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REMARKS ON DIFFERENTIAL INCLUSION LIMITS OF STOCHASTIC APPROXIMATION

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Dedicated to Professor Wendell H. Fleming on his 95th birthday

Abstract. For stochastic approximation algorithms with discontinuous dynamics, it is shown that under suitable distributional assumptions, the interpolated iterates track a Fillipov solution of the limiting differential inclusion. In addition, we give an alternative control theoretic approach to recent results of [7] on certain limiting empirical measures associated with the iteration.

1. Introduction

One popular approach to the analysis of stochastic approximation algorithms is to look at their limiting differential equations and interpret the time-asymptotic behavior of the algorithm in terms of that of the differential equation. There are, however, many situations wherein the limiting dynamics is not a differential equation, but a differential inclusion. One important situation where this happens is when the update equation for the algorithm involves a discontinuous function. This leads to a differential equation with a discontinuous right hand side. This is ill-posed in the classical sense. So one resorts to different solution concepts [10] which lead to differential inclusions. There is a considerable body of work in this direction (some of which we outline below). Our contribution here is twofold: First, to refine the set-valued map in the limiting differential equation from a Krasovskii map to a Filippov map under suitable additional conditions, and second, to provide an alternative control theoretic approach to some important recent contributions of [7], with some new insights.

We recall the basic facts about stochastic approximation iterates and their differential equation, resp., differential inclusion limits in the next section. Section 3 describes our main results.

2. Stochastic approximation: preliminaries

Here we briefly recall the relevant aspects of the stochastic approximation algorithm. Introduced by Robbins and Monro in [24], it is a scheme for finding roots of

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a nonlinear map $h : \mathcal{R}^d \mapsto \mathcal{R}^d$ given its noisy measurements. Specifically, it is the iteration

$$x(n+1) = x(n) + a(n)(h(x(n)) + M(n+1)), \; n \geq 0, \tag{2.1}$$

where $\{M(n)\}$ is a martingale difference sequence with respect to the increasing σ-fields

$$\mathcal{F}_n := \sigma(x(0), M(m), m \leq n), n \geq 0,$$

representing measurement noise, and $\{a(n)\}$ is a stepsize sequence satisfying

$$a(n) \geq 0 \; \forall \; n \geq 0, \; \sum_n a(n) = \infty, \; \sum_n a(n)^2 < \infty. \tag{2.2}$$

Typical assumptions imposed are: h is Lipschitz and $\{M(n)\}$ satisfies, for some $K > 0$,

$$\sup_n E\left[\|M(n)\|^4\right] < \infty \quad \text{and} \quad E\left[\|M(n+1)\|^2 | \mathcal{F}_n\right] \leq K\left(1 + \|x(n)\|^2\right). \tag{2.3}$$

While the initial contributions analyzed (2.1) using probabilistic techniques, another approach developed since the 70's [11, 19, 18, 2, 3] treats (2.1) as a noisy Euler scheme for the ODE (for '*Ordinary Differential Equation*')

$$\dot{x}(t) = h(x(t)). \tag{2.4}$$

Under suitable conditions, one can show that $\{x(n)\}$ a.s. (:= almost surely with respect to the underlying probability measure) tracks the asymptotic behavior of (2.4) as $n \uparrow \infty$. The argument goes as follows. (See [8], Chapter 2, for details). Define the 'algorithmic time scale' $t(n), n \geq 0$, by: $t(0) := 0, t(n) := \sum_{m=0}^{n-1} a(m)$. Then by (2.2), $t(n) \uparrow \infty$. Define $\bar{x}(t), t \geq 0$, by: $\bar{x}(t(n)) := x(n) \; \forall n$ with linear interpolation on each interval $[t(n), t(n+1)]$, so that it is a continuous piecewise linear curve. Fix $T > 0$ and for $n \geq 0$, define $m(n) := \min\{k : t(k) - t(n) \geq t(n) + T\}$. On the interval $[t(n), t(m(n))]$, define the ODE

$$\dot{y}^n(t) = h(y^n(t)), \; t \in [t(n), t(m(n))], \; y^n(t(n)) = x(n).$$

If one can establish a.s. boundedness of the iterates, i.e., $\sup_n \|x(n)\| < \infty$ a.s. (this usually needs a separate proof), then (2.3) and the square-summability of $\{a(n)\}$ ensures that the $\{\mathcal{F}_n\}$-martingale $\sum_{m=1}^{n-1} a(m)M(m+1)$ converges a.s. (Proposition VII.2.3(c), p. 149, of [20]). This and the fact $a(n) \to 0$ implied by (2.2) together imply, via the Gronwall inequality, that

$$\lim_{n \uparrow \infty} \max_{t \in [t(n), t(m(n))]} \|\bar{x}(t) - y^n(t)\| \to 0 \text{ a.s.} \tag{2.5}$$

Here the martingale convergence and vanishing stepsize lead to the vanishing of errors due to noise and discretization, respectively. A similar argument works for $T < 0$. This in turn leads to the following characterization of the asymptotic behavior of $\{x(n)\}$ due to Benaim [2, 3].

Theorem 2.1. *Almost surely, $x(n) \to$ a nonempty compact connected internally chain transitive invariant set of (2.4).*

See *ibid.* or Chapter 2 of [8] for details. A more general situation is that of a *stochastic recursive inclusion*, i.e., the iteration

$$x(n+1) = x(n) + a(n)(z(n) + M(n+1)),\ z(n) \in H(x(n))\ n \geq 0, \tag{2.6}$$

where $H : x \in \mathcal{R}^d \mapsto H(x) \subset \mathcal{R}^d$ is a set-valued map and $\{M(n)\}$ is as above. We assume that $x \mapsto H(x)$ is a nonempty compact and convex valued map which is upper semicontinuous (i.e., its graph $\{(x, z) : z \in H(x)\}$ is closed) and satisfies the linear growth condition

$$\max_{z \in H(x)} \|z\| \ \leq \ K'(1 + \|x\|)$$

for some $K' > 0$. Then one looks at a *differential inclusion* limit instead of an o.d.e. This differential inclusion is

$$\dot{x}(t) \in H(x(t)). \tag{2.7}$$

Following the pioneering work of Benaim, Hofbauer and Sorin [4, 5], which, among other things, establish a counterpart of Theorem 2.1 for this case, this iteration has been extensively studied in literature, some of it motivated by applications to reinforcement learning [7, 13, 14, 22, 23, 27, 28, 29]. The aforementioned extension of Theorem 2.1 is as follows.

Theorem 2.2. *Almost surely, $\bar{x}(\cdot)$ asymptotically tracks a solution of (2.7) in the sense that (2.5) holds with $y^n(\cdot) :=$ a solution of (2.7) on $[t(n), t(m(n))]$ with $y^n(t(n)) = x(n)\ \forall n$. Furthermore, $x(n) \to$ a nonempty compact connected internally chain transitive invariant set of (2.4).*

Here, an invariant set is a set of points x such that *some* solution of (2.7) passing through x remains in the set for all time.

A special case of interest is (2.4) when the map h is discontinuous and the usual theory for well-posedness of (2.4) does not apply. The standard approach (see [21] and its references) has been to treat (2.4) as a special case of (2.7) by setting $z(n) = h(x(n))$ and

$$H(x) = K_h(x) := \cap_{\delta > 0} \overline{co}\left(\{h(y) : \|y - x\| < \delta\}\right).$$

Any trajectory thereof is known as a Krasovskii solution to (2.4) [17], one of the many solution concepts for differential equations driven by discontinuous vector fields. (See [10] for a recent survey of the various solution concepts and their interrelationships.)

3. Main results

We assume throughout that

$$\sup_n \|x(n)\| < \infty \text{ a.s.} \tag{3.1}$$

We shall denote by $\mathcal{P}(S)$ the Polish space of probability measures on a Polish space S with the Prohorov topology. $C_b(S)$ will denote the space of bounded continuous functions $S \mapsto \mathcal{R}$.

We present two results in this section. The first is the observation that if the laws of $(x(n), M(n+1))$ are Lebesgue-continuous, then we may replace the commonly used Krasovskii solution by the more restrictive Filippov solution [15] for the limiting differential equation. This is facilitated by a remarkable result from [9].

The second result is in the spirit of another remarkable piece of work [7] that characterizes the limiting behavior of suitably averaged 'occupation measures' $\mu(dxdz|t)$ defined below, as $n \to \infty$. While [7] uses classical analytic tools such as Young measures and concepts from topological dynamics, we take a control theoretic view based on relaxed controls and Stockbridge's extension of Echeverria's theorem [12] to controlled martingale problems [25]. (See [6] for a further extension and [1], Chapter 6, for a detailed exposition.) In addition to recovering variants of the results of [7], this also gives some additional insights.

3.1. **Filippov solution.** Let $\mathcal{N}$ denote the collection of Lebesgue-null sets in $\mathcal{R}^d$. A Fillipov solution to (2.4) replaces $K_h(x)$ above by

$$F_h(x) := \cap_{N\in\mathcal{N}} \cap_{\delta>0} \overline{co}\left(\{h(y) : \|y-x\| < \delta\}\backslash N\right).$$

That is, we consider the differential inclusion limit given by

$$\dot{x}(t) \in F_h(x(t)). \tag{3.2}$$

The elimination of sets of zero Lebesgue measure does matter. Consider, e.g., the following example.

Example 1: Consider the two dimensional case where $h : \mathcal{R}^2 \mapsto \mathcal{R}^2$ is given by

$$\begin{aligned} h(x,y) &= [1,-1], \quad y>0,\\ &= [1,1], \quad y<0,\\ &= [-1,0], \quad y=0. \end{aligned}$$

One can easily see that the Filippov solution will be

$$\begin{aligned} \dot{x}(t) &= 1,\\ \dot{y}(t) &= -1, \quad y>0,\\ &= 1, \quad y<0,\\ &= 0, \quad y=0. \end{aligned}$$

In particular, letting $[x,y]$ denote a generic vector in $\mathcal{R}^2$, we have for $y(t) = 0$, $\frac{d}{dt}[x(t), y(t)] = [1,0] \neq [-1,0]$. A Krasovskii solution would only tell us that for $y(t) = 0$, $\frac{d}{dt}[x(t), y(t)] \in \overline{co}\Big([1,-1],[1,1],[-1,0]\Big)$.

Assume that:

(†) The laws of $(x(n), M(n+1))$ are absolutely continuous with respect to the Lebesgue measure for all n.

This holds, e.g., if the law of $x(0)$ and regular conditional law of $M(n+1)$ given $x(n)$ is absolutely continuous w.r.t. the Lebesgue measure for all n ('a.s.' in the latter case). Our first main result is:

Theorem 3.1. *The $\{x(n)\}$ generated by (2.1) with a discontinuous h will asymptotically track a Filippov solution of (3.2), a.s.*

Proof. We use the following result from [9].

Lemma 3.2. *Let $f : \mathcal{R}^d \mapsto \mathcal{R}^d$ be measurable and locally bounded. Then,*

(1) *for a.e. (:= almost everywhere with respect to the Lebesgue measure) x, $f(x) \in F_f(x)$ and $F_f(\cdot)$ is the smallest closed convex valued upper semicontinuous set valued map for which this holds, and,*
(2) *there exists a measurable $\tilde{f} = f$ a.e. such that $F_f(x) = K_{\tilde{f}}(x)$ $\forall x \in \mathcal{R}^d$.*

See Prop. 2, pp. 231-234, [9], for a proof. In fact, it is easy to see that if one considers the convolutions $f^\delta := f * \varphi^\delta$ where $\varphi^\delta, \delta > 0$, are smooth approximations to the Dirac measure at the origin that are supported in the ball of radius $\delta > 0$ centered at the origin, then $f = \lim_{\delta\downarrow 0} f^\delta$ at all Lebesgue points of f [16], and therefore $f(x) \in F_f(x)$ at all Lebesgue points of f.

By the above lemma, there exists a measurable $\tilde{h} : \mathcal{R}^d \mapsto \mathcal{R}^d$ such that $\tilde{h} = h$ a.e. and $F_h(\cdot) = K_{\tilde{h}}(\cdot)$. Thus the iteration

$$\tilde{x}(n+1) = \tilde{x}(n) + a(n)(\tilde{h}(x(n)) + M(n+1)), \ n \geq 0,$$

with $\tilde{x}(0) = x(0)$ a.s., can be shown to satisfy $\tilde{x}(n) = x(n)$ a.s. by induction, in view of (†). But by standard theory for stochastic recursive inclusions (see, e.g., [4] or Chapter 5 of [8]), $\{\tilde{x}(n)\}$ asymptotically track a.s. the differential inclusion

$$\dot{x}(t) \in K_{\tilde{h}}(x(t)) = F_h(x(t)),$$

in view of the foregoing. Then so does $\{x(n)\}$. The claim follows. □

3.2. **Limiting empirical measures.** In this section we take a control theoretic view of (2.7). Let $f_i : \mathcal{R}^d \mapsto \mathcal{R}, i \geq 1$, be a countable family of bounded twice continuously differentiable functions that has bounded first and second partial derivatives, chosen such that it forms a convergence determining class for $\mathcal{P}(\mathcal{R}^d)$. Note that for each $i \geq 1$,

$$\xi_i(n) := \sum_{m=0}^{n-1} a(m)\langle \nabla f_i, M(m+1)\rangle, \ n \geq 1,$$

is a square-integrable martingale w.r.t. $\{\mathcal{F}_n\}$.

Lemma 3.3. *$\xi_i(n)$ converges a.s. as $n \to \infty$, $\forall i$.*

Proof. By (3.1), the quadratic variation process $\{\langle\xi\rangle_n\}$ of $\{\xi_i(n)\}$ satisfies

$$\langle\xi_i\rangle_n \leq K_i \sum_{m=0}^{n-1} a(m)^2 E\left[\|M(m+1)\|^2|\mathcal{F}_m\right] \leq K_i K \sum_n a(n)^2(1 + \|x(n)\|^2) < \infty \text{ a.s.}$$

for a suitable bound K_i on $\|\nabla f_i(\cdot)\|$ and K as in (2.3). By Proposition VII.2.3(c), p. 149, of [20], $\xi_i(n)$ converges a.s. □

Likewise, we have:

Lemma 3.4. $\sum_n a(n)^2\|M(n+1)\|^2 < \infty$ *a.s.*

Proof. Let

$$Z(n) := \sum_{m=0}^{n-1} a(n)^2 \left(\|M(m+1)\|^2 - E\left[\|M(m+1)\|^2|\mathcal{F}_m\right]\right), \ n \geq 1.$$

By our assumption (2.3), $\sup_n E\left[\|M(n)\|^4\right] < \infty$. From this and (2.2), it follows that the quadratic variation process $\langle Z\rangle_n$ of $\{Z_n\}$ is uniformly bounded in mean square and therefore, is bounded a.s. In turn, this and Proposition VII.2.3(c), p. 149, of [20], imply that $Z(n)$ converges a.s. as $n \uparrow \infty$. Since

$$\sum_{m=0}^{n-1} a(n)^2 E\left[\|M(m+1)\|^2|\mathcal{F}_m\right] \leq K \sum_{m=0}^{n-1} a(n)^2 \left(1 + \|x(m)\|^2\right) < \infty$$

a.s. (where $K > 0$ is as in (2.3)), the claim follows. □

Next, let B, D be closed convex subsets of $\mathcal{R}^d$. We define $\mathcal{U} :=$ the space of measurable maps $[0, \infty) \mapsto \mathcal{P}(B \times D)$ with the coarsest topology that renders continuous the maps

$$\mu(dxdy|\cdot) \in \mathcal{U} \mapsto \int_s^t g(y) \int_{B\times D} f(x,u)\mu(dxdu|y)dy,$$
$$f \in C(B \times D), g \in L_2[s,t], t > s \geq 0.$$

This is a compact metrizable topology, hence $\mathcal{U}$ is Polish (See, e.g., pp. 71-73 of [8]). In fact this is a special case of the standard topology for relaxed (or 'chattering') controls introduced by L. C. Young that later led to the more general notion of 'Young measures' in calculus of variations [30].

Let $\Phi :=$ a zero probability set outside which (3.1) holds and the conclusions of Lemmas 3.3 and 3.4 hold. Define a $\mathcal{P}((\mathcal{R}^d)^2)$-valued process $\mu(dxdz|t), t \in [0, \infty)$, by

$$\mu(dxdz|t) := \delta_{(x(n),z(n))}(dxdz), \ t(n) \leq t < t(n+1), n \geq 0,$$

where $\{t(n)\}$ are defined as in the preceding section and $\delta_{(x',z')}(dxdz)$ denotes the Dirac measure at (x', z'). Fix a sample point in Φ^c. Choose $B, D \subset \mathcal{R}^d$ such that $(x(n), h(x(n))) \in B \times D \ \forall n$. This is possible by (3.1) and the linear growth condition on $F_h(\cdot)$. Define $\breve{\mu}(dxdz|\cdot) \in \mathcal{U}$ by:

$$\int f(x,z)\breve{\mu}(dxdz|t) := \frac{1}{t}\int_0^t \int f(x,z)\mu(dxdz|s)ds$$

for twice continuously differentiable $f \in C_b(B \times D)$ with bounded first and second partial derivatives. In particular,

$$\frac{1}{t(n)} \int_0^{t(n)} \int \langle \nabla f_i(x), z\rangle \mu(dxdz|t)dt = \frac{\sum_{k=0}^{n-1} a(k)\langle \nabla f_i(x(k)), z(k)\rangle}{\sum_{k=0}^{n-1} a(k)}.$$

Note that we work with a fixed sample path and therefore the sets B, D, though sample path dependent, are fixed.

Theorem 3.5. *For the chosen sample path, any limit point of* $\mu(dxdz|t + \cdot)$ *in* $\mathcal{U}$ *as* $t \uparrow \infty$ *is of the form* $\mu^*(dxdz)$ *such that,*

(1) *μ^* is supported on the graph of $K_h(\cdot)$ and if (†) holds, on the graph of $F_h(\cdot)$, and,*
(2) *there exists a stationary process $(X(\cdot), Z(\cdot))$ satisfying $\dot{X}(t) = Z(t)$ for which the marginal law of $(X(t), Z(t))$ is μ^* for all t. Furthermore, one may view the control $Z(\cdot)$ as corresponding to a stationary Markov relaxed control $v(X(\cdot))$ for some measurable $v : x \in \mathcal{R}^d \mapsto \mathcal{P}(K_h(x))$, ($v : x \in \mathcal{R}^d \mapsto \mathcal{P}(F_h(x))$ if (†) holds), i.e., $v(X_t)$ is the law of Z_t for all $t \geq 0$.*

Proof. Recall from the preceding section the definitions of $t(n), n \geq 0$, and $\bar{x}(\cdot)$. Since $t(n) \to \infty$ and $t(n+1) - t(n) \to 0$, it suffices to consider a subsequence along $\{t(n)\}$ as $n \uparrow \infty$. For $i \geq 1$,

$$\begin{aligned} f_i(\bar{x}(t(n)) - f_i(\bar{x}(0)) &= \sum_{k=0}^{n-1} (f_i(\bar{x}(t(k+1)) - f_i(\bar{x}(t(k)))) \\ &= \sum_{k=0}^{n-1} a(k) \langle \nabla f_i(x(k)), z(k) \rangle \\ &\quad + \sum_{k=0}^{n-1} a(k) \langle \nabla f_i(x(k)), M(k+1) \rangle + \sum_{k=0}^{n-1} \zeta(k) \end{aligned}$$

by Taylor formula, where $\|\zeta(k)\| \leq Ca(k)^2 \left(1 + \|M(k+1)\|^2\right)$ for some constant $C > 0$. By our choice of the sample path, the conclusion of Lemma 3.3 applies and the second term is bounded. Likewise by Lemma 3.4, so is the third term. Then dividing both sides by $t(n)$ and letting $n \uparrow \infty$, we have $t(n) \uparrow \infty$ and therefore

$$\frac{1}{n} \sum_{k=0}^{n-1} a(k) \langle \nabla f_i(x(k)), z(k) \rangle = \frac{1}{t(n)} \int_0^{t(n)} \int \langle \nabla f_i(x), z \rangle d\mu(dxdz|t) dt \to 0.$$

Suppose $\mu^*(dxdz)$ is a limit point of $\check{\mu}(dxdz|t)$ as $t \to \infty$. Then by passing to the limit along an appropriate subsequence, we have

$$\int \langle \nabla f_i(x), z \rangle \mu^*(dxdz) = 0 \ \forall i.$$

Then the first claim follows from Theorem 2.2 and Theorem 3.1. By Theorem 4.7 of [25], there exists a stationary process $(X(\cdot), Z(\cdot))$ such that $\dot{X}(t) = Z(t)$ $\forall t$ and the marginal of $(X(t), Z(t))$ is μ^* for all t. This implies the first part of the second claim. The second part of the second claim follows from Corollary 2.1 of [6], see also Corollary 6.3.7, p. 230, of [1]. □

Remark 3.6. Note that the control being a stationary relaxed Markov process does not imply that the state process itself is time-homogeneous Markov, or even Markov, in this case. This is due to nonuniqueness of solutions: for some $t > 0$, we can take some solution up to t and then choose the regular conditional law of the process after t from the set of solution measures initiated at $X(t)$ as a non-trivial function of the entire trajectory up to time t. This yields a non-Markov solution. More generally, one can follow the recipe of Section 12.3 of [26] to generate non-Markov solutions.

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