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DRAWDOWN CONSTRAINT FOR LONG-TERM INVESTMENTS UNDER PARTIAL INFORMATION

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Abstract. In this paper we study a problem of determining the optimal growth rate of the certainty equivalent under partial information with portfolios satisfying a drawdown constraint. Asset prices are modeled through a Brownian diffusion with coefficients governed by stochastic exogenous factors. These factors are non-observable directly and decisions are taken only with information about prices. The goal of investors is to outperform a drawdown constraint and choose the optimal investment policy in closed form. Using the Kalman filter we deal with the source of incomplete information, and through the theory of Riccati algebraic equation an explicit form for the optimal growth rate is given. Finally, an explicit form of the optimal investment strategy is established adapting the theory of Azema-Yor processes.

1. Introduction

In this paper we study a problem of utility maximization of terminal wealth in infinite horizon. The criterion that we maximize is the asymptotic growth rate of expected utility for power utility functions. In contrast to the extensive literature in this topic, the present work has two main distinctive elements, introducing a drawdown constraint in the wealth portfolio and partial information in the evolution of the processes involved in the market model. Each one of these elements have been studied individually but not in combination. We work in a specific model where prices S_t are modeled by a diffusion with coefficients depending on some Markov process X_t of underlying factors. A general assumption is that X_t is known when the portfolio π_t is chosen, and there are some factors such as the long or short rate which are observable, but models in which the economic factors are hidden might be of interest. In our case, partial information derives from the fact that strategies are constructed on the exclusive observation of asset prices S_t, and the factors X_t are not directly observable, leading to an optimal control problem under partial information for the case of power utility.

As early as the start of the 80's, problems of stochastic control under partial information were studied by [12]. Their presentation is general with the possibility to be formulated for a wide class of applications. Indeed, studies oriented to finance took up the theme of partial information and the list of papers dedicated to the classical problem of maximization of expected utility under partial information grew up.

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We review a few of them without intentions of being exhaustive. Papers studying utility maximization problems with partial information include among others: [32], [2], [8], [22], [24], [36], [30], [23], [33], [31], [29]. Most of these works attempted to solve the problem through the dynamical programming principle, introducing a filter and obtaining the Hamilton-Jacobi-Bellman (HJB) partial differential equation. An important challenge in this approach is then to prove the existence of a classical solution of the HJB equation and characterize optimal strategies through it. This task is solved for special linear dynamics and quadratic cost structure through the solution of a Riccati differential equation. It is also possible to rely on the solution of a BSDEs, as has been done by e.g., [22], [36], [30]. Still another approach with minimal restrictions on the underlying dynamic of asset prices, different from HJB, is presented by [29] who study the problem through the insights of convex duality and the projection of density processes of martingale measures.

The natural question of quantifying the loss in expected utility due to partial information has only been investigated by a few authors including [8], [36], [30], and [23]. Surprisingly, there are non trivial examples where there is no loss in optimal expected utility despite maximizing under restricted information. However, a precise formulation of this loss, characterized in explicit form in terms of the coefficients of the model, is still a loose end. For the long-term portfolio management presented in this paper, there is an interesting connection between optimal growth rate and risk-sensitivy control. In fact, some of the papers cited below studied finite horizon problems as an intermediate step to solve an asymptotic problem in which the horizon goes to infinity.

The risk-sensitivity formulation for infinite horizon utility maximization was introduced by [4] and [13]. Further developed by [14, 15]. Infinite horizon problems under partial information has been studied among other authors by [32, 23, 33]. The goal of [23] is to solve the maximization of the asymptotic probability of terminal wealth remaining above a threshold. By duality considerations they study an ergodic risk-sensitivity stochastic control problem. They find a condition under which there is no loss in the optimal growth rate under partial information; see their Proposition 7.2, part 2(ii). Their approach strongly depends on the convergence of a time dependent differential Riccati equation to an algebraic equation that does not depend on time. This is a special form of an important and recurrent topic, the convergence, in a specific sense, of a dynamical programming equation in finite horizon to an infinite horizon counterpart for a so-called ergodic equation in infinite horizon. Such an equation has been considered in the papers [35], [26], [27], [15], [11], [3].

A drawdown constraint requires that the value process of a strategy has to remain above a fraction of its current maximum. Drawdown constraints were first considered by [19] and has become quite popular due to its financial appeal as an effective risk control. A generalization to a multi-asset framework with a simplified proof was provided by [9]. Their approach advances a key concept in the solution, now known as Azema-Yor semimartingale processes; see [5]. The relevance in the problem of expected utility maximization under drawdown constraints has been systematically developed in [37] and [7]. The use of the dynamic programming principle to solve problems of portfolio selection under drawdown constraints has been considered, among others, by [25], [1], [39], [20], [21], [6], [10]. The only paper we are aware that also considers partial information with drawdown constraints is [18]. In this work the authors consider a discrete-time framework in a finite horizon. Starting with dynamical programming techniques, they investigate numerical approximations by deep learning considerations. Another formulation to portfolio selection under drawdown constraints that explicitly consider the estimation of parameters can be found in [34].

The contributions of the present paper are as follows. The main goal is to maximize the long-run growth rate of expected utility of wealth for an investor with partial information about the evolution of asset prices and drawdown constraint. We solve this problem for a model with stochastic economic factors explicitly affecting the mean return of stocks, providing explicit solutions for both, the optimal investment strategy and the critical growth rate. These results require the study of the finite horizon problem as well as the asymptotic analysis as the time horizon converges to infinity. The solution of the problem with drawdown constraints is based in the theory of Azema-Yor processes, taking advantage of the bijection between the set of strategies satisfying this constraint and those satisfying a non bankruptcy constraint. Our contributions rely heavily on previous results developed in [33], [37].

The structure of the present paper is organized as follows. In Section 2 we present our model, which is basically a Markovian asset price model with invertible volatility matrix, where the asset prices are driven by a multi dimensional Wiener process and the mean return rate processes depends on stochastic factors, and hence it is adapted to a larger filtration. Section 3 is devoted to a fairly detailed study of the optimal investment problem in the special cases of finite horizon and partial observation. The asymptotic limit as time goes to infinity are presented in Section 4. Finally, once the preliminary results have been established, in Section 5 we present our main theorem, showing how a complex problem can be solved, given the proper perspective.

2. The optimal certainty equivalent under partial information in finite horizon: No drawdown constraints

We consider a market whose evolution runs in the time interval $[0,T]$ and in which a non-risky asset S^0 is available together with m risky assets $S=(S^1,\dots,S^m)$. There are n factor processes $X=(X^1,X^2,\dots,X^n)$ influencing the performance of the market. They evolve on a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ supporting an $\mathbb{R}^d$ Brownian motion W according with the dynamics given by

$$\begin{pmatrix} dX^1 \\ dX^2 \\ \vdots \\ dX^n \end{pmatrix} = \begin{pmatrix} \beta^1(X) \\ \beta^2(X) \\ \vdots \\ \beta^n(X) \end{pmatrix} dt + \begin{pmatrix} \lambda^{1,1}(X) & \dots & \lambda^{1,d}(X) \\ \lambda^{2,1}(X) & \dots & \lambda^{2,d}(X) \\ \vdots & & \\ \lambda^{n,1}(X) & \dots & \lambda^{n,d}(X) \end{pmatrix} \begin{pmatrix} dW^1 \\ dW^2 \\ \vdots \\ dW^d \end{pmatrix}, \tag{2.1}$$

with initial condition $X_0=x_0$. The bond price S^0 is assumed to satisfy the ordinary differential equation:

$$dS^0 = S^0 r(X)dt, \quad S^0_0 = s_0,$$

and the risky assets $S = (S^1, \ldots, S^m)$ are assumed to be governed by the stochastic differential equation

$$\begin{pmatrix} \frac{dS^1}{S^1} \\ \frac{dS^2}{S^2} \\ \vdots \\ \frac{dS^m}{S^m} \end{pmatrix} = \begin{pmatrix} \alpha^1(X) \\ \alpha^2(X) \\ \vdots \\ \alpha^m(X) \end{pmatrix} dt + \begin{pmatrix} \sigma^{1,1}(X) & \ldots & \sigma^{1,d}(X) \\ \sigma^{2,1}(X) & \ldots & \sigma^{2,d}(X) \\ \vdots & & \\ \sigma^{m,1}(X) & \ldots & \sigma^{m,d}(X) \end{pmatrix} \begin{pmatrix} dW^1 \\ dW^2 \\ \vdots \\ dW^d \end{pmatrix}. \tag{2.2}$$

In this paper we consider the linear model, taking $r(x) = r$, $\alpha(x) = a + Ax$, $\sigma(x) = \Sigma$, $\beta(x) = b + Bx$ and $\lambda(x) = \Lambda$, for matrices $A \in \mathbb{R}^{m\times n}$, $\Sigma \in \mathbb{R}^{m\times d}$, $B \in \mathbb{R}^{n\times n}$ and $\Lambda \in \mathbb{R}^{n\times d}$. We present the case when the factor process $\{X_t\}$ is not directly observed and its values have to be estimated observing the prices of risky assets $\{S^i\}$. The information generated by the stock prices is the filtration $\mathbb{G} := \{\mathcal{G}_t\}$ set at time t by the $\sigma-$ algebra $\mathcal{G}_t := \sigma\{S_u, \quad u \leq t\}$. An investor trades in the market using $\mathbb{R}^m-$valued $\mathbb{G}$ progressively measurable self-financing strategies $\pi = \{\pi^i\}$, for $i = 1, \ldots, m$, representing the proportion of wealth invested in asset i, and the rest of the wealth is invested in the asset S^0. Hence, defining $\pi^0 := 1 - \sum_{i=1}^m \pi^i$ we obtain the proportion invested in the reference asset S^0. For a self-financing investment strategy $\pi = (\pi^1, \ldots, \pi^m)$, its wealth process $V = V^\pi$ satisfies the equation

$$\frac{dV}{V} = r\pi^0 dt + \pi \cdot \frac{dS}{S} = r\pi^0 dt + \sum_{i=1}^m \pi^i \frac{dS^i}{S^i}. \tag{2.3}$$

We assume, without loss of generality, that the initial capital is fixed at $V_0 = 1$. Admissible investment strategies π must satisfy the integrability condition $\int_0^T \|\pi_t\|^2 dt < \infty$, $\mathbb{P} - a.s.$, among other technical conditions required below.

Our optimization problem concerns the study of the exponential growth rate of the certainty equivalent for a power utility. However, we have to deal with partial information under which decisions are taken. Concerning this point, the transformation of the optimization problem into one completely observed requires the solution of a filtering problem for the estimate $\hat{X}_t := \mathbb{E}[X_t \mid \mathcal{G}_t]$, which evolves linearly under a new Brownian motion. This requires that we linearize the (observable) dynamics of the risky price process $\{S_t\}_{0\leq t\leq T}$, defining $\xi_t^i := \log S_t^i$, which evolves as

$$d\xi_t = (\delta + AX_t)dt + \Sigma\, dW_t,$$

with $\delta = (\delta^i)$, $\delta^i = a^i - \frac{1}{2}(\Theta)^{ii}$, where

$$\Theta := \Sigma\Sigma^*; \tag{2.4}$$

throughout we always assume that $\Theta > 0$. Observe that the conditional distribution of X_t given $\mathcal{G}_t$ has normal distribution with mean $\hat{X}$ and conditional covariance matrix Π given by

$$\hat{X}_t := \mathbb{E}[X_t \mid \mathcal{G}_t] \tag{2.5}$$

$$\Pi_t := \mathbb{E}[(X_t - \hat{X}_t)(X_t - \hat{X}_t)^* | \mathcal{G}_t]. \tag{2.6}$$

Now we are interested in describing the evolution of the Kalman filter $\hat{X}_t$, in terms of the innovation process I defined by

$$I_t := \int_0^t \Theta^{-1/2}[d\xi_u - (\delta + A\hat{X}_u)du], \quad I_0 = 0, \tag{2.7}$$

which is a $\mathbb{G}-$Brownian motion under $\mathbb{P}$; see [3]. Then, the Kalman filter $\hat{X}$ is the unique solution of the linear SDE:

$$\begin{cases} d\hat{X}_t = \beta(\hat{X}_t)dt + \tilde{\Lambda} dI_t \\ \text{with initial condition } \hat{X}_0 = x_0. \end{cases} \tag{2.8}$$

Let us define

$$\tilde{\Lambda}_t := \tilde{\Lambda}(\Pi_t) := (\Pi_t A^* + \Lambda\Sigma^*)(\Theta)^{-1/2}, \tag{2.9}$$

where the covariance matrix Π_t is the unique non-negative definite symmetric solution of the matrix Riccati equation (see Fleming and Rishel [17]):

$$\dot{\Pi}_t + (\Pi_t A^* + \Lambda\Sigma^*)\Theta^{-1}(A\Pi_t + \Sigma\Lambda) - \Lambda\Lambda^* - B\Pi_t - \Pi_t B^* = 0, \quad \Pi_0 = 0. \tag{2.10}$$

Thus, the dynamics of the log prices process ξ_t can be written in terms of the estimate Kalman filter and the innovation process as

$$d\xi_t = (\delta + A\hat{X}_t)dt + \Theta^{1/2}\, dI_t, \quad \xi_0 = \ln S_0.$$

We can write $\frac{dS^i}{S^i}$ in term of $d\xi^i$ as

$$\frac{dS^i}{S^i} = d\xi^i + \frac{1}{2}d\left\langle \xi^i \right\rangle = d\xi^i + \frac{1}{2}\Theta^{i,i}dt.$$

Hence, we can rewrite the dynamics (2.3) of the value process V in terms of the innovation process I by

$$\frac{dV}{V} = (r\pi^0 + \pi^*\alpha(\hat{X}))dt + \pi^*\Theta^{1/2}\, dI. \tag{2.11}$$

Notice that this SDE is written in terms of *completely* observed processes, in comparison with (2.3).

3. The optimal certainty equivalent as a risk sensitivity problem

We focus in the power utility function

$$U(x) = \frac{1}{p}x^p, \text{ with } p < 0.$$

For a random variable Z, the certainty equivalent of Z with respect to U is

$$\text{CE}(Z) := U^{-1}(\mathbb{E}[U(Z)]) = \mathbb{E}^{\frac{1}{p}}[Z^p].$$

Let the process $M^{\pi,p}$ be defined by

$$M_t^{\pi,p} := \exp\left\{p\int_0^t \pi_s^*\Theta^{\frac{1}{2}}dI_s - \frac{1}{2}p^2\int_0^t \pi_s^*\Theta\pi_s ds\right\}, \tag{3.1}$$

where I is the innovation process defined above in (2.7). The set $\mathcal{A}(T)$ consists of those investment strategies π such that $\{M_t^{\pi,p}\}_{0\le t\le T}$, is a genuine martingale. Our first goal is to characterize the value function

$$\zeta_T(p,x_0,0) := \max_{\pi\in\mathcal{A}(T)} \log \mathrm{CE}(V_T^\pi), \tag{3.2}$$

by a dynamical programming PDE from which an optimal strategy $\hat{\pi}$ can be determined. The notation for ζ deserves a clarification. Dependence on x_0 is due to the initial condition on the factor process. We have introduced an extra argument evaluated at zero in order to incorporate latter in the same notation a parameter for drawdown constraints, the value zero corresponds to a condition of no bankruptcy. It is well known from Fleming and Sheu [14] that the problem can be transformed into a risk sensitivity control problem. Indeed, let us review following Fleming and Soner [16, Chapter VI] how this transformation is achieved.

For an admissible π, let $Q^{\pi,p}$ be the probability measure defined by $\mathbb{E}_{Q^{\pi,p}}[X] = \mathbb{E}[XM_T^{\pi,p}]$ where X is a bounded $\mathcal{F}_T$-measurable function. The process $I^{\pi,p}$ defined by

$$dI^{\pi,p} := dI - p\Theta^{\frac{1}{2}}\pi dt,$$

is a $Q^{\pi,p}$-Brownian motion by Girsanov's transformation theorem. Now we consider the expected utility $\mathbb{E}[V_T^p]$ of terminal wealth and express it in the following form:

$$\begin{aligned}\mathbb{E}[V_T^p] =& \mathbb{E}[\exp\{p\ln V_T\}] = V_0^p\mathbb{E}\left[\exp\left\{p\int_0^T\{r\pi_t^0+\pi_t^*\alpha(\hat{X}_t)\}dt\right.\right.\\ &\left.\left.+p\int_0^T \pi_t^*\Theta^{\frac{1}{2}}\,dI_t - \frac{p}{2}\int_0^T \pi_t^*\Theta\pi_t\,dt\right\}\right]\\ =&\mathbb{E}_p^\pi\left[\exp\left\{p\int_0^T l^p(\hat{X}_t,\pi_t)dt\right\}\right].\end{aligned} \tag{3.3}$$

Here $l^p(x,\pi) := r\pi^0 + \pi^*\alpha(x) + \frac{1}{2}(p-1)\pi^*\Theta\pi$ and $\mathbb{E}_p^\pi[\cdot]$ is expected value with respect to the probability measure $Q^{\pi,p}$. Hence,

$$\begin{aligned}\zeta_T(p,x,0) &= \max_{\pi\in\mathcal{A}(T)} \frac{1}{p}\log\mathbb{E}[(V_T^\pi)^p]\\ &= \max_{\pi\in\mathcal{A}(T)} \frac{1}{p}\log\mathbb{E}_p^\pi\left[\exp\left\{p\int_0^T l^p(\hat{X}_t,\pi_t)dt\right\}\right].\end{aligned} \tag{3.4}$$

For our goal of characterizing the value function ζ_T and determining an optimal strategy $\hat{\pi}$ through a dynamical programming PDE we introduce notation preliminaries in the next section.

3.1. Preliminaries for the dynamical programming equation.

For a function $f:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ we use the notation

$$Df := \begin{pmatrix} \frac{\partial f}{\partial x_1}\\ \frac{\partial f}{\partial x_2}\\ \vdots \\ \frac{\partial f}{\partial x_n}\end{pmatrix}$$

For a vector $v = (v_1, \dots, v_n)^*$ the expression $(Df)^* v$ means

$$(Df)^* v = \sum_{i=1}^{n} \frac{\partial f}{\partial x_n} v_i.$$

For the matrix of second order derivatives we use the notation

$$D^2 f := \begin{pmatrix} \frac{\partial f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial f}{\partial x_1 \partial x_n} \\ \frac{\partial f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial f}{\partial x_2 \partial x_n} \\ \vdots & & \\ \frac{\partial f}{\partial x_n \partial x_1} & \cdots & \frac{\partial f}{\partial x_n \partial x_n} \end{pmatrix}.$$

For matrices $M = (M_{i,j})_{i,j}$ and $N = (N_{i,j})_{i,j}$ of dimension $n \times n$ we define $\operatorname{tr}(NM)$ as in [16, IV (3.1)]. Hence, the expression $\operatorname{tr}(D^2 f\, M)$ means

$$\operatorname{tr}(D^2 f M) := \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i \partial x_j} M_{i,j}.$$

For the verification Theorem 3.2 below we consider the following construction. Define the smooth function $g : [0,T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by $g(t,x,y) := y f(t,x)$. Note that under $Q^{\pi,p}$ the Kalman filter $\hat{X}$ follows the dynamics

$$d\hat{X}_t = B(\hat{X}_t, \pi)dt + \tilde{\Lambda} dI_t^{\pi,p}, \quad \hat{X}_0 = x_0 \tag{3.5}$$

where

$$B(x,\pi) := \beta(x) + p\tilde{\Lambda}\Theta^{\frac{1}{2}}\pi.$$

We will make constant use of the matrix product $\tilde{\Lambda}\tilde{\Lambda}^*$ (see (2.9) for its definition) and therefore we introduce the notation:

$$\Upsilon := \tilde{\Lambda}\tilde{\Lambda}^*. \tag{3.6}$$

Observe that this is a matrix-valued stochastic process, depending on the covariance matrix Π_t at time t. The process Y is defined as the solution of $dY_t = Y_t p l^p(\hat{X}_t, \pi_t)dt$. An application of Itô's formula to $G_t := g(t, \hat{X}_t, Y_t)$ leads to

$$\frac{1}{Y_t} dG_t = \left[\frac{\partial f}{\partial t} + p l^p f + (Df)^* B + \frac{1}{2} \operatorname{tr}(D^2 f\, \Upsilon) \right] dt + dN_t, \tag{3.7}$$

where $dN_t = (Df)^* \tilde{\Lambda} dI^{\pi,p}$. Hence we introduce the operator

$$\mathfrak{F}[f](\cdot,\cdot,\pi) := \frac{\partial f}{\partial t} + p l^p(\cdot,\pi) f + (Df)^* B(\cdot,\pi) + \frac{1}{2} \operatorname{tr}(D^2 f\, \Upsilon). \tag{3.8}$$

3.2. **Minimizing the Hamiltonian.** Now we solve

$$\inf_{\pi \in \mathbb{R}^m} \left\{ p l^p(\cdot,\pi) f + (Df)^* B(\cdot,\pi) \right\}.$$

Recall that we defined $\Theta := \Sigma\Sigma^*$ in equation (2.4). Also recall that $l^p(x,\pi) = r\pi^0 + \pi^*\alpha(x) + \frac{1}{2}(p-1)\pi^*\Theta\pi = r + \pi^*(\alpha(x) - r\mathbf{1}^m) + \frac{1}{2}(p-1)\pi^*\Theta\pi$. The critical point $\hat{\pi}$ must satisfy the first order condition

$$f\left\{ p(\alpha - r\mathbf{1}^m)^* + p(p-1)\hat{\pi}^*\Theta \right\} + p(Df)^*\tilde{\Lambda}\Theta^{\frac{1}{2}} = 0.$$

Hence

$$\hat{\pi} = \frac{1}{1-p}\Theta^{-1}\left\{\frac{1}{f}\Theta^{\frac{1}{2}}\tilde{\Lambda}^* Df + (\alpha - r\mathbf{1}^m)\right\}. \tag{3.9}$$

The expression $\inf_{\pi\in\mathbb{R}^m}\{pl^p(\cdot,\pi)f + (Df)^* B(\cdot,\pi)\}$ is then equal to

$$\begin{aligned} pl^p(x,\hat{\pi})f(t,x) + (Df(t,x))^* B &= rpf + \frac{p}{1-p}(Df)^*\tilde{\Lambda}\Theta^{-\frac{1}{2}}(\alpha - r\mathbf{1}^m) + (Df)^*\beta \\ &+ \frac{1}{2}\frac{p}{1-p}\left\{\frac{1}{f}(Df)^*\Upsilon Df + f(\alpha - r\mathbf{1}^m)^*\Theta^{-1}(\alpha - r\mathbf{1}^m)\right\}. \end{aligned}$$

The operator (3.8) returns a function that depends on (t,x,π). Evaluated at $\hat{\pi}$ it takes the form

$$\begin{aligned} \mathfrak{F}[f](\cdot,\cdot,\hat{\pi}) &= \frac{\partial f}{\partial t} + rpf + \frac{1}{2}\operatorname{tr}(D^2 f\,\Upsilon) \\ &+ \frac{1}{2}\frac{p}{1-p}\left\{\frac{1}{f}(Df)^*\Upsilon Df + f(\alpha - r\mathbf{1}^m)^*\Theta^{-1}(\alpha - r\mathbf{1}^m)\right\} \\ &+ (Df)^*\left\{\frac{p}{1-p}\tilde{\Lambda}\Theta^{-\frac{1}{2}}(\alpha - r\mathbf{1}^m) + \beta\right\}. \end{aligned} \tag{3.10}$$

3.3. **Logarithmic transform and Riccati equation.** Let k be defined by $f = e^{ck}$ for a constant c. One can compute that $Df = cfDk$ and $D^2 f = f(cD^2 k + c^2 Dk(Dk)^*)$. Then (3.10) takes the form

$$\begin{aligned} \frac{1}{cf}\mathfrak{F}[f](\cdot,\cdot,\hat{\pi}) &= \frac{\partial k}{\partial t} + r\frac{p}{c} + \frac{1}{2}c\frac{1}{1-p}Dk^*\Upsilon Dk + \frac{1}{2}\operatorname{tr}(D^2 k\,\Upsilon) \\ &+ \frac{1}{2}\frac{1}{c}\frac{p}{1-p}(\alpha - r\mathbf{1}^m)^*\Theta^{-1}(\alpha - r\mathbf{1}^m) \\ &+ (Dk)^*\left\{\frac{p}{1-p}\tilde{\Lambda}\Theta^{-\frac{1}{2}}(\alpha - r\mathbf{1}^m) + \beta\right\}. \end{aligned} \tag{3.11}$$

Remark 3.1. If the reader wants to take the opportunity to verify (3.11), then the following elementary detail provides the main simplification. Note that for x a $n\times 1$-dim vector and M a $n\times n$ matrix $x^*Mx = \operatorname{tr}(xx^*M)$. Hence, $\operatorname{tr}[DkDk^*\,\Upsilon] = Dk^*\Upsilon Dk$.

The choice $c = p$ is the one that simplifies the most in (3.11) and is also the one that links to the value function in (3.2). Hence, we define

$$\begin{aligned} \mathfrak{L}[k] &:= \frac{\partial k}{\partial t} + \frac{1}{2}\frac{p}{1-p}Dk^*\Upsilon Dk + \frac{1}{2}\operatorname{tr}(D^2 k\,\Upsilon) \\ &+ \frac{1}{2}\frac{1}{1-p}(\alpha - r\mathbf{1}^m)^*\Theta^{-1}(\alpha - r\mathbf{1}^m) + r \\ &+ (Dk)^*\left\{\frac{p}{1-p}\tilde{\Lambda}\Theta^{-\frac{1}{2}}(\alpha - r\mathbf{1}^m) + \beta\right\}. \end{aligned} \tag{3.12}$$

We propose a solution of the equation

$$\mathfrak{L}[k] = 0 \tag{3.13}$$

in the form $k(t,x) = x^*Q(t)x + 2q(t)^*x + m(t)$ where Q is a $n \times n$-dim symmetric matrix, q is a $n \times 1$-dim vector and m is a function in $\mathbb{R}$. Then $Dk = 2Qx + 2q$, $D^2k = 2Q$, and the equation (3.13) reduces to

$$x^*Mx + Nx + R = 0, \tag{3.14}$$

where

$$\begin{cases} M := \dot{Q} + 2\frac{p}{1-p}Q\Upsilon Q + 2Q\left\{\frac{p}{1-p}\tilde{\Lambda}\Theta^{-\frac{1}{2}}A + B\right\} + \frac{1}{2}\frac{1}{1-p}A^*\Theta^{-1}A, \\ \text{with terminal condition } Q(T) = 0, \end{cases} \tag{3.15}$$

$$\begin{cases} N := \dot{q}^* + \left(\frac{p}{1-p}\tilde{a}^*\Theta^{-\frac{1}{2}}\tilde{\Lambda}^* + \frac{2p}{1-p}q^*\Upsilon + b^*\right)Q \\ \qquad + \left(\frac{1}{2(1-p)}\tilde{a}^*\Theta^{-1} + \frac{p}{1-p}q^*\tilde{\Lambda}\Theta^{-\frac{1}{2}}\right)A + q^*B, \\ \text{with terminal condition } q(T) = 0, \end{cases} \tag{3.16}$$

$$\begin{cases} R := \dot{m} + r + \mathrm{tr}[Q\Upsilon] + 2\frac{p}{1-p}q^*\Upsilon q + \frac{1}{2}\frac{1}{1-p}\tilde{a}^*\Theta^{-1}\tilde{a} \\ \qquad +2q^*\left(\frac{p}{1-p}\tilde{\Lambda}\Theta^{-\frac{1}{2}}\tilde{a} + b\right). \\ \text{with terminal condition } m(T) = 0. \end{cases} \tag{3.17}$$

Hence, the solution k is reduced to solve first a matrix Riccati differential equation for Q, and then get a solution q for the linear differential equation, and finally m is found by integration. Note that $x^*Mx = \frac{1}{2}x^*(M + M^*)x$, and then it suffices to verify that $\frac{1}{2}x^*(M + M^*)x = 0$ for each x. The requirement $\frac{1}{2}(M + M^*) = 0$ yields the following Riccati equation where Q is the unknown, assumed to be symmetric

$$\begin{aligned} \dot{Q}+2\frac{p}{1-p}Q\Upsilon Q + Q\left\{\frac{p}{1-p}\tilde{\Lambda}\Theta^{-\frac{1}{2}}A + B\right\} + \left\{\frac{p}{1-p}\tilde{\Lambda}\Theta^{-\frac{1}{2}}A + B\right\}^* Q \\ + \frac{1}{2}\frac{1}{1-p}A^*\Theta^{-1}A = 0. \end{aligned} \tag{3.18}$$

The equation (3.18) has a unique solution due to well known results; see [38] and [28].

3.4. **The optimal feedback $\hat{\pi}$ revisited.** We defined the (optimal) feedback $\hat{\pi}$ in (3.9). Considering that $\ln f = pk$ and $\frac{1}{f}Df = 2p(Qx + q)$, we obtain the explicit form

$$\hat{\pi}(x) = \frac{1}{1-p}\Theta^{-1}\left\{2p\Theta^{\frac{1}{2}}\tilde{\Lambda}^*(Qx + q) + (\alpha(x) - r\mathbf{1}^m)\right\}. \tag{3.19}$$

In particular, we obtain that $\hat{\pi}$ is linear with respect to x. Hence, there exists a constant $K > 0$ such that

$$\|\hat{\pi}(x)\| \leq K\|x\| + K.$$

This linear growth of $\hat{\pi}$ is necessary in order to show that $\{\hat{\pi}(\hat{X}_t)\}$ belongs to the set of admissible strategies $\mathcal{A}(T)$; see the verification Theorem 3.2.

3.5. **Verification theorem in finite horizon.** In order to apply the dynamical programming principle we introduce a dynamic version of the value function ζ_T in (3.4). It is given by

$$\zeta_T(p,x,0,t) = \max_{\pi\in\mathcal{A}(T)} \frac{1}{p}\log \mathbb{E}^{\pi}_p\left[\exp\left\{p\int_t^T l^p(\hat{X}_s,\pi_s)ds\right\} \mid \hat{X}_t = x\right]. \tag{3.20}$$

Our goal is to characterize the function ζ_T as the solution of a dynamical programming equation. In the next result we proceed as in the proof of [16, Theorem VI.8.1] by considering an additional state component $dY_s = pY_s l^p(\hat{X}_s,\pi_s)ds$.

Theorem 3.2. *The function ζ_T is of the form $\zeta(p,x,0,t) = k(p,x,t)$ where k is the unique solution of* (3.13). *The strategy $\hat{\pi}(\hat{X})$ determined by the system* (3.5) *and* (3.19) *is optimal.*

Proof. There exists a solution to the Riccati equation (3.18) and (3.16)-(3.17) see e.g., [17, Theorem IV.5.2]. Then, the equation $\mathfrak{L}[k] = 0$ in (3.13) has the solution $k(t,x) = x^*Q(t)x + 2q(t)^*x + m(t)$ with terminal condition $k(x,T) = 0$. Hence, the function $f := e^{pk}$ is a solution to $\mathfrak{F}[f](\cdot,\cdot,\hat{\pi}) = 0$ with terminal condition $f(x,T) = 1$, where $\mathfrak{F}$ is defined in (3.8). Indeed, $\mathfrak{F}$ and $\mathfrak{L}$ are connected by the logarithmic transformation; see (3.11). The claim of the theorem will be proved after we verify that

$$k = \max_{\pi\in\mathcal{A}(T)} \frac{1}{p}\log \mathbb{E}^{\pi}_p\left[\exp\left\{p\int_t^T l^p(\hat{X}_s,\pi_s)ds\right\} \mid \hat{X}_t = x\right].$$

Equivalently

$$f = \min_{\pi\in\mathcal{A}(T)} \mathbb{E}^{\pi}_p\left[\exp\left\{p\int_t^T l^p(\hat{X}_s,\pi_s)ds\right\} \mid \hat{X}_t = x\right]. \tag{3.21}$$

We start with the inequality $\leq$) in (3.21). Take an admissible control π and consider an additional state component $dY_s = pY_s l^p(\hat{X}_s,\pi)ds$ in the interval $[t,T]$ with initial condition $Y_t = y$, for $y > 0$, where $\hat{X}$ satisfies (3.5). Further, we define the function

$$\tilde{f}(x,y,t) := yf(x,t),\ y > 0.$$

Apply Itô's formula for $s \in [t,T]$

$$\tilde{f}(\hat{X}_s,Y_s,s) = \tilde{f}(x,y,t) + \int_t^s Y_u\mathfrak{F}[f](\cdot,\cdot,\pi)du + \frac{1}{2}\int_t^s Y_u Df^*\tilde{\Lambda}dI^{\pi,p}_u.$$

Then, for an initial condition $y > 0$

$$\begin{aligned}\tilde{f}(\hat{X}_s,Y_s,s) &= yf(x,t) + \int_t^s Y_u\mathfrak{F}[f](\cdot,\cdot,\pi)du + \frac{1}{2}\int_t^s Y_u Df^*\tilde{\Lambda}dI^{\pi,p}_u \\ &\geq yf(x,t) + \frac{1}{2}\int_t^s Y_u Df^*\tilde{\Lambda}dI^{\pi,p}_u.\end{aligned}$$

We assume without loss of generality that $\int_t^s Y_u Df^*\tilde{\Lambda}dI^{\pi,p}_u$ is a Q^{π}_p-martingale by a localization argument as in the proof of Fleming and Soner [16, Lemma IV.3.1]. As a consequence

$$\mathbb{E}^{\pi}_p[Y_T] = \mathbb{E}^{\pi}_p[\tilde{f}(\hat{X}_T,Y_T,T)] \geq yf(x,t).$$

The desired inequality $\leq$) in (3.21) follows taking $y = 1$ and the infimum over admissible π. For the converse inequality, we must verify that for the feedback $\hat{\pi}$, the process $M^{\hat{\pi},p}$ is a martingale. In this case the inequality in the previous display is an equality since the integral $\int_t^T Y_u Df^* \tilde{\Lambda} dI_u^{\pi,p}$ is a $Q_p^{\hat{\pi}}$ supermartingale. It can be shown that $M^{\hat{\pi},p}$ is a martingale with similar arguments as in [3, Lemma 4.1.1]. $\square$

4. Solving the infinite time horizon problem

Next we look at the investor's optimal certainty equivalent rate, reviewing the solution of the problem of maximizing the exponential rate of growth of the expected utility of terminal wealth, under power utility function, as $T \to \infty$ in (3.2). Let

$$
\begin{aligned}
(4.1) \quad \zeta_\infty(p, x_0, 0) &:= \max_{\pi \in \mathcal{A}} \limsup_{T \to \infty} \frac{1}{T} \log \mathrm{CE}(V_T^\pi) \\
&= \max_{\pi \in \mathcal{A}} \limsup_{T \to \infty} \frac{1}{Tp} \log \mathbb{E}\left[(V_T^\pi)^p \,|\hat{X}_0 = x_0 \right] \\
&= \max_{\pi \in \mathcal{A}} \limsup_{T \to \infty} \frac{1}{Tp} \log \mathbb{E}_p^\pi \left[\exp\left\{ p \int_0^T l^p(\hat{X}_t, \pi_t) dt \right\} |\hat{X}_0 = x_0 \right].
\end{aligned}
$$

We define $\mathcal{A}$ as the set of investment strategies π in the interval $[0, \infty)$ such that $\pi \in \mathcal{A}(T)$ for each positive T; this set describes the set of admissible strategies throughout this section. The above value function can be interpreted as the optimal long-term relative growth rate of a partially observed risk-sensitivity control problem [3], and its analysis is based on the previous results for the finite horizon case.

Remark 4.1. Long-term growth rate of the expected utility U of wealth can be studied maximizing

$$
\limsup_{T \to \infty} \frac{1}{T} \log \mathbb{E}[U(V_T^\pi)].
$$

Defining $\log x = -\log(-x)$, for $x < 0$, this criterion is equivalent to the one defined in (4.1) modulus some multiplicative positive factor for the power utility function. This critical rate shall be analyzed in the next section introducing drawdown restrictions in the wealth process V^π.

One of the main difficulties to adapt known results on this regard to our framework is that the diffusion coefficient of the dynamics of the Kalman filter $\tilde{\Lambda}$ in (3.5) depends on the information up to time t through the covariance matrix Π_t; see (2.9). Indeed, from the finite-time dynamic programming equation (3.10), we can write formally the PDE satisfied by $(\rho(p), W)$ as

$$
\begin{aligned}
\rho(p) &= \frac{1}{2} \mathrm{tr}(D^2 W \, (\hat{\Lambda}\hat{\Lambda}^*)) + \frac{p}{2} (DW)^* \hat{\Lambda}\hat{\Lambda}^* DW \\
&\quad + \inf_{\pi \in \mathbb{R}^m} \{ l^p(\cdot, \pi) + (DW)^* B(\cdot, \pi) \} \\
&= \frac{1}{2} \mathrm{tr}(D^2 W \, (\hat{\Lambda}\hat{\Lambda}^*)) + \frac{1}{2} \frac{1}{1-p} (\alpha - r\mathbf{1}^m)^* \Theta^{-1} (\alpha - r\mathbf{1}^m) + r \\
&\quad + \frac{p}{2(1-p)} (DW)^* \hat{\Lambda}\hat{\Lambda}^* DW + (Dk)^* \left\{ \frac{p}{1-p} \hat{\Lambda} \Theta^{-\frac{1}{2}} (\alpha - r\mathbf{1}^m) + \beta \right\},
\end{aligned}
\tag{4.2}
$$

where the second equality is obtained substituting the value of $\hat{\pi}$ where the infimum is achieved, which is given by

$$\hat{\pi}(x) = \frac{1}{1-p}\Theta^{-1}\left\{p\Theta^{\frac{1}{2}}\hat{\Lambda}^* DW(x) + (\alpha(x) - r\mathbf{1}^m)\right\}. \tag{4.3}$$

Notice that the sign of the nonlinear term in (4.2) depends explicitly on p and then will influence the drift term of $\hat{X}$ in (3.5).

Here $\rho(p) \in \mathbb{R}$ and the function W are unknowns and explicit formulas for them would be expected, following similar arguments given for the finite horizon case, since the control set is the Euclidean space $\mathbb{R}^m$ and the dynamics involved are linear with a quadratic exponential-type structure. Then, we are tempt to consider a quadratic form for W, i.e.

$$W(x) = x^*\hat{Q}x + 2\hat{q}^*x, \tag{4.4}$$

with $\hat{Q}$ a symmetric $n\times n$ matrix and $\hat{q} \in \mathbb{R}^n$. This argument has been developed for the completely observed version of our problem by Fleming and Shue [13, Theorem 3.5].

However, formalize the previous equation is an open question, as far as we know, since there are several technical issues involved. One of the most relevant consists in defining properly matrix $\hat{\Lambda}$ in (4.2). These important features were first investigated by Nagai and Peng in their fundamental work [33], and next we quote some results obtained by them. Observe that, in the finite horizon problem, the HJB equation depends on matrix $\tilde{\Lambda} := \tilde{\Lambda}(\Pi_t)$, which is the linear transformation of variance matrix Π_t defined as $\tilde{\Lambda}(\Pi_t) = (\Pi_t A^* + \Lambda\Sigma^*)\Theta^{-1/2}$, and Π_t is the unique solution of the ODE (2.10). For the model under consideration, in which the diffusion coefficient matrix of the Kalman filter $\hat{X}_t$ turns out to be $\tilde{\Lambda}$ (see (2.8)), it is natural to analyze first the asymptotic limit of Π_t as $T-t \to \infty$, in order to adapt the approach followed for solving the finite horizon problem. Under the condition that the matrix

$$B - \Lambda\Sigma^*\Theta^{-1}A \tag{4.5}$$

is stable, Nagai and Peng [33, Lemma 4.1] proved that Π_t converges exponentially fast to $\bar{\Pi} \geq 0$, and the limit matrix is the unique solution of the algebraic Riccati equation:

$$\begin{aligned}(B - \Lambda\Sigma^*\Theta^{-1}A)\bar{\Pi} + \bar{\Pi}(B - \Lambda\Sigma^*\Theta^{-1}A)^* - \bar{\Pi}A^*\Theta^{-1}A\bar{\Pi}&\\ + \Lambda(I_{m+n} - \Sigma^*\Theta^{-1}\Sigma)\Lambda^* = 0.&\end{aligned}$$

The attractor $\bar{\Pi}$ allow us to describe asymptotic limits for the funtions involved in the description of the finite time value function k, and now we present the analogous versions of equations (3.15)-(3.17). First, we define $\hat{\Lambda} := \hat{\Lambda}(\bar{\Pi}) := (\bar{\Pi}A^* + \Lambda\Sigma^*)\Theta^{-1/2}$ and $\hat{\Upsilon} := \hat{\Lambda}\hat{\Lambda}^*$. Let

$$\begin{cases}\dot{\bar{Q}} + \frac{2p}{1-p}\bar{Q}\hat{\Upsilon}\bar{Q} + \bar{Q}\left\{\frac{p}{1-p}\hat{\Lambda}\Theta^{-\frac{1}{2}}A + B\right\} + \left\{\frac{p}{1-p}\hat{\Lambda}\Theta^{-\frac{1}{2}}A + B\right\}^*\bar{Q}\\ \qquad + \frac{1}{2(1-p)}A^*\Theta^{-1}A = 0,\\ \text{with final condition } \bar{Q}(T) = 0.\end{cases}$$

Since we are interested in the long-time behaviour of this equation, its asymptotic limit $\hat{Q}$, as $T-t$ goes to infinity, corresponds to the unique nonnegative definite solution of the algebraic Riccati equation

$$\frac{2p}{1-p}\hat{Q}\hat{\Upsilon}\hat{Q} + \hat{Q}\left\{\frac{p}{1-p}\hat{\Lambda}\Theta^{-\frac{1}{2}}A + B\right\} + \left\{\frac{p}{1-p}\hat{\Lambda}\Theta^{-\frac{1}{2}}A + B\right\}^{*}\hat{Q} + \frac{1}{2(1-p)}A^{*}\Theta^{-1}A = 0. \tag{4.6}$$

Moreover,

$$\frac{p}{1-p}\hat{\Lambda}\Theta^{-\frac{1}{2}}A + B + \frac{2p}{1-p}\hat{\Upsilon}\hat{Q} \tag{4.7}$$

is a stable matrix.

Remark 4.2. Existence and uniqueness of solution of the algebraic Riccati equation (4.6) has been rarely analyzed explicitly in the literature, and its solution depends on the sign of the parameter p and condition (4.5). Arguments to prove uniqueness are based on ergodic properties of the linear diffusion $\hat{X}_t$ when the candidate for being optimal control $\hat{\pi}$ in (4.3) is applied; see [13, Lemma 3.3] for the completely observed case.

In order to describe the steady limit of the analogous equation (3.16), substituting first in that equation $\tilde{\Lambda}(\Pi_t)$ by $\hat{\Lambda}(\bar{\Pi})$, and writing

$$\begin{cases} \dot{\bar{q}}^{*} + \left(\frac{p}{1-p}a^{*}\Theta^{-\frac{1}{2}}\hat{\Lambda}^{*} + b^{*} + \frac{2p}{1-p}\bar{q}^{*}\hat{\Upsilon}\right)\hat{Q} \\ \quad + \left(\frac{1}{2(1-p)}a^{*}\Theta^{-1} + \frac{p}{1-p}\bar{q}^{*}\hat{\Lambda}\Theta^{-\frac{1}{2}}\right)A + \bar{q}^{*}B = 0, \\ \text{with terminal condition } \bar{q}(T) = 0. \end{cases}$$

Then, as $T-t$ goes to infinity and $t \to \infty$, $\bar{q}$ converges to $\hat{q}$, which solves

$$\left(\frac{p}{1-p}a^{*}\Theta^{-\frac{1}{2}}\hat{\Lambda}^{*} + b^{*} + \frac{2p}{1-p}\hat{q}^{*}\hat{\Upsilon}\right)\hat{Q} + \left(\frac{1}{2(1-p)}a^{*}\Theta^{-1} + \frac{p}{1-p}\hat{q}^{*}\hat{\Lambda}\Theta^{-\frac{1}{2}}\right)A + \hat{q}^{*}B = 0. \tag{4.8}$$

Finally, the asymptotic limit as $T-t \to \infty$ of $\dot{m}$ in (3.17) is

$$\rho(p) := \frac{1}{2}tr[\hat{Q}\hat{\Upsilon}] + \frac{2p}{1-p}\hat{q}^{*}\hat{\Upsilon}\hat{q} + \frac{1}{2(1-p)}a^{*}\Theta^{-1}a + \frac{2p}{1-p}\hat{q}^{*}\hat{\Lambda}\Theta^{-\frac{1}{2}}a + \hat{q}^{*}b + r. \tag{4.9}$$

We expect to have an explicit solution to the optimal investment policy, analogous to the one described in (3.9) for the finite horizon problem. The following results summarize some conclusions in that direction. The first one is obtained substituting the form proposed for $\rho(p)$ and W, and following the arguments given in [38] together with (4.7).

Proposition 4.3. *Let W be defined as in (4.4) with $\hat{Q}$ and $\hat{q}$ as in (4.6) and (4.8), respectively. Then, $(\rho(p), W)$ solves equation (4.4), with $\rho(p)$ given by (4.9).*

The main merit of the previous result is to show that there is a connection between the solution of the ergodic HJB equation (4.2) and the long-run optimal investment problem through a verification result, analogous to the finite horizon case studied in the previous section. A natural approach consists in approximating the infinite horizon problem through the finite solution, taking advantage of the analytical and explicit solution provided by the theory of the Riccati differential equation. Its asymptotic behavior requires some balance between the quadratic coefficient and the independent coefficient in (4.6), as shown below in (4.10). This approach was implemented successfully by Nagai and Peng [33, Theorem 6.1].

Proposition 4.4. *Assume that*

$$\hat{Q}\hat{\Upsilon}\hat{Q} < \frac{1}{4p^2}A^*\Theta^{-1}A. \tag{4.10}$$

Then, $\rho(p)$ corresponds to the value function $\zeta_\infty(p, x_0, 0)$ and

$$\begin{aligned} \hat{\pi}_t &= \frac{1}{1-p}\Theta^{-1}\left\{p\Theta^{\frac{1}{2}}\hat{\Lambda}^* DW(\hat{X}_t) + \left(\alpha(\hat{X}_t) - r\mathbf{1}^m\right)\right\} \\ &= \frac{1}{1-p}\Theta^{-1}\left\{[A + 2p\Theta^{-\frac{1}{2}}\hat{\Lambda}^*\hat{Q}]\hat{X}_t + \{2p\Theta^{\frac{1}{2}}\hat{\Lambda}^*\hat{q}^* + a - r\mathbf{1}^m\}\right\} \end{aligned} \tag{4.11}$$

is an optimal feedback investment strategy, obtained from (4.3)*, where $\hat{X}_t$ follows the dynamics described by the linear SDE* (3.5)*.*

5. Solution to our maximization problem under partial information and drawdown constraints

In this section we solve the problem of maximizing the exponential growth rate as the horizon T goes to infinity in (3.4) in which information is restricted to the filtration $\mathbb{G}$ generated by asset prices and in which the wealth process of admissible strategies satisfy a condition known as drawdown constraint. One of the first works solving a portfolio selection problem with drawdown constraints is [19]. The solution to this class of problems involves elements of singular control in the sense that the corresponding dynamical programming equation takes the form of a variational inequality; see e.g., [10]. Here, however, we follow an alternative approach taking advantage of the explicit form of the optimal growth rate as well as the optimal investment strategy found in the previous section. Instead of following a direct approach of incorporating the drawdown constraint into the dynamical programming equation, we consider the correspondence of value functions of problems with and without constraints as presented in [7] and [37], which are based on the Azema-Yor semimartingale processes systematically studied by [5] and from which we give below the necessary preliminaries.

Fix $\kappa \in [0, 1)$. By $\mathcal{A}_\kappa$ we denote the family of admissible strategies $\pi \in \mathcal{A}$ such that almost surely

$$\tilde{V}_t > \kappa\tilde{V}_t^*, \text{ for } t \in [0,\infty), \tag{5.1}$$

where $\tilde{V} := \frac{V^\pi}{S^0}$ is the discounted wealth process of π and we use the notation of the maximum to date of process $\tilde{V}$:

$$\tilde{V}_t^* := \sup_{0\le s\le t} \tilde{V}_s.$$

Throughout we are concerned with a portfolio management problem where the goal is to exceed the performance of a fraction of the last-record maximum of the discounted wealth process at any time. The value function of our problem is then

$$\begin{aligned}\zeta_\infty(p,x,\kappa) &:= \max_{\pi\in\mathcal{A}_\kappa}\limsup_{T\to\infty}\frac{1}{T}\log \mathrm{CE}(V_T^\pi)\\ &= \max_{\pi\in\mathcal{A}_\kappa}\limsup_{T\to\infty}\frac{1}{Tp}\log \mathbb{E}_p^\pi\left[\exp\left\{p\int_0^T l^p(\hat{X}_t,\pi_t)dt\right\}\mid \hat{X}_0=x\right].\end{aligned}$$

The exponent $p\in(-\infty,0)$ which defines the utility function becomes a fixed risk-sensitivity parameter and the investor's portfolio benchmark is represented as a constraint that capital cannot drop below a fraction κ of the last recorded maximum; see (5.1). Note that $\zeta_\infty(p,x,0)$ is the classical problem on infinite horizon, with bankruptcy restriction, and without drawdown constraint.

Remark 5.1. The observant reader will notice that utility is being assessed from non discounted portfolios while the drawdown constraint is formulated for discounted portfolios. It is also possible to assess utility from discounted portfolios. However, dealing with the optimization problem under drawdown constraints formulated directly on non discounted portfolios is a more subtle topic. Indeed, to a large extent it is an open problem; see e.g., [7, Remark 4.8] and [37, Remark 2.3].

There are two well known techniques to deal with the drawdown constraint, one based on Lagrange multipliers and optimal control theory, and the approach based on the study of the Azema-Yor semimartingale processes. As we mentioned before, we focus in this last one. The problem is to characterize the value function $\zeta_\infty(p,x,\kappa)$ by a dynamical programming PDE from which an optimal strategy $\hat{\pi}$ can be determined. Its solution is presented in Theorem 5.3 below. Before that, we give a few necessary preliminaries on the Azema-Yor semimartingale processes. The key observation consists in noting that there is a bijection between wealth processes associated to portfolios $\pi\in\mathcal{A}$ and those associated with elements of $\mathcal{A}_\kappa$. Hence, each discounted wealth process $\tilde{V}^\pi$, with $\pi\in\mathcal{A}$, can be transformed into ${}^\kappa\tilde{V}^\pi := M^{F,\kappa}(\tilde{V}^\pi)$, where

$$M^{F,\kappa}(\tilde{V}_t^\pi) := F(Z_t) - F'(Z_t)(Z_t-\tilde{V}_t^\pi),\ \text{with } Z := (\tilde{V}^\pi)^*. \tag{5.2}$$

Here F' is a locally bounded function and $F(z)=F(z_0)+\int_{z_0}^z F'(u)du$. For the drawdown function $w(v)=\kappa v$, the election of F is given by $F(v)=v^{1-\kappa}$; see [7, Example 3.3] with $v_0=1$. Moreover, given $\tilde{V}^\pi$, for $\pi\in\mathcal{A}$, its transformation into $\tilde{V}^{\pi,\kappa} := M^{F,\kappa}(\tilde{V}^\pi)$ corresponds to a process satisfying the drawdown constraint (5.1); cf. [7, Proposition 3.2].

Now, there is a relation between the value functions $\zeta_\infty(p,x_0,\kappa)$ and $\zeta_\infty(p(1-\kappa),x_0,0)$ formulated in Theorem 5.2 of [7] since we are considering power utilities. Summarizing, in order to be able to solve the partially observed portfolio optimization problem with drawdown restrictions it is sufficient to solve a portfolio optimization with non bankrupcy restrictions using as state variables the Kalman filter $\hat{X}_t$ in the evolution of the portfolio process associated with admissible strategies, and a power utility function with parameter $\gamma := p(1-\kappa)$. Having these conclusions in mind, we proceed to solve first the finite horizon risk-sensitivity control problem arising from this formulation, and then pass to the limit when T goes to infinity. This will allow us to determine the optimal strategies for the investment strategies, and then proceed to get the explicit form of the investment portfolio π^* for the original problem with drawdown constraints.

A general form of the Azema-Yor process is analyzed deeply in [5] . The transformation given by Azema-Yor processes allows to characterize portfolios satisfying the constraint (5.1). The process defined in (5.2) can also be written as

$$\frac{d\,{}^{\kappa}\tilde{V}^{\pi}}{{}^{\kappa}\tilde{V}^{\pi}} = \frac{{}^{\kappa}\tilde{V}^{\pi} - \kappa({}^{\kappa}\tilde{V}^{\pi})^*}{{}^{\kappa}\tilde{V}^{\pi}} \frac{d\tilde{V}^{\pi}}{\tilde{V}^{\pi}}, \tag{5.3}$$

see the Proposition 2.2 and Corollary 2.4 in [5]. A process with dynamic (5.3) satisfies the constraint (5.1). Let

$$R(v, v^*) := \frac{(1-\kappa)\frac{v}{v^*}}{\kappa + (1-\kappa)\frac{v}{v^*}}.$$

We have a further expression of ${}^{\kappa}\tilde{V}^{\pi}$ in terms of R:

$$\frac{d\,{}^{\kappa}\tilde{V}^{\pi}}{{}^{\kappa}\tilde{V}^{\pi}} = R(\tilde{V}^{\pi}, (\tilde{V}^{\pi})^*) \frac{d\tilde{V}^{\pi}}{\tilde{V}^{\pi}}. \tag{5.4}$$

In combination with equation (2.11) we can express the dynamic (5.3) in terms of the Kalman filter as

$$\begin{aligned} \frac{d\,{}^{\kappa}\tilde{V}^{\pi}}{{}^{\kappa}\tilde{V}^{\pi}} &= R(\tilde{V}^{\pi}, (\tilde{V}^{\pi})^*)\pi \cdot \left(d\xi + \frac{1}{2}\Theta dt\right) \\ &= R(\tilde{V}^{\pi}, (\tilde{V}^{\pi})^*)\pi \cdot \left(\alpha(\hat{X})dt + \Theta^{1/2} dI\right). \end{aligned} \tag{5.5}$$

Hence, for a $\mathbb{G}$-adapted strategy π the solution to the dynamic (5.5) defines a wealth process that satisfies the drawdown constraint and is expressed in terms of observables in our problem, namely $\mathbb{G}$-adapted processes.

Remark 5.2. We defined in equation (2.3) the wealth process, also called value process, of a strategy of proportions $\pi = (\pi^1, \dots, \pi^d)$ as

$$\frac{dV}{V} = \bar{\pi}\frac{d\bar{S}}{\bar{S}} = r\pi^0 dt + \pi \cdot \frac{dS}{S} = r\pi^0 dt + \sum_{i=1}^{m} \pi^i \frac{dS^i}{S^i}, \quad V_0 = 1,$$

where $\bar{S} = (S^0, S^1, \dots, S^m)$ and $\bar{\pi} = (\pi^0, \pi^1, \dots, \pi^d.)$ We recall that in this dynamic, π^i is the proportion in monetary units allocated to the asset S^i and $\pi^0 = 1 - \sum_{i=1}^m \pi^i$. In the proof of Theorem 5.3 below, we require to move from proportions allocated to each asset to the corresponding proportion on discounted assets, and here we recall the elementary and necessary preliminary. Let us denote by $\tilde{S}^i$ the discounted price $\tilde{S}^i = \frac{S^i}{S^0}$ of the i asset and by $\tilde{V} = \frac{V}{S^0}$ the discounted value of a wealth process. One easily verifies that the discounted process $\tilde{V}$ shares the same proportions as V in that $\frac{d\tilde{V}}{\tilde{V}} = \pi\frac{d\tilde{S}}{\tilde{S}}$.

Theorem 5.3. *Assume the condition* (4.10) *of Proposition 4.4. The value function* ζ_∞ *does not depend on* x*, hence for* $x_0 \in \mathbb{R}^n$ *we have* $\zeta_\infty(p,x,\kappa) = \zeta_\infty(p,x_0,\kappa)$*. It satisfies* $\zeta_\infty(p,x_0,\kappa) = (1-\kappa)\zeta_\infty(\gamma,x_0,0) + \kappa r$*, where* $\gamma := p(1-\kappa)$*.*

There exists a quadratic function $W(x) = x^*\hat{Q}x + 2\hat{q}^*x$ *such that* $(\zeta_\infty(\gamma,x_0,0), W)$ *is a solution to the ergodic equation*

$$\begin{aligned}\zeta_\infty(\gamma,x_0,0) = \frac{1}{2}\operatorname{tr}(D^2W\ (\hat{\Lambda}\hat{\Lambda}^*)) + \frac{\gamma}{2(1-\gamma)}(DW)^*\hat{\Lambda}\hat{\Lambda}^*DW \\ + \frac{1}{2}\frac{1}{1-\gamma}(\alpha - r\mathbf{1}^m)^*\Theta^{-1}(\alpha - r\mathbf{1}^m) + r \\ + (Dk)^*\left\{\frac{\gamma}{1-\gamma}\hat{\Lambda}\Theta^{-\frac{1}{2}}(\alpha - r\mathbf{1}^m) + \beta\right\}.\end{aligned}$$

Let $\hat{\pi}$ *be defined in* (4.11) *of Proposition 4.4 with* p *replaced by* γ*. Let* V *satisfy* $\frac{dV}{V} = \breve{\pi}\frac{dS}{S}$*. Let* $\tilde{V} := \frac{V}{S^0}$*. An optimal strategy is given by*

$$\breve{\pi}_t := R(\tilde{V}_t, \tilde{V}_t^*)\hat{\pi}.$$

Proof. Our function $\zeta_\infty(p,x_0,\kappa)$ is in the notation of [7, equation (5.1)] equal to $|p|\mathrm{CER}^{w_\kappa}_{H^{(p)}}$ with $w_\kappa(x) = \kappa x$ and $H^{(p)}(x) = \frac{1}{p}x^p$. Note that we have written the function $\zeta_\infty(p,x_0,\kappa)$ as depending on a fixed x_0, indeed it is independent on the initial condition of the factor process. Hence, we have that

$$|p|\zeta_\infty(p,x_0,\kappa) = |p|(1-\kappa)\zeta_\infty(p(1-\kappa),x_0,0) + |p|\kappa r,$$

due to [7, Theorem 5.2 and Remark 5.4]. From this follows the first part of the Theorem concerning the relationship of $\zeta_\infty(p,x_0,\kappa)$ and $\zeta_\infty(\gamma,x_0,0)$.

The solution of $\zeta_\infty(\gamma,x_0,0)$ is given by $\hat{\pi}$ defined in (4.11) as we have proved in Proposition 4.4. Moreover, the value process V defined by $\frac{dV}{V} = \breve{\pi}\frac{dS}{S}$ is an element of $\mathcal{A}_\kappa$ by the general properties of Azema-Yor processes presented before the statement of the Theorem. It is optimal for $\zeta_\infty(p,x_0,\kappa)$ again due to [7, Theorem 5.2], part 2.

Note that the coefficient $R(\tilde{V}, \tilde{V}^*)$ is positive and bounded by $1-\kappa$, and then the structural properties used to verify that $\hat{\pi}$ is admissible remain valid to prove that $\breve{\pi}$ belongs to $\mathcal{A}_\kappa$. ☐

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