e.\, s \in [0,t],$$

where ϕ is from (2.1). Note that $AC^1[0,t]$ is the space of absolutely continuous functions on $[0,t]$ with bounded derivatives.

We prepare several results on Riemann-Louville integrals. We denote a function on $[0,t]$, for instance, by x_t whose subscript indicates its domain.

Proposition 2.1 (cf. [27] Theorem 3.6 and Remark 3.3, [4] Theorem 2.6)**.** *Let $0 < \alpha \le 1$. Let $x_t \in L^\infty[0,t]$. Then, there exists $C = C_\alpha > 0$, which does not depend on x_t, such that*

$$|(I_0^\alpha x_t)(s) - (I_0^\alpha x_t)(r)| \le C\|x_t\|_\infty|s-r|^\alpha,\ \ s,r \in [0,t],$$

where $\|x_t\|_\infty = \text{ess.sup}_{0\le r\le t}|x_t(r)|$. More precisely, C can be taken by $2/\Gamma(\alpha+1)$.

Proposition 2.2. *Let $0 < \alpha \le 1$. For $t,s \in [0,T]$, $x_t \in C[0,t]$ and $y_s \in C[0,s]$,*

$$\begin{aligned} &\max_{0\le r\le T}\left|(I_0^{1-\alpha}x_t)(r\wedge t) - (I_0^{1-\alpha}y_s)(r\wedge s)\right| \\ &\le \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\max_{0\le r\le T}|x_t(r\wedge t) - y_s(r\wedge s)| \\ &\qquad + \frac{2}{\Gamma(2-\alpha)}(\|x_t\|_\infty + \|y_s\|_\infty)|t-s|^{1-\alpha}1_{\{0<\alpha<1\}}, \end{aligned} \tag{2.2}$$

where

$$1_{\{0<\alpha<1\}} = \begin{cases} 1 & (0<\alpha<1) \\ 0 & (\alpha = 1), \end{cases}$$

and $a \wedge b = \min\{a, b\}$ *for* $a, b \in \mathbb{R}$.

See Appendix A for the proof of Proposition 2.2.

Proposition 2.3 (cf. [4] Lemma 2.7)**.** *Let* $0 < \alpha \leq 1$. *Let* $x_t \in C[0, t]$ *and* $x_t^n \in C[0, t]$ $(n = 1, 2, \dots)$. *If* x_t^n *converges to* x_t *as* $n \to \infty$ *uniformly on* $[0, t]$, *then*

$$(I_0^\alpha x_t^n)(s) \to (I_0^\alpha x_t)(s) \ (n \to \infty) \text{ uniformly on } [0, t].$$

As explained in section 3, Hamilton-Jacobi equations studied in this paper cover Hamilton-Jacobi-Bellman (HJB) equations of optimal control and Isaacs equations of differential games for fractional order systems governed by Caputo differential equations. Due to the dependence of Caputo derivatives on past state trajectories, fractional order systems cannot be Markovian. Hence calculus over past trajectories is required to be developed. We introduce path spaces for fractional order systems and recall a derivative notion on the path spaces proposed in [13].

Let $0 < \alpha \leq 1$. Let $T > 0$ be a time horizon. We define $\mathbf{X}_{0,T}^\alpha$ by

$$\mathbf{X}_{0,T}^\alpha = \{(t, x_t);\ 0 \leq t \leq T,\ x_t \in AC^\alpha[0, t]\}.$$

We also need a subspace of $\mathbf{X}_{0,T}^\alpha$ given by

$$\mathbf{X}_{0,T-}^\alpha = \{(t, x_t) \in \mathbf{X}_{0,T}^\alpha;\ t < T\}.$$

We consider metric ρ on $\mathbf{X}_{0,T}^\alpha$ given by

$$\rho((t, x_t), (s, y_s)) = |t - s|^\alpha + \max_{0 \leq r \leq T} |x_t(r \wedge t) - y_s(r \wedge s)|,\ (t, x_t), (s, y_s) \in \mathbf{X}_{0,T}^\alpha.$$

Note that α-Hölder continuity of ρ on the time variable appears in regularity of value functionals of optimal control and differential games for fractional order systems of Caputo differential equations (see (3.12)).

Let $(t, x_t) \in \mathbf{X}_{0,T}^\alpha$. Define $\mathcal{P}(t, x_t)$ by

$$\mathcal{P}(t, x_t) = \{y \in AC^\alpha[0, T];\ x_t(s) = y(s)\ (0 \leq s \leq t)\}.$$

Let $\varphi : \mathbf{X}_{0,T}^\alpha \to \mathbb{R}$ and $(t, x_t) \in \mathbf{X}_{0,T-}^\alpha$. φ *is α-co-invariant (α-ci-) differentiable at* (t, x_t) if there exist $c \in \mathbb{R}$ and $p \in \mathbb{R}^n$ such that the following holds (cf. [13]): For any $y \in \mathcal{P}(t, x_t)$,

$$\begin{aligned} \varphi(t+h, y_{t+h}) &= \varphi(t, x_t) + ch \\ &\quad + p \cdot \{I_0^{1-\alpha}(y(\cdot) - y(0))(t+h) - I_0^{1-\alpha}(x(\cdot) - x(0))(t)\} + o_y(h) \\ &= \varphi(t, x_t) + ch + p \cdot \int_t^{t+h} {}^C D_0^\alpha y(r) dr + o_y(h)\ (h \to 0+), \end{aligned} \tag{2.3}$$

where $o_y(h) = h\omega(h; y)$ with some function $\omega(\cdot; y) : (0, T - t] \to \mathbb{R}$ satisfying $\lim_{h \to 0+} \omega(h; y) = 0$. c and p are called *α-ci-derivatives of φ at* (t, x_t) and denoted by

$$\partial_t^\alpha \varphi(t, x_t) = c,\ \nabla_{x_t}^\alpha \varphi(t, x_t) = p.$$

For $\alpha = 1$, we note that 1-ci-derivatives is nothing but ci-derivatives for functional differential equations (cf. [20]).

φ is a C^1-function on $\mathbf{X}^\alpha_{0,T}$ if φ is α-ci-differentiable at any $(t, x_t) \in \mathbf{X}^\alpha_{0,T-}$ and $\varphi : \mathbf{X}^\alpha_{0,T} \to \mathbb{R}$, $\partial^\alpha_t \varphi : \mathbf{X}^\alpha_{0,T-} \to \mathbb{R}$, $\nabla^\alpha_{x_t} \varphi : \mathbf{X}^\alpha_{0,T-} \to \mathbb{R}^n$ are continuous functions. We denote by $\mathcal{C}^1_\alpha(\mathbf{X}^\alpha_{0,T})$ the set of C^1-functions on $\mathbf{X}^\alpha_{0,T}$.

3. Hamilton-Jacobi equations with α-ci-derivatives and viscosity solutions

Let $v : \mathbf{X}^\alpha_{0,T} \to \mathbb{R}$. We consider the following partial differential equation (PDE) with α-ci-derivatives:

$$\partial^\alpha_t v(t, x_t) + F(t, x_t, \nabla^\alpha_{x_t} v(t, x_t)) = 0, \ (t, x_t) \in \mathbf{X}^\alpha_{0,T-}, \tag{3.1}$$

where $F : \mathbf{X}^\alpha_{0,T} \times \mathbb{R}^n \to \mathbb{R}$. We call general nonlinear PDEs (3.1) *Hamilton-Jacobi equations*. We explain that Isaacs equations for fractional order systems have the form of (3.1) by a formal way (see [13], [14], [24] for detailed discussions).

Let $(t, x_t) \in \mathbf{X}^\alpha_{0,T}$ be an initial condition. Let $\xi \in AC^\alpha[0, T]$ be a state trajectory satisfying the following Caputo differential equation:

$$\begin{aligned} {}^C D^\alpha_0 \xi(s) &= G(s, \xi(s), a(s), b(s)) \quad (t \le s \le T), \\ \xi(s) &= x_t(s) \quad (0 \le s \le t). \end{aligned} \tag{3.2}$$

which is equivalent to the integral equation

$$\begin{aligned} \xi(s) = x_t(0) &+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{({}^C D^\alpha_0 x_t)(r)}{(s-r)^{1-\alpha}} dr \\ &+ \frac{1}{\Gamma(\alpha)} \int_t^s \frac{G(r, \xi(r), a(r), b(r))}{(s-r)^{1-\alpha}} dr \quad (t \le s \le T), \\ \xi(s) = x_t(s) &\quad (0 \le s \le t), \end{aligned}$$

where $G : [0, T] \times \mathbb{R}^n \times A \times B \to \mathbb{R}^n$ with metric spaces A and B, and $a : [t, T] \to A$, $b : [t, T] \to B$ are measurable controls of two players, respectively. The game payoff is given by

$$J(t, x_t; a, b) = \int_t^T L(s, \xi(s), a(s), b(s)) ds + \Psi(\xi(T)), \tag{3.3}$$

where $L : [0, T] \times \mathbb{R}^n \times A \times B \to \mathbb{R}$ and $\Psi : \mathbb{R}^n \to \mathbb{R}$. If we start with $t = 0$ and $\xi(0) = x$ in (3.2), a Markovian (flow) property does not hold for (3.2), thus we cannot apply conventional dynamic programming methods. In [13] and [14], the initial value problem (3.2) for fractional order systems is suggested to obtain dynamic programming principle for value functionals of past trajectory x_t, not value functions of current state $x_t(t)$. To be more precise, consider a value functional of the zero-sum game given by

$$V(t, x_t) = \inf_\theta \sup_b J(t, x_t; \theta[b], b),$$

where θ is a mapping of $b : [t, T] \to B$ to $a : [t, T] \to A$ which corresponds to a strategy of the minimizing player. Under a suitable choice of a strategy class, the value functional satisfies dynamic programming principle: For $t < t + h \le T$,

$$V(t, x_t) = \inf_\theta \sup_b \left\{ \int_t^{t+h} L(s, \xi(s), \theta[b](s), b(s)) ds + V(t+h, \xi_{t+h}) \right\},$$

where ξ_{t+h} is the restriction of the solution of (3.2) on $[0, t+h]$ with $a(s) = \theta[b](s)$ and $b(s)$ (see [13, Theorem 6.1] for optimal control). Suppose V is a C^1-function on $\mathbf{X}_{0,T}^{\alpha}$. Using a chain rule for α-ci-derivatives (cf. [13, Lemma 9.2]), we formally obtain

$$\partial_t^{\alpha} V(t, x_t) + \sup_{b \in B} \inf_{a \in A} \{G(t, x_t(t), a, b) \cdot \nabla_{x_t}^{\alpha} V(t, x_t) + L(t, x_t(t), a, b)\} = 0, \tag{3.4}$$

which is an Isaacs equation of the differential game for fractional order system (3.2) with payoff (3.3) (cf. [13, Theorem 10.1] for a rigorous deduction of HJB equations). We see that (3.4) has the form of (3.1). Note that we have terminal condition $V(T, x_T) = \Psi(x_T(T))$ $(x_T \in AC^{\alpha}[0, T])$.

In the above discussion, G, L and Ψ can be generalized to path-dependent functionals $g : \mathbf{X}_{0,T}^{\alpha} \times A \times B \to \mathbb{R}^n$, $l : \mathbf{X}_{0,T}^{\alpha} \times A \times B \to \mathbb{R}$ and $\Phi : AC^{\alpha}[0, T] \to \mathbb{R}$, respectively. For initial condition $(t, x_t) \in \mathbf{X}_{0,T}^{\alpha}$, consider the path-dependent Caputo differential equation

$$\begin{aligned} {}^{C}D_0^{\alpha} \xi(s) &= g(s, \xi_s, a(s), b(s)) \quad (t \le s \le T), \\ \xi(s) &= x_t(s) \quad (0 \le s \le t), \end{aligned} \tag{3.5}$$

which is equivalent to

$$\begin{aligned} \xi(s) = x_t(0) &+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{({}^{C}D_0^{\alpha} x_t)(r)}{(s-r)^{1-\alpha}} dr \\ &+ \frac{1}{\Gamma(\alpha)} \int_t^s \frac{g(r, \xi_r, a(r), b(r))}{(s-r)^{1-\alpha}} dr \quad (t \le s \le T), \end{aligned} \tag{3.6}$$

$$\xi(s) = x_t(s) \quad (0 \le s \le t), \tag{3.7}$$

where ξ_s is the restriction of ξ on $[0, s]$. ξ is denoted by $\xi = \xi^{t, x_t, a, b}$ if we need to specify the dependence of initial condition (t, x_t) and controls a, b. We introduce the payoff with path-dependent costs

$$\mathcal{J}(t, x_t; a, b) = \int_t^T l(s, \xi_s^{t, x_t, a, b}, a(s), b(s)) ds + \Phi(\xi_T^{t, x_t, a, b}). \tag{3.8}$$

Then the value functional is defined by

$$W(t, x_t) = \inf_{\theta} \sup_{b} \mathcal{J}(t, x_t; \theta[b], b). \tag{3.9}$$

Using the dynamic programming principle, we can derive Isaacs equation

$$\partial_t^{\alpha} W(t, x_t) + \sup_{b \in B} \inf_{a \in A} \{g(t, x_t, a, b) \cdot \nabla_{x_t}^{\alpha} W(t, x_t) + l(t, x_t, a, b)\} = 0, \tag{3.10}$$

with terminal condition $W(T, x_T) = \Phi(x_T)$ $(x_T \in AC^{\alpha}[0, T])$.

We assumed that the value functionals are α-ci-differentiable to derive (3.4) and (3.10). However it is known that value functionals in optimal control and differential games are not necessarily α-ci-differentiable on $\mathbf{X}_{0,T-}$, thus we need a weak solution notion. In this paper, we consider a viscosity solution with α-ci-derivatives given in [14] (also, see [24]) which is a generalization of the viscosity solution notion of [23] in functional Hamilton-Jacobi equations ($\alpha = 1$) to fractional cases ($0 < \alpha < 1$).

To define a viscosity solution notion for (3.1), we need a sequence of subsets of $\mathbf{X}^\alpha_{0,T}$. Let $\nu \geq 1$ be given. Let D_k $(k = 1, 2, \ldots)$ be subsets of $\mathbf{X}^\alpha_{0,T}$ given by

$$D_k = \{(t, x_t) \in \mathbf{X}^\alpha_{0,T};\ |x_t(0)| \leq k,\\ |{}^C D^\alpha_0 x_t(s)| \leq k\nu(1 + \|x_s\|_\infty),\ \ a.e.\, s \in [0, t]\},$$

where x_s is the restriction of x_t on $[0, s]$ and $\|x_s\|_\infty = \max_{0\leq r\leq s} |x_s(r)|$.

The following proposition is crucial for defining viscosity solutions.

Proposition 3.1 (cf. [12], [14], [24])**.** (i) D_k $(k = 1, 2, \ldots)$ *are compact in* $\mathbf{X}^\alpha_{0,T}$.
(ii) $\mathbf{X}^\alpha_{0,T} = \bigcup_{k=1}^\infty D_k$.

Definition 3.2 (cf. [14], [24])**.** Let $v : \mathbf{X}^\alpha_{0,T} \to \mathbb{R}$ be a continuous function on $\mathbf{X}^\alpha_{0,T}$. v is a *viscosity subsolution (resp. viscosity supersolution)* of (3.1) if the following condition holds: Let $\varphi \in \mathcal{C}^1_\alpha(\mathbf{X}^\alpha_{0,T})$ and $k \in \mathbb{N}$. If $(\hat{t}, \hat{x}_{\hat{t}}) \in \mathbf{X}^\alpha_{0,T-}$ is a maximum point (*resp.* minimum point) of $v - \varphi$ on D_k, then

$$\partial^\alpha_t \varphi(\hat{t}, \hat{x}_{\hat{t}}) + F(\hat{t}, \hat{x}_{\hat{t}}, \nabla^\alpha_{x_t} \varphi(\hat{t}, \hat{x}_{\hat{t}})) \geq 0,$$
$$\left(\textit{resp.}\ \partial^\alpha_t \varphi(\hat{t}, \hat{x}_{\hat{t}}) + F(\hat{t}, \hat{x}_{\hat{t}}, \nabla^\alpha_{x_t} \varphi(\hat{t}, \hat{x}_{\hat{t}})) \leq 0\right).$$

v is a *viscosity solution* of (3.1) if v is a viscosity sub- and supersolution of (3.1).

Remark 3.3. Viscosity solutions given in Definition 3.2 are weak solutions under suitable conditions. More precisely, suppose condition (A2) in section 4 holds with a specific growth condition of the Lipschitz constant, that is, there exists $L_2 > 0$ such that for any $(t, x_t) \in \mathbf{X}^\alpha_{0,T}$, $p, q \in \mathbb{R}^n$,

$$|F(t, x_t, p) - F(t, x_t, q)| \leq L_2(1 + \|x_t\|_\infty)|p - q|.$$

v is a C^1-function satisfying (3.1) if and only if $v \in \mathcal{C}^1_\alpha(\mathbf{X}^\alpha_{0,T})$ is a viscosity sub and supersolution of (3.1) for $\nu \geq L_2$. This can be seen in a way similar to [23, Assertion 1].

To see metric ρ is suitable for the regularity of value functionals, we give a continuity result of value functional (3.9) of differential games. Since the argument is standard (but not obvious) and lengthy, we will give a sketch of the proof in Appendix B. We assume g, l and Φ are continuous. Suppose the following conditions hold:

i) A and B are compact metric spaces
ii) Let $h = g, l$. There exist $L, K > 0$ such that for any $(t, x_t), (t, y_t) \in \mathbf{X}^\alpha_{0,T}$, $x_T, y_T \in AC^\alpha[0, T]$, $a \in A$, $b \in B$,

$$|h(t, x_t, a, b) - h(t, y_t, a, b)| \leq L\|x_t - y_t\|_\infty,$$
$$|\Phi(x_T) - \Phi(y_T)| \leq L\|x_T - y_T\|_\infty,$$
$$|h(t, x_t, a, b)| \leq K\left(1 + \|x_t\|_\infty\right),\ |\Phi(x_T)| \leq K\left(1 + \|x_T\|_\infty\right).$$

Suppose the class of the strategies θ of the minimizing player is that of non-anticipative strategies (cf. [11]). To be precise, let $\mathcal{A}(t, T) = L^\infty([t, T]; A)$ and $\mathcal{B}(t, T) = L^\infty([t, T]; B)$, which denote the sets of controls of minimizing and maximizing players, respectively. $\theta : \mathcal{B}(t, T) \to \mathcal{A}(t, T)$ is *a non-anticipative strategy* *of the minimizing player* if the following condition holds: Let $b, \tilde{b} \in \mathcal{B}(t,T)$ and $t \leq s \leq T$. If $b(r) = \tilde{b}(r)$ a.e. $r \in [t,s]$, then

$$\theta[b](r) = \theta[\tilde{b}](r), \text{ a.e. } r \in [t,s].$$

Letting $\Theta(t,T)$ be the set of non-anticipative strategies of the minimizing player, the value functional is defined by

$$W(t,x_t) = \inf_{\theta \in \Theta(t,T)} \sup_{b \in \mathcal{B}(t,T)} \mathcal{J}(t,x_t;\theta[b],b). \tag{3.11}$$

Under i) and ii), there exists $C > 0$ such that for any $(t,x_t),(s,y_s) \in \mathbf{X}^\alpha_{0,T}$,

$$\begin{aligned} |W(t,x_t) - W(s,y_s)| \leq &\, C\left(1 + \|x_t\|_\infty + \|{}^C D^\alpha_0 x_t\|_\infty\right)|t-s|^\alpha \\ &+ C \max_{0 \leq r \leq T} |x_t(r \wedge t) - y_s(r \wedge s)|, \end{aligned} \tag{3.12}$$

which implies

$$|W(t,x_t) - W(s,y_s)| \leq C\left(1 + \|x_t\|_\infty + \|{}^C D^\alpha_0 x_t\|_\infty\right)\rho((t,x_t),(s,y_s)).$$

See Appendix B for the sketch of the proof of (3.12). Hence W is continuous on $\mathbf{X}^\alpha_{0,T}$.

Under suitable conditions, it is known that value functionals of optimal control and differential games for fractional order systems are viscosity solutions of HJB equations and Isaacs equations (cf. [24, Theorem 6.3] for HJB equations and [14, Theorem 4.2] for Isaacs equations). In [14], a minimax solution characterization is used to show value functionals are viscosity solutions of the Isaacs equations. By following classical arguments such as [9] (see also [16] for $\alpha = 1$ case), we can directly prove that W is a viscosity solution of (3.10) under conditions i) and ii) for large ν in D_k. Since the arguments are quite similar to those of [9] except for using a chain rule for α-ci-derivatives (cf. [13, Lemma 9.2]) and our main concern is a comparison result for general Hamilton-Jacobi equations, we will just give a sketch of the proof in Appendix C.

4. Comparison theorems for viscosity solutions

We assume F satisfies the following conditions:

(A1) For each D_k $(k = 1,2,\ldots)$, there exists $L_{1,k} > 0$ such that

$$|F(t,x_t,p) - F(s,y_s,p)| \leq L_{1,k}\rho((t,x_t),(s,y_s)),$$
$$\forall (t,x_t),(s,y_s) \in D_k, \ \forall p \in \mathbb{R}^n.$$

(A2) For each D_k $(k = 1,2,\ldots)$, there exists $L_{2,k} > 0$ such that

$$|F(t,x_t,p) - F(t,x_t,q)| \leq L_{2,k}|p-q|, \ \forall (t,x_t) \in D_k, \forall p,q \in \mathbb{R}^n.$$

Theorem 4.1. *Suppose* (A1) *and* (A2) *hold. Let v and w be a viscosity subsolution and viscosity supersolution of* (3.1)*, respectively. If $v(T,x_T) \leq w(T,x_T)$ $(x_T \in AC^\alpha[0,T])$, then*

$$v(t,x_t) \leq w(t,x_t), \ (t,x_t) \in \mathbf{X}^\alpha_{0,T}. \tag{4.1}$$

We prepare a key lemma for the proof of Theorem 4.1. The proof is given in Appendix D.

Lemma 4.2. *Let $(s, y_s) \in \mathbf{X}^\alpha_{0,T}$ be given. Define $\varphi : \mathbf{X}^\alpha_{0,T} \to \mathbb{R}$ by*

$$\varphi(t, x_t) = \frac{|t-s|^2}{2} + \frac{|x_t(0) - y_s(0)|^2}{2} + \frac{1}{2}\int_0^T \left|(I_0^{1-\alpha}\bar{x}_t)(r \wedge t) - (I_0^{1-\alpha}\bar{y}_s)(r \wedge s)\right|^2 dr, \ (t, x_t) \in \mathbf{X}^\alpha_{0,T},$$

where for $z_u : [0, u] \to \mathbb{R}^n$, $\bar{z}_u : [0, u] \to \mathbb{R}^n$ denotes

$$\bar{z}_u(r) = z_u(r) - z_u(0) \quad (0 \le r \le u). \tag{4.2}$$

Then, $\varphi \in \mathcal{C}^1_\alpha(\mathbf{X}^\alpha_{0,T})$ and

$$\partial_t^\alpha \varphi(t, x_t) = t - s,$$

$$\nabla^\alpha_{x_t}\varphi(t, x_t) = \int_t^T \{(I_0^{1-\alpha}\bar{x}_t)(t) - (I_0^{1-\alpha}\bar{y}_s)(r \wedge s)\}dr = (T-t)(I_0^{1-\alpha}\bar{x}_t)(t) - \int_t^T (I_0^{1-\alpha}\bar{y}_s)(r \wedge s)dr.$$

Proof of Theorem 4.1. Noting that $\mathbf{X}^\alpha_{0,T} = \bigcup_{k=1}^\infty D_k$ from Proposition 3.1 (ii), (4.1) is equivalent to

$$v(t, x_t) \le w(t, x_t), \ (t, x_t) \in D_k, \ k = 1, 2, \ldots. \tag{4.3}$$

Suppose that (4.3) does not hold. By the continuity of v and w and Proposition 3.1 (i), there exists k such that

$$\delta := \max_{D_k}(v - w) > 0. \tag{4.4}$$

Take $\gamma > 0$ such that $0 < \gamma < \delta/(4T)$. For $\epsilon > 0$, consider $\Phi_\epsilon : \mathbf{X}^\alpha_{0,T} \times \mathbf{X}^\alpha_{0,T} \to \mathbb{R}$ given by

$$\Phi_\epsilon((t, x_t), (s, y_s)) = v(t, x_t) - w(s, y_s) - \gamma(2T - t - s) - \frac{1}{\epsilon}\nu((t, x_t), (s, y_s)), \ (t, x_t), (s, y_s) \in \mathbf{X}^\alpha_{0,T}, \tag{4.5}$$

where

$$\nu((t, x_t), (s, y_s)) = \frac{|t-s|^2}{2} + \frac{|x_t(0) - y_s(0)|^2}{2} + \frac{1}{2}\int_0^T \left|(I_0^{1-\alpha}\bar{x}_t)(r \wedge t) - (I_0^{1-\alpha}\bar{y}_s)(r \wedge s)\right|^2 dr.$$

See (4.2) for the definitions of $\bar{x}_t$ and $\bar{y}_s$

Let $((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon}))$ be a maximum point of Φ_ϵ on $D_k \times D_k$, that is,

$$\Phi_\epsilon((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) \ge \Phi_\epsilon((t, x_t), (s, y_s)), \ (t, x_t), (s, y_s) \in D_k. \tag{4.6}$$

Since $(T, 0) \in D_k$, we have

$$\Phi_\epsilon((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) \ge \Phi_\epsilon((T, 0), (T, 0)) = v(T, 0) - w(T, 0).$$

We note that

$$\Phi_\epsilon((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) \le \max_{D_k} v - \min_{D_k} w - \frac{1}{\epsilon}\nu((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})).$$

Thus we obtain

$$\frac{1}{\epsilon}\nu((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) \leq \max_{D_k} v - \min_{D_k} w - v(T,0) + w(T,0). \tag{4.7}$$

We prove that

$$\rho((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) \to 0 \ (\epsilon \to 0) \tag{4.8}$$

via a subsequence. Since D_k is compact, there exist $(\tau, \xi_\tau), (\sigma, \eta_\sigma) \in D_k$ such that

$$\rho((t_\epsilon, x^\epsilon_{t_\epsilon}), (\tau, \xi_\tau)) \to 0, \ \ \rho((s_\epsilon, y^\epsilon_{s_\epsilon}), (\sigma, \eta_\sigma)) \to 0 \ (\epsilon \to 0) \tag{4.9}$$

by taking a subsequence, which are equivalent to

$$t_\epsilon \to \tau, \ \max_{0\leq r\leq T} |x^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - \xi_\tau(r\wedge\tau)| \to 0 \ (\epsilon\to 0), \tag{4.10}$$

$$s_\epsilon \to \sigma, \ \max_{0\leq r\leq T} |y^\epsilon_{s_\epsilon}(r\wedge s_\epsilon) - \eta_\sigma(r\wedge\sigma)| \to 0 \ (\epsilon\to 0). \tag{4.11}$$

In the rest of the present proof, we omit subsequences to simplify notations. Note that (4.7) implies

$$t_\epsilon - s_\epsilon \to 0, \ x^\epsilon_{t_\epsilon}(0) - y^\epsilon_{s_\epsilon}(0) \to 0 \ (\epsilon \to 0), \tag{4.12}$$

$$\int_0^T \left|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(r\wedge t_\epsilon) - (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(r\wedge s_\epsilon)\right|^2 dr \to 0 \ (\epsilon\to 0). \tag{4.13}$$

From (4.10), (4.11) and (4.12), we have

$$\tau = \sigma. \tag{4.14}$$

Since $x^\epsilon_{t_\epsilon}(0) \to \xi_\tau(0)$ and $y^\epsilon_{s_\epsilon}(0) \to \eta_\sigma(0)$ by (4.10) and (4.11), we have from (4.12)

$$\xi_\tau(0) = \eta_\sigma(0). \tag{4.15}$$

By Proposition 2.2, we have

$$\begin{aligned}
&\max_{0\leq r\leq T} |(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(r\wedge t_\epsilon) - (I_0^{1-\alpha}\bar{\xi}_\tau)(r\wedge\tau)| \\
&\leq \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \max_{0\leq r\leq T} |\bar{x}^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - \bar{\xi}_\tau(r\wedge\tau)| \\
&\quad + \frac{2}{\Gamma(2-\alpha)}(\|\bar{x}^\epsilon_{t_\epsilon}\|_\infty + \|\bar{\xi}_\tau\|_\infty)|t_\epsilon - \tau|^{1-\alpha} 1_{\{0<\alpha<1\}}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\max_{0\leq r\leq T} |\bar{x}^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - \bar{\xi}_\tau(r\wedge\tau)| \\
&\leq \max_{0\leq r\leq T} |x^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - \xi_\tau(r\wedge\tau)| + |x^\epsilon_{t_\epsilon}(0) - \xi_\tau(0)| \\
&\leq 2\max_{0\leq r\leq T} |x^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - \xi_\tau(r\wedge\tau)|.
\end{aligned}$$

Also, note that

$$\begin{aligned}
\|\bar{x}^\epsilon_{t_\epsilon}\|_\infty &\leq \max_{0\leq r\leq T} |\bar{x}^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - \bar{\xi}_\tau(r\wedge\tau)| + \max_{0\leq r\leq T} |\bar{\xi}_\tau(r\wedge\tau)| \\
&\leq 2\max_{0\leq r\leq T} |x^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - \xi_\tau(r\wedge\tau)| + \|\bar{\xi}_\tau\|_\infty.
\end{aligned}$$

Thus we have

$$\begin{aligned}&\max_{0\le r\le T}|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(r\wedge t_\epsilon)-(I_0^{1-\alpha}\bar{\xi}_\tau)(r\wedge\tau)|\\&\le\frac{2T^{1-\alpha}}{\Gamma(2-\alpha)}\max_{0\le r\le T}|x^\epsilon_{t_\epsilon}(r\wedge t_\epsilon)-\xi_\tau(r\wedge\tau)|\\&\quad+\frac{4}{\Gamma(2-\alpha)}\left\{\max_{0\le r\le T}|x^\epsilon_{t_\epsilon}(r\wedge t_\epsilon)-\xi_\tau(r\wedge\tau)|+\|\bar{\xi}_\tau\|_\infty\right\}\\&\qquad\qquad\times|t_\epsilon-\tau|^{1-\alpha}1_{\{0<\alpha<1\}}.\end{aligned}$$

By (4.10), we have

$$\max_{0\le r\le T}|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(r\wedge t_\epsilon)-(I_0^{1-\alpha}\bar{\xi}_\tau)(r\wedge\tau)|\to 0\ (\epsilon\to 0).\tag{4.16}$$

Similarly, we obtain

$$\max_{0\le r\le T}|(I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(r\wedge s_\epsilon)-(I_0^{1-\alpha}\bar{\eta}_\sigma)(r\wedge\sigma)|\to 0\ (\epsilon\to 0).\tag{4.17}$$

By (4.16) and (4.17), we have

$$\begin{aligned}&\int_0^T\left|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(r\wedge t_\epsilon)-(I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(r\wedge s_\epsilon)\right|^2dr\\&\qquad\to\int_0^T\left|(I_0^{1-\alpha}\bar{\xi}_\tau)(r\wedge\tau)-(I_0^{1-\alpha}\bar{\eta}_\sigma)(r\wedge\sigma)\right|^2dr\ (\epsilon\to 0).\end{aligned}$$

Noting (4.13) with the above convergence, we have

$$\int_0^T\left|(I_0^{1-\alpha}\bar{\xi}_\tau)(r\wedge\tau)-(I_0^{1-\alpha}\bar{\eta}_\sigma)(r\wedge\sigma)\right|^2dr=0,$$

which implies with $\tau=\sigma$

$$(I_0^{1-\alpha}\bar{\xi}_\tau)(r)=(I_0^{1-\alpha}\bar{\eta}_\sigma)(r),\ r\in[0,\tau]=[0,\sigma].\tag{4.18}$$

Note that

$$\bar{\xi}_\tau=I_0^{\alpha}{}^CD_0^\alpha\xi_\tau,\ \bar{\eta}_\sigma=I_0^{\alpha}{}^CD_0^\alpha\eta_\sigma\text{ on }[0,\tau]=[0,\sigma].$$

Plugging these into (4.18) and using the semi-group property of Riemann-Louville integral operators (cf. [4, Theorem 2.4]), we have

$$I_0({}^CD_0^\alpha\xi_\tau)(r)=I_0({}^CD_0^\alpha\eta_\sigma)(r),\ r\in[0,\tau]=[0,\sigma].$$

Differentiating the above equation on r, we have

$${}^CD_0^\alpha\xi_\tau(r)={}^CD_0^\alpha\eta_\sigma(r),\ a.e.\,r\in[0,\tau]=[0,\sigma].$$

Thus we obtain

$$\begin{aligned}\xi_\tau(r)&=\xi_\tau(0)+I_0^\alpha({}^CD_0^\alpha\xi_\tau)(r)\\&=\eta_\sigma(0)+I_0^\alpha({}^CD_0^\alpha\eta_\sigma)(r)=\eta_\sigma(r),\ r\in[0,\tau]=[0,\sigma].\end{aligned}$$

Thus we have

$$(\tau,\xi_\tau)=(\sigma,\eta_\sigma).\tag{4.19}$$

Hence, by (4.9), we obtain (4.8).

By (4.6), we have

$$\begin{aligned}\Phi_\epsilon((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) &\geq \Phi_\epsilon((\tau, \xi_\tau), (\tau, \xi_\tau)) \\ &= v(\tau, \xi_\tau) - w(\tau, \xi_\tau) - \gamma(2T - 2\tau),\end{aligned}$$

from which, we obtain

$$\begin{aligned}&\frac{1}{\epsilon}\nu((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) \\ &\leq v(t_\epsilon, x^\epsilon_{t_\epsilon}) - v(\tau, \xi_\tau) + w(\tau, \xi_\tau) - w(s_\epsilon, y^\epsilon_{s_\epsilon}) - \gamma(2\tau - t_\epsilon - s_\epsilon).\end{aligned} \tag{4.20}$$

Since $\rho((t_\epsilon, x^\epsilon_{t_\epsilon}), (\tau, \xi_\tau)) \to 0$, $\rho((s_\epsilon, y^\epsilon_{s_\epsilon}), (\sigma, \eta_\sigma)) = \rho((s_\epsilon, y^\epsilon_{s_\epsilon}), (\tau, \xi_\tau)) \to 0$ as $\epsilon \to 0$, we have

$$\frac{1}{\epsilon}\nu((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) \to 0 \ (\epsilon \to 0). \tag{4.21}$$

We will show that there exist $\vartheta > 0$ and $\epsilon_0 > 0$ such that

$$t_\epsilon, s_\epsilon \leq T - \vartheta < T, \ 0 < \forall\epsilon < \epsilon_0. \tag{4.22}$$

Let $(\hat{t}, \hat{x}_{\hat{t}}) \in D_k$ be a maximum point of $v(t, x_t) - w(t, x_t)$ on D_k. Taking $(t, x_t) = (s, y_s) = (\hat{t}, \hat{x}_{\hat{t}})$ at (4.6), we have

$$\begin{aligned}\Phi_\epsilon((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) &\geq \Phi_\epsilon((\hat{t}, \hat{x}_{\hat{t}}), (\hat{t}, \hat{x}_{\hat{t}})) \\ &= v(\hat{t}, \hat{x}_{\hat{t}}) - w(\hat{t}, \hat{x}_{\hat{t}}) - \gamma(2T - 2\hat{t}) \\ &= \delta - \gamma(2T - 2\hat{t}) \geq \delta - 2\gamma T.\end{aligned}$$

Here we recall δ is defined in (4.4). Since we have

$$\Phi_\epsilon((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) \leq v(t_\epsilon, x^\epsilon_{t_\epsilon}) - w(s_\epsilon, y^\epsilon_{s_\epsilon}),$$

we obtain

$$\delta - 2\gamma T \leq v(t_\epsilon, x^\epsilon_{t_\epsilon}) - w(s_\epsilon, y^\epsilon_{s_\epsilon}).$$

Since $0 < \gamma < \delta/(4T)$, we have

$$\frac{\delta}{2} \leq v(t_\epsilon, x^\epsilon_{t_\epsilon}) - w(s_\epsilon, y^\epsilon_{s_\epsilon}).$$

Taking the limit as $\epsilon \to 0$, we have

$$\frac{\delta}{2} \leq v(\tau, \xi_\tau) - w(\tau, \xi_\tau).$$

Suppose $\tau = T$. Since $v(T, x_T) \leq w(T, x_T)$ $(x_T \in AC^\alpha[0, T])$, we have

$$\frac{\delta}{2} \leq v(T, \xi_\tau) - w(T, \xi_\tau) \leq 0,$$

which contradicts to $\delta > 0$. Thus $\tau < T$. Since $t_\epsilon, s_\epsilon \to \tau$ as $\epsilon \to 0$, (4.22) holds for some $\vartheta > 0$ and $\epsilon_0 > 0$.

Suppose $0 < \epsilon < \epsilon_0$. By (4.6), note that

$$\Phi_\epsilon((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) \geq \Phi_\epsilon((t, x_t), (s_\epsilon, y^\epsilon_{s_\epsilon})), \ (t, x_t) \in D_k,$$

which is equivalent to

$$v(t, x_t) - \phi(t, x_t) \leq v(t_\epsilon, x^\epsilon_{t_\epsilon}) - \phi(t_\epsilon, x^\epsilon_{t_\epsilon}), \ (t, x_t) \in D_k, \tag{4.23}$$

where

$$\phi(t, x_t) = w(s_\epsilon, y^\epsilon_{s_\epsilon}) + \gamma(2T - t - s_\epsilon) + \frac{1}{\epsilon}\nu((t, x_t), (s_\epsilon, y^\epsilon_{s_\epsilon})).$$

We also note that

$$\Phi_\epsilon((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) \geq \Phi_\epsilon((t_\epsilon, x^\epsilon_{t_\epsilon}), (s, y_s)),\ (s, y_s) \in D_k,$$

which is equivalent to

$$w(s_\epsilon, y^\epsilon_{s_\epsilon}) - \psi(s_\epsilon, y^\epsilon_{s_\epsilon}) \leq w(s, y_s) - \psi(s, y_s),\ (s, y_s) \in D_k, \tag{4.24}$$

where

$$\psi(s, y_s) = v(t_\epsilon, x^\epsilon_{t_\epsilon}) - \gamma(2T - t_\epsilon - s) - \frac{1}{\epsilon}\nu((t_\epsilon, x^\epsilon_{t_\epsilon}), (s, y_s)).$$

Note that ϕ, $\psi \in \mathcal{C}^1_\alpha(\mathbf{X}^\alpha_{0,T})$ from Lemma 4.2. Calculating α-ci-derivatives of ϕ and ψ by using Lemma 4.2, we have

$$\partial^\alpha_t \phi(t_\epsilon, x^\epsilon_{t_\epsilon}) = -\gamma + \frac{1}{\epsilon}(t_\epsilon - s_\epsilon),\ \partial^\alpha_t \psi(s_\epsilon, y^\epsilon_{s_\epsilon}) = \gamma + \frac{1}{\epsilon}(t_\epsilon - s_\epsilon),$$

$$\begin{aligned} &\nabla^\alpha_{x_t}\phi(t_\epsilon, x^\epsilon_{t_\epsilon}) \\ &= \frac{1}{\epsilon}\left\{(T - t_\epsilon)(I^{1-\alpha}_0 \bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) - \int_{t_\epsilon}^T (I^{1-\alpha}_0 \bar{y}^\epsilon_{s_\epsilon})(r \wedge s_\epsilon)dr\right\} =: p_\epsilon, \end{aligned} \tag{4.25}$$

$$\begin{aligned} &\nabla^\alpha_{y_s}\psi(s_\epsilon, y^\epsilon_{s_\epsilon}) \\ &= \frac{1}{\epsilon}\left\{-(T - s_\epsilon)(I^{1-\alpha}_0 \bar{y}^\epsilon_{s_\epsilon})(s_\epsilon) + \int_{s_\epsilon}^T (I^{1-\alpha}_0 \bar{x}^\epsilon_{t_\epsilon})(r \wedge t_\epsilon)dr\right\} =: q_\epsilon. \end{aligned} \tag{4.26}$$

Since v is a viscosity subsolution of (3.1), we have

$$-\gamma + \frac{1}{\epsilon}(t_\epsilon - s_\epsilon) + F(t_\epsilon, x^\epsilon_{t_\epsilon}, p_\epsilon) \geq 0. \tag{4.27}$$

Since w is a viscosity supersolution of (3.1), we obtain

$$\gamma + \frac{1}{\epsilon}(t_\epsilon - s_\epsilon) + F(s_\epsilon, y^\epsilon_{s_\epsilon}, q_\epsilon) \leq 0. \tag{4.28}$$

Subtracting (4.27) from (4.28), we have

$$2\gamma \leq F(t_\epsilon, x^\epsilon_{t_\epsilon}, p_\epsilon) - F(s_\epsilon, y^\epsilon_{s_\epsilon}, q_\epsilon),$$

from which, we have

$$\begin{aligned} 2\gamma &\leq F(t_\epsilon, x^\epsilon_{t_\epsilon}, p_\epsilon) - F(s_\epsilon, y^\epsilon_{s_\epsilon}, q_\epsilon) \\ &\leq |F(t_\epsilon, x^\epsilon_{t_\epsilon}, p_\epsilon) - F(s_\epsilon, y^\epsilon_{s_\epsilon}, p_\epsilon)| + |F(s_\epsilon, y^\epsilon_{s_\epsilon}, p_\epsilon) - F(s_\epsilon, y^\epsilon_{s_\epsilon}, q_\epsilon)|. \end{aligned}$$

By (A1) and (A2) with D_k, we have

$$2\gamma \leq L_{1,k}\rho((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) + L_{2,k}|p_\epsilon - q_\epsilon|. \tag{4.29}$$

We estimate $|p_\epsilon - q_\epsilon|$. Suppose $t_\epsilon \le s_\epsilon$. Then we have

$$\begin{aligned}|p_\epsilon - q_\epsilon| &= \frac{1}{\epsilon}\Big|(T-t_\epsilon)(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) - \int_{t_\epsilon}^T (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(r\wedge s_\epsilon)dr \\ &\qquad + (T-s_\epsilon)(I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon) - \int_{s_\epsilon}^T (I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(r\wedge t_\epsilon)dr\Big| \\ &= \frac{1}{\epsilon}\Big|(T-t_\epsilon)(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) \\ &\qquad - \int_{t_\epsilon}^{s_\epsilon} (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(r)dr - \int_{s_\epsilon}^T (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon)dr \\ &\qquad + (T-s_\epsilon)(I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon) - \int_{s_\epsilon}^T (I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon)dr\Big| \\ &= \frac{1}{\epsilon}\Big|(s_\epsilon-t_\epsilon)(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) - \int_{t_\epsilon}^{s_\epsilon} (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(r)dr\Big|.\end{aligned}$$

For $t_\epsilon \le r \le s_\epsilon$, we have

$$\begin{aligned}(I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(r) &= I_0^{1-\alpha}I_0^\alpha({}^CD_0^\alpha y^\epsilon_{s_\epsilon})(r) = I_0({}^CD_0^\alpha y^\epsilon_{s_\epsilon})(r) = \int_0^r {}^CD_0^\alpha y^\epsilon_{s_\epsilon}(u)du \\ &= \int_0^{s_\epsilon} {}^CD_0^\alpha y^\epsilon_{s_\epsilon}(u)du - \int_r^{s_\epsilon} {}^CD_0^\alpha y^\epsilon_{s_\epsilon}(u)du \\ &= I_0({}^CD_0^\alpha y^\epsilon_{s_\epsilon})(s_\epsilon) - \int_r^{s_\epsilon} {}^CD_0^\alpha y^\epsilon_{s_\epsilon}(u)du \\ &= I_0^{1-\alpha}I_0^\alpha({}^CD_0^\alpha y^\epsilon_{s_\epsilon})(s_\epsilon) - \int_r^{s_\epsilon} {}^CD_0^\alpha y^\epsilon_{s_\epsilon}(u)du \\ &= (I^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon) - \int_r^{s_\epsilon} {}^CD_0^\alpha y^\epsilon_{s_\epsilon}(u)du.\end{aligned}$$

Thus, we have

$$\begin{aligned}&|p_\epsilon - q_\epsilon| \\ &= \frac{1}{\epsilon}\Big|(s_\epsilon-t_\epsilon)(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) - \int_{t_\epsilon}^{s_\epsilon}\Big\{(I^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon) - \int_r^{s_\epsilon} {}^CD_0^\alpha y^\epsilon_{s_\epsilon}(u)du\Big\}dr\Big| \\ &= \frac{1}{\epsilon}\Big|(s_\epsilon-t_\epsilon)\{(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) - (I^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon)\} - \int_{t_\epsilon}^{s_\epsilon}\int_r^{s_\epsilon} {}^CD_0^\alpha y^\epsilon_{s_\epsilon}(u)dudr\Big| \\ &\le \frac{1}{\epsilon}|s_\epsilon-t_\epsilon||(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) - (I^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon)| + \frac{1}{\epsilon}\int_{t_\epsilon}^{s_\epsilon}\int_{t_\epsilon}^{s_\epsilon} |{}^CD_0^\alpha y^\epsilon_{s_\epsilon}(u)|dudr.\end{aligned} \tag{4.30}$$

Since D_k is compact, D_k is bounded, that is, there exists $C_k > 0$ such that

$$D_k \subset \{(u, z_u) \in \mathbf{X}^\alpha_{0,T};\ \rho((u,z_u),(0,0)) \le C_k\}.$$

Noting that $(s_\epsilon, y^\epsilon_{s_\epsilon}) \in D_k$, we have

$$\|y^\epsilon_{s_\epsilon}\|_\infty = \max_{0\le r\le s_\epsilon} |y^\epsilon_{s_\epsilon}(r)| \le C_k. \tag{4.31}$$

Also, since $(s_\epsilon, y^\epsilon_{s_\epsilon}) \in D_k$, we have

$$\operatorname*{ess.sup}_{0\le r\le s_\epsilon} |{}^C D_0^\alpha y^\epsilon_{s_\epsilon}(r)| = \|{}^C D_0^\alpha y^\epsilon_{s_\epsilon}\|_\infty \le k\nu(1+\|y^\epsilon_{s_\epsilon}\|_\infty)$$

which implies from (4.31) that

$$\|{}^C D_0^\alpha y^\epsilon_{s_\epsilon}\|_\infty \le k\nu(1+C_k) =: \tilde{C}_k. \tag{4.32}$$

Hence we have from (4.30)

$$|p_\epsilon - q_\epsilon| \le \frac{1}{\epsilon}|s_\epsilon - t_\epsilon| \left|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) - (I^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon)\right| + \tilde{C}_k \frac{|s_\epsilon - t_\epsilon|^2}{\epsilon}. \tag{4.33}$$

In the case where $s_\epsilon < t_\epsilon$, we can have (4.33) in a manner similar to the case where $t_\epsilon \le s_\epsilon$.

Recalling (4.21), we have

$$\frac{1}{\epsilon}|t_\epsilon - s_\epsilon|^2 \to 0 \ (\epsilon \to 0), \tag{4.34}$$

$$\frac{1}{\epsilon}\int_0^T \left|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(r\wedge t_\epsilon) - (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(r\wedge s_\epsilon)\right|^2 dr \to 0 \ (\epsilon \to 0). \tag{4.35}$$

Suppose $t_\epsilon \le s_\epsilon$. Then, we have

$$\begin{aligned}
&\frac{1}{\epsilon}\int_0^T |(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(r\wedge t_\epsilon) - (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(r\wedge s_\epsilon)|^2 dr \\
&= \frac{1}{\epsilon}\int_0^{t_\epsilon} \left|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(r) - (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(r)\right|^2 dr \\
&\quad + \frac{1}{\epsilon}\int_{t_\epsilon}^{s_\epsilon} \left|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) - (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(r)\right|^2 dr \\
&\quad + \frac{1}{\epsilon}\int_{s_\epsilon}^{T} \left|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) - (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon)\right|^2 dr \\
&\ge \frac{1}{\epsilon}(T-s_\epsilon)\left|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) - (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon)\right|^2.
\end{aligned}$$

Supposing $s_\epsilon < t_\epsilon$, we have

$$\begin{aligned}
&\frac{1}{\epsilon}\int_0^T \left|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(r\wedge t_\epsilon) - (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(r\wedge s_\epsilon)\right|^2 dr \\
&= \frac{1}{\epsilon}\int_0^{s_\epsilon} \left|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(r) - (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(r)\right|^2 dr \\
&\quad + \frac{1}{\epsilon}\int_{s_\epsilon}^{t_\epsilon} \left|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(r) - (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon)\right|^2 dr \\
&\quad + \frac{1}{\epsilon}\int_{t_\epsilon}^{T} \left|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) - (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon)\right|^2 dr \\
&\ge \frac{1}{\epsilon}(T-t_\epsilon)\left|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) - (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon)\right|^2.
\end{aligned}$$

Noting (4.22), we have from (4.35)

$$\frac{1}{\epsilon}|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) - (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon)|^2 \to 0 \ (\epsilon \to 0). \tag{4.36}$$

By (4.33), we see that

$$|p_\epsilon - q_\epsilon| \le \frac{1}{2\epsilon}|s_\epsilon - t_\epsilon|^2 + \frac{1}{2\epsilon}\left|(I_0^{1-\alpha}\bar{x}^\epsilon_{t_\epsilon})(t_\epsilon) - (I_0^{1-\alpha}\bar{y}^\epsilon_{s_\epsilon})(s_\epsilon)\right|^2 + \tilde{C}_k\frac{|s_\epsilon - t_\epsilon|^2}{\epsilon}.$$

Taking the limit as $\epsilon \to 0$ with (4.34) and (4.36), we obtain

$$|p_\epsilon - q_\epsilon| \to 0 \ (\epsilon \to 0). \tag{4.37}$$

Letting $\epsilon \to 0$ in (4.29), we have

$$2\gamma \le 0,$$

which contradicts to $\gamma > 0$. Hence (4.3) holds. □

Remark 4.3. Aiming at applications of Theorem 4.1 for HJB-equations and Isaacs equations, (A1) is restrictive. To cover general vector fields for (3.5), (A1) should be the following condition (see Example 4.4 below): For each D_k $(k = 1, 2, \ldots)$, there exists $L_{1,k} > 0$ such that

$$\begin{aligned} &|F(t, x_t, p) - F(s, y_s, p)| \le L_{1,k}(1 + |p|)\rho((t, x_t), (s, y_s)), \\ &\qquad \forall (t, x_t), (s, y_s) \in D_k,\ \forall p \in \mathbb{R}^n. \end{aligned} \tag{4.38}$$

Following the argument of the proof of Theorem 4.1, we need to show that

$$|p_\epsilon|\rho((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) \to 0 \ (\epsilon \to 0).$$

However it is not clear whether it holds or not. There might be additional conditions on F and/or viscosity sub/super solutions. We leave this problem for the future research. For $\alpha = 1$, we will see in the next section that a comparison theorem holds under an L^2-continuity condition on F by using our distance functional.

Example 4.4. Consider $H : \mathbf{X}^\alpha_{0,T} \times \mathbb{R}^n \to \mathbb{R}$ given by

$$H(t, x_t, p) = \sup_{b\in B}\inf_{a\in A}\{g(t, x_t, a, b) \cdot p + l(t, x_t, a, b)\},$$

where A and B are compact metric spaces, $g : \mathbf{X}^\alpha_{0,T} \times A \times B \to \mathbb{R}^n$ and $l : \mathbf{X}^\alpha_{0,T} \times A \times B \to \mathbb{R}$ are continuous. Suppose that the following condition holds: for each D_k $(k = 1, 2, \ldots)$, there exist $L_{g,k}$, $L_{l,k} \ge 0$ such that for any $(t, x_t), (s, y_s) \in D_k$, $a \in A$, $b \in B$,

$$|g(t, x_t, a, b) - g(s, y_s, a, b)| \le L_{g,k}\rho((t, x_t), (s, y_s)), \tag{4.39}$$

$$|l(t, x_t, a, b) - l(s, y_s, a, b)| \le L_{l,k}\rho((t, x_t), (s, y_s)). \tag{4.40}$$

Then, for $(t, x_t), (s, y_s) \in D_k$ and $p, q \in \mathbb{R}^n$, we have

$$|H(t, x_t, p) - H(s, y_s, p)| \le (L_{l,k} + L_{g,k}|p|)\rho((t, x_t), (s, y_s)), \tag{4.41}$$

$$|H(t, x_t, p) - H(t, x_t, q)| \le \max_{(t,x_t)\in D_k, a\in A, b\in B}|g(t, x_t, a, b)||p - q|. \tag{4.42}$$

Consider the case where $g(t, x_t, a, b) = c(a, b)$ for some $c : A \times B \to \mathbb{R}^n$, then (4.41) can be

$$|H(t, x_t, p) - H(s, y_s, p)| \le L_{l,k}\rho((t, x_t), (s, y_s)).$$

Hence (A1) and (A2) hold for this particular case. In this case, we see that W given by (3.11) is the unique viscosity solution of (3.10) satisfying $W(T, x_T) = \Phi(x_T)$ $(x_T \in AC^\alpha[0, T])$ under conditions i) and ii) with (4.40).

Lastly we give examples of $h = g, l : \mathbf{X}^\alpha_{0,T} \times A \times B \to \mathbb{R}^\kappa$ ($\kappa = n$ or 1) satisfying (4.39) and (4.40) (which imply (4.41) and (4.42)) with specific delay structures. Let $\mu_h : [0,T] \times \mathbb{R}^n \times A \times B \to \mathbb{R}^\kappa$ be continuous. Suppose that there exists $L_{\mu_h} > 0$ such that for any $x, y \in \mathbb{R}^n$, $t, s \in [0,T]$, $a \in A$ and $b \in B$,

$$|\mu_h(t,x,a,b) - \mu_h(t,y,a,b)| \le L_{\mu_h}|x-y|,$$
$$|\mu_h(t,x,a,b) - \mu_h(s,x,a,b)| \le L_{\mu_h}(1+|x|)|t-s|.$$

i) Discrete delay. Let $h : \mathbf{X}^\alpha_{0,T} \times A \times B \to \mathbb{R}^\kappa$ be given by

$$h(t,x_t,a,b) = \mu_h(t, x_t(\sigma_h(t)), a, b),\ (t,x_t) \in \mathbf{X}^\alpha_{0,T}, a \in A, b \in B,$$

where $\sigma_h : [0,T] \to [0,T]$ is a Lipschitz continuous function satisfying $0 \le \sigma_h(t) \le t$ ($0 \le t \le T$). Note that $\|x_t\|_\infty$ and $\|{}^C D^\alpha_0 x_t\|_\infty$ are bounded on $(t,x_t) \in D_k$ (see, for instance, (4.31) and (4.32)). Using Proposition 2.1 (also, [13, (2.1)]) and [13, Lem. 7.2], we see that there exists $\tilde{C}_k > 0$ such that for any $(t,x_t),(s,y_s) \in D_k$, $a \in A$, $b \in B$,

$$|\mu_h(t,x_t(\sigma_h(t)),a,b) - \mu_h(s,y_s(\sigma_h(s)),a,b)| \le \tilde{C}_k\left\{|t-s|^\alpha + \max_{0\le r\le T}|x_t(r\wedge t) - y_s(r\wedge s)|\right\}.$$

ii) Distributed delay. Let $h : \mathbf{X}^\alpha_{0,T} \times A \times B \to \mathbb{R}^\kappa$ be given by

$$h(t,x_t,a,b) = \mu_h\left(t, \int_0^t x_t(r)dr, a, b\right),\ (t,x_t) \in \mathbf{X}^\alpha_{0,T}, a \in A, b \in B.$$

Noting that

$$\left|\int_0^t x_t(r)dr - \int_0^s y_s(r)dr\right| \le T\max_{0\le r\le T}|x_t(r\wedge t) - y_s(r\wedge s)| + (\|x_t\|_\infty + \|y_s\|_\infty)|t-s|$$

and the uniform bound $\|x_t\|_\infty$ for $(t,x_t) \in D_k$, there exists $\bar{C}_k > 0$ such that for any $(t,x_t),(s,y_s) \in D_k$, $a \in A$, $b \in B$,

$$\left|\mu_h\left(t, \int_0^t x_t(r)dr, a, b\right) - \mu_h\left(s, \int_0^s y_s(r)dr, a, b\right)\right| \le \bar{C}_k\left\{|t-s| + \max_{0\le r\le T}|x_t(r\wedge t) - y_s(r\wedge s)|\right\}.$$

5. A remark on functional Hamilton-Jacobi equations with ci-derivatives

We consider the case where $\alpha = 1$ in (3.1), that is,

$$\partial_t v(t,x_t) + F(t,x_t,\nabla_{x_t}v(t,x_t)) = 0,\ (t,x_t) \in \mathbf{X}^1_{0,T-}, \tag{5.1}$$

where we denote ∂^1_t and $\nabla^1_{x_t}$ by ∂_t and ∇_{x_t}, respectively. (3.1) is a functional Hamilton-Jacobi equation studied in [23] which typically includes Hamilton-Jacobi-Bellman equations of optimal control and Isaacs equations of differential games for path-dependent systems governed by ordinary differential equations (cf. [21], [22],

[23]). In this section, we will remark that our test function used in the proof of Theorem 4.1 is valid for the comparison arguments under L^2-continuity conditions on trajectory-spaces as assumed in [23]. This means that the test function in Theorem 4.1 is a natural extension of the case $\alpha = 1$ to $0 < \alpha < 1$. In order to avoid technical issues, we suppose that the domain of functionals is $\mathbf{X}^1_{0,T}$ which is smaller than $\mathbf{X}_{0,T} := \bigcup_{t\in[0,T]}(\{t\} \times C[0,t])$ treated in [23].

Instead of (A1), suppose that $F : \mathbf{X}^1_{0,T} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following condition:

(A1)' For each D_k $(k = 1, 2, \ldots)$, there exists $L'_{1,k} > 0$ such that

$$\begin{aligned}
&|F(t, x_t, p) - F(s, y_s, p)| \\
&\le L'_{1,k}(1 + |p|) \\
&\quad \times \left\{ |t - s| + |x_t(t) - y_s(s)| + \left(\int_0^T |x_t(r \wedge t) - y_s(r \wedge s)|^2 dr \right)^{1/2} \right\}, \\
&\qquad \forall (t, x_t), (s, y_s) \in D_k,\ \forall p \in \mathbb{R}^n.
\end{aligned}$$

Theorem 5.1 (cf. Theorem 2 of [23])**.** *Suppose* (A1)' *and* (A2) *hold. Let v and w be a viscosity subsolution and viscosity supersolutions of* (5.1)*, respectively. If $v(T, x_T) \le w(T, x_T)$ $(x_T \in AC^1[0, T])$, then*

$$v(t, x_t) \le w(t, x_t),\ (t, x_t) \in \mathbf{X}^1_{0,T}.$$

Proof. The proof of Theorem 5.1 proceeds in a way similar to that of [23, Theorem 2] with the choice of the test function of Theorem 4.1. In the rest of the current proof, we take $\alpha = 1$. Suppose $v \le w$ in $\mathbf{X}^1_{0,T} = \bigcup_{k=1}^{\infty} D_k$ does not hold, that is, there exists $k \in \mathbb{N}$ such that

$$\delta := \max_{D_k}(v - w) > 0.$$

Let $\epsilon > 0$ and $0 < \gamma < \delta/(4T)$. We consider $\Phi_\epsilon : \mathbf{X}^1_{0,T} \times \mathbf{X}^1_{0,T} \to \mathbb{R}$ as given in (4.5). Let $((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) \in D_k \times D_k$ be a maximum point of Φ_ϵ on $D_k \times D_k$. In the proof of Theorem 4.1, we see that

$$\frac{1}{\epsilon}\nu((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon})) \to 0\ (\epsilon \to 0), \tag{5.2}$$

by taking a subsequence. We suppose the subsequence of the limits in the above is a full sequence for the simplicity of notations. We also have (4.22).

Let $0 < \epsilon < \epsilon_0$ where ϵ_0 is taken from (4.22). Since $((t_\epsilon, x^\epsilon_{t_\epsilon}), (s_\epsilon, y^\epsilon_{s_\epsilon}))$ is a maximum point of Φ_ϵ on $D_k \times D_k$, we have (4.23) and (4.24). By using the definitions of viscosity subsolutions and supersolutions, we have (4.27) and (4.28), which implies

$$2\gamma \le F(t_\epsilon, x^\epsilon_{t_\epsilon}, p_\epsilon) - F(s_\epsilon, y^\epsilon_{s_\epsilon}, q_\epsilon),$$

where p_ϵ and q_ϵ are given in (4.25) and (4.26), respectively. Thus we have

$$\begin{aligned}
2\gamma &\le F(t_\epsilon, x^\epsilon_{t_\epsilon}, p_\epsilon) - F(s_\epsilon, y^\epsilon_{s_\epsilon}, q_\epsilon) \\
&\le |F(t_\epsilon, x^\epsilon_{t_\epsilon}, p_\epsilon) - F(s_\epsilon, y^\epsilon_{s_\epsilon}, p_\epsilon)| + |F(s_\epsilon, y^\epsilon_{s_\epsilon}, p_\epsilon) - F(s_\epsilon, y^\epsilon_{s_\epsilon}, q_\epsilon)|.
\end{aligned}$$

Using (A1)' and (A2) with D_k, we have

$$2\gamma \le L'_{1,k}(1+|p_\epsilon|)\left\{|t_\epsilon - s_\epsilon| + |x^\epsilon_{t_\epsilon}(t_\epsilon) - y^\epsilon_{s_\epsilon}(s_\epsilon)| + \left(\int_0^T |x^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - y^\epsilon_{s_\epsilon}(r\wedge s_\epsilon)|^2 dr\right)^{1/2}\right\} \tag{5.3}$$

$$+ L_{2,k}|p_\epsilon - q_\epsilon|. \tag{5.4}$$

By (4.37), we have

$$|p_\epsilon - q_\epsilon| = o(1)\ (\epsilon \to 0). \tag{5.5}$$

By (5.2), we have

$$\frac{|t_\epsilon - s_\epsilon|^2}{\epsilon} = o(1),\ \frac{|x^\epsilon_{t_\epsilon}(0) - y^\epsilon_{s_\epsilon}(0)|^2}{\epsilon} = o(1)\ (\epsilon \to 0) \tag{5.6}$$

$$\frac{1}{\epsilon}\int_0^T |\bar{x}^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - \bar{y}^\epsilon_{s_\epsilon}(r\wedge s_\epsilon)|^2 dr = o(1)\ (\epsilon \to 0), \tag{5.7}$$

where $\bar{z}_u(r) = z_u(r) - z_u(0)$ $(0 \le r \le u)$ for $z_u : [0,u] \to \mathbb{R}^n$. We note that from (5.6) and (5.7)

$$\frac{1}{\epsilon}\int_0^T |x^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - y^\epsilon_{s_\epsilon}(r\wedge s_\epsilon)|^2 dr = o(1)\ (\epsilon \to 0). \tag{5.8}$$

Since (4.36) holds for $\alpha = 1$, that is,

$$\frac{1}{\epsilon}|\bar{x}^\epsilon_{t_\epsilon}(t_\epsilon) - \bar{y}^\epsilon_{s_\epsilon}(s_\epsilon)|^2 = o(1)\ (\epsilon \to 0),$$

we obtain with (5.6)

$$\frac{1}{\epsilon}|x^\epsilon_{t_\epsilon}(t_\epsilon) - y^\epsilon_{s_\epsilon}(s_\epsilon)|^2 = o(1)\ (\epsilon \to 0). \tag{5.9}$$

Thus we have from (5.6), (5.8) and (5.9)

$$|t_\epsilon - s_\epsilon| + |x^\epsilon_{t_\epsilon}(t_\epsilon) - y^\epsilon_{s_\epsilon}(s_\epsilon)| + \left(\int_0^T |x^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - y^\epsilon_{s_\epsilon}(r\wedge s_\epsilon)|^2 dr\right)^{1/2} = o(\epsilon^{1/2})\ (\epsilon \to 0). \tag{5.10}$$

Recalling p_ϵ from (4.25) with $\alpha = 1$, we have

$$\begin{aligned} p_\epsilon &= \frac{1}{\epsilon}\left\{(T - t_\epsilon)\bar{x}^\epsilon_{t_\epsilon}(t_\epsilon) - \int_{t_\epsilon}^T \bar{y}^\epsilon_{s_\epsilon}(r\wedge s_\epsilon)dr\right\} \\ &= \frac{1}{\epsilon}\int_{t_\epsilon}^T \bar{x}^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - \bar{y}^\epsilon_{s_\epsilon}(r\wedge s_\epsilon)dr, \end{aligned}$$

from which, we have

$$\begin{aligned} |p_\epsilon| &\le \frac{1}{\epsilon}\int_0^T |\bar{x}^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - \bar{y}^\epsilon_{s_\epsilon}(r\wedge s_\epsilon)|dr \\ &\le \frac{\sqrt{T}}{\epsilon}\left(\int_0^T |\bar{x}^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - \bar{y}^\epsilon_{s_\epsilon}(r\wedge s_\epsilon)|^2 dr\right)^{1/2}. \end{aligned}$$

Using (5.7), we have

$$|p_\epsilon| = \frac{1}{\epsilon^{1/2}} o(1) \ (\epsilon \to 0). \tag{5.11}$$

By (5.10) and (5.11), we have

$$\begin{aligned}(5.12)\quad & (1+|p_\epsilon|) \\ & \times \left\{ |t_\epsilon - s_\epsilon| + |x^\epsilon_{t_\epsilon}(t_\epsilon) - y^\epsilon_{s_\epsilon}(s_\epsilon)| + \left(\int_0^T |x^\epsilon_{t_\epsilon}(r\wedge t_\epsilon) - y^\epsilon_{s_\epsilon}(r \wedge s_\epsilon)|^2 dr \right)^{1/2} \right\} \\ & = \left(1 + \frac{1}{\epsilon^{1/2}} o(1) \right) \times o(\epsilon^{1/2}) = o(\epsilon^{1/2}) + o(1) = o(1) \ (\epsilon \to 0).\end{aligned}$$

Taking the limit as $\epsilon \to 0$ in (5.4), we have with (5.5) and (5.12)

$$2\gamma \le 0,$$

which contradicts to $\gamma > 0$. Hence we have

$$v(t, x_t) \le w(t, x_t), \ (t, x_t) \in \mathbf{X}^1_{0,T} = \bigcup_{k=1}^\infty D_k. \quad \square$$

Appendix A. Proof of Proposition 2.2

If $\alpha = 1$, (2.2) is immediate because $I_0^{1-\alpha} = I_0^0$ is an identity mapping. Suppose $0 < \alpha < 1$. Let $t, s \in [0, T]$, $x_t \in C[0, t]$ and $y_s \in C[0, s]$. Let $0 \le r \le T$. Suppose $t \le s$. Then, we have

$$\begin{aligned}& \left| (I_0^{1-\alpha} x_t)(r\wedge t) - (I_0^{1-\alpha} y_s)(r \wedge s) \right| \\ & \le \left| (I_0^{1-\alpha} x_t)(r\wedge t) - (I_0^{1-\alpha} y_s)(r \wedge t) \right| + \left| (I_0^{1-\alpha} y_s)(r\wedge t) - (I_0^{1-\alpha} y_s)(r \wedge s) \right|.\end{aligned}$$

Estimating the first term of the above inequality, we have

$$\begin{aligned}& \left| (I_0^{1-\alpha} x_t)(r\wedge t) - (I_0^{1-\alpha} y_s)(r \wedge t) \right| \\ & \le \frac{1}{\Gamma(1-\alpha)} \int_0^{r\wedge t} \frac{|x_t(u) - y_s(u)|}{(r\wedge t - u)^\alpha} du \\ & \le \frac{1}{\Gamma(1-\alpha)} \max_{0\le u \le r\wedge t} |x_t(u) - y_s(u)| \int_0^{r\wedge t} \frac{du}{(r\wedge t - u)^\alpha} \\ & \le \frac{1}{\Gamma(1-\alpha)} \max_{0 \le u \le T} |x_t(u\wedge t) - y_s(u \wedge s)| \frac{(r\wedge t)^{1-\alpha}}{1-\alpha} \\ & \le \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \max_{0\le u \le T} |x_t(u\wedge t) - y_s(u\wedge s)|.\end{aligned}$$

By Proposition 2.1, we have

$$\begin{aligned}& \left| (I_0^{1-\alpha} y_s)(r\wedge t) - (I_0^{1-\alpha} y_s)(r \wedge s) \right| \\ & \qquad \le \frac{2}{\Gamma(2-\alpha)} \|y_s\|_\infty |r\wedge t - r\wedge s|^{1-\alpha} \le \frac{2\|y_s\|_\infty}{\Gamma(2-\alpha)} |t-s|^{1-\alpha}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\left|(I_0^{1-\alpha}x_t)(r\wedge t)-(I_0^{1-\alpha}y_s)(r\wedge s)\right|\\&\qquad\le\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\max_{0\le u\le T}|x_t(u\wedge t)-y_s(u\wedge s)|+\frac{2\|y_s\|_\infty}{\Gamma(2-\alpha)}|t-s|^{1-\alpha}.\end{aligned}$$

Suppose $s\le t$. In a manner similar to the above case, we have

$$\begin{aligned}&\left|(I_0^{1-\alpha}x_t)(r\wedge t)-(I_0^{1-\alpha}y_s)(r\wedge s)\right|\\&\qquad\le\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\max_{0\le u\le T}|x_t(u\wedge t)-y_s(u\wedge s)|+\frac{2\|x_s\|_\infty}{\Gamma(2-\alpha)}|t-s|^{1-\alpha}.\end{aligned}$$

Hence, we obtain (2.2). □

Appendix B. Regularity of value functionals of differential games

In this section, we give a sketch of the proof for (3.12). Using standard splitting and pasting arguments of strategies and controls (cf.[9], [16]), note that W given by (3.11) satisfies DPP.

Proposition B.1. *For $(t,x_t)\in\mathbf{X}_{0,T}^\alpha$ and $t\le s\le T$,*

$$\begin{aligned}&W(t,x_t)\\&=\inf_{\theta\in\Theta(t,T)}\sup_{b\in\mathcal{B}(t,T)}\left\{\int_t^s l(r,\xi_r^{t,x_t,\theta[b],b},\theta[b](r),b(r))dr+W(s,\xi_s^{t,x_t,\theta[b],b})\right\},\end{aligned}\tag{B.1}$$

where $\xi=\xi^{t,x_t,\theta[b],b}:[0,T]\to\mathbb{R}^n$ is the solution of (3.5) with $\theta[b]$ and b, and $\xi_s^{t,x_t,\theta[b],b}$ is the restriction of $\xi=\xi^{t,x_t,\theta[b],b}$ on $[0,s]$.

Using Gronwall-type inequality (cf. [4, Lem. 6.19]) and an estimate of Caputo derivatives by L^∞-norm (cf.[27, Thm. 14.10 and Cor. 1], [13, Lem. 7.2]) under conditions i) and ii), the following result is standard.

Lemma B.2. *Let $(t,x_t),(t,y_t)\in\mathbf{X}_{0,T}^\alpha$, $a\in\mathcal{A}(t,T)$, $b\in\mathcal{B}(t,T)$. Then there exists $C>0$, which does not depend on $(t,x_t),(t,y_t),a,b$, such that*

$$\|\xi_r^{t,x_t,a,b}\|_\infty\le C(1+\|x_t\|_\infty),\ \ 0\le r\le T,\tag{B.2}$$

$$\|\xi_r^{t,x_t,a,b}-\xi_r^{t,y_t,a,b}\|_\infty\le C\|x_t-y_t\|_\infty,\ \ 0\le r\le T.\tag{B.3}$$

Using conditions i),ii) and Lemma B.2, we obtain:

Proposition B.3. *There exists $L>0$ such that*

$$|W(t,x_t)-W(t,y_t)|\le L\|x_t-y_t\|_\infty,\ \ \forall(t,x_t),(t,y_t)\in\mathbf{X}_{0,T}^\alpha.$$

Proof of (3.12). Let $(t,x_t),(s,y_s)\in\mathbf{X}_{0,T}^\alpha$.

1) Suppose $t\le s$. By Proposition B.1, we have

$$\begin{aligned}&W(t,x_t)\\&=\inf_{\theta\in\Theta(t,T)}\sup_{b\in\mathcal{B}(t,T)}\left\{\int_t^s l(r,\xi_r^{t,x_t,\theta[b],b},\theta[b](r),b(r))dr+W(s,\xi_s^{t,x_t,\theta[b],b})\right\}.\end{aligned}$$

Subtracting $W(s,y_s)$ in the above equation, we have

$$
\begin{aligned}
&|W(t,x_t)-W(s,y_s)| \\
&= \left| \inf_{\theta\in\Theta(t,T)} \sup_{b\in\mathcal{B}(t,T)} \left\{ \int_t^s l(r,\xi_r^{t,x_t,\theta[b],b},\theta[b](r),b(r))dr \right.\right. \\
&\qquad\qquad\qquad \left.\left. + W(s,\xi_s^{t,x_t,\theta[b],b}) - W(s,y_s) \right\} \right| \\
&\le \sup_{\substack{a\in\mathcal{A}(t,T)\\ b\in\mathcal{B}(t,T)}} \int_t^s |l(r,\xi_r^{t,x_t,a,b},a(r),b(r))|dr \\
&\qquad\qquad + \sup_{\substack{a\in\mathcal{A}(t,T)\\ b\in\mathcal{B}(t,T)}} |W(s,\xi_s^{t,x_t,a,b}) - W(s,y_s)|.
\end{aligned}
\tag{B.4}
$$

By condition ii) and (B.2), there exists $\hat{C}_1 > 0$ such that

$$
|l(r,\xi_r^{t,x_t,a,b},a(r),b(r))| \le \hat{C}_1(1+\|x_t\|_\infty),
$$

from which, we obtain

$$
\sup_{\substack{a\in\mathcal{A}(t,T)\\ b\in\mathcal{B}(t,T)}} \int_t^s |l(r,\xi_r^{t,x_t,a,b},a(r),b(r))|dr \le \hat{C}_1(1+\|x_t\|_\infty)(s-t).
\tag{B.5}
$$

By Proposition B.3, we have

$$
\begin{aligned}
|W(s,\xi_s^{t,x_t,a,b}) - W(s,y_s)| &\le L\|\xi_s^{t,x_t,a,b} - y_s\|_\infty \\
&\le L\|\xi_s^{t,x_t,a,b} - \tilde{x}_s\|_\infty + L\|\tilde{x}_s - y_s\|_\infty,
\end{aligned}
$$

where $\tilde{x}_s \in AC^\alpha[0,s]$ is given by

$$
\tilde{x}_s(r) = x_t(0) + \frac{1}{\Gamma(\alpha)} \int_0^r \frac{\tilde{\phi}(\tau)}{(r-\tau)^{1-\alpha}} d\tau,\ 0 \le r \le s,
\tag{B.6}
$$

where

$$
\tilde{\phi}(\tau) = \begin{cases} {}^C D_0^\alpha x(\tau) & (0 \le \tau \le t), \\ 0 & (t < \tau \le s). \end{cases}
$$

Noting that $\tilde{x}_s(r) = x_t(r)$ $(0 \le r \le t)$, we have

$$
\|\xi_s^{t,x_t,a,b} - \tilde{x}_s\|_\infty = \max_{t\le r\le s} |\xi_s^{t,x_t,a,b}(r) - \tilde{x}_s(r)|.
$$

By (3.6) and (B.6), we have

$$
\xi_s^{t,x_t,a,b}(r) - \tilde{x}_s(r) = \frac{1}{\Gamma(\alpha)} \int_t^r \frac{g(\tau,\xi_\tau^{t,x_t,a,b},a(\tau),b(\tau))}{(r-\tau)^{1-\alpha}} d\tau
$$

from which, with (B.2), there exists $\hat{C}_2 > 0$ such that

$$
\|\xi_s^{t,x_t,a,b} - \tilde{x}_s\|_\infty = \max_{t\le r\le s} |\xi_s^{t,x_t,a,b}(r) - \tilde{x}_s(r)| \le \hat{C}_2(1+\|x_t\|_\infty)(s-t)^\alpha.
\tag{B.7}
$$

Note that

$$\begin{aligned}\|\tilde{x}_s - y_s\|_\infty &\le \max_{0\le r\le t} |\tilde{x}_s(r) - y_s(r)| + \max_{t\le r\le s} |\tilde{x}_s(r) - y_s(r)| \\ &\le \max_{0\le r\le T} |x_t(r\wedge t) - y_s(r\wedge s)| \\ &\qquad + \max_{t\le r\le s} |\tilde{x}_s(r) - x_t(t)| + \max_{t\le r\le s} |x_t(t) - y_s(r)| \\ &\le 2 \max_{0\le r\le T} |x_t(r\wedge t) - y_s(r\wedge s)| + \max_{t\le r\le s} |\tilde{x}_s(r) - x_t(t)|.\end{aligned}$$

Recalling $\tilde{x}_s$ is given by (B.6), we have

$$\begin{aligned}|\tilde{x}_s(r) - x_t(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{{}^C D_0^\alpha x_t(\tau)}{(r-\tau)^{1-\alpha}} d\tau - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{{}^C D_0^\alpha x_t(\tau)}{(t-\tau)^{1-\alpha}} d\tau \right| \\ &\le \frac{\|{}^C D_0^\alpha x_t\|_\infty}{\Gamma(\alpha)} \int_0^t \left\{ \frac{1}{(t-\tau)^{1-\alpha}} - \frac{1}{(r-\tau)^{1-\alpha}} \right\} d\tau \\ &\le \frac{\|{}^C D_0^\alpha x_t\|_\infty}{\alpha\Gamma(\alpha)} (r-t)^\alpha.\end{aligned}$$

Thus we have

$$\|\tilde{x}_s - y_s\|_\infty \le 2 \max_{0\le r\le T} |x_t(r\wedge t) - y_s(r\wedge s)| + \frac{\|{}^C D_0^\alpha x_t\|_\infty}{\alpha\Gamma(\alpha)} (s-t)^\alpha. \tag{B.8}$$

By (B.4), (B.5), (B.7) and (B.8), there exists $\hat{C}_3 > 0$ such that

$$\begin{aligned}&|W(t, x_t) - W(s, y_s)| \\ &\qquad \le \hat{C}_3 (1 + \|x_t\|_\infty + \|{}^C D_0^\alpha x_t\|_\infty)(s-t)^\alpha + \hat{C}_3 \max_{0\le r\le T} |x_t(r\wedge t) - y_s(r\wedge s)|.\end{aligned}$$

2) Suppose $s < t$. Let x_s be the restriction of x_t on $[0, s]$. Note that

$$|W(s, y_s) - W(t, x_t)| \le |W(s, y_s) - W(s, x_s)| + |W(s, x_s) - W(t, x_t)|. \tag{B.9}$$

By Proposition B.3, we have

$$|W(s, y_s) - W(s, x_s)| \le L\|y_s - x_s\|_\infty \le L \max_{0\le r\le T} |x_t(r\wedge t) - y_s(r\wedge s)|. \tag{B.10}$$

As in the proof of (B.4) using the DPP for $W(s,x_s)$, we have

$$
\begin{aligned}
&|W(s,x_s)-W(t,x_t)| \\
&= \left| \inf_{\theta\in\Theta(s,T)} \sup_{b\in\mathcal{B}(s,T)} \left\{ \int_s^t l(r,\xi_r^{s,x_s,\theta[b],b},\theta[b](r),b(r))dr \right.\right. \\
&\qquad\qquad \left.\left. + W(t,\xi_t^{s,x_s,\theta[b],b}) - W(t,x_t) \right\} \right| \\
&\le \sup_{\substack{a\in\mathcal{A}(s,T)\\ b\in\mathcal{B}(s,T)}} \int_s^t |l(r,\xi_r^{s,x_s,a,b},a(r),b(r))|dr \\
&\quad + \sup_{\substack{a\in\mathcal{A}(s,T)\\ b\in\mathcal{B}(s,T)}} |W(t,\xi_t^{s,x_s,a,b}) - W(t,x_t)| \\
&\le \hat{C}_1(1+\|x_t\|_\infty)(t-s) + \sup_{\substack{a\in\mathcal{A}(s,T)\\ b\in\mathcal{B}(s,T)}} |W(t,\xi_t^{s,x_s,a,b}) - W(t,x_t)|.
\end{aligned} \tag{B.11}
$$

By Proposition B.3, we have

$$|W(t,\xi_t^{s,x_s,a,b}) - W(t,x_t)| \le L\|\xi_t^{s,x_s,a,b} - x_t\|_\infty = L \max_{s\le r\le t} |\xi^{s,x_s,a,b}(r) - x_t(r)|.$$

Note that $\xi^{s,x_s,a,b}(r)$ and $x_t(r)$ $(s \le r \le t)$ satisfy

$$
\begin{aligned}
&\xi^{s,x_s,a,b}(r) \\
&= x_s(0) + \frac{1}{\Gamma(\alpha)}\int_0^s \frac{{}^C D_0^\alpha x_s(\tau)}{(r-\tau)^{1-\alpha}}d\tau + \frac{1}{\Gamma(\alpha)}\int_s^r \frac{g(\tau,\xi_\tau^{s,x_s,a,b},a(\tau),b(\tau))}{(r-\tau)^{1-\alpha}}d\tau \\
&= x_t(0) + \frac{1}{\Gamma(\alpha)}\int_0^s \frac{{}^C D_0^\alpha x_t(\tau)}{(r-\tau)^{1-\alpha}}d\tau + \frac{1}{\Gamma(\alpha)}\int_s^r \frac{g(\tau,\xi_\tau^{s,x_s,a,b},a(\tau),b(\tau))}{(r-\tau)^{1-\alpha}}d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
x_t(r) &= x_t(0) + \frac{1}{\Gamma(\alpha)}\int_0^r \frac{{}^C D_0^\alpha x_t(\tau)}{(r-\tau)^{1-\alpha}}d\tau \\
&= x_t(0) + \frac{1}{\Gamma(\alpha)}\int_0^s \frac{{}^C D_0^\alpha x_t(\tau)}{(r-\tau)^{1-\alpha}}d\tau + \frac{1}{\Gamma(\alpha)}\int_s^r \frac{{}^C D_0^\alpha x_t(\tau)}{(r-\tau)^{1-\alpha}}d\tau,
\end{aligned}
$$

which imply

$$
\begin{aligned}
|\xi^{s,x_s,a,b}(r) - x_t(r)| \le \frac{1}{\Gamma(\alpha)}\int_s^r \frac{|g(\tau,\xi_\tau^{s,x_s,a,b},a(\tau),b(\tau))|}{(r-\tau)^{1-\alpha}}d\tau& \\
+ \frac{1}{\Gamma(\alpha)}\int_s^r \frac{|{}^C D_0^\alpha x_t(\tau)|}{(r-\tau)^{1-\alpha}}d\tau&.
\end{aligned}
$$

By (B.2) with condition ii), there exists $\hat{C}_4 > 0$ such that

$$|g(\tau,\xi_\tau^{s,x_s,a,b},a(\tau),b(\tau))| \le \hat{C}_4(1+\|x_s\|_\infty) \le \hat{C}_4(1+\|x_t\|_\infty).$$

Thus we have

$$\frac{1}{\Gamma(\alpha)}\int_s^r \frac{|g(\tau,\xi_\tau^{s,x_s,a,b},a(\tau),b(\tau))|}{(r-\tau)^{1-\alpha}}d\tau \leq \frac{\hat{C}_4(1+\|x_t\|_\infty)}{\alpha\Gamma(\alpha)}(t-s)^\alpha.$$

Note that

$$\frac{1}{\Gamma(\alpha)}\int_s^r \frac{|{}^C D_0^\alpha x_t(\tau)|}{(r-\tau)^{1-\alpha}}d\tau \leq \frac{\|{}^C D_0^\alpha x_t\|_\infty}{\alpha\Gamma(\alpha)}(t-s)^\alpha.$$

Thus we have

$$|W(t,\xi_t^{s,x_s,a,b})-W(t,x_t)| \leq \frac{L\hat{C}_4(1+\|x_t\|_\infty)}{\alpha\Gamma(\alpha)}(t-s)^\alpha + \frac{L\|{}^C D_0^\alpha x_t\|_\infty}{\alpha\Gamma(\alpha)}(t-s)^\alpha. \tag{B.12}$$

Hence, by (B.9), (B.10), (B.11), (B.12), there exists $\hat{C}_5 > 0$ such that

$$\begin{aligned}&|W(s,y_s)-W(t,x_t)| \\ &\quad \leq \hat{C}_5(1+\|x_t\|_\infty+\|{}^C D_0^\alpha x_t\|_\infty)(t-s)^\alpha + \hat{C}_5 \max_{0\leq r\leq T}|x_t(r\wedge t)-y_s(r\wedge s)|.\end{aligned}$$

Appendix C. Viscosity characterization of value functionals of differential games

We take ν in D_k such that $\nu \geq K$ where K is the constant in condition ii).

Proof of a viscosity supersolution. Let $\varphi \in \mathcal{C}_\alpha^1(\mathbf{X}_{0,T}^\alpha)$ and $(t,x_t) \in \mathbf{X}_{0,T-}^\alpha$ be a minimum point of $W-\varphi$ on D_k. Noting $(s,\xi_s) \in D_k$ $(t \leq s \leq t+h)$, we have from (B.1) and the chain rule of α-ci-derivatives (cf. [13, Lemma 9.2])

$$\begin{aligned}0 &\geq \inf_{\theta\in\Theta(t,T)}\sup_{b\in\mathcal{B}(t,T)}\left\{\int_t^{t+h} l(s,\xi_s,\theta[b](s),b(s))ds + \varphi(t+h,\xi_{t+h})-\varphi(t,x_t)\right\}\\ &= \inf_{\theta\in\Theta(t,T)}\sup_{b\in\mathcal{B}(t,T)}\left\{\int_t^{t+h}\{l(s,\xi_s,\theta[b](s),b(s))\right.\\ &\qquad\left. + \partial_t^\alpha\varphi(s,\xi_s)+\nabla_{x_t}^\alpha\varphi(s,\xi_s)\cdot g(s,\xi_s,\theta[b](s),b(s))\}ds\right\}.\end{aligned}$$

For any $\epsilon > 0$, take $\theta^* = \theta^{*,\epsilon,h} \in \Theta(t,T)$ satisfying

$$\begin{aligned}\epsilon h \geq \int_t^{t+h} &\{l(s,\xi_s^*,\theta^*[\bar{b}](s),\bar{b})\\ &+\partial_t^\alpha\varphi(s,\xi_s^*)+\nabla_{x_t}^\alpha\varphi(s,\xi_s^*)\cdot g(s,\xi_s^*,\theta^*[\bar{b}](s),\bar{b})\}ds,\ \forall \bar{b}\in B\end{aligned}$$

where $\xi^*(r)$ $(0 \leq r \leq t+h)$ is the solution of (3.5) for $a(\cdot) = \theta^*[\bar{b}](\cdot)$ and $b(\cdot) \equiv \bar{b}$. Noting that there exists $C = C(\|x_t\|_\infty, \|{}^C D_0^\alpha x_t\|_\infty) > 0$ such that $\max_{t\leq r\leq s}|\xi^*(r) - x_t(t)| \leq C|s-t|^\alpha$, we have

$$\begin{aligned}\epsilon h \geq \int_t^{t+h} &\{l(t,x_t,\theta^*[\bar{b}](s),\bar{b})\\ &+\partial_t^\alpha\varphi(t,x_t)+\nabla_{x_t}^\alpha\varphi(t,x_t)\cdot g(t,x_t,\theta^*[\bar{b}](s),\bar{b})\}ds + o(h),\end{aligned}$$

where $o(h)$ is uniform on ϵ and $\bar{b}$. Thus we have

$$\epsilon h \geq \int_t^{t+h}\min_{a\in A}\{l(t,x_t,a,\bar{b})+\partial_t^\alpha\varphi(t,x_t)+\nabla_{x_t}^\alpha\varphi(t,x_t)\cdot g(t,x_t,a,\bar{b})\}ds + o(h),$$

which implies

$$\epsilon h \geq h \max_{b\in B}\min_{a\in A}\{l(t,x_t,a,b)+\partial_t^\alpha\varphi(t,x_t)+\nabla_{x_t}^\alpha\varphi(t,x_t)\cdot g(t,x_t,a,b)\}+o(h).$$

Dividing the above inequality by h, letting $h\to 0$ and then letting $\epsilon\to 0$, we obtain

$$0\geq \partial_t^\alpha\varphi(t,x_t)+\max_{b\in B}\min_{a\in A}\left\{g(t,x_t,a,b)\cdot\nabla_{x_t}^\alpha\varphi(t,x_t)+l(t,x_t,a,b)\right\}.$$

Proof of a viscosity subsolution. Let $\varphi\in\mathcal{C}_\alpha^1(\mathbf{X}_{0,T}^\alpha)$ and $(t,x_t)\in\mathbf{X}_{0,T-}^\alpha$ be a maximum point of $W-\varphi$ on D_k. As in the proof of a viscosity subsolution, we have

$$\begin{aligned}0&\leq \inf_{\theta\in\Theta(t,T)}\sup_{b\in\mathcal{B}(t,T)}\left\{\int_t^{t+h} l(s,\xi_s,\theta[b](s),b(s))ds+\varphi(t+h,\xi_{t+h})-\varphi(t,x_t)\right\}\\&=\inf_{\theta\in\Theta(t,T)}\sup_{b\in\mathcal{B}(t,T)}\left\{\int_t^{t+h}\{l(s,\xi_s,\theta[b](s),b(s))\right.\\&\qquad\left.+\partial_t^\alpha\varphi(s,\xi_s)+\nabla_{x_t}^\alpha\varphi(s,\xi_s)\cdot g(s,\xi_s,\theta[b](s),b(s))\}ds\right\}.\end{aligned}\tag{C.1}$$

For any $b\in B$, take $\hat a=\hat a(b)\in A$ such that

$$\begin{aligned}&\min_{a\in A}\{g(t,x_t,a,b)\cdot\nabla_{x_t}^\alpha\varphi(t,x_t)+l(t,x_t,a,b)\}\\&\qquad=g(t,x_t,\hat a(b),b)\cdot\nabla_{x_t}^\alpha\varphi(t,x_t)+l(t,x_t,\hat a(b),b).\end{aligned}$$

Let $\epsilon>0$. Since we consider $t<T$ and sufficiently small $h>0$, we may suppose $t+h\leq(t+T)/2$. Let $D_k'\subset\mathbf{X}_{0,T-}^\alpha$ be given by

$$D_k'=\{(s,y_s)\in D_k;\,0\leq s\leq(t+T)/2\}.$$

Note that D_k' is compact. Since g, l (*resp.* $\partial_t^\alpha\varphi$, $\nabla_{x_t}^\alpha\varphi$) are uniformly continuous on $D_k'\times A\times B$ (*resp.* on D_k'), there exists $\delta=\delta(\epsilon)>0$ such that if $\rho((s,y_s),(u,z_u))+d_A(a,a')+d_B(b,b')<\delta$, $(s,y_s),(u,z_u)\in D_k'$, $a,a'\in A,b,b'\in B$,

$$\begin{aligned}&|l(s,y_s,a,b)-l(u,z_u,a',b')|<\epsilon,\ |\partial_t^\alpha\varphi(s,y_s)-\partial_t^\alpha\varphi(u,z_u)|<\epsilon,\\&|g(s,y_s,a,b)\cdot\nabla_{x_t}^\alpha\varphi(s,y_s)-g(u,z_u,a',b')\cdot\nabla_{x_t}^\alpha\varphi(u,z_u)|<\epsilon,\end{aligned}\tag{C.2}$$

where d_A and d_B are metrics on A and B, respectively. We denote by $B(b,\delta)$ open ball in B centered at b with radius δ. Noting that $B=\bigcup_{b\in B}B(b,\delta)$ and B is compact, there exists a finite subset $\{b_1,b_1,\ldots,b_N\}\subset B$ such that $B=\bigcup_{i=1}^N B(b_i,\delta)$. For $b\in\mathcal{B}(t,T)$, define $\hat\theta[b]\in\mathcal{A}(t,T)$ by

$$\hat\theta[b](r)=\begin{cases}\hat a(a_1), & b(r)\in B_1:=B(b_1,\delta),\\ \hat a(a_i), & b(r)\in B_i:=B(b_i,\delta)\backslash\bigcup_{j=1}^{i-1}B(b_j,\delta),\ i=2,3,\ldots,N.\end{cases}$$

Note that $\hat\theta\in\Theta(t,T)$. Taking $\theta=\hat\theta$ in (C.1), we have

$$\begin{aligned}0\leq\sup_{b\in\mathcal{B}(t,T)}&\left\{\int_t^{t+h}\{l(s,\xi_s,\hat\theta[b](s),b(s))\right.\\&\left.+\partial_t^\alpha\varphi(s,\xi_s)+\nabla_{x_t}^\alpha\varphi(s,\xi_s)\cdot g(s,\xi_s,\hat\theta[b](s),b(s))\}ds\right\}.\end{aligned}$$

Take $\hat{b} = \hat{b}^{\epsilon,h} \in \mathcal{B}(t,T)$ such that

$$-\epsilon h \le \int_t^{t+h} \{l(s,\hat{\xi}_s,\hat{\theta}[\hat{b}](s),\hat{b}(s)) + \partial_t^\alpha \varphi(s,\hat{\xi}_s) + \nabla_{x_t}^\alpha \varphi(s,\hat{\xi}_s)\cdot g(s,\hat{\xi}_s,\hat{\theta}[\hat{b}](s),\hat{b}(s))\}ds,$$

where $\hat{\xi}(r)$ $(0 \le r \le t+h)$ is the solution of (3.5) with $\hat{\theta}[\hat{b}]$ and $\hat{b}(\cdot)$. Recall that there exists $C = C(\|x_t\|_\infty, \|{}^C D_0^\alpha x_t\|_\infty) > 0$ such that $\max_{t\le r\le s} |\hat{\xi}(r) - x_t(t)| \le C|s-t|^\alpha$. By using (C.2), there exists $h_0 > 0$ such that for any $0 < h < h_0$

$$\begin{aligned}\int_t^{t+h} &\{l(s,\hat{\xi}_s,\hat{\theta}[\hat{b}](s),\hat{b}(s)) + \partial_t^\alpha \varphi(s,\hat{\xi}_s) + \nabla_{x_t}^\alpha \varphi(s,\hat{\xi}_s)\cdot g(s,\hat{\xi}_s,\hat{\theta}[\hat{b}](s),\hat{b}(s))\}ds \\ &\le \int_t^{t+h} \sum_{i=1}^N \{l(t,x_t,\hat{\theta}[\hat{b}](s),b_i) + \partial_t^\alpha \varphi(t,x_t) + \nabla_{x_t}^\alpha \varphi(t,x_t)\cdot g(t,x_t,\hat{\theta}[\hat{b}](s),b_i)\} 1_{B_i}(\hat{b}(s))ds + 3\epsilon h.\end{aligned} \tag{C.3}$$

Note that

$$\begin{aligned}&\text{RHS of (C.3)} \\ &= \int_t^{t+h} \sum_{i=1}^N \Big\{ \partial_t^\alpha \varphi(t,x_t) + \min_{a\in A} \big\{ \nabla_{x_t}^\alpha \varphi(t,x_t)\cdot g(t,x_t,a,b_i) + l(t,x_t,a,b_i)\big\}\Big\} 1_{B_i}(\hat{b}(s))ds + 3\epsilon h \\ &\le \Big\{ \partial_t^\alpha \varphi(t,x_t) + \max_{b\in B}\min_{a\in A} \big\{ \nabla_{x_t}^\alpha \varphi(t,x_t)\cdot g(t,x_t,a,b) + l(t,x_t,a,b)\big\}\Big\} h \\ &\quad + 3\epsilon h.\end{aligned}$$

Hence we have

$$\begin{aligned}&-\epsilon h \\ &\le \Big\{ \partial_t^\alpha \varphi(t,x_t) + \max_{b\in B}\min_{a\in A} \big\{ \nabla_{x_t}^\alpha \varphi(t,x_t)\cdot g(t,x_t,a,b) + l(t,x_t,a,b)\big\}\Big\} h \\ &\quad + 3\epsilon h,\end{aligned}$$

which implies

$$-\epsilon \le \partial_t^\alpha \varphi(t,x_t) + \max_{b\in B}\min_{a\in A} \big\{ \nabla_{x_t}^\alpha \varphi(t,x_t)\cdot g(t,x_t,a,b) + l(t,x_t,a,b)\big\} + 3\epsilon.$$

Letting $\epsilon \to 0$, we obtain

$$0 \le \partial_t^\alpha \varphi(t,x_t) + \max_{b\in B}\min_{a\in A} \big\{ \nabla_{x_t}^\alpha \varphi(t,x_t)\cdot g(t,x_t,a,b) + l(t,x_t,a,b)\big\}. \quad \square$$

Appendix D. Proof of Lemma 4.2

Let $(t, x_t) \in \mathbf{X}^\alpha_{0,T-}$. Let $\eta \in \mathcal{P}(t, x_t)$ and $0 < h \le T - t$. Noting $x_t(0) = \eta(0)$, we have

$$\begin{aligned}\text{(D.1)}\quad \varphi(t+h, \eta_{t+h}) - \varphi(t, x_t) &= \frac{1}{2}\left\{|t+h-s|^2 - |t-s|^2\right\} \\ &\quad + \frac{1}{2}\int_0^T \left|(I_0^{1-\alpha}\bar{\eta})(r \wedge (t+h)) - (I_0^{1-\alpha}\bar{y}_s)(r \wedge s)\right|^2 \\ &\quad - \left|(I_0^{1-\alpha}\bar{x}_t)(r \wedge t) - (I_0^{1-\alpha}\bar{y}_s)(r \wedge s)\right|^2 dr.\end{aligned}$$

We see that the first term of (D.1) is

$$\frac{1}{2}\left\{|t+h-s|^2 - |t-s|^2\right\} = (t-s)h + \frac{1}{2}h^2 = (t-s)h + O(h^2)\ (h \to 0+).$$

Using $|a|^2 - |b|^2 = (a+b)\cdot(a-b)$ $(a, b \in \mathbb{R}^n)$, we have

$$\begin{aligned}&\left|(I_0^{1-\alpha}\bar{\eta})(r \wedge (t+h)) - (I_0^{1-\alpha}\bar{y}_s)(r \wedge s)\right|^2 \\ &\qquad\qquad - \left|(I_0^{1-\alpha}\bar{x}_t)(r \wedge t) - (I_0^{1-\alpha}\bar{y}_s)(r \wedge s)\right|^2 \\ &= \left((I_0^{1-\alpha}\bar{\eta})(r \wedge (t+h)) + (I_0^{1-\alpha}\bar{x}_t)(r \wedge t) - 2(I_0^{1-\alpha}\bar{y}_s)(r \wedge s)\right) \\ &\quad \cdot \left((I_0^{1-\alpha}\bar{\eta})(r \wedge (t+h)) - (I_0^{1-\alpha}\bar{x}_t)(r \wedge t)\right) \\ &= 2\left((I_0^{1-\alpha}\bar{x}_t)(r \wedge t) - (I_0^{1-\alpha}\bar{y}_s)(r \wedge s)\right) \\ &\quad \cdot \left((I_0^{1-\alpha}\bar{\eta})(r \wedge (t+h)) - (I_0^{1-\alpha}\bar{x}_t)(r \wedge t)\right) \\ &\quad + \left|(I_0^{1-\alpha}\bar{\eta})(r \wedge (t+h)) - (I_0^{1-\alpha}\bar{x}_t)(r \wedge t)\right|^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}\text{(D.2)}\qquad &\frac{1}{2}\int_0^T \{|(I_0^{1-\alpha}\bar{\eta})(r \wedge (t+h)) - (I_0^{1-\alpha}\bar{y}_s)(r \wedge s)|^2 \\ &\qquad\qquad - |(I_0^{1-\alpha}\bar{x}_t)(r \wedge t) - (I_0^{1-\alpha}\bar{y}_s)(r \wedge s)|^2\}dr \\ &= \int_0^T \{\left((I_0^{1-\alpha}\bar{x}_t)(r \wedge t) - (I_0^{1-\alpha}\bar{y}_s)(r \wedge s)\right) \\ &\qquad \cdot \left((I_0^{1-\alpha}\bar{\eta})(r \wedge (t+h)) - (I_0^{1-\alpha}\bar{x}_t)(r \wedge t)\right)\}dr \\ &\quad + \frac{1}{2}\int_0^T \left|(I_0^{1-\alpha}\bar{\eta})(r \wedge (t+h)) - (I_0^{1-\alpha}\bar{x}_t)(r \wedge t)\right|^2 dr.\end{aligned}$$

Since $\eta \in AC^\alpha[0, T]$ and $x_t \in AC^\alpha[0, t]$, we have

$$\bar{\eta}(\cdot) = \eta(\cdot) - \eta(0) = I_0^\alpha {}^C D_0^\alpha \eta \text{ on } [0, T],$$
$$\bar{x}_t(\cdot) = x_t(\cdot) - x_t(0) = I_0^\alpha {}^C D_0^\alpha x_t \text{ on } [0, t].$$

Applying $I_0^{1-\alpha}$ to the above equations and using the semi-group property, we have

$$I_0^{1-\alpha}\bar{\eta} = I_0 {}^C D_0^\alpha \eta \text{ on } [0, T],\ I_0^{1-\alpha}\bar{x}_t = I_0 {}^C D_0^\alpha x_t \text{ on } [0, t].$$

Thus, we have

$$(I_0^{1-\alpha}\bar{\eta})(r\wedge(t+h)) - (I_0^{1-\alpha}\bar{x}_t)(r\wedge t)$$
$$= \int_0^{r\wedge(t+h)} {}^C D_0^\alpha \eta(u)du - \int_0^{r\wedge t} {}^C D_0^\alpha x_t(u)du.$$

Noting that ${}^C D_0^\alpha \eta(u) = {}^C D_0^\alpha x_t(u)$ $a.e.\, u \in [0, r\wedge t]$ because $\eta(u) = x_t(u)$ $(0 \le u \le t)$, we have

$$\int_0^{r\wedge(t+h)} {}^C D_0^\alpha \eta(u)du - \int_0^{r\wedge t} {}^C D_0^\alpha x_t(u)du = \int_{r\wedge t}^{r\wedge(t+h)} {}^C D_0^\alpha \eta(u)du.$$

Hence, we have

$$\text{(D.3)} \qquad \text{RHS of (D.2)}$$
$$= \int_0^T \big((I_0^{1-\alpha}\bar{x}_t)(r\wedge t) - (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)\big) \cdot \int_{r\wedge t}^{r\wedge(t+h)} {}^C D_0^\alpha \eta(u)dudr$$
$$+ \frac{1}{2}\int_0^T \left| \int_{r\wedge t}^{r\wedge(t+h)} {}^C D_0^\alpha \eta(u)du \right|^2 dr.$$

Splitting the interval $[0,T]$ of the integral into $[0,t]$ and $[t,T]$, we have

$$\text{(D.4)} \qquad \text{RHS of (D.3)}$$
$$= \left(\int_0^t + \int_t^T\right) \big((I_0^{1-\alpha}\bar{x}_t)(r\wedge t) - (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)\big)$$
$$\cdot \int_{r\wedge t}^{r\wedge(t+h)} {}^C D_0^\alpha \eta(u)dudr$$
$$+ \frac{1}{2}\int_0^T \left| \int_{r\wedge t}^{r\wedge(t+h)} {}^C D_0^\alpha \eta(u)du \right|^2 dr$$
$$= \int_t^T \big((I_0^{1-\alpha}\bar{x}_t)(t) - (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)\big) \cdot \int_t^{r\wedge(t+h)} {}^C D_0^\alpha \eta(u)dudr$$
$$+ \frac{1}{2}\int_0^T \left| \int_{r\wedge t}^{r\wedge(t+h)} {}^C D_0^\alpha \eta(u)du \right|^2 dr.$$

Noting that

$$\int_t^{r\wedge(t+h)} {}^C D_0^\alpha \eta(u)du = \int_t^{t+h} {}^C D_0^\alpha \eta(u)du - \int_{r\wedge(t+h)}^{t+h} {}^C D_0^\alpha \eta(u)du,$$

we have

$$\text{1st term of RHS of (D.4)}$$
$$= \int_t^T (I_0^{1-\alpha}\bar{x}_t)(t) - (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)dr \cdot \int_t^{t+h} {}^C D_0^\alpha \eta(u)du$$
$$- \int_t^T \big((I_0^{1-\alpha}\bar{x}_t)(t) - (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)\big) \cdot \int_{r\wedge(t+h)}^{t+h} {}^C D_0^\alpha \eta(u)dudr.$$

Estimating the second term of the RHS of the above inequality, we have

$$\left|\int_t^T \big((I_0^{1-\alpha}\bar{x}_t)(t) - (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)\big)\cdot \int_{r\wedge(t+h)}^{t+h} {}^C D_0^\alpha \eta(u)dudr\right|$$
$$\le \big(|(I_0^{1-\alpha}\bar{x}_t)(t)| + \|I_0^{1-\alpha}\bar{y}_s(\cdot\wedge s)\|_\infty\big)\,\|{}^C D_0^\alpha\eta\|_\infty \int_t^T\int_{r\wedge(t+h)}^{t+h} dudr.$$

Note that

$$\int_t^T\int_{r\wedge(t+h)}^{t+h} dudr$$
$$= \int_t^{t+h}\int_{r\wedge(t+h)}^{t+h} dudr + \int_{t+h}^T\int_{r\wedge(t+h)}^{t+h} dudr = \int_t^{t+h}\int_r^{t+h} dudr$$
$$= \int_t^{t+h} (t+h-r)dr = \frac{h^2}{2}.$$

Therefore, we have

$$\int_t^T \big((I_0^{1-\alpha}\bar{x}_t)(t) - (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)\big)\cdot \int_t^{r\wedge(t+h)} {}^C D_0^\alpha \eta(u)dudr$$
$$= \int_t^T \big((I_0^{1-\alpha}\bar{x}_t)(t) - (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)\big)\,dr\cdot \int_t^{t+h} {}^C D_0^\alpha \eta(u)du$$
$$+ O_\eta(h^2)\ (h\to 0)$$

where $O_\eta(h^2)$ is a function $\omega(h;\eta)$ $(0 < h \le T-t)$ satisfying

$$\sup_{0<h\le T-t} |\omega(h;\eta)|/h^2 < \infty.$$

Estimating the second term of RHS of (D.4), we have

$$\frac{1}{2}\int_0^T \left|\int_{r\wedge t}^{r\wedge(t+h)} {}^C D_0^\alpha \eta(u)du\right|^2 dr$$
$$\le \frac{1}{2}\|{}^C D_0^\alpha\eta\|_\infty^2 \int_0^T \left(\int_{r\wedge t}^{r\wedge(t+h)} du\right)^2 dr \le \frac{T}{2}\|{}^C D_0^\alpha\eta\|_\infty^2 h^2.$$

Thus we have

$$\frac{1}{2}\int_0^T \left|\int_{r\wedge t}^{r\wedge(t+h)} {}^C D_0^\alpha \eta(u)du\right|^2 dr = O_\eta(h^2)\ (h\to 0+).$$

Thus, we obtain

$$\varphi(t+h,\eta_{t+h}) - \varphi(t,x_t)$$
$$= (t-s)h + \int_t^T (I_0^{1-\alpha}\bar{x}_t)(t) - (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)dr\cdot \int_t^{t+h} {}^C D_0^\alpha \eta(u)du$$
$$+ O_\eta(h^2)\ (h\to 0+).$$

Hence φ is α-ci-differentiable at (t, x_t) and

$$\begin{aligned}\partial_t^\alpha \varphi(t,x_t) &= t-s,\\ \nabla_{x_t}^\alpha \varphi(t,x_t) &= \int_t^T (I_0^{1-\alpha}\bar{x}_t)(t) - (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)dr\\ &= (T-t)(I_0^{1-\alpha}\bar{x}_t)(t) - \int_t^T (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)dr.\end{aligned}$$

We show that φ is continuous on $\mathbf{X}_{0,T}^\alpha$. Let $(t, x_t) \in \mathbf{X}_{0,T}^\alpha$ and $\{(t_n, x_{t_n}^n)\}$ be a sequence of $\mathbf{X}_{0,T}^\alpha$ satisfying $\rho((t_n, x_{t_n}^n), (t, x_t)) \to 0$ $(n \to \infty)$, that is,

$$t_n \to t,\ \max_{0\le r\le T} |x_{t_n}^n(r\wedge t_n) - x_t(r\wedge t)| \to 0\ (n\to\infty). \tag{D.5}$$

Recalling the definition of $\varphi(t_n, x_{t_n}^n)$,

$$\begin{aligned}\varphi(t_n,x_{t_n}^n) = \frac{1}{2}|t_n-s|^2 + \frac{1}{2}|x_{t_n}^n(0)-y_s(0)|^2 \\ + \frac{1}{2}\int_0^T \left|(I_0^{1-\alpha}\bar{x}_{t_n}^n)(r\wedge t_n) - (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)\right|^2 dr.\end{aligned} \tag{D.6}$$

By (D.5), we have

$$\frac{1}{2}|t_n-s|^2 \to \frac{1}{2}|t-s|^2,\ \frac{1}{2}|x_{t_n}^n(0)-y_s(0)|^2 \to \frac{1}{2}|x_t(0)-y_s(0)|^2 \to 0\ (n\to\infty).$$

By Proposition 2.2, we have

$$\begin{aligned}&\max_{0\le r\le T}|(I_0^{1-\alpha}\bar{x}_{t_n}^n)(r\wedge t_n) - (I_0^{1-\alpha}\bar{x}_t)(r\wedge t)|\\ &\le \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\max_{0\le r\le T}|\bar{x}_{t_n}^n(r\wedge t_n) - \bar{x}_t(r\wedge t)|\\ &\qquad + \frac{2}{\Gamma(2-\alpha)}(\|\bar{x}_{t_n}^n\|_\infty + \|\bar{x}_t\|_\infty)|t_n-t|^{1-\alpha}1_{\{0<\alpha<1\}}\\ &\le \frac{2T^{1-\alpha}}{\Gamma(2-\alpha)}\max_{0\le r\le T}|x_{t_n}^n(r\wedge t_n) - x_t(r\wedge t)|\\ &\qquad + \frac{2}{\Gamma(2-\alpha)}(\|\bar{x}_{t_n}^n\|_\infty + \|\bar{x}_t\|_\infty)|t_n-t|^{1-\alpha}1_{\{0<\alpha<1\}}.\end{aligned}$$

Noting that

$$\begin{aligned}\|\bar{x}_{t_n}^n\|_\infty &= \max_{0\le r\le T}|\bar{x}_{t_n}^n(r\wedge t_n)|\\ &\le \max_{0\le r\le T}|\bar{x}_{t_n}^n(r\wedge t_n) - \bar{x}_t(r\wedge t)| + \max_{0\le r\le T}|\bar{x}_t(r\wedge t)|\\ &\le 2\max_{0\le r\le T}|x_{t_n}^n(r\wedge t_n) - x_t(r\wedge t)| + \|\bar{x}_t\|_\infty,\end{aligned}$$

we have

$$
\begin{aligned}
\text{(D.7)} \quad & \max_{0\le r\le T} |(I_0^{1-\alpha}\bar{x}^n_{t_n})(r\wedge t_n) - (I_0^{1-\alpha}\bar{x}_t)(r\wedge t)| \\
&\le \frac{2T^{1-\alpha}}{\Gamma(2-\alpha)} \max_{0\le r\le T} |x^n_{t_n}(r\wedge t_n) - x_t(r\wedge t)| \\
&\quad + \frac{4}{\Gamma(2-\alpha)} \left(\max_{0\le r\le T} |x^n_{t_n}(r\wedge t_n) - x_t(r\wedge t)| + \|\bar{x}_t\|_\infty \right) \\
&\qquad \times |t_n - t|^{1-\alpha} 1_{\{0<\alpha<1\}}.
\end{aligned}
$$

By (D.5), we obtain

$$\max_{0\le r\le T} |(I_0^{1-\alpha}\bar{x}^n_{t_n})(r\wedge t_n) - (I_0^{1-\alpha}\bar{x}_t)(r\wedge t)| \to 0 \ (n\to\infty).$$

Thus we have

$$
\begin{aligned}
&\frac{1}{2}\int_0^T \left|(I_0^{1-\alpha}\bar{x}^n_{t_n})(r\wedge t_n) - (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)\right|^2 dr \\
&\qquad \to \frac{1}{2}\int_0^T \left|(I_0^{1-\alpha}\bar{x}_t)(r\wedge t) - (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)\right|^2 dr \ (n\to\infty).
\end{aligned}
$$

Taking the limit of (D.6) as $n\to\infty$, we have

$$
\begin{aligned}
\varphi(t_n, x^n_{t_n}) \to &\frac{1}{2}|t-s|^2 + \frac{1}{2}|x_t(0) - y_s(0)|^2 \\
&+ \frac{1}{2}\int_0^T \left|(I_0^{1-\alpha}\bar{x}_t)(r\wedge t) - (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)\right|^2 dr = \varphi(t, x_t).
\end{aligned}
$$

Hence φ is continuous at (t, x_t).

Next, we show that $\partial_t^\alpha \varphi$ and $\nabla_{x_t}^\alpha \varphi$ are continuous on $\mathbf{X}^\alpha_{0,T-}$. Let $(t, x_t) \in \mathbf{X}^\alpha_{0,T-}$ and $\{(t_n, x^n_{t_n})\}_{n=1}^\infty \subset \mathbf{X}^\alpha_{0,T-}$ converge to (t, x_t). By the definition of metric ρ, we see that

$$t_n \to t, \quad \max_{0\le r\le T} |x^n_{t_n}(r\wedge t_n) - x_t(r\wedge t)| \to 0 \ \ (n\to\infty).$$

It is immediate to see that

$$\partial_t^\alpha \varphi(t_n, x_{t_n}) = t_n - s \to t - s = \partial_t^\alpha \varphi(t, x_t) \ (n\to\infty).$$

Hence, $\partial_t^\alpha \varphi$ is continuous at (t, x_t). To prove $\nabla_{x_t}^\alpha \varphi$ is continuous, suppose that $t_n \le t$. Then, we have

$$
\begin{aligned}
|(I_0^{1-\alpha}\bar{x}^n_{t_n})(t_n) - (I_0^{1-\alpha}\bar{x}_t)(t)| &= |(I_0^{1-\alpha}\bar{x}^n_{t_n})(t\wedge t_n) - (I_0^{1-\alpha}\bar{x}_t)(t\wedge t)| \\
&\le \max_{0\le r\le T} |(I_0^{1-\alpha}\bar{x}^n_{t_n})(r\wedge t_n) - (I_0^{1-\alpha}\bar{x}_t)(r\wedge t)|.
\end{aligned}
$$

Supposing $t < t_n$, we have

$$
\begin{aligned}
|(I_0^{1-\alpha}\bar{x}^n_{t_n})(t_n) - (I_0^{1-\alpha}\bar{x}_t)(t)| &= |(I_0^{1-\alpha}\bar{x}^n_{t_n})(t_n\wedge t_n) - (I_0^{1-\alpha}\bar{x}_t)(t_n\wedge t)| \\
&\le \max_{0\le r\le T} |(I_0^{1-\alpha}\bar{x}^n_{t_n})(r\wedge t_n) - (I_0^{1-\alpha}\bar{x}_t)(r\wedge t)|.
\end{aligned}
$$

By (D.7), we have

$$\begin{aligned}
&|(I_0^{1-\alpha}\bar{x}^n_{t_n})(t_n) - (I_0^{1-\alpha}\bar{x}_t)(t)| \\
&\leq \frac{2T^{1-\alpha}}{\Gamma(2-\alpha)} \max_{0\leq r\leq T} |x^n_{t_n}(r\wedge t_n) - x_t(r\wedge t)| \\
&\quad + \frac{4}{\Gamma(2-\alpha)} \left(\max_{0\leq r\leq T} |x^n_{t_n}(r\wedge t_n) - x_t(r\wedge t)| + \|\bar{x}_t\|_\infty \right) \\
&\qquad\qquad \times |t_n - t|^{1-\alpha} 1_{\{0<\alpha<1\}}.
\end{aligned}$$

Letting $n\to\infty$, we have

$$(I_0^{1-\alpha}\bar{x}^n_{t_n})(t_n) \to (I_0^{1-\alpha}\bar{x}_t)(t) \ (n\to\infty).$$

Hence we obtain

$$\begin{aligned}
\nabla^\alpha_{x_t}\varphi(t_n, x^n_{t_n}) &= (T-t_n)(I_0^{1-\alpha}\bar{x}^n_{t_n})(t_n) - \int_{t_n}^T (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)dr \\
&\to (T-t)(I_0^{1-\alpha}\bar{x}_t)(t) - \int_t^T (I_0^{1-\alpha}\bar{y}_s)(r\wedge s)dr = \nabla^\alpha_{x_t}\varphi(t, x_t) \ (n\to\infty). \qquad \square
\end{aligned}$$

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