Pure and Applied Functional Analysis
Volume 9, Number 2, 2024, 555–600

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

ON THE CURVATURES OF IRREGULAR CURVES IN EUCLIDEAN SPACES AND RIEMANNIAN SURFACES

DOMENICO MUCCI AND ALBERTO SARACCO

Dedicated to the memory of Professor Yurii Grigorievich Reshetnyak.

Abstract. In this survey paper we report our recent results concerning a notion of curvatures for irregular curves defined in Euclidean spaces or in Riemannian surfaces. Some of our ideas are inspired among the others by the work of Professor Yurii G. Reshetnyak and of the Russian school (especially A. D. Alexandrov), who e.g. pointed out the relevant notion of modulus of a curve, the properties of one sidedly smooth curves, and the role of integral geometric formulas for the rotation of spherical curves.

1. Introduction

The theory of irregular curves in Euclidean spaces and Riemannian manifolds owes a lot, among the others, to the contribution of the Russian school, especially to the work by A. D. Alexandrov and Yu. G. Reshetnyak.

Some crucial results contained in the treatise [3], such has the notion of modulus of a curve, the properties of one sidedly smooth curves, and the role of integral geometric formulas for the rotation of spherical curves, have guided us in the proof of our results contained in [22, 23, 24], that are reported in this survey paper.

After collecting in Sec. 2 some background material and preliminary facts, in Sec. 3 we analyze irregular curves in the Euclidean space $\mathbb{R}^3$, see [22]. In Sec. 4, we treat a notion of weak curvatures in high dimension Euclidean spaces, see [23]. Finally, Sec. 5 concerns the intrinsic curvature of irregular curves supported in a Riemannian surface, see [24].

It is a great honor for us to give a contribution to this Special Issue on Analysis, Geometry and PDE in memory of Professor Yurii Reshetnyak.

2. Background material and preliminary results

This preliminary section contains some well-known facts about length and total curvature of curves in Euclidean spaces. We then briefly discuss some relevant integral-geometric formulas. Finally, we deal with the geodesic curvature of smooth curves into Riemannian surfaces.

2020 *Mathematics Subject Classification.* 53A04, 53A35, 26A45, 49J45.

Key words and phrases. Non-smooth curves; curvatures; integral-geometric formulas; surfaces; geodesic curvature; weak parallel transport.

The research of D.M. was partially supported by the GNAMPA of INDAM. The research of A.S. was partially supported by the GNSAGA of INDAM.

2.1. **Total variation.** We refer to Secs. 3.1 and 3.2 of [4] for the following notation.

Let $I \subset \mathbb{R}$ be a bounded open interval, and $N \in \mathbb{N}^+$. A vector-valued summable function $u : I \to \mathbb{R}^{N+1}$ is said to be of *bounded variation* if its distributional derivative Du is a finite $\mathbb{R}^{N+1}$-valued measure in I.

The *total variation* $|Du|(I)$ of a function $u \in \mathrm{BV}(I, \mathbb{R}^{N+1})$ is given by

$$|Du|(I) := \sup\Big\{ \int_I \varphi'(s) \bullet u(s)\, ds \mid \varphi \in C_c^\infty(I, \mathbb{R}^{N+1})\,, \quad \|\varphi\|_\infty \le 1 \Big\}$$

and hence it does not depend on the choice of the representative in the equivalence class of the functions that agree $\mathcal{L}^1$-a.e. in I with u, where $\mathcal{L}^1$ is the Lebesgue measure in $\mathbb{R}$.

A sequence $\{u_n\} \subset \mathrm{BV}(I, \mathbb{R}^{N+1})$ converges to $u \in \mathrm{BV}(I, \mathbb{R}^{N+1})$ *weakly-* in* BV if $u_n \to u$ strongly in $L^1(I, \mathbb{R}^{N+1})$ and $\sup_n |Du_n|(I) < \infty$. In this case, the lower semicontinuity inequality holds:

$$|Du|(I) \le \liminf_{n\to\infty} |Du_h|(I)\,.$$

If in addition $|Du_h|(I) \to |Du|(I)$, we say that $\{u_h\}$ *strictly converges* to u.

The *weak-* compactness* theorem yields that if $\{u_n\} \subset \mathrm{BV}(I, \mathbb{R}^{N+1})$ converges $\mathcal{L}^1$-a.e. on I to a function u, and if $\sup_h |Du_n|(I) < \infty$, then $u \in \mathrm{BV}(I, \mathbb{R}^{N+1})$ and a subsequence of $\{u_n\}$ weakly-* converges to u.

Let $u \in \mathrm{BV}(I, \mathbb{R}^{N+1})$. Since each component of u is the difference of two monotone functions, it turns out that u is continuous outside an at most countable set, and that both the right and left limits $u(s\pm) := \lim_{t\to s^\pm} u(t)$ exist for every $s \in I$. Also, u is an L^∞ function that is differentiable $\mathcal{L}^1$-a.e. on I, with derivative $\dot u$ in $L^1(I, \mathbb{R}^{N+1})$.

The total variation of u agrees with the *essential variation* $\mathrm{Var}_{\mathbb{R}^{N+1}}(u)$, which is equal to the pointwise variation of any *good representative* of u in its equivalence class. A good (or precise) representative is e.g. given by choosing $u(s) = (u(s+) + u(s-))/2$ at the discontinuity points. Letting $u_\pm(s) := u(s\pm)$ for every $s \in I$, both the left- and right-continuous functions $u_\pm$ are good representatives.

If $u \in \mathrm{BV}(I, \mathbb{R}^{N+1})$, the decomposition into the *absolutely continuous*, *Jump*, and *Cantor* parts holds:

$$Du = D^a u + D^J u + D^C u\,, \quad |Du|(I) = |D^a u|(I) + |D^J u|(I) + |D^C u|(I)\,.$$

More precisely, one splits $Du = D^a u + D^s u$ into the absolutely continuous and singular parts w.r.t. Lebesgue measure $\mathcal{L}^1$. The Jump set J_u being the (at most countable) set of discontinuity points of any good representative of u, and δ_s denoting the unit Dirac mass at $s \in I$, one has:

$$D^a u = \dot u\, \mathcal{L}^1\,, \quad D^J u = \sum_{s\in J_u} [u(s+) - u(s-)]\, \delta_s\,, \quad D^C u = D^s u \llcorner (I \setminus J_u)\,.$$

Also, any $u \in \mathrm{BV}(I, \mathbb{R}^{N+1})$ can be represented by $u = u^a + u^J + u^C$, where u^a is a Sobolev function in $W^{1,1}(I, \mathbb{R}^{N+1})$, u^J is a Jump function, and u^C is a Cantor function, so that

$$|D^a u|(I) = |Du^a|(I)\,, \qquad |D^J u|(I) = |Du^J|(I)\,, \qquad |D^C u|(I) = |Du^C|(I)\,.$$

Finally, we recall that if $u, v \in \mathrm{BV}(I) := \mathrm{BV}(I, \mathbb{R})$, the product $uv \in \mathrm{BV}(I)$. In the particular case in which the Jump sets coincide, $J_u = J_v = J$, the chain rule formula (cf. [4, Sec. 3.10]) yields:

$$D^a(uv) = (\dot{u}v + u\dot{v})\,\mathcal{L}^1\,, \quad D^C(uv) = uD^Cv + vD^Cu\,,$$
$$D^J(uv) = \sum_{s\in J}[u(s+)v(s+) - u(s-)v(s-)]\,\delta_s$$

where we can choose any good representatives of u and v in the second equality.

2.2. **Length.** Consider a curve $\mathbf{c}$ in the Euclidean space $\mathbb{R}^{N+1}$ parameterized by the continuous map $\mathbf{c} : [a, b] \to \mathbb{R}^{N+1}$, with components $\mathbf{c}(t) = (\mathbf{c}^1(t), \ldots, \mathbf{c}^{N+1}(t))$. Any polygonal curve P *inscribed* in $\mathbf{c}$, say $P \ll \mathbf{c}$, is obtained by choosing a finite partition $\mathcal{D} := \{a = t_0 < t_1 < \ldots < t_{m-1} < t_m = b\}$ of $[a, b]$, say $P = P(\mathcal{D})$, and letting $P : [a, b] \to \mathbb{R}^{N+1}$ such that $P(t_i) = \mathbf{c}(t_i)$ for $i = 0, \ldots, m$, and $P(t)$ affine on each interval $I_i := [t_{i-1}, t_i]$. We call $\operatorname{mesh} P$ the maximum lenght of its edges.

The *length* $\mathcal{L}(\mathbf{c})$ of $\mathbf{c}$ is defined by

$$\mathcal{L}(\mathbf{c}) := \sup\{\mathcal{L}(P) \mid P \ll \mathbf{c}\}$$

and $\mathbf{c}$ is said to be *rectifiable* if $\mathcal{L}(\mathbf{c}) < \infty$. By uniform continuity, for each $\varepsilon > 0$ we can find $\delta > 0$ such that $\operatorname{mesh} P < \varepsilon$ if $\operatorname{mesh} \mathcal{D} < \delta$ and $P = P(\mathcal{D})$. As a consequence, taking $P_n = P(\mathcal{D}_n)$, where $\{\mathcal{D}_n\}$ is any sequence of partitions of I such that $\operatorname{mesh} \mathcal{D}_n \to 0$, we get $\operatorname{mesh} P_n \to 0$ and hence the convergence $\mathcal{L}(P_n) \to \mathcal{L}(\mathbf{c})$ of the length functional. Finally, the curve $\mathbf{c}$ is rectifiable if and only if $\mathbf{c} \in \mathrm{BV}(I, \mathbb{R}^{N+1})$, and in that case

$$\mathcal{L}(\mathbf{c}) = \operatorname{Var}_{\mathbb{R}^{N+1}}(\mathbf{c}) = |D\mathbf{c}|(I)\,.$$

2.3. **Total curvature.** We call *rotation* $\mathbf{k}^*(P)$ of a polygonal curve P in $\mathbb{R}^{N+1}$ the sum of the exterior angles between consecutive segments. Milnor [20] defined *total curvature* $\mathrm{TC}(\mathbf{c})$ of a curve $\mathbf{c}$ in $\mathbb{R}^{N+1}$ by

$$\mathrm{TC}(\mathbf{c}) := \sup\{\mathbf{k}^*(P) \mid P \ll \mathbf{c}\}\,.$$

Then $\mathrm{TC}(P) = \mathbf{k}^*(P)$ for each polygonal P. Moreover, if a curve $\mathbf{c}$ has compact support and finite total curvature, $\mathrm{TC}(\mathbf{c}) < \infty$, then it is a rectifiable curve, see Example 2.4 below for a proof.

Assume now that a rectifiable curve $\mathbf{c}$ is parameterized by arc-length, so that $\mathbf{c} = \mathbf{c}(s)$, with $s \in [0, L] = \overline{I}_L$, where $I_L := (0, L)$ and $L = \mathcal{L}(\mathbf{c})$. If $\mathbf{c}$ is smooth and regular, one has $\mathrm{TC}(\mathbf{c}) = \int_0^L |\mathbf{k}|\, ds$, where $\mathbf{k}(s) := \ddot{\mathbf{c}}(s)$ is the curvature vector. More generally, since $\mathbf{c}$ is a Lipschitz function, by Rademacher's theorem (cf. [4, Thm. 2.14]) it is differentiable $\mathcal{L}^1$-a.e. in I_L. Denoting by $\dot{f} := \frac{d}{ds}f$ the derivative w.r.t. arc-length parameter s, the tantrix $\mathbf{t} = \dot{\mathbf{c}}$ exists a.e., and actually $\mathbf{t} : I_L \to \mathbb{R}^{N+1}$ is a function of bounded variation. Since moreover $\mathbf{t}(s) \in \mathbb{S}^N$ for a.e. s, where $\mathbb{S}^N$ is the Gauss hyper-sphere

$$\mathbb{S}^N := \{y \in \mathbb{R}^{N+1} \,:\, |y| = 1\}$$

we shall write $\mathbf{t} \in \mathrm{BV}(I_L, \mathbb{S}^N)$. The essential variation $\operatorname{Var}_{\mathbb{S}^N}(\mathbf{t})$ of $\mathbf{t}$ in $\mathbb{S}^N$ differs from $\operatorname{Var}_{\mathbb{R}^{N+1}}(\mathbf{t})$, as its definition involves the geodesic distance $d_{\mathbb{S}^N}$ in $\mathbb{S}^N$ instead of the Euclidean distance in $\mathbb{R}^{N+1}$. Therefore, $\operatorname{Var}_{\mathbb{R}^{N+1}}(\mathbf{t}) \le \operatorname{Var}_{\mathbb{S}^N}(\mathbf{t})$, and equality holds if and only if $\mathbf{t}$ has a continuous representative. More precisely, by decomposing $\mathbf{t} = \mathbf{t}^a + \mathbf{t}^J + \mathbf{t}^C$, one obtains:

$$\mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}) = \int_0^L |\dot{\mathbf{t}}|\, ds + \sum_{s \in J_{\mathbf{t}}} d_{\mathbb{S}^N}(\mathbf{t}(s+), \mathbf{t}(s-)) + |D^C \mathbf{t}|(I_L) \tag{2.1}$$

whereas in the formula for $\mathrm{Var}_{\mathbb{R}^{N+1}}(\mathbf{t})$, that is equal to $|D\mathbf{t}|(I_L)$, one has to replace in (2.1) the geodesic distance $d_{\mathbb{S}^N}(\mathbf{t}(s+), \mathbf{t}(s-))$ with the Euclidean distance $|\mathbf{t}(s+) - \mathbf{t}(s-)|$ at each Jump point $s \in J_{\mathbf{t}}$.

Notice moreover that the Cantor component $D^C\mathbf{t}$ is non-trivial, in general.

Example 2.1. Let e.g. $\gamma : \overline{I} \to \mathbb{R}^2$, where $I = (0,1)$, denote the Cartesian curve $\gamma(t) := (t, u(t))$ in $\mathbb{R}^2$ given by the graph of the primitive $u(t) := \int_0^t v(\lambda)\, d\lambda$ of the classical Cantor-Vitali function $v : \overline{I} \to \mathbb{R}$ associated to the "middle thirds" Cantor set. It turns out that $\mathbf{t} = (1+v^2)^{-1/2}(1, v)$, whence $\mathbf{t}$ is a Cantor function, i.e., $D^a\mathbf{t} = D^J\mathbf{t} = 0$, and

$$D\mathbf{t}(I) = D^C\mathbf{t}(I) = \int_I \frac{1}{(1+v^2)^{3/2}}\, (-v, 1)\, dD^C v\,.$$

The angle ω between the unit vectors $(1,0)$ and $\mathbf{t}$ satisfies $\omega = \arctan v \in \mathrm{BV}(I)$. Therefore, $D\omega(I) = D^C\omega(I) = \int_I \frac{1}{1+v^2}\, dD^C v$, which yields

$$|D\omega|(I) = \int_I \frac{1}{1+v^2}\, d|D^C v| = |D\mathbf{t}|(I) = \mathrm{TC}(\gamma) = \frac{\pi}{4}\,.$$

The following facts hold:

(1) if P and P' are inscribed polygonals and P' is obtained by adding a vertex in $\mathbf{c}$ to the vertices of P, then $\mathbf{k}^*(P) \le \mathbf{k}^*(P')$;
(2) if $\mathbf{c}$ has finite total curvature, for each point v in $\mathbf{c}$, small open arcs of $\mathbf{c}$ with an end point equal to v have small total curvature.

As a consequence, compare [28], it turns out that $\mathrm{TC}(\mathbf{c}) = \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t})$, see (2.1), and the total curvature of $\mathbf{c}$ is equal to the limit of $\mathbf{k}^*(P_n)$ for *any* sequence $\{P_n\}$ of polygonals in $\mathbb{R}^{N+1}$ inscribed in $\mathbf{c}$ such that $\mathrm{mesh}\, P_n \to 0$. More precisely, if $\mathbf{t}_n$ is the tantrix of P_n, then $\mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}_n) \to \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t})$.

Remark 2.2. For future use, we recall how equality

$$\mathrm{TC}(\mathbf{c}) = \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t})\,, \quad \mathbf{t} = \dot{\mathbf{c}} \tag{2.2}$$

is checked for rectifiable curves $\mathbf{c}$ in $\mathbb{R}^{N+1}$ with finite total curvature (and parameterized in arc-length). In case $N = 1$, we apply a "planar" version of the Gauss-Bonnet theorem, see Theorem 5.8 below.

Let P_n be an inscribed polygonal to the curve $\mathbf{c} : [0, L] \to \mathbb{R}^{N+1}$ and generated by the consecutive vertices $\mathbf{c}(s_i)$, where $0 = s_0 < s_1 < \cdots < s_m = L$, and call $\mathbf{v}_i$ the oriented segment of P_n from $\mathbf{c}(s_{i-1})$ to $\mathbf{c}(s_i)$. If $\mathbf{t}_n$ is the tantrix of P_n in $\mathbb{S}^N$, the value of $\mathbf{t}_n$ in $\mathbf{v}_i$ is an average of the values of the restriction of the tantrix $\mathbf{t}$ of $\mathbf{c}$ to (s_{i-1}, s_i), when completed to a continuous curve in $\mathbb{S}^N$ by connecting with geodesic arcs the points $\mathbf{t}(s-)$ and $\mathbf{t}(s+)$ for each $s \in J_{\mathbf{t}} \cap (s_{i-1}, s_i)$, in the sense of Alexandrov-Reshetnyak [3]. This property implies that $\mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}_h) \le \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t})$. If $\{P_n\}$ is an inscribed sequence satisfying $\mathrm{mesh}\, P_n \to 0$, the weak BV convergence of

$\mathbf{t}_n$ to $\mathbf{t}$ implies the lower semicontinuity inequality $\mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}) \leq \liminf_n \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}_n)$, yielding the strict convergence $\mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}_n) \to \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t})$. Using that $\mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}_n) \to \mathrm{TC}(\mathbf{c})$, one gets (2.2).

When $\mathbf{c}$ is a *planar curve*, i.e., when $N = 1$, the value of $\mathbf{t}_n \in \mathbb{S}^1$ on the segment $\mathbf{v}_i$ is equal to one of the values of the "completion" in $\mathbb{S}^1$ of the restriction of the tantrix $\mathbf{t}$ to the interval $]s_{i-1}, s_i[$.

Actually, this property can be rewritten in terms of angle functions, and hence of the "planar" version of the Gauss-Bonnet theorem 5.8.

2.4. **Curvature force.** The *curvature force* was introduced in [6] by J. M. Sullivan and his collaborators as the distributional derivative of the tangent indicatrix of rectifiable curves $\mathbf{c}$ in $\mathbb{R}^{N+1}$ with finite total curvature.

The curvature force $\mathrm{TC}^*(P)$ of a polygonal is the total variation in $\mathbb{R}^{N+1}$ of the tantrix $\mathfrak{t}_P$:

$$\mathrm{TC}^*(P) := \mathrm{Var}_{\mathbb{R}^{N+1}}(\mathfrak{t}_P)\,.$$

In particular, if $P \ll \mathbf{c}$, with the previous notation one has:

$$\mathrm{TC}^*(P) = \sum_{i=1}^{m-1} 2\sin(\theta_i/2)$$

where θ_i is the i-th turning angle. Defining the Euclidean total curvature, or curvature force, of $\mathbf{c}$ by

$$\mathrm{TC}^*(\mathbf{c}) := \sup\{\mathrm{TC}^*(P) \mid P \ll \mathbf{c}\}$$

then $\mathbf{c}$ has finite curvature force if and only if it has finite total curvature. In addition, compare [21], if a rectifiable curve $\mathbf{c}$ is parameterized in arc-length, and $\mathfrak{t} := \dot{\mathbf{c}}(s)$, $s \in I_L$, we recover the definition by Sullivan [28]:

Proposition 2.3. *If* $\mathrm{TC}^*(\mathbf{c}) < \infty$*, then* $\mathfrak{t}$ *is a function of bounded variation in* $\mathrm{BV}(I_L, \mathbb{S}^N)$*, and its total variation in* $\mathbb{R}^{N+1}$ *is equal to the curvature force, i.e.*

$$|D\mathfrak{t}|(I_L) = \mathrm{Var}_{\mathbb{R}^{N+1}}(\mathfrak{t}) = \mathrm{TC}^*(\mathbf{c})\,.$$

Notice that the curve $\mathbf{c}$ from Example 2.1 satisfies $\mathrm{TC}^*(\mathbf{c}) = \mathrm{TC}(\mathbf{c})$, compare [1]. Therefore, the occurrence of a Cantor-part in the derivative of the tantrix does not change the computation when considering the total variation in $\mathbb{S}^1$ or in $\mathbb{R}^2$.

The curvature force comes into play when computing the first variation of length. In fact, let $\mathbf{c} : [0, L] \to \mathbb{R}^{N+1}$ parameterized in arc length, and let $\mathbf{c}_\varepsilon$ a variation of $\mathbf{c}$ under which the motion of each point $\mathbf{c}(s)$ is smooth in time and with initial velocity $\xi(s)$, where $\xi : [0, L] \to \mathbb{R}^{N+1}$ is a Lipschitz continuous function of arc length. The first variation formula gives

$$\delta_\xi \mathcal{L}(\mathbf{c}) := \frac{d}{d\varepsilon}\,\mathcal{L}(\mathbf{c}_\varepsilon)_{|\varepsilon=0} = \int_0^L \mathfrak{t}(s) \bullet \dot{\xi}(s)\,ds$$

where $\mathfrak{t}(s) = \dot{\mathbf{c}}(s)$ and $\dot{\xi}(s)$ are defined for a.e. s, by Rademacher's theorem. If $\mathbf{c}$ is of class C^2, integrating by parts one gets

$$\delta_\xi \mathcal{L}(c) = -\int_0^L \dot{\mathfrak{t}}(s) \bullet \xi(s)\,ds + \big(\mathfrak{t}(L) \bullet \xi(L) - \mathfrak{t}(0) \bullet \xi(0)\big)$$

where in terms of the (positive) first curvature $\mathbf{k}$ and first unit normal $\mathfrak{n}(s)$ one has $\dot{\mathfrak{t}}(s) = \mathbf{k}(s)\,\mathfrak{n}(s)$.

More generally, if $\mathbf{c}$ is a rectifiable curve with finite total curvature, then $\mathfrak{t}$ is a function of bounded variation, the right and left limits $\mathfrak{t}(s\pm) \in \mathbb{S}^{N-1}$ are well defined for each $s \in]0, L[$, and the distributional derivative $D\mathfrak{t}$ is a finite vector-valued measure. Therefore, if in addition $\xi(0) = \xi(L) = 0$

$$\delta_\xi \mathcal{L}(c) = \int_0^L \mathfrak{t}(s) \bullet \dot{\xi}(s)\, ds = -\langle D\mathfrak{t}, \xi\rangle$$

whence the first variation $\delta_\xi \mathcal{L}(\mathbf{c})$ of the length has distributional order one.

The measure $\mathcal{K} := D\mathfrak{t}$ is called in [6] curvature force, and in the smooth case one has $\mathcal{K} = \mathbf{k}\,\mathfrak{n}\, ds$. If $\mathbf{c}$ is a piecewise smooth function, one has the decomposition $\mathcal{K} = \mathcal{K}^a + \mathcal{K}^s$, where the absolutely continuous component $\mathcal{K}^a$ is equal to $\mathbf{k}\,\mathfrak{n}\, d\mathcal{L}^1 \llcorner]0, L[$, whereas the singular component $\mathcal{K}^s$ is given by a sum of Dirac masses concentrated at the corner points of the curve $\mathbf{c}$.

More precisely, if $s \in]0, L[$ is such that $\mathfrak{t}(s-) \neq \mathfrak{t}(s+)$, then $\mathcal{K}(\{c(s)\}) = (\mathfrak{t}(s+) - \mathfrak{t}(s-))\,\delta_{c(s)}$. Therefore, if $\theta \in]0, \pi]$ is the shortest angle in the Gauss sphere $\mathbb{S}^N$ between $\mathfrak{t}(s\pm)$, so that $d_{\mathbb{S}^2}(\mathfrak{t}(s+), \mathfrak{t}(s-)) = \theta$, one has $|\mathcal{K}|(\{c(s)\}) = \|\mathfrak{t}(s+) - \mathfrak{t}(s-)\| = 2\sin(\theta/2)$.

2.5. **Integral-geometric formulas.** Several classical properties of curves in Euclidean spaces can be proved in a somewhat cleaner way by exploiting suitable *integral-geometric formulas*, that we now recall.

For $0 \leq j \leq N-1$ integer, denote by $G_{j+1}\mathbb{R}^{N+1}$ the Grassmannian of unoriented $(j+1)$-planes in $\mathbb{R}^{N+1}$. It is a compact group, and it can be equipped with a unique rotationally invariant probability measure μ_{j+1}. For $p \in G_{j+1}\mathbb{R}^{N+1}$, we denote by π_p the orthogonal projection of $\mathbb{R}^{N+1}$ onto p.

If $\mathbf{c}$ is a (rectifiable) curve in $\mathbb{R}^{N+1}$, the integral-geometric formula for the length reads as

$$\mathcal{L}(\mathbf{c}) = \frac{\sigma_j}{\sigma_N} \cdot \int_{G_{j+1}\mathbb{R}^{N+1}} \mathcal{L}(\pi_p(\mathbf{c}))\, d\mu_{j+1}(p)$$

where σ_j and σ_N are positive constants only depending on j and N, respectively, see e.g. [3, Sec. 4.8].

Let us also recall the average result due to Fáry [13], who showed that the total curvature of a curve (with finite total curvature) is the average of the total curvatures of all its projections onto $(j+1)$-planes:

$$\mathrm{TC}(\mathbf{c}) = \int_{G_{j+1}\mathbb{R}^{N+1}} \mathrm{TC}(\pi_p(\mathbf{c}))\, d\mu_{j+1}(p) \qquad \forall j = 0, \ldots, N-1\,.$$

Following [28, Prop. 4.1], it suffices to prove the average formula for an angle, hence for the rotation $\mathbf{k}^*(P)$ of a polygonal P, and then use the monotone convergence theorem.

Example 2.4. We e.g. readily check that if a curve $\mathbf{c}$ in $\mathbb{R}^{N+1}$ has compact support and finite total curvature, then $\mathbf{c}$ is a rectifiable curve. In fact, one has $\mathcal{L}(\pi_p(\mathbf{c})) \leq d\,(\mathrm{TC}(\pi_p(\mathbf{c})) + 1)$ for μ_1-a.e. $p \in G_1\mathbb{R}^{N+1}$, where d is the diameter of $\mathbf{c}$. Therefore, the previous average formulas (with $j = 0$) yield

$$\begin{aligned}\mathcal{L}(\mathbf{c}) = &\ \frac{\sigma_0}{\sigma_N}\int_{G_1\mathbb{R}^{N+1}} \mathcal{L}(\pi_p(\mathbf{c}))\, d\mu_1(p) \\ \leq &\ \frac{\sigma_0\, d}{\sigma_N}\int_{G_1\mathbb{R}^{N+1}} (\mathrm{TC}(\pi_p(\mathbf{c}))+1)\, d\mu_1(p) = \frac{\sigma_0\, d}{\sigma_N}(\mathrm{TC}(\mathbf{c})+1) < \infty\,.\end{aligned}$$

We now deal with polygonal curves in the sphere $\mathbb{S}^N$. Following [3], given $x \in \mathbb{S}^N$ we denote by $\eta_p(x)$ the nearest point to x on the j-dimensional sphere $\mathbb{S}^j_p := \mathbb{S}^N \cap p$. It is well-defined by

$$\eta_p(x) := \frac{\pi_p(x)}{|\pi_p(x)|} \tag{2.3}$$

provided that x is not orthogonal to the $(j+1)$-plane p, i.e., if x does not belong to the $(N-j-1)$-sphere $\mathbb{S}^{j\perp}_p$ of $\mathbb{S}^N$ given by the *polar* to $\mathbb{S}^j_p$. Therefore, if γ is a polygonal curve in $\mathbb{S}^N$, it turns out that the projected curve $\eta_p(\gamma)$ is well-defined for μ_{j+1}-a.e. $p \in G_{j+1}\mathbb{R}^{N+1}$.

The *geodesic rotation* $\mathbf{K}_g(\gamma)$ of a polygonal curve γ in $\mathbb{S}^N$ is given by the sum of the turning angles at the edges of γ, see [3], so that clearly $\mathrm{TC}(\gamma) = \mathcal{L}_{\mathbb{S}^N}(\gamma) + \mathbf{K}_g(\gamma)$. The following integral-geometric formulas, that are proved in [3, Thm. 6.2.2, p. 190] for $j = 1$, actually hold true for larger ranges of values of j.

Theorem 2.5. *Given a polygonal curve γ in $\mathbb{S}^N$, for any integer $1 \leq j \leq N-1$ one has*

$$\begin{aligned}\mathcal{L}_{\mathbb{S}^N}(\gamma) = &\ \int_{G_{j+1}\mathbb{R}^{N+1}} \mathcal{L}_{\mathbb{S}^j_p}(\eta_p(\gamma))\, d\mu_{j+1}(p) \\ \mathbf{K}_g(\gamma) = &\ \int_{G_{j+1}\mathbb{R}^{N+1}} \mathbf{K}_g(\eta_p(\gamma))\, d\mu_{j+1}(p)\,.\end{aligned}$$

As a consequence, since $\mathrm{TC}(\eta_p(\gamma)) = \mathcal{L}_{\mathbb{S}^j_p}(\eta_p(\gamma)) + \mathbf{K}_g(\eta_p(\gamma))$, one gets:

$$\mathrm{TC}(\gamma) = \int_{G_{j+1}\mathbb{R}^{N+1}} \mathrm{TC}(\eta_p(\gamma))\, d\mu_{j+1}(p)\,.$$

The average formula concerning the length of spherical curves was proved in [3, Thm. 4.8.3, p. 108].

Proposition 2.6. *Given a rectifiable curve $\mathbf{c}$ in $\mathbb{S}^N$, for any integer $1 \leq j \leq N-1$ one has*

$$\mathcal{L}(\mathbf{c}) = \int_{G_{j+1}\mathbb{R}^{N+1}} \mathcal{L}(\eta_p(\mathbf{c}))\, d\mu_{j+1}(p)\,.$$

In the sequel, we shall also consider polygonal curves in $\mathbb{RP}^N$, the *real projective space*. It is given by the quotient $\mathbb{RP}^N := \mathbb{S}^N/\sim$, the equivalence relation being $y \sim \widetilde{y} \iff y = \widetilde{y}$ or $y = -\widetilde{y}$. We denote by $[y]$ an element of $\mathbb{RP}^N$, and by $\Pi : \mathbb{S}^N \to \mathbb{RP}^N$ the canonical projection $\Pi(y) := [y]$. Recall that $\mathbb{RP}^N$ is a complete metric space, when equipped with the induced metric

$$d_{\mathbb{RP}^2}([y],[\widetilde{y}]) := \min\{d_{\mathbb{S}^N}(y,\widetilde{y}), d_{\mathbb{S}^N}(y,-\widetilde{y})\}\,.$$

Now, denote by $\mathbb{RP}^j_p$ the projective j-space corresponding to the j-sphere $\mathbb{S}^j_p$, for any $p \in G_{j+1}\mathbb{R}^{N+1}$, and let $\widetilde{\eta}_p$ denote the nearest point projection of $\mathbb{RP}^N$ onto $\mathbb{RP}^j_p$, i.e., $\widetilde{\eta}_p([x]) := [\eta_p(x)]$, for $x \in \mathbb{S}^N \setminus \mathbb{S}^{j\,\perp}_p$, where η_p is given by (2.3). Following the proof of Theorem 2.5 from [23], one similarly obtains:

Proposition 2.7. *Given a polygonal curve γ in $\mathbb{RP}^N$, for any integer $1 \le j \le N-1$ we have*

$$\begin{aligned}\mathcal{L}_{\mathbb{RP}^N}(\gamma) &= \int_{G_{j+1}\mathbb{R}^{N+1}} \mathcal{L}_{\mathbb{RP}^j_p}(\widetilde{\eta}_p(\gamma))\, d\mu_{j+1}(p) \\ \mathbf{K}_g(\gamma) &= \int_{G_{j+1}\mathbb{R}^{N+1}} \mathbf{K}_g(\widetilde{\eta}_p(\gamma))\, d\mu_{j+1}(p)\end{aligned}$$

and hence

$$\mathrm{TC}(\gamma) = \int_{G_{j+1}\mathbb{R}^{N+1}} \mathrm{TC}(\widetilde{\eta}_p(\gamma))\, d\mu_{j+1}(p)\,.$$

2.6. **Curves into Riemannian surfaces.** Let now $\mathcal{M}$ be a smooth (of class C^3), closed, and compact immersed surface in $\mathbb{R}^{N+1}$, with $N \ge 2$.

If $\mathbf{c}$ is a smooth and regular curve in $\mathcal{M}$, parameterized by arc-length, the unit tangent vector $\mathbf{t}(s) := \dot{\mathbf{c}}(s)$ satisfies $\dot{\mathbf{t}} \bullet \mathbf{t} \equiv 0$, whence the curvature vector $\mathbf{k}(s) := \dot{\mathbf{t}}(s)$ is orthogonal to $\mathbf{t}(s)$, and decomposes as

$$\mathbf{k}(s) = \mathfrak{K}_g(s)\,\mathbf{u}(s) + \mathfrak{K}_n(s)\,\mathbf{n}(s)\,.$$

When $N = 2$, the triad $(\mathbf{t}, \mathbf{n}, \mathbf{u})$, where $\mathbf{n}(s) := \nu(\mathbf{c}(s))$, $\nu(p)$ being the unit normal to the tangent 2-space $T_p\mathcal{M}$ in $\mathbb{R}^3$, and $\mathbf{u}(s) := \mathbf{n}(s) \times \mathbf{t}(s)$ is the unit conormal, is called the Darboux frame along $\mathbf{c}$, whereas $\mathfrak{K}_g := \mathbf{k} \bullet \mathbf{u}$ and $\mathfrak{K}_n := \mathbf{k} \bullet \mathbf{n}$ are called the *geodesic* and *normal curvature* of $\mathbf{c}$, respectively. The Frenet-Serret formulas in $\mathbb{R}^3$, see (3.4), yield to the Darboux system:

$$\dot{\mathbf{t}} = \mathfrak{K}_g\mathbf{u} + \mathfrak{K}_n\mathbf{n}\,, \quad \dot{\mathbf{n}} = -\mathfrak{K}_n\mathbf{t} - \mathfrak{T}_g\mathbf{u}\,, \quad \dot{\mathbf{u}} = -\mathfrak{K}_g\mathbf{t} + \mathfrak{T}_g\mathbf{n} \tag{2.4}$$

where $\mathfrak{T}_g := \dot{\mathbf{n}} \bullet (\mathbf{t} \times \mathbf{n})$ is the geodesic torsion of the curve. If $\mathbf{c}$ is a geodesic on $\mathcal{M}$, we have $\mathfrak{K}_g \equiv 0$, whence the Darboux frame agrees (up to the sign) with the Frenet frame, and the conormal $\mathbf{u}$ with the bi-normal vector. In particular, the normal curvature $\mathfrak{K}_n$ and the geodesic torsion $\mathfrak{T}_g$ are equal (up to the sign) to the scalar curvature and to the torsion of $\mathbf{c}$ in $\mathbb{R}^3$, respectively.

If $N \ge 3$, the unit conormal $\mathbf{u} : [0, L] \to \mathbb{S}^N$ is obtained through a positive rotation of $\mathbf{t}$ on the tangent space $T_{\mathbf{c}}\mathcal{M}$ along $\mathbf{c}$, so that $\mathbf{t} \bullet \mathbf{u} \equiv 0$, and $\mathbf{n} : [0, L] \to \mathbb{S}^N$ is a smooth normal unit vector field (a section of the normal bundle of $\mathcal{M}$). Finally, the projection $\mathfrak{K}_g\mathbf{u}$ of $\mathbf{k}$ onto the tangent bundle is an intrinsic object.

Let X denote a tangent vector field along the smooth curve $\mathbf{c}$ in $\mathcal{M}$, so that $X : [0, L] \to \mathbb{R}^{N+1}$ satiisfies $X(s) \in T_{\mathbf{c}(s)}\mathcal{M}$ for each s. Then, X is a *parallel transport* along $\mathbf{c}$ if $\dot{X}(s) \perp T_{\mathbf{c}(s)}\mathcal{M}$ for each s. Since $\frac{d}{ds}|X(s)|^2 = 2X(s) \bullet \dot{X}(s) = 0$, a parallel transport preserves the length of the initial tangent vector $X(0)$. We shall then assume $|X(0)| = 1$, so that $|X(s)| = 1$ for each s.

It is well-known that the geodesic curvature of $\mathbf{c}$ satisfies

$$\mathfrak{K}_g(s) = \dot{\Theta}(s) \qquad \forall\, s \in [0, L] \tag{2.5}$$

where $\Theta(s)$ is the oriented angle from the parallel transport $X(s)$ to the tangent vector $\mathbf{t}(s)$ to $\mathbf{c}$, so that

$$X(s) = \cos\Theta(s)\,\mathbf{t}(s) - \sin\Theta(s)\,\mathbf{u}(s)\,, \qquad s \in [0, L] \tag{2.6}$$

compare e.g. [26, 13.6.1] for a proof. We thus get the formula for the *total intrinsic curvature* of $\mathbf{c}$, namely:

$$\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) = \int_0^L |\mathfrak{K}_g(s)|\,ds = \int_0^L |\dot\Theta(s)|\,ds \tag{2.7}$$

compare e.g. [7]. In particular, when $N = 2$ and $X(s) \bullet \mathbf{t}(s) \neq 0$, by (2.6) one gets

$$\tan\Theta(s) = -\frac{X(s) \bullet \mathbf{u}(s)}{X(s) \bullet \mathbf{t}(s)}\,. \tag{2.8}$$

The parallel transport (2.6) is a well-defined smooth vector field for each regular and piecewise smooth curve $\mathbf{c}$, once the initial position $X(0)$ is prescribed. Moreover, the angle Θ is a function of bounded variation, with a finite number of Jump points in correspondence to the values $\{s_i \mid i = 1, \dots m\}$ of the arc-length parameter $s \in I_L$ where $\mathbf{c}(s)$ fails to be smooth, the corner points $\mathbf{c}(s_i)$ of $\mathbf{c}$. More precisely, Θ is a special function of bounded variation in $\mathrm{SBV}(I_L)$, i.e., $D^C\Theta = 0$, and its distributional derivative decomposes as $D\Theta = \dot\Theta\,\mathcal{L}^1 + D^J\Theta$. The derivative $\dot\Theta$ agrees with the geodesic curvature $\mathfrak{K}_g$ outside the corner points of $\mathbf{c}$, and the Jump component $D^J\Theta$ is a sum of Dirac masses centered at the related points s_i, with weight given by the oriented turning angles α_i between the incoming and outcoming unit tangent vectors at each corner point of $\mathbf{c}$, i.e.

$$D\Theta = \mathfrak{K}_g\,\mathcal{L}^1 + \sum_{i=1}^n \alpha_i\,\delta_{s_i}\,, \qquad |D\Theta|(I_L) = \int_0^L |\mathfrak{K}_g|\,ds + \sum_{i=1}^n |\alpha_i|$$

If $N = 2$, since the Darboux formulas (2.4) hold true outside the points s_i, by the smoothness of X

$$\dot X = -\sin\Theta\,\dot\Theta\,\mathbf{t} - \cos\Theta\,\dot\Theta\,\mathbf{u} + \cos\Theta\,\dot{\mathbf{t}} - \sin\Theta\,\dot{\mathbf{u}}$$

and the parallel transport of piecewise smooth curves satisfies, for $s \neq s_i$,

$$\dot X = (\cos\Theta\,\mathfrak{K}_n - \sin\Theta\,\mathfrak{T}_g)\,\mathbf{n}\,.$$

If $N \geq 3$, on account of (2.6), by decomposing the derivative

$$\dot{\mathbf{u}} = (\dot{\mathbf{u}} \bullet \mathbf{t})\,\mathbf{t} + \dot{\mathbf{u}}^\perp$$

of the unit conormal into the tangential and normal component to $\mathcal{M}$, and recalling that $\dot{\mathbf{u}} \bullet \mathbf{t} = -\mathbf{t} \bullet \mathbf{u} = -\dot\Theta$, the parallel transport of (piecewise) smooth curves this time satisfies

$$\dot X = \cos\Theta\,\mathfrak{K}_n\,\mathbf{n} - \sin\Theta\,\dot{\mathbf{u}}^\perp\,,$$

where $\dot{\mathbf{u}}^\perp = \dot{\mathbf{u}}$ when $\mathbf{c}$ is a geodesic arc.

Example 2.8. If $\mathcal{M} = \mathcal{S}^2$, the unit sphere in $\mathbb{R}^3$, taking polar coordinates

$$\mathbf{r}(\theta, \varphi)^T = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,, \qquad \theta \in [0, \pi]\,, \quad \varphi \in [0, 2\pi]$$

a smooth spherical curve $\mathbf{c}$ can be parameterized by $\mathbf{c}(s) = \mathbf{r}(\theta(s), \varphi(s))^T$ for suitable angle functions $\theta(s)$ and $\varphi(s)$. In terms of the usual frame $\mathbf{e}_\theta$, $\mathbf{e}_\varphi$, and

$\mathbf{n}(\theta,\varphi)=\mathbf{e}_\theta\times\mathbf{e}_\varphi$, the outward unit normal, and letting $\mathbf{v}(s):=\mathbf{v}(\theta(s),\varphi(s))$, for $\mathbf{v}=\mathbf{e}_\theta$, $\mathbf{e}_\varphi$, or $\mathbf{n}$, we thus have for any $s\in[0,L]$

$$\begin{gathered}\mathbf{t}(s):=\dot{\mathbf{c}}(s)=\dot\theta(s)\,\mathbf{e}_\theta(s)+\sin\theta(s)\,\dot\varphi(s)\,\mathbf{e}_\varphi(s)\,,\\ \dot\theta(s)^2+\sin^2\theta(s)\,\dot\varphi(s)^2=1\,.\end{gathered}\tag{2.9}$$

Taking a tangent vector field X along $\mathbf{c}$, say $X(s):=\alpha(s)\,\mathbf{e}_\theta(s)+\beta(s)\,\mathbf{e}_\varphi(s)$, where $s\in[0,L]$, the condition for a parallel transport turns out to be equivalent to the first order system for the unknown coefficients:

$$\begin{cases}\dot\alpha(s)=\cos\theta(s)\,\dot\varphi(s)\,\beta(s)\\ \dot\beta(s)=-\cos\theta(s)\,\dot\varphi(s)\,\alpha(s)\end{cases}\qquad s\in[0,L]\tag{2.10}$$

which has a unique solution for any given initial position $X(0)\in T_{\mathbf{c}(0)}\mathcal{S}^2$.

On account of (2.8), and since by (2.9) the unit conormal along $\mathbf{c}$ is

$$\mathbf{u}(s):=\mathbf{n}(s)\times\mathbf{t}(s)=-\sin\theta(s)\,\dot\varphi(s)\,\mathbf{e}_\theta(s)+\dot\theta(s)\,\mathbf{e}_\varphi(s)\tag{2.11}$$

one then computes

$$\dot\Theta=\sin\theta\,(\ddot\varphi\,\dot\theta-\ddot\theta\,\dot\varphi)+\cos\theta\,\dot\varphi\,(\sin^2\theta\,\dot\varphi^2+2\dot\theta^2)\,.$$

On the other hand, recalling formula (2.9), the curvature vector of $\mathbf{c}$ is

$$\mathbf{k}=\dot{\mathbf{t}}=(\ddot\theta-\sin\theta\,\cos\theta\,\dot\varphi^2)\,\mathbf{e}_\theta+(2\cos\theta\,\dot\theta\,\dot\varphi+\sin\theta\,\ddot\varphi)\,\mathbf{e}_\varphi-\mathbf{n}\tag{2.12}$$

and hence by (2.11) the geodesic curvature becomes

$$\mathfrak{K}_g=\mathbf{k}\bullet\mathbf{u}=\sin\theta\,(\ddot\varphi\,\dot\theta-\ddot\theta\,\dot\varphi)+\cos\theta\,\dot\varphi\,(\sin^2\theta\,\dot\varphi^2+2\dot\theta^2)\tag{2.13}$$

where $(\sin^2\theta\,\dot\varphi^2+2\dot\theta^2)=(1+\dot\theta^2)$, so that one recovers equation (2.5).

Example 2.9. If $\mathbf{c}=\mathbf{c}_{\theta_0}$ is the parallel with constant co-latitude $\theta_0\in]0,\pi/2]$, we choose $\theta(s)\equiv\theta_0$ and $\varphi(s)=s/\sin\theta_0$, where $s\in[0,L]$, with $L:=\mathcal{L}(\mathbf{c}_{\theta_0})=2\pi\,\sin\theta_0$. By (2.9) and (2.11), one has

$$\mathbf{t}(s)=\mathbf{e}_\varphi(\theta_0,s/\sin\theta_0)\,,\quad \mathbf{u}(s)=-\mathbf{e}_\theta(\theta_0,s/\sin\theta_0)\qquad\forall\,s$$

and by solving the system (2.10) as above, on account of (2.12) and (2.13) one obtains

$$\Theta(s)=\cot\theta_0\cdot s\,,\qquad \mathfrak{K}_g=\dot\Theta\equiv\cot\theta_0\qquad\forall\,s\,.$$

Therefore, according to (2.7) one recovers for any $\theta_0\in]0,\pi/2]$ the formula

$$\mathrm{TC}_{\mathcal{S}^2}(\mathbf{c}_{\theta_0})=\int_0^{2\pi\,\sin\theta_0}|\dot\Theta(s)|\,ds=2\pi\,\cos\theta_0$$

for the total intrinsic curvature of the parallel, compare e.g. [7]. In particular, $\mathrm{TC}_{\mathcal{S}^2}(\mathbf{c}_{\theta_0})$ is equal to zero when $\theta_0=\pi/2$, i.e., when $\mathbf{c}_{\theta_0}$ is a great circle, whence a geodesic in $\mathcal{S}^2$.

3. Weak binormal and total absolute torsion

In this section, we collect our results from [22] concerning irregular curves in the Euclidean space $\mathbb{R}^3$.

3.1. **Total absolute torsion of polygonal curves.** Assume now $N = 2$, and let P a polygonal curve in $\mathbb{R}^3$ with consecutive vertices v_i, $i = 0, \dots, m$, where $m \geq 3$ and P is not closed, i.e., $v_0 \neq v_m$. Without loss of generality, we assume that every oriented segment $\sigma_i := [v_{i-1}, v_i]$ has positive length $\mathcal{L}(\sigma_i) := \|v_i - v_{i-1}\|$, for $i = 1, \dots, m$, and that two consecutive segments are never aligned, i.e., the vector product $\sigma_i \times \sigma_{i+1} \neq 0_{\mathbb{R}^3}$ for each $i = 1, \dots, m-1$. If the vector product $\sigma_i \times \sigma_{i+1}$ is null, we replace σ_{i+1} with the oriented segment $[v_i, v_{j+1}]$, where j is the first index greater than i such that $\sigma_j \times \sigma_{j+1} \neq 0_{\mathbb{R}^3}$. If $\sigma_j \times \sigma_{j+1} = 0_{\mathbb{R}^3}$ for each $j > i$, we set $b_i = b_{i-1}$ in (3.1) below.

In the definition by M. A. Penna [25], the *discrete unit binormal* is the unit vector given at each interior vertex v_i of P by the formula:

$$b_i := \frac{\sigma_i \times \sigma_{i+1}}{\|\sigma_i \times \sigma_{i+1}\|}, \qquad i = 1, \dots, m-1 . \tag{3.1}$$

The *torsion of* P is a function $\boldsymbol{\tau}(\sigma_i)$ of the interior oriented segments σ_i defined as follows. Let $i = 2, \dots, m-1$. If the three segments σ_{i-1}, σ_i, σ_{i+1} are coplanar, i.e., if $b_{i-1} \times b_i = 0_{\mathbb{R}^3}$, one sets $\boldsymbol{\tau}(\sigma_i) = 0$. Otherwise,

$$\boldsymbol{\tau}(\sigma_i) := \frac{\theta_i}{\mathcal{L}(\sigma_i)}$$

where θ_i denotes the angle between $-\pi/2$ and $\pi/2$ whose magnitude is the undirected angle between the binormals b_{i-1} and b_i, and whose sign is equal to the sign of the scalar product between the linearly independent vectors $b_{i-1} \times b_i$ and σ_i. Penna then defined the total torsion of P through the sum:

$$\sum_{i=2}^{m-1} \boldsymbol{\tau}(\sigma_i) \cdot \mathcal{L}(\sigma_i) = \sum_{i=2}^{m-1} \theta_i .$$

In a similar way, we define the *total absolute torsion* of P by:

$$\mathrm{TAT}(P) := \sum_{i=2}^{m-1} |\boldsymbol{\tau}(\sigma_i)| \cdot \mathcal{L}(\sigma_i) = \sum_{i=2}^{m-1} |\theta_i| .$$

We thus consider angles between unoriented osculating planes. In fact, it may happen that the planes $\mathrm{span}\,(\sigma_{i-1}, \sigma_i)$ and $\mathrm{span}\,(\sigma_i, \sigma_{i+1})$ are almost parallel, but the directed angle between the binormal vectors b_i and b_{i+1} is equal to $\pi - \varepsilon$ for $\varepsilon > 0$ small. However, one gets $|\theta_i| = \varepsilon$, since in general

$$|\theta_i| = \min\{\arccos(b_{i-1} \bullet b_i), \arccos(-b_{i-1} \bullet b_i)\} \in [0, \pi/2] .$$

Notice that the total absolute torsion of P can be equivalently defined through the formula:

$$\mathrm{TAT}(P) := \sum_{i=2}^{m-1} \widetilde{\theta}_i$$

where $\widetilde{\theta}_i \in [0, \pi/2]$ is the shortest angle in $\mathbb{S}^2$ between the unoriented geodesic arcs γ_{i-1} and γ_i meeting at the edge t_i of $\mathfrak{t}_P$. Therefore, any reasonable notion of binormal (for non-smooth curves) naturally lives in the projective plane $\mathbb{RP}^2$.

3.2. **Binormal indicatrix of polygonal curves.** Approaching the matter from another viewpoint, W. Fenchel [14] in the 1950's exploited the *spherical polarity* of the tangent and binormal indicatrix in order to analyze differential geometric properties of smooth curves in $\mathbb{R}^3$. In his survey, Fenchel proposed a general method that gathers several results on curves in a unified scheme. We point out that Fenchel deals with C^4 rectifiable curves (parameterized by arc-length) such that at each point it is well-defined the osculating plane, that is, a plane containing the linearly independent vectors $\mathfrak{t} := \dot{c}$ and $\ddot{c}$, such that its suitably oriented normal unit vector $\mathfrak{b}$, the binormal vector, is of class C^2, and the two vectors $\dot{\mathfrak{t}}$ and $\dot{\mathfrak{b}}$ never vanish simultaneously. He then defines the principal normal by the vector product

$$\mathfrak{n} := \mathfrak{b} \times \mathfrak{t} \,. \tag{3.2}$$

Since the derivatives of $\mathfrak{t}$ and $\mathfrak{b}$ are perpendicular to both $\mathfrak{t}$ and $\mathfrak{b}$, the curvature $\mathbf{k}$ and torsion $\boldsymbol{\tau}$ are well-defined through the formulas:

$$\dot{\mathfrak{t}} = \mathbf{k}\, \mathfrak{n} \,, \qquad \dot{\mathfrak{b}} = -\boldsymbol{\tau}\, \mathfrak{n} \,.$$

As a consequence, one has

$$\dot{\mathfrak{n}} = -\mathbf{k}\, \mathfrak{t} + \boldsymbol{\tau}\, \mathfrak{b}$$

and hence the Frenet-Serret formulas hold true, but Fenchel allows both curvature and torsion to be zero or negative. Related arguments have been treated in [5, 11, 12, 19, 29].

By melting together the approaches due to Penna and Fenchel, we define *binormal indicatrix* $\mathfrak{b}_P$ of a polygonal P in $\mathbb{R}^3$ as the arc-length parameterization $\mathfrak{b}_P$ of the *polar* in $\mathbb{RP}^2$ of the tangent indicatrix $\mathfrak{t}_P$.

For this purpose, we recall that the support of $\mathfrak{t}_P$ is the union of $m-1$ geodesic arcs γ_i, where γ_i has initial point t_i and end point t_{i+1}, for $i = 1, \ldots, m-1$. Since we assumed that consecutive segments of P are never aligned, each arc γ_i is non-trivial and well-defined. Then, the discrete unit binormal $b_i \in \mathbb{S}^2$ from definition (3.1) is the "north pole" corresponding to the great circle passing through γ_i and with the same orientation as γ_i.

For any $i = 2, \ldots, m-1$, we denote by Γ_i the geodesic arc in $\mathbb{RP}^2$ with initial point $[b_{i-1}]$ and end point $[b_i]$. Then Γ_i is degenerate when $b_{i-1} = \pm b_i$, i.e., when the three segments σ_{i-1}, σ_i, σ_{i+1} are coplanar. We thus have $\mathcal{L}_{\mathbb{RP}^2}(\Gamma_i) = \widetilde{\theta}_i = |\theta_i|$ for each i, and hence

$$\sum_{i=2}^{m-1} \mathcal{L}_{\mathbb{RP}^2}(\Gamma_i) = \mathrm{TAT}(P) \,.$$

Also, for $i < m-2$ the end point of Γ_i is equal to the initial point of Γ_{i+1}. Finally, if $\mathrm{TAT}(P) = 0$, i.e., if the polygonal P is coplanar, all the arcs Γ_i degenerate to a point $[b] \in \mathbb{RP}^2$, which actually identifies the binormal to P.

Definition 3.1. *Polar* of the tangent indicatrix $\mathfrak{t}_P$ is the oriented curve in $\mathbb{RP}^2$ obtained by connecting the consecutive geodesic arcs Γ_i, for $i = 2, \ldots, m-1$.

The polar of $\mathfrak{t}_P$ connects by geodesic arcs in $\mathbb{RP}^2$ the consecutive discrete binormals $[b_i]$ of the polygonal P, and its length is equal to the total absolute torsion

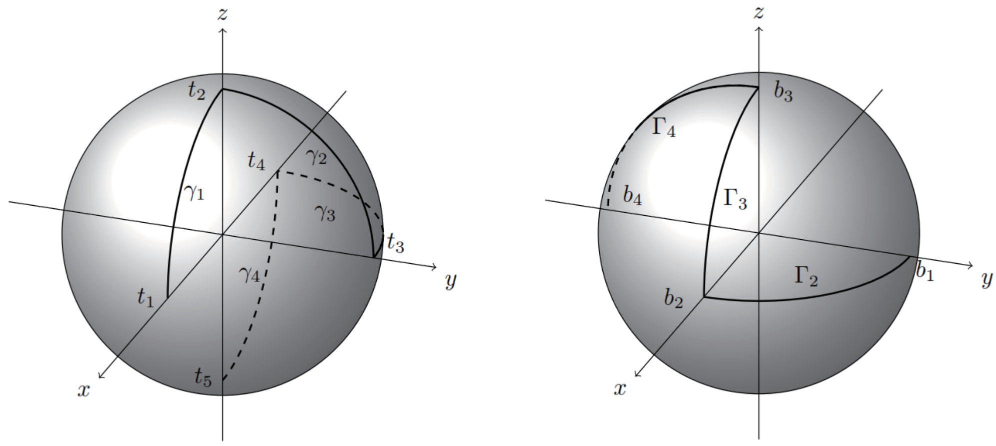

FIGURE 1. An example of a polygonal curve with tangent indicatrix moving as in the left figure. The weak binormal indicatrix moves as in the right figure. Since the weak binormal indicatrix lives in the projective space $\mathbb{RP}^2$, in the figure we have drawn one of its two possible liftings into $\mathbb{S}^2$.

$\mathrm{TAT}(P)$ of P. In particular, it is a rectifiable curve. This property allows us to introduce a suitable weak notion of binormal.

Definition 3.2. We denote *binormal indicatrix* of the polygonal P the arc-length parameterization $\mathfrak{b}_P$ of the polar in $\mathbb{RP}^2$ of the tangent indicatrix $\mathfrak{t}_P$ (see Figure 1).

We thus have $\mathfrak{b}_P : [0,T] \to \mathbb{RP}^2$, where $T := \mathcal{L}_{\mathbb{RP}^2}(\mathfrak{b}_P) = \mathrm{TAT}(P)$. Moreover, $\mathfrak{b}_P$ is Lipschitz-continuous and piecewise smooth, with $|\dot{\mathfrak{b}}_P| = 1$ everywhere except to a finite number of points. Therefore, the total absolute torsion $\mathrm{TAT}(P)$ of P is equal to the length of the curve $\mathfrak{b}_P$. We remark that a similar definition has been introduced by T. F. Banchoff in his paper [5] on space polygons.

However, differently from what happens for length and total curvature, the monotonicity formula fails to hold. More precisely, if P' is a polygonal inscribed in P, by the triangular inequality we have $\mathcal{L}(P') \leq \mathcal{L}(P)$ and $\mathrm{TC}(P') \leq \mathrm{TC}(P)$, compare e.g. [28, Cor. 2.2], but it may happen that $\mathrm{TAT}(P') > \mathrm{TAT}(P)$. This is due to the fact that the total absolute torsion of a polygonal P can be computed as the sum of $\min\{\theta_i, \pi - \theta_i\}$, where θ_i is the turning angle of the tantrix $\mathfrak{t}_P$ at the i-th vertex.

Example 3.3. The polygonal P is made of six segments σ_i, for $i = 1, \dots, 6$, where the first three ones and the last three ones lay on two different planes Π_1 and Π_2. Then the tantrix $\mathbf{t}_P$ connects with geodesic arcs in $\mathbb{S}^2$ the consecutive points $v_i := \sigma_i/\mathcal{L}(\sigma_i)$, for $i = 1, \dots, 6$, where the triplets v_1, v_2, v_3 and v_4, v_5, v_6 lay on two geodesic arcs, which are inscribed in the great circles corresponding to the vector spaces spanning the planes Π_1 and Π_2, respectively. If both the angles α and β of $\mathbf{t}_P$ at the points v_3 and v_4 are small, then $\mathrm{TAT}(P) = \alpha + \beta$.

Let P' be the inscribed polygonal obtained by replacing the segments σ_3 and σ_4 of P with the segment σ between the first point of σ_3 and the last point of σ_4. The tantrix $\mathbf{t}_{P'}$ connects with geodesic arcs the consecutive points $v_1, v_2, w, v_5, v_6,$

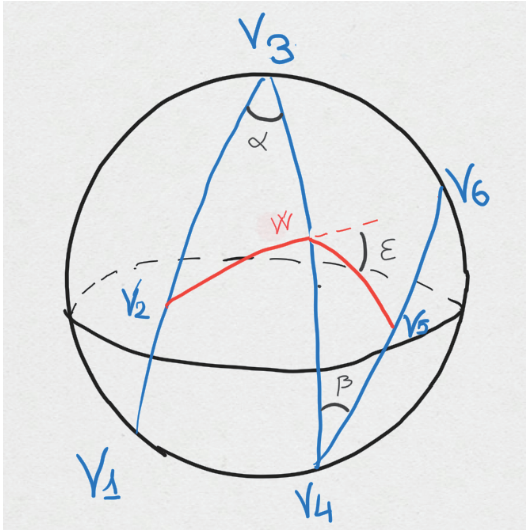

FIGURE 2. The tantrix of the polygonal P, in blue color, and of the inscribed polygonal P', in red color. The drawing is courtesy offered by the young artist Sofia Saracco.

where the point w lays in the minimal geodesic arc between v_3 and v_4. Now, assume that the turning angle ε of $\mathbf{t}_{P'}$ at the point v_5 satisfies $\alpha < \varepsilon < \pi/2$, and that the two geodesic triangles with vertices v_2, v_3, w and w, v_4, v_5 have the same area. By suitably choosing the position of the involved vertices, and by using Gauss-Bonnet theorem in the computation, it turns out that $\mathrm{TAT}(P') - \mathrm{TAT}(P) = 2(\varepsilon - \alpha) > 0$, see Figure 2.

3.3. **Total absolute torsion and weak binormal.** For the above reasons, the *total absolute torsion* $\mathrm{TAT}(\mathbf{c})$ of a curve $\mathbf{c}$ in $\mathbb{R}^3$ is defined by following the approach due to Alexandrov-Reshetnyak [3], that involves the notion of *modulus* $\mu_{\mathbf{c}}(P)$, namely:

$$\mathrm{TAT}(\mathbf{c}) := \lim_{\varepsilon \to 0^+} \sup\{\mathrm{TAT}(P) \mid P \ll \mathbf{c}\,,\ \mu_{\mathbf{c}}(P) < \varepsilon\}\,.$$

The modulus $\mu_{\mathbf{c}}(P)$ of a polygonal P inscribed in $\mathbf{c}$ is the maximum of the diameter of the arcs of $\mathbf{c}$ determined by the consecutive vertices in P. Notice that if $\mathbf{c}$ is a polygonal curve itself, there exists $\varepsilon > 0$ such that any polygonal P inscribed in $\mathbf{c}$ and with modulus $\mu_{\mathbf{c}}(P) < \varepsilon$ satisfies $\mathfrak{t}_P = \mathfrak{t}_{\mathbf{c}}$, whence $\mathfrak{b}_P = \mathfrak{b}_{\mathbf{c}}$ and definitely we get $\mathrm{TAT}(P) = \mathrm{TAT}(\mathbf{c})$. It suffices indeed to take ε lower than half of the mesh of the polygonal $\mathbf{c}$, so that in every segment of $\mathbf{c}$ there are at least two vertices of P.

Therefore, if $\mathrm{TAT}(\mathbf{c}) < \infty$, for any sequence $\{P_h\}$ of polygonal curves inscribed in $\mathbf{c}$ and satisfying $\mu_{\mathbf{c}}(P_h) \to 0$, one has $\sup_h \mathrm{TAT}(P_h) < \infty$, and one can also find an optimal sequence as above in such a way that $\mathrm{TAT}(P_h) \to \mathrm{TAT}(\mathbf{c})$. The following result is proved in [22].

Theorem 3.4. *Let* $\mathbf{c}$ *be a rectifiable curve in* $\mathbb{R}^3$ *with finite total curvature* $\mathrm{TC}(\mathbf{c})$ *and finite (and non-zero) total absolute torsion* $T := \mathrm{TAT}(\mathbf{c})$*. Then, there exists a* *rectifiable curve* $\mathfrak{b}_{\mathbf{c}} : [0, T] \to \mathbb{RP}^2$ *parameterized by arc-length, so that*

$$\mathcal{L}_{\mathbb{RP}^2}(\mathfrak{b}_{\mathbf{c}}) = \mathrm{TAT}(\mathbf{c}) \tag{3.3}$$

holds, satisfying the following property. For any sequence $\{P_n\}$ *of inscribed polygonal curves, let* $b_n : [0, T] \to \mathbb{RP}^2$ *denote for each* h *the parameterization with constant velocity of the binormal indicatrix* $\mathfrak{b}_{P_n}$ *of* P_n*. If* $\mu_{\mathbf{c}}(P_n) \to 0$*, then* $b_n \to \mathfrak{b}_c$ *uniformly on* $[0, T]$ *and* $\mathcal{L}_{\mathbb{RP}^2}(b_n) \to \mathcal{L}_{\mathbb{RP}^2}(\mathfrak{b}_{\mathbf{c}})$*.*

Our weak binormal $\mathfrak{b}_{\mathbf{c}}$ only depends on the curve $\mathbf{c}$. Recalling that $\mathcal{L}_{\mathbb{RP}^2}(b_h) = \mathrm{TAT}(P_h)$, we indeed obtain:

Proposition 3.5. *Let* $\mathbf{c}$ *be a rectifiable curve in* $\mathbb{R}^3$ *with both finite total curvature* $\mathrm{TC}(\mathbf{c})$ *and total absolute torsion* $\mathrm{TAT}(\mathbf{c})$*. Then for any sequence* $\{P_n\}$ *of inscribed polygonal curves such that* $\mu_{\mathbf{c}}(P_n) \to 0$*, one has* $\mathrm{TAT}(P_n) \to \mathrm{TAT}(\mathbf{c})$*.*

3.4. **Relationship with the smooth binormal.** Let now $\mathbf{c}$ be a smooth regular curve in $\mathbb{R}^3$ defined through arc-length parameterization. Assuming $\ddot{\mathbf{c}} \neq 0$ everywhere, and letting $\mathfrak{t} := \dot{\mathbf{c}}$, $\mathfrak{n} := \dot{\mathfrak{t}}/|\dot{\mathfrak{t}}|$, $\mathbf{k} := |\dot{\mathfrak{t}}|$, $\mathfrak{b} := \mathfrak{t} \times \mathfrak{n}$, the classical Frenet-Serret formulas for the spherical frame $(\mathfrak{t}, \mathfrak{n}, \mathfrak{b})$ of $\mathbf{c}$ give:

$$\dot{\mathfrak{t}} = \mathbf{k}\,\mathfrak{n}\,, \qquad \dot{\mathfrak{n}} = -\mathbf{k}\,\mathfrak{t} + \boldsymbol{\tau}\,\mathfrak{b}\,, \qquad \dot{\mathfrak{b}} = -\boldsymbol{\tau}\,\mathfrak{n} \tag{3.4}$$

where $\mathbf{k}$ is the (positive) curvature and $\boldsymbol{\tau}$ the torsion of the curve.

Remark 3.6. Notice that a rectifiable curve may have unbounded total curvature but zero torsion (just consider a planar curve). Conversely, by taking $s \in [0, 1]$ and letting $\mathbf{k}(s) \equiv 1$ and $\boldsymbol{\tau}(s) = (1 - s)^{-1}$, solutions to the Frenet-Serret system (3.4) are rectifiable curves $\mathbf{c}$ such that $\int_{\mathbf{c}} \mathbf{k}\, ds = 1$ but $\int_{\mathbf{c}} |\boldsymbol{\tau}|\, ds = +\infty$.

For smooth curves, the total absolute torsion, which agrees with the length of the smooth binormal curve $\mathfrak{b}$ in the Gauss sphere $\mathbb{S}^2$, actually agrees with the total *geodesic curvature* of the smooth tantrix $\mathfrak{t}$ in $\mathbb{S}^2$. In fact, on account of the density result from [25, Prop. 4], by Proposition 3.5 one readily obtains

$$\mathrm{TAT}(\mathbf{c}) = \int_{\mathbf{c}} |\boldsymbol{\tau}|\, ds\,. \tag{3.5}$$

As the following example shows, the (absolute value of the) torsion may be seen as the curvature of the tantrix, when computed in the sense of the spherical geometry.

Example 3.7. Given $R > 0$ and $K \geq 0$, we let $\mathbf{c} : [-L/2, L/2] \to \mathbb{R}^3$ denote the helicoidal curve

$$\mathbf{c}(s) := (R\cos(s/v), R\sin(s/v), Ks/(2\pi v))\,, \quad s \in [-L/2, L/2]$$

where we denote $v := (R^2 + (K/2\pi)^2)^{1/2}$ and choose $L := 2\pi v$, so that $|\dot{\mathbf{c}}| \equiv 1$ and the length $\mathcal{L}(\mathbf{c}) = L$. Moreover, $\mathbf{c}(\pm L/2) = (\pm R, 0, \pm K/2)$, and $\mathbf{c}(0) = (R, 0, 0)$. We thus have

$$\begin{aligned}
\mathfrak{t}(s) = &\quad v^{-1}(-R\sin(s/v), R\cos(s/v), K/2\pi) \\
\mathfrak{n}(s) = &\quad (-\cos(s/v), -\sin(s/v), 0) \\
\mathfrak{b}(s) = &\quad v^{-1}((K/2\pi)\sin(s/v), -(K/2\pi)\cos(s/v), R)
\end{aligned}$$

so that both curvature and torsion are constant, $\mathbf{k} \equiv Rv^{-2}$, $\boldsymbol{\tau} \equiv v^{-2}(K/2\pi)$. Therefore, the integral of the curvature and of the torsion of $\mathbf{c}$ are:

$$\int_{\mathbf{c}} \mathbf{k}\, ds = L \cdot \mathbf{k} = \frac{2\pi R}{v}\,, \qquad \int_{\mathbf{c}} |\boldsymbol{\tau}|\, ds = L \cdot \boldsymbol{\tau} = \frac{K}{v}\,, \qquad v := (R^2 + (K/2\pi)^2)^{1/2}.$$

We can compute the *spherical curvature* $\mathbf{k}_{\mathbb{S}^2}(\mathfrak{t})$ of the tantrix $\mathfrak{t}$, a closed curve embedded in the Gauss sphere $\mathbb{S}^2$ and parameterizing (when $K > 0$) a small circle whose radius depends on R and K. To this aim, we first consider a sequence of (strongly converging) polygonal curves $\{\mathfrak{t}_n\}$ in $\mathbb{S}^2$ inscribed in the tantrix $\mathfrak{t}$. Namely, for each $n \in \mathbb{N}^+$ we let $t_n(i) := \mathfrak{t}(s_i)$, where $s_i = (L/n)i$ and $i \in \mathbb{Z} \cap [-n, n]$, and consider the closed spherical polygonal generated by the consecutive points $t_n(i) \in \mathbb{S}^2$. The total curvature of $\mathfrak{t}_n$ is equal to the sum of the width in $\mathbb{S}^2$ of the angles between consecutive segments. The turning angle in $\mathbb{S}^2$ of two consecutive geodesic segments $t_n(i-1)t_n(i)$ and $t_n(i)t_n(i+1)$, agrees with the angle between the two planes in $\mathbb{R}^3$ spanned by $0_{\mathbb{R}^3}$ and the end points of the above segments, i.e., between the normals $t_n(i-1) \times t_n(i)$ and $t_n(i) \times t_n(i+1)$. By symmetry, such an angle θ_n does not depend on the choice of i. The total spherical curvature of the polygonal being equal to $n \cdot \theta_n$, one obtains:

$$\lim_{n\to\infty} n \cdot \theta_n = \frac{K}{v}\,.$$

Here, we have considered a sequence $\{\mathfrak{t}_n\}$ of polygonal curves in $\mathbb{S}^2$ inscribed in the tantrix $\mathfrak{t}$ of $\mathbf{c}$ and converging to $\mathfrak{t}$ in the sense of the Hausdorff distance. In general, each $\mathfrak{t}_n$ is not the tangent indicatrix of a polygonal inscribed in $\mathbf{c}$. However, the total spherical curvature $n \cdot \theta_n$ of $\mathfrak{t}_n$ clearly agrees with the length in $\mathbb{RP}^2$ of the polar of $\mathfrak{t}_n$, which is constructed as above, see Definition 3.1.

Now, one may similarly consider a sequence $\{P_n\}$ of polygonals inscribed in $\mathbf{c}$, each one made of n segments with the same length, so that mesh $P_n \to 0$. The total absolute torsion $\mathrm{TAT}(P_n)$ of P_n agrees with the length in $\mathbb{RP}^2$ of the binormal indicatrix $\mathfrak{b}_{P_n}$, see Definition 3.2. One can similarly show that $\mathcal{L}_{\mathbb{RP}^2}(\mathfrak{b}_{P_n}) \to K/v$ as $n \to \infty$, in accordance with the formula in (3.5). By uniform convergence, we have thus obtained the total curvature of $\mathfrak{t}$ in $\mathbb{S}^2$. In conclusion, we have:

$$\int_{\mathfrak{t}} \mathbf{k}_{\mathbb{S}^2}(\mathfrak{t})\, ds = \frac{K}{v} = \int_c |\boldsymbol{\tau}|\, ds\,.$$

We now see that the binormal $\mathfrak{b}(s)$ of $\mathbf{c}$ agrees with the value of a suitable lifting of the weak binormal $\mathfrak{b}_c$ in $\mathbb{S}^2$, when computed at the expected point.

Theorem 3.8. *In the latter smoothness hypotheses, for each $s \in]0, L[$ there exists $t(s) \in [0, T]$ such that*

$$\mathfrak{b}(s) = \widetilde{\mathfrak{b}}_{\mathbf{c}}(t(s))$$

for a unique lifting $\widetilde{\mathfrak{b}}_{\mathbf{c}}$ of $\mathfrak{b}_{\mathbf{c}}$ in $\mathbb{S}^2$. Moreover, $t(s)$ is equal to the total absolute torsion $\mathrm{TAT}(\mathbf{c}_{|[0,s]})$ of the curve $\mathbf{c}_{|[0,s]} : [0, s] \to \mathbb{R}^3$. In particular, we have:

$$t(s) = \int_0^s |\boldsymbol{\tau}(\lambda)|\, d\lambda \qquad \forall\, s \in [0, L]\,. \tag{3.6}$$

Notice that if in particular the torsion $\boldsymbol{\tau}$ of $\mathbf{c}$ (almost) never vanishes, the function $t(s) : [0, L] \to [0, T]$ in equation (3.6) is strictly increasing, and its inverse $s(t)$: $[0, T] \to [0, L]$ gives

$$\widetilde{\mathfrak{b}}_{\mathbf{c}}(t) = \mathfrak{b}(s(t)) \qquad \forall\, t \in [0, T]\,, \qquad T = \mathrm{TAT}(\mathbf{c})\,.$$

Therefore, in this case, the weak binormal $\mathfrak{b}_{\mathbf{c}}$ in $\mathbb{RP}^2$, when suitably lifted to $\mathbb{S}^2$, agrees with the *arc-length parameterization of the binormal* $\mathfrak{b}$ of $\mathbf{c}$.

Remark 3.9. The hypothesis $\mathrm{TC}(\mathbf{c}) < \infty$ in Theorem 3.4 may sound a bit unnatural, and actually a technical point, since it allows us to prove that $\mathfrak{b}_{\mathbf{c}}$ has constant velocity one, so that (3.3) holds true.

To this purpose, we recall that the definition of *complete torsion* $\mathrm{CT}(P)$ of polygonals P given by Alexandrov-Reshetnyak [3], who essentially take the distance in $\mathbb{S}^2$ between consecutive discrete binormals, implies that planar polygonals may have positive torsion at "inflections points". Defining the *complete torsion* $\mathrm{CT}(\mathbf{c})$ of curves $\mathbf{c}$ in $\mathbb{R}^3$ as the supremum of the complete torsion of the inscribed polygonals, they obtain in [3, p. 244] that any curve with finite complete torsion and with no points of return must have finite total curvature.

With our definition of torsion, the above implication clearly fails to hold, see Remark 3.6. On the other hand, equality (3.5) is violated if one considers the complete torsion from [3], since for a smooth planar curve with inflection points, one has $\mathrm{CT}(\mathbf{c}) > 0$.

We finally notice that a curve with finite total curvature and total absolute torsion may have infinite complete torsion in the sense of [3]: just take a smooth planar curve with a countable set of inflection points.

3.5. **Complete tangent indicatrix.** Similar features concerning the tantrix hold. Our curve $\mathfrak{t}_{\mathbf{c}}$ is strictly related with the *complete tangent indicatrix* in the sense of Alexandrov-Reshetnyak [3].

Proposition 3.10. *Let* $\mathbf{c}$ *be a rectifiable curve in* $\mathbb{R}^3$ *with finite total curvature* $C := \mathrm{TC}(\mathbf{c})$ *and with no points of return. Then, there exists a rectifiable curve* $\mathfrak{t}_{\mathbf{c}} : [0, C] \to \mathbb{S}^2$*, parameterized by arc-length, so that* $\mathcal{L}_{\mathbb{S}^2}(\mathfrak{t}_{\mathbf{c}}) = \mathrm{TC}(\mathbf{c})$*, satisfying the following property. For any sequence* $\{P_n\}$ *of inscribed polygonal curves such that* $\operatorname{mesh} P_n \to 0$*, denoting by* $t_n : [0, C] \to \mathbb{S}^2$ *the parameterization with constant velocity of the tangent indicatrix* $\mathfrak{t}_{P_n}$ *of* P_n*, then* $t_n \to \mathfrak{t}_{\mathbf{c}}$ *uniformly on* $[0, C]$ *and* $\mathcal{L}_{\mathbb{S}^2}(t_n) \to \mathcal{L}_{\mathbb{S}^2}(\mathfrak{t}_{\mathbf{c}})$.

If $\mathbf{c}$ has points of return, i.e., if e.g. for some $s \in]0, L[$ we have $\mathfrak{t}(s-) = -\mathfrak{t}(s+)$, the curve $\mathfrak{t}_{\mathbf{c}}$ is uniquely determined up to the choice of the geodesic arc in $\mathbb{S}^2$ connecting $\mathfrak{t}(s-)$ and $\mathfrak{t}(s+)$. In the smooth case, we also have:

Proposition 3.11. *Let* $\mathbf{c} : [0, L] \to \mathbb{R}^3$ *be a curve of class* C^2 *parameterized in arc-length, so that* $L = \mathcal{L}(\mathbf{c})$*, and let* $\mathfrak{t}_{\mathbf{c}} : [0, C] \to \mathbb{S}^2$ *the rectifiable curve in* $\mathbb{S}^2$ *defined in Proposition* 3.10*, so that* $C = \mathrm{TC}(\mathbf{c})$*. Then, for each* $s \in]0, L[$ *there exists* $k(s) \in [0, C]$ *such that the tangent indicatrix* $\mathfrak{t} := \dot{\mathbf{c}}$ *satisfies*

$$\mathfrak{t}(s) = \mathfrak{t}_{\mathbf{c}}(k(s))\,.$$

Moreover, $k(s)$ is equal to the total curvature $\mathrm{TC}(c_{|[0,s]})$ *of the curve* $c_{|[0,s]} : [0,s] \to \mathbb{R}^3$*, whence:*

$$k(s) = \int_0^s \mathbf{k}(\lambda)\, d\lambda \qquad \forall\, s \in [0,L] \tag{3.7}$$

where $\mathbf{k}(\lambda) := \|\ddot{\mathbf{c}}(\lambda)\|$ *is the curvature of* $\mathbf{c}$ *at the point* $\mathbf{c}(\lambda)$*.*

As before, if the curvature $\mathbf{k}$ of $\mathbf{c}$ (almost) never vanishes, the function $k(s) : [0,L] \to [0,C]$ in equation (3.7) is strictly increasing, and its inverse $s(k) : [0,C] \to [0,L]$ gives

$$\mathfrak{t}_{\mathbf{c}}(k) = \mathfrak{t}(s(k)) \qquad \forall\, k \in [0,C]\,, \qquad C = \mathrm{TC}(\mathbf{c})\,.$$

3.6. **Weak principal normal.** When looking for a possible weak notion of principal normal, a drawback appears. In fact, in Penna's approach [25], the curvature of an open polygonal P is a non-negative measure μ_P concentrated at the interior vertices, whereas the torsion is a signed measure ν_P concentrated at the interior segments. Since these two measures are mutually singular, in principle there is no way to extend Fenchel's formula (3.2) in order to define the principal normal.

To overcome this problem, we proceed as follows. Firstly, for a polygonal P we choose two suitable curves $\widetilde{\mathfrak{t}}_P, \widetilde{\mathfrak{b}}_P : [0, C+T] \to \mathbb{RP}^2$, where $C = \mathrm{TC}(P)$ and $T = \mathrm{TAT}(P)$, that on one side inherit the properties of the tangent and binormal indicatrix $\mathfrak{t}_P$ and $\mathfrak{b}_P$, respectively, and on the other side take account of the order in which curvature and torsion are defined along P. More precisely, one of the two curves is constant when the other one parameterizes a geodesic arc, whose length is equal to the curvature or to the (absolute value of the) torsion at one vertex or segment of P, respectively. As in Fenchel's approach, by exploiting the polarity of the curves $\widetilde{\mathfrak{t}}_P$ and $\widetilde{\mathfrak{b}}_P$, the weak normal of the polygonal is well-defined by the inner product

$$\mathfrak{n}_P(s) := \widetilde{\mathfrak{b}}_P(s) \times \widetilde{\mathfrak{t}}_P(s) \in \mathbb{RP}^2\,, \qquad s \in [0, T+C]$$

see Figure 3. Notice that by our definition we have:

$$\mathcal{L}_{\mathbb{RP}^2}(\mathfrak{n}_P) = \mathrm{TC}(P) + \mathrm{TAT}(P)\,.$$

As a consequence, using again an approximation procedure, the *weak principal normal* of a rectifiable curve $\mathbf{c}$ with finite total curvature and finite complete torsion is well-defined as a rectifiable curve $\mathfrak{n}_{\mathbf{c}}$ in $\mathbb{RP}^2$. We recall that condition $\mathrm{CT}(\mathbf{c}) < \infty$ is stronger than the more natural assumption $\mathrm{TAT}(\mathbf{c}) < \infty$. Moreover, it turns out that the product formula (3.2) continues to hold in a suitable sense, and we also have:

$$\mathcal{L}_{\mathbb{RP}^2}(\mathfrak{n}_{\mathbf{c}}) = \mathrm{TC}(\mathbf{c}) + \mathrm{TAT}(\mathbf{c})\,.$$

In particular, for smooth curves whose curvature (almost) never vanishes, the principal normal $\mathfrak{n}$ agrees with a lifting of a suitable parameterization of the weak normal $\mathfrak{n}_{\mathbf{c}}$. More precisely, we obtain:

$$[\mathfrak{n}(s(t))] = \mathfrak{n}_c(t) \in \mathbb{RP}^2 \qquad \forall\, t \in [0, \mathrm{TC}(c) + \mathrm{TAT}(c)]$$

where $s(t)$ is the inverse of the increasing and bijective function

$$t(s) := \int_0^s (\mathbf{k}(\lambda) + |\boldsymbol{\tau}(\lambda)|)\, d\lambda\,, \qquad s \in [0, \mathcal{L}(\mathbf{c})]\,.$$

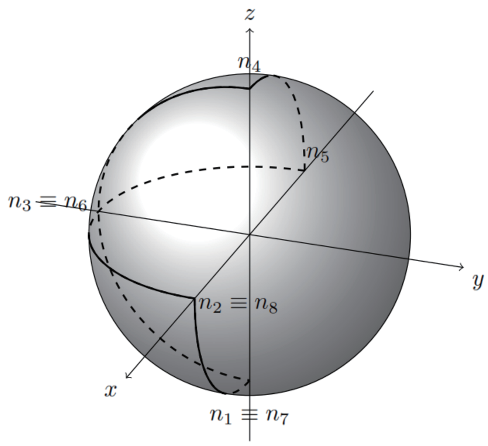

FIGURE 3. The weak normal indicatrix of the curve whose tangent and binormal indicatrix are those in Figure 1 of page 567. Again, for the sake of the illustration we consider one of the two liftings of the normal indicatrix into the sphere $\mathbb{S}^2$.

3.7. **Spherical indicatrices of smooth curves.** The trihedral $(\mathfrak{t}, \mathfrak{n}, \mathfrak{b})$ is well-defined everywhere in case of regular curves γ in $\mathbb{R}^3$ of class C^2 such that $\gamma'(t)$ and $\gamma''(t)$ are always independent vectors, and the Frenet-Serret formulas (3.4) hold true if in addition γ is of class C^3. On the other hand, Fenchel in [14] used a geometric approach in order to define (under weaker hypotheses on the curve) the osculating plane. He chooses the binormal $\mathfrak{b}$ as a smooth function. Therefore, the principal normal is the smooth function given by $\mathfrak{n} = \mathfrak{b} \times \mathfrak{t}$. The Frenet-Serret formulas continue to hold, but this time the curvature may vanish and even be negative. He also calls $\mathbf{k}$-inflection or $\boldsymbol{\tau}$-inflection a point of the curve where the curvature or the torsion changes sign, respectively.

Assume now that $\gamma : [a, b] \to \mathbb{R}^3$ satisfies the following properties:

(1) γ is differentiable at each $t \in [a, b]$ and $\gamma'(t) \neq 0_{\mathbb{R}^3}$, i.e., γ is a regular curve;
(2) for each $t_0 \in]a, b[$, the function γ is of class C^n in a neighborhood of t_0, for some $n \geq 2$, and $\gamma^{(n)}(t_0) \neq 0_{\mathbb{R}^3}$, but $\gamma^{(k)}(t_0) = 0_{\mathbb{R}^3}$ for $2 \leq k \leq n - 1$, if $n \geq 3$.

In that case, denoting as above by $\mathbf{c}(s)$ the arc-length parameterization of the curve γ, it turns out that the Frenet-Serret frame $(\mathfrak{t}, \mathfrak{b}, \mathfrak{n})$ is well-defined for each $s_0 \in [0, L]$ by:

$$\begin{aligned} \mathfrak{t}(s_0) &:= \dot{\mathbf{c}}(s_0), \qquad \mathfrak{b}(s_0) := \frac{\dot{\mathbf{c}}(s_0) \times \mathbf{c}^{(n)}(s_0)}{\|\mathbf{c}^{(n)}(s_0)\|}, \\ \mathfrak{n}(s_0) &:= \mathfrak{b}(s_0) \times \mathfrak{t}(s_0) = \frac{\mathbf{c}^{(n)}(s_0)}{\|\mathbf{c}^{(n)}(s_0)\|} \end{aligned} \tag{3.8}$$

where $s_0 = s(t_0)$ and $n \geq 2$ as above. Furthermore, $\ddot{\mathbf{c}}(s_0) = 0_{\mathbb{R}^3}$ at a finite or countable set of points, and if $\ddot{\mathbf{c}}(s_0) \neq 0_{\mathbb{R}^3}$, then $\mathfrak{n}(s_0) = \ddot{\mathbf{c}}(s_0)/\|\ddot{\mathbf{c}}(s_0)\|$. Finally, $[\mathfrak{b}]$ and $[\mathfrak{n}]$ are continuous functions with values in $\mathbb{RP}^2$.

If in addition we assume that γ is of class C^3, it turns out that the Frenet-Serret formulas (3.4) hold true outside the at most countable set of inflection points. In fact, $\ddot{c}(s) = 0_{\mathbb{R}^3}$ only at isolated points $s \in [0, L]$.

Example 3.12. Let $\mathbf{c} : [-1, 1] \to \mathbb{R}^3$ be a regular curve with derivative

$$\dot{\mathbf{c}}(s) = \frac{1}{\sqrt{2}} \big(1, s^2, \sqrt{1 - s^4}\big), \quad s \in [-1, 1]$$

so that $\|\dot{\mathbf{c}}(s)\| \equiv 1$ and hence $\mathfrak{t}(s) = \dot{\mathbf{c}}(s)$. For $s \in]-1, 1[$, we compute

$$\ddot{\mathbf{c}}(s) = \frac{\sqrt{2}s}{\sqrt{1 - s^4}} \left(0, \sqrt{1 - s^4}, -s^2\right), \quad \mathbf{c}^{(3)}(s) = \sqrt{2} \left(0, 1, \frac{s^2(s^4 - 3)}{(1 - s^4)^{3/2}}\right).$$

Therefore, if $0 < |s| < 1$ we have $\ddot{\mathbf{c}}(s) \neq 0_{\mathbb{R}^3}$ and hence

$$\mathfrak{n}(s) = \frac{s}{|s|} \left(0, \sqrt{1 - s^4}, -s^2\right), \quad \mathfrak{b}(s) = \frac{s}{|s|} \frac{1}{\sqrt{2}} \left(-1, s^2, \sqrt{1 - s^4}\right).$$

In particular, the normal and binormal can be extended by continuity at $s = \pm 1$ by letting $\mathfrak{n}(\pm 1) := (0, 0, \mp 1)$ and $\mathfrak{b}(\pm 1) := 2^{-1/2}(\mp 1, \pm 1, 0)$. Furthermore, for $0 < |s| < 1$ we get:

$$\mathbf{k}(s) := \|\ddot{\mathbf{c}}(s)\| = \frac{\sqrt{2}|s|}{\sqrt{1 - s^4}}, \quad \boldsymbol{\tau}(s) := \big(\dot{\mathbf{c}}(s) \times \ddot{\mathbf{c}}(s)\big) \bullet \frac{\mathbf{c}^{(3)}(s)}{\|\ddot{\mathbf{c}}(s)\|^2} = -\frac{\sqrt{2}s}{\sqrt{1 - s^4}}$$

and hence $\mathbf{k}(s) \to 0$ and $\boldsymbol{\tau}(s) \to 0$ as $s \to 0$, whereas both $\mathbf{k}$ and $\boldsymbol{\tau}$ are summable functions in $L^1(-1, 1)$. Moreover, the Frenet-Serret formulas (3.4) hold true in the open intervals $]-1, 0[$ and $]0, 1[$.

Since $\mathfrak{t}(0) = 2^{-1/2}(1, 0, 1)$, $\ddot{\mathbf{c}}(0) = 0_{\mathbb{R}^3}$, and $\mathbf{c}^{(3)}(0) = 2^{-1/2}(0, 1, 0)$, by the formulas in (3.8) we get:

$$\mathfrak{b}(0) := \frac{\dot{\mathbf{c}}(0) \times \mathbf{c}^{(3)}(0)}{\|\mathbf{c}^{(3)}(0)\|} = \frac{1}{\sqrt{2}} (-1, 0, 1), \quad \mathfrak{n}(0) := \mathfrak{b}(0) \times \mathfrak{t}(0) = (0, 1, 0)$$

and hence both the unit normal and binormal are not continuous at $s = 0$. However, since $[\mathfrak{n}(s)] \to [\mathfrak{n}(0)]$ and $[\mathfrak{b}(s)] \to [\mathfrak{b}(0)]$ as $s \to 0$, they are both continuous as functions with values in $\mathbb{RP}^2$. Notice also that

$$\frac{\dot{\mathfrak{n}}(s)}{\|\dot{\mathfrak{n}}(s)\|} = \frac{s}{|s|} \left(0, -s^2, -\sqrt{1 - s^4}\right), \qquad s \neq 0. \tag{3.9}$$

We finally compute the total curvature and the total absolute torsion of $\mathbf{c}$. With $t = s^2$, we have:

$$\mathrm{TC}(\mathbf{c}) = \int_{-1}^{1} \mathbf{k}(s)\, ds = \int_{-1}^{1} \frac{\sqrt{2}|s|}{\sqrt{1 - s^4}}\, ds = \sqrt{2} \int_{0}^{1} \frac{1}{\sqrt{1 - t^2}}\, dt = \frac{\pi}{\sqrt{2}}$$

and similarly

$$\mathrm{TAT}(\mathbf{c}) = \int_{-1}^{1} |\boldsymbol{\tau}(s)|\, ds = \int_{-1}^{1} \frac{\sqrt{2}|s|}{\sqrt{1 - s^4}}\, ds = \frac{\pi}{\sqrt{2}}.$$

In fact, $\mathbf{c}$ is regular at $s = 0$, whence there is no turning angle at $\mathbf{c}(0)$, whereas $\mathfrak{b}(0-) = -\mathfrak{b}(0+)$, so that also the total absolute torsion is zero at $\mathbf{c}(0)$. On the other hand, due to the occurrence of an inflection point at $\mathbf{c}(0)$, the complete torsion in the sense of Alexandrov-Reshetnyak [3] yields a contribution equal to π at $\mathbf{c}(0)$, so that $\mathrm{CT}(\mathbf{c}) = \mathrm{TAT}(\mathbf{c}) + \pi$.

With the assumptions written above, the statements of Theorem 3.8 and Proposition 3.11 continue to hold. More precisely, using that the non-negative curvature $\mathbf{k}(\lambda)$ and the torsion $\boldsymbol{\tau}(\lambda)$ may vanish only at a negligible set of inflection points, we readily obtain the following relations concerning the trihedral $(\mathfrak{t}, \mathfrak{b}, \mathfrak{n})$:

(1) $\mathfrak{t}(s_1(k)) = \mathfrak{t}_c(k) \in \mathbb{S}^2$ for $k \in [0, C]$, where $s_1 : [0, C] \to [0, L]$ is the inverse of the function

$$k(s) := \int_0^s \mathbf{k}(\lambda)\, d\lambda\,, \qquad s \in [0, L]\,; \tag{3.10}$$

(2) $[\mathfrak{b}(s_2(t))] = \mathfrak{b}_c(t) \in \mathbb{RP}^2$ for $t \in [0, T]$, where $s_2 : [0, T] \to [0, L]$ is the inverse of the function

$$t(s) := \int_0^s |\boldsymbol{\tau}(\lambda)|\, d\lambda\,, \qquad s \in [0, L]\,;$$

(3) $[\mathfrak{n}(s_3(\rho))] = \mathfrak{n}_c(\rho) \in \mathbb{RP}^2$ for $\rho \in [0, C + T]$, where $s_3 : [0, C + T] \to [0, L]$ is the inverse of the function

$$\rho(s) := \int_0^s (\mathbf{k}(\lambda) + |\boldsymbol{\tau}(\lambda)|)\, d\lambda\,, \qquad s \in [0, L]\,.$$

Example 3.13. Going back to Example 3.12, we compute

$$k(s) := \int_{-1}^s \mathbf{k}(\lambda)\, d\lambda = \frac{1}{\sqrt{2}} \Big(\frac{\pi}{2} + \frac{s}{|s|} \arcsin(s^2) \Big)\,, \quad s \in [-1, 1]$$

and hence $s_1(k) = |\cos(\sqrt{2}k)|^{1/2}$, where $k \in [0, C]$, with $C = \pi/\sqrt{2}$, so that

$$\mathfrak{t}_{\mathbf{c}}(k) := \mathfrak{t}(s_1(k)) = \frac{1}{\sqrt{2}} \big(1, |\cos(\sqrt{2}k)|, \sin(\sqrt{2}k)\big)\,, \qquad k \in [0, \pi/\sqrt{2}]$$

with $k(0) = \pi/(2\sqrt{2})$ and $\mathfrak{t}_{\mathbf{c}}(k(0)) = 2^{-1/2}(1, 0, 1)$. Notice moreover that

$$\dot{\mathfrak{t}}_{\mathbf{c}}(k) = \begin{cases} \big(0, -\sin(\sqrt{2}k), \cos(\sqrt{2}k)\big) & \text{if } k \in [0, \pi/(2\sqrt{2})[\\ \big(0, \sin(\sqrt{2}k), \cos(\sqrt{2}k)\big) & \text{if } k \in]\pi/(2\sqrt{2}), \pi/\sqrt{2}] \end{cases}$$

so that $\dot{\mathfrak{t}}_{\mathbf{c}}(k(0)\pm) = (0, \pm 1, 0)$. We also get

$$\mathfrak{b}_{\mathbf{c}}(t) = \big[2^{-1/2}\big(-1, |\cos(\sqrt{2}t)|, \sin(\sqrt{2}t)\big)\big]\,, \qquad k \in [0, T]\,, \quad T = \pi/\sqrt{2}$$

where $t(0) = \pi/(2\sqrt{2})$ and $\mathfrak{b}_{\mathbf{c}}(t(0)) = \big[2^{-1/2}\,(-1, 0, 1)\big]$. Finally,

$$\dot{\mathfrak{b}}_{\mathbf{c}}(t) = \begin{cases} \big[\big(0, -\sin(\sqrt{2}k), \cos(\sqrt{2}k)\big)\big] & \text{if } t \in [0, \pi/(2\sqrt{2})[\\ \big[(0, \sin(\sqrt{2}k), \cos(\sqrt{2}k)\big] & \text{if } t \in]\pi/(2\sqrt{2}), \pi/\sqrt{2}] \end{cases}$$

so that $\dot{\mathfrak{b}}_{\mathbf{c}}(t(0)+) = \dot{\mathfrak{b}}_{\mathbf{c}}(t(0)-) = [(0, 1, 0)]$, whence $\mathfrak{b}_{\mathbf{c}}$ has no corner points.

3.8. **Torsion force.** Similarly to the curvature force $\mathcal{K}$, a *torsion force* measure $\mathcal{T}$ can be obtained by means of tangential variations of the length $\mathcal{L}_{\mathbb{S}^2}(\mathfrak{t}_{\mathbf{c}})$ of the tangent indicatrix $\mathfrak{t}_{\mathbf{c}}$ that we built up in Proposition 3.10, for any rectifiable curve $\mathbf{c}$ with finite total curvature.

For this purpose, we assume that $\mathfrak{t}_{c,\varepsilon}$ is a variation of $\mathfrak{t}_c$ under which the motion of each point $\mathfrak{t}_c(k)$ is smooth in time and with initial velocity $\xi(s)$, where $\xi : [0, C] \to \mathbb{R}^3$ is a Lipschitz continuous function of arc length k, with $\xi(0) = \xi(C) = 0$. Since we deal with tangential variations, *we assume in addition that* $\xi(k) \in T_{\mathfrak{t}_c(k)}\mathbb{S}^2$ *for each* k. The first variation formula gives:

$$\delta_\xi \mathcal{L}_{\mathbb{S}^2}(\mathfrak{t}_c) := \frac{d}{d\varepsilon}\,\mathcal{L}_{\mathbb{S}^2}(\mathfrak{t}_{c,\varepsilon})_{|\varepsilon=0} = \int_0^C \dot{\mathfrak{t}}_c(k) \bullet \dot{\xi}(k)\,dk$$

where $\dot{\mathfrak{t}}_c(k)$ and $\dot{\xi}(k)$ are defined for a.e. k. Therefore, in general we obtain:

$$\delta_\xi \mathcal{L}_{\mathbb{S}^2}(\mathfrak{t}_c) = \int_0^C \dot{\mathfrak{t}}_c(k) \bullet \dot{\xi}(k)\,dk =: -\langle D\dot{\mathfrak{t}}_c, \xi\rangle \tag{3.11}$$

and hence the first variation $\delta_\xi \mathcal{L}_{\mathbb{S}^2}(\mathfrak{t}_c)$ has distributional order one if and only if the arc-length derivative $\dot{\mathfrak{t}}_c$ of the tantrix $\mathfrak{t}_c$ is a function of bounded variation. By the way, this condition is satisfied if in addition the curve $\mathbf{c}$ has finite complete torsion, $\mathrm{CT}(\mathbf{c}) < \infty$. In this case, there exists a finite measure $\mathcal{T}$, that we call torsion force, such that $\langle \mathcal{T}, \xi\rangle = \langle D\dot{\mathfrak{t}}_c, \xi\rangle$ for each smooth tangential vector field ξ along $\mathfrak{t}_c$.

If $\mathbf{c}$ is of class C^3 and $\ddot{\mathbf{c}}(s) \neq 0_{\mathbb{R}^3}$ for each $s \in]0, L[$, we compute

$$\ddot{\mathfrak{t}}_c(k) = \mathfrak{n}'(s_1)\,\dot{s}_1(k) = -\mathfrak{t}(s_1) + \frac{\boldsymbol{\tau}(s_1)}{\mathbf{k}(s_1)}\,\mathfrak{b}(s_1)\,, \qquad s_1 = s_1(k)$$

for each $k \in [0, C]$, where $s_1 : [0, C] \to [0, L]$ is the inverse of the function $k(s)$ in (3.10). Moreover, the tangential component to $\mathbb{S}^2$ of the second derivative $\ddot{\mathfrak{t}}_{\mathbf{c}}(k)$, i.e., the *geodesic curvature* of $\mathfrak{t}_{\mathbf{c}}$ at the point $\mathfrak{t}_{\mathbf{c}}(k)$, agrees with the quotient between the torsion and the scalar curvature of the initial curve $\mathbf{c}$ at the point $\mathbf{c}(s_1)$, where $s_1 = s_1(k)$.

In fact, the Darboux frame along $\mathfrak{t}_c$ is the triad $(\mathbf{T}, \mathbf{N}, \mathbf{U})$, where $\mathbf{T}(k) := \dot{\mathfrak{t}}_c(k)$, $\mathbf{N}(k) := \nu(\mathfrak{t}_c(k))$, $\nu(p)$ being the unit normal to the tangent 2-space $T_p\mathbb{S}^2$, and $\mathbf{U}(k) := \mathbf{N}(k) \times \mathbf{T}(k)$ is the unit conormal. The curvature vector $\mathbf{K}(k) := \dot{\mathbf{T}}(k) = \ddot{\mathfrak{t}}_c(k)$ is orthogonal to $\mathbf{T}(k)$, and thus decomposes as

$$\mathbf{K}(k) = \mathfrak{K}_g(k)\,\mathbf{U}(k) + \mathfrak{K}_n(k)\,\mathbf{N}(k)$$

where $\mathfrak{K}_g := \mathbf{K} \bullet \mathbf{U}$ and $\mathfrak{K}_n := \mathbf{K} \bullet \mathbf{N}$ denote the geodesic and normal curvature of $\mathfrak{t}_c$, respectively. By changing variable, we get

$$\mathbf{T}(k) = \mathfrak{n}(s_1)\,, \quad \mathbf{N}(k) = \mathfrak{t}(s_1)\,, \quad \mathbf{U}(k) = \mathfrak{b}(s_1)$$

and hence we obtain for each $k \in [0, C]$

$$\mathfrak{K}_g(k) = \frac{\boldsymbol{\tau}(s_1)}{\mathbf{k}(s_1)}\,, \qquad \mathfrak{K}_n(k) \equiv -1\,, \qquad s_1 = s_1(k)\,.$$

As a consequence, integrating by parts in (3.11) we get

$$\langle D\dot{\mathfrak{t}}_c, \xi\rangle = \int_0^C \mathfrak{K}_g(k)\,\mathfrak{b}(s_1(k)) \bullet \xi(k)\,dk = \int_0^C \frac{\boldsymbol{\tau}(s_1)}{\mathbf{k}(s_1)}\,\mathfrak{b}(s_1) \bullet \xi(k)\,dk$$

where, we recall, $\xi(k) \in T_{\mathfrak{t}_c(k)}\mathbb{S}^2$ for each k. Therefore, by changing variable $s = s_1(k)$, since $ds = \mathbf{k}(s_1(k))^{-1}\,dk$ we recover the expected formula:

$$\langle D\dot{\mathfrak{t}}_c, \xi\rangle = \int_0^L \boldsymbol{\tau}(s)\,\mathfrak{b}(s) \bullet \xi(k(s))\,ds\,.$$

Therefore, if $\mathbf{c}$ is smooth we have obtained

$$\mathcal{T} = k_{\#}\big(\boldsymbol{\tau}\,\mathfrak{b}\,d\mathcal{L}^1 \llcorner\,]0, L[\big) \tag{3.12}$$

i.e., $\mathcal{T}$ is the push forward of the measure $\boldsymbol{\tau}\,\mathfrak{b}\,d\mathcal{L}^1 \llcorner\,]0, L[$ by the function $k(s)$ defined in (3.10), and its total mass is equal to $\int_{\mathbf{c}} |\boldsymbol{\tau}|\,ds$.

If $\mathbf{c}$ is piecewise smooth, we obtain again the decomposition $\mathcal{T} = \mathcal{T}^a + \mathcal{T}^s$. By Proposition 3.11, the absolutely continuous component $\mathcal{T}^a$ takes the same form as in the right-hand side of (3.12), where this time $k(s) := \mathrm{TC}(c_{|[0,s]})$. Moreover, using that $\mathfrak{t}(s) = \mathfrak{t}_{\mathbf{c}}(k(s))$, if $\mathbf{c}$ is smooth at s we have $\mathfrak{t}'(s) = \dot{\mathfrak{t}}_{\mathbf{c}}(k(s)) \cdot k'(s)$, with $k'(s) = \mathbf{k}(s)$, hence by the first formula in (3.4) we get $\dot{\mathfrak{t}}_{\mathbf{c}}(k(s)) = \mathfrak{n}(s)$. If in addition $\mathbf{c}$ has no points of return, the torsion force $\mathcal{T}$ only depends on $\mathbf{c}$, and the singular component $\mathcal{T}^s$ is a sum of Dirac masses concentrated at the corner points $x = \mathfrak{t}_{\mathbf{c}}(k)$ of the curve $\mathfrak{t}_{\mathbf{c}}$, with weight $\dot{\mathfrak{t}}_{\mathbf{c}}(k+) - \dot{\mathfrak{t}}_{\mathbf{c}}(k-)$. If θ is the turning angle of $\mathfrak{t}_{\mathbf{c}}$ at x, then $\|\dot{\mathfrak{t}}_{\mathbf{c}}(k+) - \dot{\mathfrak{t}}_{\mathbf{c}}(k-)\| = 2\sin(\theta/2)$.

In Example 3.13, at $x = \mathfrak{t}_c(k(0)) = 2^{-1/2}(1, 0, 1)$ we have $\dot{\mathfrak{t}}_c(k(0)\pm) = (0, \pm 1, 0)$, so that $\theta = \pi$ and $\|\dot{\mathfrak{t}}_c(k(0)+) - \dot{\mathfrak{t}}_c(k(0)-)\| = 2$.

4. Weak curvatures of high order

In this section, we survey our results from [23] concerning weak curvatures of rectifiable curves in high dimension Euclidean spaces $\mathbb{R}^{N+1}$, where $N \geq 3$.

4.1. **Gram-Schmidt procedure.** When dealing with polygonal curves P in high dimension Euclidean spaces, the polarity argument we exploited in the previous section fails to hold. Therefore, we follow a different approach, based on the *orthonormalization procedure.*

To this purpose, we recall that the extension of the classical notions by Frenet-Serret to smooth curves $\mathbf{c}$ in $\mathbb{R}^{N+1}$, where $N \geq 3$, goes back to the contribution by C. Jordan [17]. He noticed that by applying the Gram-Schmidt procedure to the independent vectors $\dot{\mathbf{c}}(s)$, $\mathbf{c}^{(2)}(s), \ldots, \mathbf{c}^{(N)}(s)$ one obtains a moving frame $\mathbf{e}(s) := (\mathbf{t}(s), \mathbf{n}_1(s), \ldots, \mathbf{n}_N(s))$ along the curve, where $\mathbf{t}$ is the tantrix and $\mathbf{n}_j$ is the j-th curvature, for $j = 1, \ldots, N$. Assuming $\mathbf{c}$ parameterized by arc-length s, the Jordan system $\dot{\mathbf{e}}(s) = F(s)\,\mathbf{e}(s)$ involves a skew-symmetric and tri-diagonal square matrix $F(s)$ of order $N+1$, whose entries depend on the curvature functions $\mathbf{k}_j(s)$, where $j = 1, \ldots, N$.

In this framework, H. Gluck [15] produced an algorithm for computing the higher order curvatures, whereas more recently E. Gutkin [16] studied curvature estimates, natural invariants, and discussed the case of curves contained in Riemannian manifolds and homogeneous spaces. Finally, in the last section of his more recent survey paper [27], Reshetnyak also discussed possible ways to extend their theory of irregular curves to the high codimension case.

Coming bach to Jordan's approach, we now consider a curve $\mathbf{c} : [a,b] \to \mathbb{R}^{N+1}$ of class C^3 parameterized by arc-length, so that $\|\dot{\mathbf{c}}\| = 1$. Denoting by $\mathbf{c}^{(k)}$ the k-th arc-length derivative of $\mathbf{c}$, assume that the triplet $(\dot{\mathbf{c}}(s), \mathbf{c}^{(2)}(s), \mathbf{c}^{(3)}(s))$ is linearly independent for each s. The first two Jordan formulas give

$$\dot{\mathbf{t}} = \mathbf{k}_1\, \mathbf{n}_1\,, \quad \dot{\mathbf{n}}_1 = -\mathbf{k}_1\, \mathbf{t} + \mathbf{k}_2\, \mathbf{n}_2$$

where $\mathbf{t} := \dot{\mathbf{c}} \in \mathbb{S}^N$ is the unit tangent vector, $\mathbf{k}_1 := \|\mathbf{c}^{(2)}\|$ the first curvature, $\mathbf{n}_1 := \mathbf{c}^{(2)}/\|\mathbf{c}^{(2)}\| \in \mathbb{S}^N$ the first unit normal, $\mathbf{k}_2 \in \mathbb{R}$ the second curvature, and $\mathbf{n}_2 \in \mathbb{S}^N$ the second unit normal. Notice that when $N = 2$ one has $\mathbf{k}_2 = \boldsymbol{\tau}$, the torsion of the curve, and $\mathbf{n}_2 = \mathbf{b}$, the binormal vector $\mathbf{b} := \mathbf{t} \times \mathbf{n}$. We thus compute

$$\begin{aligned} \mathbf{k}_2\, \mathbf{n}_2 = &\quad \mathbf{k}_1\, \mathbf{t} + \dot{\mathbf{n}}_1 = \|\mathbf{c}^{(2)}\|\, \dot{\mathbf{c}} + \frac{d}{ds}\Big(\frac{\mathbf{c}^{(2)}}{\|\mathbf{c}^{(2)}\|} \Big) \\ = &\quad \frac{1}{\|\mathbf{c}^{(2)}\|} \Big(\|\mathbf{c}^{(2)}\|^2\, \dot{\mathbf{c}} + \mathbf{c}^{(3)} - \frac{\mathbf{c}^{(2)} \bullet \mathbf{c}^{(3)}}{\|\mathbf{c}^{(2)}\|^2}\, \mathbf{c}^{(2)} \Big)\,. \end{aligned}$$

Recalling that $\dot{\mathbf{c}} \bullet \mathbf{c}^{(2)} = 0$ and $\dot{\mathbf{c}} \bullet \mathbf{c}^{(3)} = -\|\mathbf{c}^{(2)}\|^2$, according to the Gram-Schmidt procedure one has:

$$\mathbf{n}_2 = \frac{\mathbf{c}^{(3)\perp}}{\|\mathbf{c}^{(3)\perp}\|}\,, \qquad \mathbf{c}^{(3)\perp} := \mathbf{c}^{(3)} - \frac{\mathbf{c}^{(3)} \bullet \dot{\mathbf{c}}}{\|\dot{\mathbf{c}}\|^2}\, \dot{\mathbf{c}} - \frac{\mathbf{c}^{(3)} \bullet \mathbf{c}^{(2)}}{\|\mathbf{c}^{(2)}\|^2}\, \mathbf{c}^{(2)}\,.$$

We wish to write Taylor expansions at a given point $s \in]a,b[$. Therefore, for each $h > 0$ small enough we consider the three vectors

$$\begin{gathered} \mathbf{v}_0(h) := \frac{\mathbf{c}(s+h) - \mathbf{c}(s-h)}{2h}\,, \qquad \mathbf{v}_1(h) := \frac{\mathbf{c}(s-3h) - \mathbf{c}(s-h)}{2h}\,, \\ \mathbf{v}_2(h) := \frac{\mathbf{c}(s+3h) - \mathbf{c}(s+h)}{2h}\,. \end{gathered} \tag{4.1}$$

In the sequel, we omit to write the dependence on s, and denote by $\mathbf{o}(h^n)$ a continuous vector function such that $\|\mathbf{o}(h^n)\| = o(h^n)$, for each $n \in \mathbb{N}$, i.e., $\|\mathbf{o}(h^n)\|/h^n \to 0$ as $h \to 0$.

By taking the third order expansions of $\mathbf{c}(s)$ and by applying the Gram-Schmidt procedure, we obtain the following formulas:

$$\mathbf{t}(h) := \frac{\mathbf{v}_0(h)}{\|\mathbf{v}_0(h)\|} = \dot{\mathbf{c}} + \frac{1}{6} \big(\|\mathbf{c}^{(2)}\|^2\, \dot{\mathbf{c}} + \mathbf{c}^{(3)} \big)\, h^2 + \mathbf{o}(h^2)$$

$$\mathbf{N}_1(h) := \mathbf{v}_1(h) - \frac{\mathbf{v}_1(h) \bullet \mathbf{v}_0(h)}{\|\mathbf{v}_0(h)\|^2}\, \mathbf{v}_0(h) = 2\mathbf{c}^{(2)}\, h - 2\big(\|\mathbf{c}^{(2)}\|^2 \dot{\mathbf{c}} + \mathbf{c}^{(3)} \big)\, h^2 + \mathbf{o}(h^2)$$

$$\mathbf{n}_1(h) := \frac{\mathbf{N}_1(h)}{\|\mathbf{N}_1(h)\|} = \frac{\mathbf{c}^{(2)}}{\|\mathbf{c}^{(2)}\|} + \Big(-\|\mathbf{c}^{(2)}\|\, \dot{\mathbf{c}} + \frac{\mathbf{c}^{(3)} \bullet \mathbf{c}^{(2)}}{\|\mathbf{c}^{(2)}\|^3}\, \mathbf{c}^{(2)} - \frac{1}{\|\mathbf{c}^{(2)}\|}\, \mathbf{c}^{(3)} \Big)\, h + \mathbf{o}(h)$$

$$\begin{aligned} \mathbf{N}_2(h) &:= \mathbf{v}_2(h) - \frac{\mathbf{v}_2(h) \bullet \mathbf{v}_0(h)}{\|\mathbf{v}_0(h)\|^2}\, \mathbf{v}_0(h) - \frac{\mathbf{v}_2(h) \bullet \mathbf{n}_1(h)}{\|\mathbf{n}_1(h)\|^2}\, \mathbf{n}_1(h) \\ &= 4 \Big(\|\mathbf{c}^{(2)}\|^2 \dot{\mathbf{c}} - \frac{\mathbf{c}^{(3)} \bullet \mathbf{c}^{(2)}}{\|\mathbf{c}^{(2)}\|^2}\, \mathbf{c}^{(2)} + \mathbf{c}^{(3)} \Big)\, h^2 + \mathbf{o}(h^2) = 4\mathbf{c}^{(3)\perp} h^2 + \mathbf{o}(h^2) \end{aligned}$$

$$\mathbf{n}_2(h) := \frac{\mathbf{N}_2(h)}{\|\mathbf{N}_2(h)\|} = \frac{\mathbf{c}^{(3)\perp}}{\|\mathbf{c}^{(3)\perp}\|} + \mathbf{o}(h^0)\,.$$

In case of high codimension $N \geq 3$, we wish to extend the previous formulas to the higher normals. For this purpose, the curve $\mathbf{c}$ is said to be *smoothly turning at order* $j+1$, for $j \in \{1, \dots, N\}$, if $\mathbf{c}$ is of class C^{j+2} and at any point $s \in [a,b]$ the vectors $(\dot{\mathbf{c}}(s), \mathbf{c}^{(2)}(s), \dots, \mathbf{c}^{(j+1)}(s))$ are linearly independent. When $j = N$, the curve is said to be *smoothly turning*. If the curve $\mathbf{c}$ is closed, the same condition is required at any $s \in \mathbb{R}$, once the curve is extended by periodicity.

If a curve is smoothly turning, we set:

$$\mathbf{t} = \mathbf{n}_0 := \dot{\mathbf{c}}\,, \qquad \mathbf{n}_1 := \frac{\mathbf{c}^{(2)}}{\|\mathbf{c}^{(2)}\|}\,,$$

$$\mathbf{c}^{(j+1)\perp} := \mathbf{c}^{(j+1)} - \sum_{k=0}^{j-1} \big(\mathbf{c}^{(j+1)} \bullet \mathbf{n}_k\big)\, \mathbf{n}_k\,, \;\; \mathbf{n}_j := \frac{\mathbf{c}^{(j+1)\perp}}{\|\mathbf{c}^{(j+1)\perp}\|}\,, \;\; j = 2, \dots, N\,.$$

The Jordan frame $(\mathbf{t}, \mathbf{n}_1, \dots, \mathbf{n}_N)$ of the curve $\mathbf{c}$ satisfies the system:

$$\dot{\mathbf{t}} = \mathbf{k}_1\, \mathbf{n}_1\,, \quad \dot{\mathbf{n}}_1 = -\mathbf{k}_1\, \mathbf{t} + \mathbf{k}_2\, \mathbf{n}_2\,, \quad \dot{\mathbf{n}}_j = -\mathbf{k}_j\, \mathbf{n}_{j-1} + \mathbf{k}_{j+1}\, \mathbf{n}_{j+1} \tag{4.2}$$

for $j = 2, \dots, N-1$, where $\mathbf{k}_j$ is the j-th curvature of the curve at $\mathbf{c}(s)$.

The last equation $\dot{\mathbf{n}}_N = -\mathbf{k}_N\, \mathbf{n}_{N-1}$ holds true since the curve $\mathbf{c}$ is differentiable $(N+2)$-times at the point s. When $N = 2$, it agrees with the third Frenet-Serret equation, $\dot{\mathbf{b}} = -\boldsymbol{\tau}\, \mathbf{n}$. Since moreover the vectors $(\dot{\mathbf{c}}(s), \mathbf{c}^{(2)}(s), \dots, \mathbf{c}^{(N+1)}(s))$ are linearly independent, the last curvature $\mathbf{k}_N$ is always non-zero. More generally, if $\mathbf{c}$ is smoothly turning at order $j+1$, where $j < N$, only the first $j+1$ Jordan formulas in (4.2) are satisfied.

Following the notation from (4.1), for $k = 0, 1, \dots, N$ and for $h > 0$ small we define:

$$\mathbf{v}_k(h) := \begin{cases} \dfrac{\mathbf{c}(s+(k+1)h) - \mathbf{c}(s+(k-1)h)}{2h} & \text{if } k \text{ is even} \\[2ex] \dfrac{\mathbf{c}(s-(k+2)h) - \mathbf{c}(s-kh)}{2h} & \text{if } k \text{ is odd}\,. \end{cases} \tag{4.3}$$

By performing the Gram-Schmidt procedure to $(\mathbf{v}_0(h), \mathbf{v}_1(h), \dots, \mathbf{v}_N(h))$, we also denote as before

$$\mathbf{t}(h) = \mathbf{n}_0(h) := \frac{\mathbf{v}_0(h)}{\|\mathbf{v}_0(h)\|}\,,$$

$$\mathbf{N}_1(h) := \mathbf{v}_1(h) - \big(\mathbf{v}_1(h) \bullet \mathbf{t}(h)\big)\, \mathbf{t}(h)\,, \quad \mathbf{n}_1(h) := \frac{\mathbf{N}_1(h)}{\|\mathbf{N}_1(h)\|}$$

and for $j = 2, \dots, N$

$$\mathbf{N}_j(h) := \mathbf{v}_j(h) - \sum_{k=0}^{j-1} \big(\mathbf{v}_j(h) \bullet \mathbf{n}_k(h)\big)\, \mathbf{n}_k(h)\,, \quad \mathbf{n}_j(h) := \frac{\mathbf{N}_j(h)}{\|\mathbf{N}_j(h)\|}\,.$$

If $\mathbf{c}$ is a smoothly turning curve, and $(\mathbf{t}, \mathbf{n}_1, \dots, \mathbf{n}_N)$ is the Jordan frame of $\mathbf{c}$ at a given point $s \in]a,b[$, then we have:

$$\mathbf{t}(h) = \mathbf{t} + \mathbf{o}(1)\,, \quad \mathbf{n}_j(h) = \mathbf{n}_j + \mathbf{o}(1) \qquad \forall\, j = 1, \dots, N\,.$$

In general, the higher order coefficients of the expansions of the terms $\mathbf{n}_j(h)$ depend on the choice of the vectors $\mathbf{v}_k(h)$ we made in (4.3), and their existence requires more regularity on the curve $\mathbf{c}$.

4.2. **Discrete normals to polygonal curves.** Let P be a polygonal curve in $\mathbb{R}^{N+1}$, for which we follow the previous notation, and assume that P does not lay in a line segment of $\mathbb{R}^{N+1}$. For any $i = 1, \dots, m$, we let v_i^1 denote the first unit vector v_h, with $h > i$, such that $[v_h] \neq [v_i]$, so that the linearly independent vectors (v_i, v_i^1) span a 2-dimensional vector space $\Pi^2(P, v_i)$, that may be called the *discrete osculating* 2-*space* of P at v_i. We then choose the orthogonal direction to v_i^1 in $\Pi^2(P, v_i)$. Therefore, by the Gram-Schmidt procedure, we let

$$\mathbf{N}_1(P,i) := v_i - \left(v_i \bullet v_i^1\right) v_i^1 \,, \quad \mathbf{n}_1(P,i) := \frac{\mathbf{N}_1(P,i)}{\|\mathbf{N}_1(P,i)\|}$$

and consider the equivalence class $[\mathbf{n}_1(P,i)]$ in $\mathbb{RP}^N$. If P is closed, we trivially extend the notation by listing the vectors v_i in a cyclical way. If P is not closed and for some $i > 1$ there are no vectors v_h, with $h > i$, such that $[v_h] \neq [v_i]$, we let $[\mathbf{n}_1(P,i)] := [\mathbf{n}_1(P, i-1)]$.

In a similar way, if $N \geq 3$ we can define the *discrete j-th normal* of P, for each $j = 2, \dots, N-1$. We thus assume that P does not lay in an affine subspace of $\mathbb{R}^{N+1}$ of dimension lower than $j+1$. For any i, we choose v_i^1 as above. By iteration on $k = 2, \dots, j$, once we have defined $v_i^{k-1} = v_l$, we let v_i^k denote the first unit vector v_h, with $h > l$, such that $v_i^1, v_i^2, \dots, v_i^k$ are linearly independent. Therefore, the vectors $(v_i, v_i^1, v_i^2, \dots, v_i^j)$ span a $(j+1)$-dimensional vector space $\Pi^{j+1}(P, v_i)$, that may be called the *discrete osculating* $(j+1)$-*space* of P at v_i.

By means of the Gram-Schmidt procedure, we then choose the *orthogonal direction* $\mathbf{n}_j(P,i) \in \mathbb{S}^N$ *to* $(v_i^1, v_i^2, \dots, v_i^j)$ *in* $\Pi^{j+1}(P, v_i)$, and consider the equivalence class $[\mathbf{n}_j(P,i)]$. If P is closed, we trivially extend the notation by listing the vectors v_i in a cyclical way. If P is not closed and for some $i > 1$ there are no j vectors satisfying the linear independence as above, we let $[\mathbf{n}_j(P,i)] := [\mathbf{n}_j(P, i-1)]$.

Finally, if P does not lay in an affine subspace of $\mathbb{R}^{N+1}$ of dimension lower than N, the *last discrete normal* $[\mathbf{n}_N(P,i)]$ is given by the equivalence class of the *orthogonal directions to the discrete osculating N-space* $\Pi^N(P, v_i)$ of P at v_i.

Definition 4.1. With the previous notation, for any $j = 1, \dots, N$, we call *discrete j-th normal* of P the curve $[\mathbf{n}_j](P)$ in $\mathbb{RP}^N$ obtained by connecting $[\mathbf{n}_j(P,i)]$ with $[\mathbf{n}_j(P, i+1)]$ by means of a minimal geodesic arc in $\mathbb{RP}^N$, for each $i = 1, \dots, m$, and also $[\mathbf{n}_j(P,m)]$ with $[\mathbf{n}_j(P,1)]$, if P is closed.

When $N = 2$, i.e., for polygonal curves in $\mathbb{R}^3$, the last discrete normal $[\mathbf{n}_2](P)$ agrees with the discrete binormal analyzed in the previous section, whence

$$\mathcal{L}_{\mathbb{RP}^2}([\mathbf{n}_2](P)) = \mathrm{TAT}(P)\,.$$

On the other hand, the first discrete normal $[\mathbf{n}_1](P)$ is different from the weak normal from the previous section, where we exploited the polarity in the Gauss sphere $\mathbb{S}^2$.

The following convergence result implies that our notion of j-th normal to a polygonal curve is the discrete counterpart of the j-th normal to a smooth curve.

Theorem 4.2. *Let $\mathbf{c} : [a,b] \to \mathbb{R}^{N+1}$ a smoothly turning curve at order $j+1$, for some $j \in \{1, \dots, N\}$. Then, there exists a sequence $\{P_n\}$ of inscribed polygonals,* *with* mesh $P_n \to 0$*, such that the length* $\mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](P_n))$ *of the discrete* j*-th normal to* P_n *converges to the length* $\mathcal{L}_{\mathbb{S}^N}(\mathbf{n}_j)$ *of the* j*-th normal* $\mathbf{n}_j$ *to the curve* $\mathbf{c}$*, i.e.,*

$$\lim_{n\to\infty} \mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](P_n)) = \int_a^b \|\dot{\mathbf{n}}_j(s)\|\, ds\,.$$

We recall that by the Jordan formulas (4.2), for each $s \in]a,b[$ one has

$$\|\dot{\mathbf{n}}_j(s)\| = \sqrt{\mathbf{k}_j^2(s) + \mathbf{k}_{j+1}^2(s)}$$

if $j < N$, whereas for the last normal $\|\dot{\mathbf{n}}_N(s)\| = |\mathbf{k}_N(s)|$. Moreover, when $N = 2$, the last normal $\mathbf{n}_2$ and curvature $\mathbf{k}_2$ agree with the binormal and torsion of the curve $\mathbf{c}$ in $\mathbb{R}^3$, respectively.

4.3. **Total curvature estimates for the discrete normals.** Let $\mathbf{c}$ be a smoothly turning curve, so that the last Jordan equation $\dot{\mathbf{n}}_N = -\mathbf{k}_N\, \mathbf{n}_{N-1}$ holds, where the last curvature $\mathbf{k}_N$ is always non-zero. If $\mathbf{T}$ denotes the unit tangent vector to the curve $\mathbf{n}_N$ in $\mathbb{S}^N$, one has $\mathbf{T} = -\mathbf{n}_{N-1}$, whence by (4.2) we get $|\dot{\mathbf{T}}| = \sqrt{\mathbf{k}_{N-1}^2 + \mathbf{k}_N^2}$ and hence the total curvature of $\mathbf{n}_N$ is equal to the length of the $(N-1)$-th normal:

$$\mathrm{TC}(\mathbf{n}_N) = \mathcal{L}(\mathbf{n}_{N-1}) = \int_a^b \sqrt{\mathbf{k}_{N-1}^2(s) + \mathbf{k}_N^2(s)}\, ds\,.$$

If e.g. $N = 2$, then $\mathbf{n}_2 = \mathbf{b}$, $\mathbf{n}_1 = \mathbf{n}$, $\mathbf{k}_1 = \mathbf{k}$, and $\mathbf{k}_2 = \boldsymbol{\tau}$, and we thus get:

$$\mathrm{TC}(\mathbf{b}) = \mathcal{L}(\mathbf{n}) = \int_a^b \sqrt{\mathbf{k}^2(s) + \boldsymbol{\tau}^2(s)}\, ds\,.$$

We now consider polygonal curves P in $\mathbb{R}^{N+1}$. An analogous inequality concerning the discrete last curvature holds true, provided that P does not lay in an affine subspace of $\mathbb{R}^{N+1}$ of dimension lower than N, namely:

$$\mathrm{TC}([\mathbf{n}_N](P)) \le \mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_{N-1}](P)) + \mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_N](P))\,.$$

Moreover, referring to the first section for the notation, for any $1 \le j \le N-1$ and for μ_{j+1}-a.e. $p \in G_{j+1}\mathbb{R}^{N+1}$, the projection formulas

$$[\mathbf{n}_j](\pi_p(P)) = \widetilde{\eta}_p([\mathbf{n}_j](P))$$

hold, and for $j \ge 2$, also:

$$[\mathbf{n}_{j-1}](\pi_p(P)) = \widetilde{\eta}_p([\mathbf{n}_{j-1}](P))\,.$$

Using Proposition 2.7, we thus readily obtain for any $1 \le j \le N-1$

$$\mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](P)) = \int_{G_{j+1}\mathbb{R}^{N+1}} \mathcal{L}_{\mathbb{RP}_p^j}([\mathbf{n}_j](\pi_p(P)))\, d\mu_{j+1}(p)\,.$$

and for $j = 1$ also:

$$\mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_1](P)) \le \mathrm{TC}(P)\,. \tag{4.4}$$

Finally, using again Proposition 2.7, we are able to extend the total curvature estimate to the intermediate normals, namely:

$$\mathrm{TC}([\mathbf{n}_j](P)) \le \mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_{j-1}](P)) + \mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](P))$$

for every $j = 2, \dots, N$, whereas for $j = 1$

$$\mathrm{TC}([\mathbf{n}_1](P)) \le \mathcal{L}_{\mathbb{S}^N}(\mathbf{t}_P) + \mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_1](P)), \qquad \mathcal{L}_{\mathbb{S}^N}(\mathbf{t}_P) = \mathrm{TC}(P).$$

4.4. **Relaxed total variation of the normals to a curve.** We now introduce a relaxed notion of total variation of the j-th normal to a curve. Due to the lack of monotonicity, see Example 3.3, we make use again of the notion of modulus.

Definition 4.3. Let $\mathbf{c}$ be a curve in $\mathbb{R}^{N+1}$. The *relaxed total variation of the j-th normal* to $\mathbf{c}$ is given by

$$\mathcal{F}_j(\mathbf{c}) := \lim_{\varepsilon \to 0^+} \sup\{\mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](P)) \mid P \prec \mathbf{c}\,,\ \mu_c(P) < \varepsilon\} \qquad j = 1, \dots, N$$

where $[\mathbf{n}_j](P)$ is the discrete j-th normal to the inscribed polygonal P, see Definition 4.1.

Notice that when $N = 2$, the relaxed total variation of the last normal agrees with the notion of total absolute torsion for curves $\mathbf{c}$ in $\mathbb{R}^3$, namely

$$\mathcal{F}_2(\mathbf{c}) = \mathrm{TAT}(\mathbf{c}).$$

If $\mathcal{F}_j(\mathbf{c}) < \infty$ for some $j = 1, \dots, N$, one has $\sup_n \mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](P_n)) < \infty$ for any sequence $\{P_n\}$ of polygonal curves inscribed in $\mathbf{c}$ and satisfying $\mu_{\mathbf{c}}(P_n) \to 0$. Also, one can find an optimal sequence as above in such a way that $\mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](P_n)) \to \mathcal{F}_j(\mathbf{c})$ as $n \to \infty$. Moreover, by the observations that we made in the previous section, for any polygonal curve P in $\mathbb{R}^{N+1}$ we obtain:

$$\mathcal{F}_j(P) = \mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](P)) \qquad \forall\, j = 1, \dots, N$$

and hence we can re-write the integral-geometric formulas for polygonals as:

$$\mathcal{F}_j(P) = \int_{G_{j+1}\mathbb{R}^{N+1}} \mathcal{F}_j(\pi_p(P))\, d\mu_{j+1}(p)\,, \qquad 1 \le j \le N-1\,. \tag{4.5}$$

However, in order to extend formula (4.5) to the relaxed total variation of the normals to a curve $\mathbf{c}$, we cannot argue as for the total curvature, where one applies the monotone convergence theorem to a sequence of approximating polygonals with $P_n \prec P_{n+1} \prec \mathbf{c}$ for each n, compare e.g. [28, Prop. 4.1]. In fact, we have seen that the monotonicity property fails to hold.

Remark 4.4. For any curve $\mathbf{c}$ in $\mathbb{R}^{N+1}$, the relaxed total variation of the first normal is always lower than the total curvature:

$$\mathcal{F}_1(\mathbf{c}) \le \mathrm{TC}(\mathbf{c}). \tag{4.6}$$

In fact, if $\mathbf{c}$ has finite total curvature, one has:

$$\mathrm{TC}(\mathbf{c}) = \lim_{\varepsilon \to 0^+} \sup\{\mathrm{TC}(P) \mid P \prec \mathbf{c}\,,\ \mu_c(P) < \varepsilon\}$$

and hence (4.6) readily follows from (4.4). Notice moreover that in general, strict inequality holds in (4.6). In fact, for e.g. a polygonal curve P in $\mathbb{R}^2$, in the quantity $\mathcal{L}_{\mathbb{RP}^1}([\mathbf{n}_1](P))$ we take distances in the projective line, so that a contribution of $\mathrm{TC}(P)$ given by a turning angle θ greater than $\pi/2$, corresponds to a contribution $\pi - \theta$ for the length of $[\mathbf{n}_1](P)$.

4.5. **Weak normals to non-smooth curves.** For curves $\mathbf{c}$ in $\mathbb{R}^{N+1}$ such that $\mathcal{F}_j(\mathbf{c}) < \infty$, we obtain a weak notion of j-th normal.

Theorem 4.5. *Let $2 \le j \le N$, and let $\mathbf{c}$ a curve in $\mathbb{R}^{N+1}$ such that $\mathcal{F}_j(\mathbf{c}) < \infty$ and $\mathcal{F}_{j-1}(\mathbf{c}) < \infty$. There exists a rectifiable curve $[\mathbf{n}_j](\mathbf{c}) : [0, L_j] \to \mathbb{RP}^N$ parameterized by arc-length, where $L_j := \mathcal{F}_j(\mathbf{c})$, so that*

$$\mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](\mathbf{c})) = \mathcal{F}_j(\mathbf{c})$$

satisfying the following property. For any sequence $\{P_n\}$ of inscribed polygonal curves, let $\boldsymbol{\gamma}_n^j : [0, L_j] \to \mathbb{RP}^N$ denote for each n the parameterization with constant velocity of the discrete j-th normal $[\mathbf{n}_j](P_n)$ to P_n, see Definition 4.1. If $\mu_{\mathbf{c}}(P_n) \to 0$, then $\boldsymbol{\gamma}_n^j \to [\mathbf{n}_j](\mathbf{c})$ uniformly on $[0, L_j]$ and

$$\mathcal{L}_{\mathbb{RP}^N}(\boldsymbol{\gamma}_n^j) = \mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](P_n)) \to \mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](\mathbf{c}))$$

as $n \to \infty$, where, we recall, $\mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](P_n)) = \mathcal{F}_j(P_n)$. Moreover, the arc-length derivative of the curve $[\mathbf{n}_j](\mathbf{c})$ is a function of bounded variation. Finally, in the case $j = 1$, for any curve $\mathbf{c}$ in $\mathbb{R}^{N+1}$ satisfying $\mathrm{TC}(\mathbf{c}) < \infty$, one has $\mathcal{F}_1(\mathbf{c}) < \infty$ and the same conclusion as above holds true.

The curve $[\mathbf{n}_j](\mathbf{c})$ in Theorem 4.5 may be called *weak j-th normal* to the curve $\mathbf{c}$. In fact, under the hypotheses of Theorem 4.5, a continuity property holds: *for any sequence $\{P_n\}$ of inscribed polygonals satisfying $\mu_{\mathbf{c}}(P_n) \to 0$*

$$\lim_{n\to\infty} \mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](P_n)) = \mathcal{F}_j(\mathbf{c})\,.$$

Moreover, by Theorem 4.5 we get the integral-geometric formula:

$$\mathcal{F}_j(\mathbf{c}) = \int_{G_{j+1}\mathbb{R}^{N+1}} \mathcal{F}_j(\pi_p(\mathbf{c})))\, d\mu_{j+1}(p)\,.$$

In particular, if $\mathbf{c} : [a, b] \to \mathbb{R}^{N+1}$ is a smoothly turning curve at order $j + 1$, we obtain:

$$\mathcal{F}_j(\mathbf{c}) = \int_a^b \|\dot{\mathbf{n}}_j(s)\|\, ds$$

where, we recall, $\|\dot{\mathbf{n}}_j(s)\| = \sqrt{\mathbf{k}_j^2(s) + \mathbf{k}_{j+1}^2(s)}$, when $j < N$, and $\|\dot{\mathbf{n}}_N(s)\| = |\mathbf{k}_N(s)|$, when $j = N$.

4.6. **Relationship with the smooth normals.** If $\mathbf{c} : [a, b] \to \mathbb{R}^{N+1}$ is a smoothly turning curve at order $j + 1$, the weak j-th normal $[\mathbf{n}_j](\mathbf{c})$ agrees (up to a lifting from $\mathbb{RP}^N$ to $\mathbb{S}^N$) with the arc-length parameterization of the smooth j-th normal $\mathbf{n}_j$ to $\mathbf{c}$. More precisely, recalling that $\Pi : \mathbb{S}^N \to \mathbb{RP}^N$ is the canonical projection, one has

$$[\mathbf{n}_j](\mathbf{c})(t) = \Pi(\mathbf{n}_j(\psi_j(t))) \qquad \forall\, t \in [0, L_j]$$

where $\psi_j : [0, L_j] \to [a, b]$ is the inverse of the bijective and C^1-class transition function

$$\varphi_j(s) := \int_a^s \|\dot{\mathbf{n}}_j(\lambda)\|\, d\lambda\,, \qquad s \in [a, b]\,. \tag{4.7}$$

Now, we have already noticed that existence of the osculating plane to a smooth curve $\mathbf{c}$ in $\mathbb{R}^3$ is guaranteed by the requirement that at each point s there exists a non-zero higher order derivative $\mathbf{c}^{(k)}(s)$. The continuity of the osculating plane $\Pi^2(\mathbf{c},s)$ as a function of arc-length parameter, indeed, ensures that the normal vector $\mathbf{n}$ (and hence the binormal vector $\mathbf{b}=\mathbf{t}\times\mathbf{n}$, too) is continuous when seen as a function in the projective plane $\mathbb{RP}^2$, compare Example 3.12.

In order to deal with high dimension osculating spaces, the analogous sufficient condition is existence of $j+1$ independent derivatives $\mathbf{c}^{(k)}(s)$ of the curve near each point $\mathbf{c}(s)$.

To this purpose, an open rectifiable curve $\mathbf{c}:[a,b]\to\mathbb{R}^{N+1}$, parameterized in arc-length, is called *mildly smoothly turning at order* $j+1$ if for each $s\in[a,b]$ the function $\mathbf{c}$ is of class C^m in a neighborhood of s, for some integer $m\geq j+2$, and there exist j integers $1<i_2<\ldots<i_{j+1}<m$ such that the $(j+1)$-vector $(\dot{\mathbf{c}}\wedge\mathbf{c}^{(i_2)}\wedge\cdots\wedge\mathbf{c}^{(i_{j+1})})(s)$ is non-trivial. When $j=N$, the curve is said to be *mildly smoothly turning.*

Extending the approach by Jordan [17], if $1<i_2<\ldots<i_{j+1}$ are the smallest integers such that the $(j+1)$-vector $(\dot{\mathbf{c}}\wedge\mathbf{c}^{(i_2)}\wedge\cdots\wedge\mathbf{c}^{(i_{j+1})})(s)$ is non-trivial, the j-th normal $\mathbf{n}_j(s)$ is defined by the last term in the Gram-Schmidt procedure to the ordered list of independent vectors $\dot{\mathbf{c}}$, $\mathbf{c}^{(i_2)},\ldots,\mathbf{c}^{(i_{j+1})}$ computed at s. If $\mathbf{c}$ is a mildly smoothly turning curve, we also set $\mathbf{n}_N:=*(\mathbf{t}\wedge\mathbf{n}_1\wedge\cdots\wedge\mathbf{n}_{N-1})$, where $*$ is Hodge operator.

The main feature is the existence and continuity of the osculating $(j+1)$-spaces along the curve. In fact, if a curve $\mathbf{c}$ is mildly smoothly turning at order $j+1$, we have:

(1) equipping the set of *unoriented* $(j+1)$*-planes* with the canonical metric, the osculating $(j+1)$-space $\Pi^{j+1}(\mathbf{c},s)$ is well-defined and continuous, as $s\in]a,b[$;
(2) there exists a finite set Σ of points in $]a,b[$ such that the $(j+1)$-vector $(\dot{\mathbf{c}}\wedge\mathbf{c}^{(2)}\wedge\cdots\wedge\mathbf{c}^{(j+1)})(s)$ is non-trivial on $]a,b[\setminus\Sigma$;
(3) the first j formulas in the Jordan system (4.2) are satisfied in each connected component of $]a,b[\setminus\Sigma$;
(4) the corresponding curvature terms $\mathbf{k}_h$ are continuous functions on $]a,b[$, that may possibly be equal to zero only at the singular points $s_i\in\Sigma$.

Now, at each point $s_i\in\Sigma$ the normals may be discontinuous. However, the continuity of the osculating $(j+1)$-space along the curve implies that

$$\mathbf{n}_k(s_i-)=\pm\mathbf{n}_k(s_i+)\qquad\forall\,k=1,\ldots,j$$

and hence *the first j unit normals are continuous when seen as a function into the projective space* $\mathbb{RP}^N$.

Moreover, it turns out that the osculating $(j+1)$-space function $s\mapsto\Pi^{j+1}(\mathbf{c},s)$ is of class $C^1(]a,b[)$, w.r.t. the canonical metric of unoriented $(j+1)$-spaces in $\mathbb{R}^{N+1}$. In addition, the curvature terms $\mathbf{k}_{j-1}$ and $\mathbf{k}_j$ are always non-zero on $]a,b[\setminus\Sigma$. We thus obtain:

$$\frac{\dot{\mathbf{n}}_j(s_i-)}{\|\dot{\mathbf{n}}_j(s_i-)\|}=\pm\frac{\dot{\mathbf{n}}_j(s_i+)}{\|\dot{\mathbf{n}}_j(s_i+)\|}\in\mathbb{S}^N$$

according to formula (3.9) from Example 3.12. Finally, if the curve is mildly smoothly turning, the last formula in the Jordan system (4.2) holds true, too, on $]a, b[\setminus\Sigma$.

We now extend the convergence result obtained in Theorem 4.2.

Proposition 4.6. *Let* $\mathbf{c}$ *be a mildly smoothly turning curve at order* $j+1$*, for some* $1 \le j \le N$*. Then there exists a sequence* $\{P_n\}$ *of inscribed polygonals, with* $\operatorname{mesh} P_n \to 0$*, such that*

$$\lim_{n\to\infty} \mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](P_n)) = \int_a^b \|\dot{\mathbf{n}}_j(s)\|\, ds\,.$$

Moreover, we have

$$\mathcal{F}_j(\mathbf{c}) = \int_a^b \|\dot{\mathbf{n}}_j(s)\|\, ds < \infty\,.$$

Finally, denoting by $\psi_j : [0, L_j] \to [a, b]$ *the inverse of the bijective and absolutely continuous transition function* (4.7)*, we obtain:*

$$[\mathbf{n}_j](\mathbf{c})(t) = \Pi(\mathbf{n}_j(\psi_j(t))) \qquad \forall\, t \in [0, L_j]$$

We finally remark that if a smooth curve fails to satisfy the previous linear independence property, then the osculating $(j+1)$-space fails to be continuous, in general.

Example 4.7. Let $f : \mathbb{R} \to \mathbb{R}$ be the C^∞ but not analytic function

$$f(x) \ := \ \begin{cases} e^{-1/x^2} & \text{if } \ x \neq 0 \\ 0 & \text{if } \ x = 0\,. \end{cases}$$

The function f has all derivatives vanishing in zero. Let us consider the curve $\gamma : [-1, 1] \to \mathbb{R}^3$ defined as

$$\gamma(t) \ := \ \begin{cases} \big(t,\, f(t),\, 0\big) & \text{if } \ t \le 0 \\ \big(t,\, 0,\, f(t)\big) & \text{if } \ t \ge 0\,. \end{cases}$$

It is smooth (C^∞), but all its derivatives vanish in zero, whence it does not satisfy the previous assumptions. The same is true if one considers a re-parametrization $\mathbf{c}$ of γ in arc-length.

Since for $t \le 0$ the curve lies in the plane $\pi_1 = \{z = 0\}$ and for $t \ge 0$ it lies in the plane $\pi_2 = \{y = 0\}$, the torsion of the curve is always zero, $\mathbf{b}$ is constant out of $t = 0$, and $\mathbf{b}$ and $\mathbf{n}$ jump of an angle of $\pi/2$ at $t = 0$. By modifying the plane π_2, it is immediate to find an example in which the curve has both the normal $\mathbf{n}$ and binormal $\mathbf{b}$ jumping of an arbitrary angle α at $t = 0$. Notice that since $\mathbf{t}$ is continuous and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, the jump angle α must be the same for both $\mathbf{n}$ and $\mathbf{b}$.

The previous example is easily adapted to curves in $\mathbb{R}^{N+1}$ having an arbitrary number of normals jumping of arbitrary angles. Notice, though, that since the last normal $\mathbf{n}_N$ is determined by the vectors $\mathbf{t}, \mathbf{n}_1, \ldots, \mathbf{n}_{N-1}$, the angle of jump of $\mathbf{n}_N$ is determined by those of the other normals.

4.7. **Curvature measures.** Similar arguments to the ones concerning the *torsion force*, can be repeated for the weak j-th normals of open curves. To this purpose, we recall that in Theorem 4.5, we also showed that the arc-length derivative of the curve $[\mathbf{n}_j](\mathbf{c})$ in $\mathbb{RP}^N$ is a function of bounded variation. For simplicity, we denote here by $\boldsymbol{\gamma}^j : [0, L_j] \to \mathbb{S}^N$ a continuous lifting of the curve $[\mathbf{n}_j](\mathbf{c})$, so that $\dot{\boldsymbol{\gamma}}^j$ is a function of bounded variation, with $\|\dot{\boldsymbol{\gamma}}^j\| \equiv 1$. Moreover, we have:

$$\mathcal{L}_{\mathbb{S}^N}(\boldsymbol{\gamma}^j) = \mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](\mathbf{c})) = \mathcal{F}_j(\mathbf{c}) .$$

We assume that $\boldsymbol{\gamma}^j_\varepsilon$ is a variation of $\boldsymbol{\gamma}^j$ under which the motion of each point $\boldsymbol{\gamma}^j(t)$ is smooth in time and with initial velocity $\xi(t)$, where $\xi : [0, L_j] \to \mathbb{R}^{N+1}$ is a Lipschitz continuous function with $\xi(0) = \xi(L_j) = 0$, so that $\dot{\xi}(t)$ is defined for a.e. t, by Rademacher's theorem.

Denoting by $D\dot{\boldsymbol{\gamma}}^j$ the finite measure given by the distributional derivative of $\dot{\boldsymbol{\gamma}}^j$, the first variation formula of the length of the curve $\boldsymbol{\gamma}^j$ gives:

$$\delta_\xi \mathcal{L}_{\mathbb{S}^N}(\boldsymbol{\gamma}^j) := \frac{d}{d\varepsilon}\, \mathcal{L}_{\mathbb{S}^N}(\boldsymbol{\gamma}^j_\varepsilon)_{|\varepsilon=0} = \int_0^{L_j} \dot{\boldsymbol{\gamma}}^j(t) \bullet \dot{\xi}(t)\, dt =: -\langle D\dot{\boldsymbol{\gamma}}^j, \xi\rangle . \tag{4.8}$$

If $\mathbf{c}$ is a polygonal curve P, the weak j-th normal agrees with the discrete j-th normal $[\mathbf{n}_j](P)$ from Definition 4.1, obtained by connecting the consecutive points $[\mathbf{n}_j(P, i)]$ with minimal geodesic arcs in $\mathbb{RP}^N$. Therefore, the arc-length derivative of the lifting $\boldsymbol{\gamma}^j$ has a discontinuity in correspondence eventually to the points $[\mathbf{n}_j(P, i)]$, where the norm of the jump is equal to the turning angle between the consecutive geodesic arcs meeting at $[\mathbf{n}_j(P, i)]$. Therefore, the total variation of the measure $D\dot{\gamma}^j$ is equal to the total curvature of the curve $\dot{\gamma}^j$ in $\mathbb{R}^{N+1}$, and hence to the sum $\mathcal{L}_{\mathbb{RP}^N}([\mathbf{n}_j](P)) + \mathrm{TC}_{\mathbb{RP}^N}([\mathbf{n}_j](P))$, where $\mathrm{TC}_{\mathbb{RP}^N}$ is the intrinsic total curvature of the curve in $\mathbb{RP}^N$.

Assume now that the curve $\mathbf{c}$ is smoothly turning at order $j+1$. Possibly considering the antipodal continuous lifted function of $[\mathbf{n}_j](\mathbf{c})$, by Proposition 4.6, for every $t \in [0, L_j]$ we have $\boldsymbol{\gamma}^j(t) = \mathbf{n}_j(\psi_j(t))$. Then, by changing variable $t = \varphi_j(s)$ we can write

$$\langle D\dot{\boldsymbol{\gamma}}^j, \xi\rangle = -\int_a^b \dot{\boldsymbol{\gamma}}^j(\varphi_j(s)) \bullet \frac{d}{ds}\,[\xi(\varphi_j(s))]\, ds$$

and hence, using that

$$\dot{\boldsymbol{\gamma}}^j(t) = \frac{\dot{\mathbf{n}}_j(s)}{\|\dot{\mathbf{n}}_j(s)\|}\,, \qquad t = \varphi_j(s) \tag{4.9}$$

and integrating by parts, since $\xi(\varphi_j(a)) = \xi(\varphi_j(b)) = 0$ we obtain:

$$\langle D\dot{\boldsymbol{\gamma}}^j, \xi\rangle = -\int_a^b \frac{\dot{\mathbf{n}}_j(s)}{\|\dot{\mathbf{n}}_j(s)\|} \bullet \frac{d}{ds}\,[\xi(\varphi_j(s))]\, ds = \int_a^b \frac{d}{ds}\, \frac{\dot{\mathbf{n}}_j(s)}{\|\dot{\mathbf{n}}_j(s)\|} \bullet \xi(\varphi_j(s))\, ds .$$

Therefore, the function $\dot{\boldsymbol{\gamma}}^j$ is of class $C^1(]a, b[)$, and the distributional derivative of $\dot{\gamma}^j$ is an absolutely continuous measure

$$D\dot{\boldsymbol{\gamma}}^j = \varphi_{j\,\#}\mu_j\,, \qquad \mu_j := \frac{d}{ds}\, \frac{\dot{\mathbf{n}}_j(s)}{\|\dot{\mathbf{n}}_j(s)\|}\, \mathcal{L}^1 \llcorner\,]a, b[\tag{4.10}$$

given by the push forward of the measure μ_j by the function $t = \varphi_j(s)$.

In general, when $j < N$ the denominator $\|\dot{\mathbf{n}}_j\|$ in formula (4.9) involves two curvatures. Therefore, the explicit computation of the density of the measure μ_j involves five normals and four curvatures. We now consider in particular the simpler case of the last normal.

Example 4.8. When $j = N$, we recall the last two Jordan formulas:

$$\dot{\mathbf{n}}_{N-1} = -\mathbf{k}_{N-1}\,\mathbf{n}_{N-2} + \boldsymbol{\tau}\,\mathbf{n}_N\,, \qquad \dot{\mathbf{n}}_N = -\boldsymbol{\tau}\,\mathbf{n}_{N-1}$$

where we have denoted $\boldsymbol{\tau} := \mathbf{k}_N$, the last curvature (that is, the torsion, when $N = 2$, in which case the Frenet-Serret formulas give $\mathbf{n}_0 = \mathbf{t}$, $\mathbf{n}_1 = \mathbf{n}$, $\mathbf{k}_1 = \mathbf{k}$, and $\mathbf{n}_2 = \mathbf{b}$). Denoting by $\operatorname{sgn}\boldsymbol{\tau}$ the constant sign of the non-zero smooth function $\boldsymbol{\tau}(s)$, we thus obtain:

$$\begin{gathered}\frac{\dot{\mathbf{n}}_N(s)}{\|\dot{\mathbf{n}}_N(s)\|} = -\operatorname{sgn}\boldsymbol{\tau}\cdot\mathbf{n}_{N-1}(s)\,,\\ \frac{d}{ds}\,\frac{\dot{\mathbf{n}}_N(s)}{\|\dot{\mathbf{n}}_N(s)\|} = \operatorname{sgn}\boldsymbol{\tau}\cdot\Big(\mathbf{k}_{N-1}\,\mathbf{n}_{N-2} - \boldsymbol{\tau}\,\mathbf{n}_N\Big)(s)\,.\end{gathered}$$

Now, we restrict to consider tangential variations in formula (4.8), i.e., *we assume in addition that* $\xi(t) \in T_{\boldsymbol{\gamma}^j(t)}\mathbb{S}^N$ *for each* t. We correspondingly deduce that the *tangential component* $D^\top\boldsymbol{\gamma}^N$ of the measure $D\boldsymbol{\gamma}^N$ satisfies:

$$D^\top\boldsymbol{\gamma}^N = \operatorname{sgn}\boldsymbol{\tau}\,\cdot\varphi_{N\,\#}\big(\mathbf{k}_{N-1}\,\mathbf{n}_{N-2}\,d\mathcal{L}^1 \llcorner\,]a,b[\big)$$

where, we recall, $\varphi_N(s) := \int_a^s \|\dot{\mathbf{n}}_N(\lambda)\|\,d\lambda = \int_a^s |\boldsymbol{\tau}(\lambda)|\,d\lambda$.

If the curve $\mathbf{c}$ is mildly smoothly turning at order $j+1$, then the distributional derivative of the arc-length derivative of $[\mathbf{n}_j](\mathbf{c})$ is an absolutely continuous measure, and on account of (4.10) we get to:

$$D\frac{d}{dt}[\mathbf{n}_j](\mathbf{c}) = \varphi_j^{\#}\widetilde{\mu}_j\,, \qquad \widetilde{\mu}_j := \frac{d}{ds}\Big(\Pi\circ\frac{\dot{\mathbf{n}}_j(s)}{\|\dot{\mathbf{n}}_j(s)\|}\Big)\,\mathcal{L}^1 \llcorner\,]a,b[$$

a formula that makes sense by means of an isometric embedding of $\mathbb{RP}^N$ into some Euclidean space.

5. Intrinsic curvature of curves into Riemannian surfaces

In this section we collect our results from [24] concerning the intrinsic curvature of irregular curves supported in a Riemannian surface.

We thus let $\mathcal{M}$ be a smooth (at least of class C^3), closed, and compact immersed surface in $\mathbb{R}^{N+1}$, with $N \geq 2$. We remark that $\mathcal{M}$ is not assumed to be oriented, when $N \geq 3$.

5.1. **Total intrinsic curvature.** The *(intrinsic) rotation* $\mathbf{k}_{\mathcal{M}}(P)$ of a polygonal P in $\mathcal{M}$ is the sum of the turning angles between the consecutive geodesic arcs of P. The polygonal P is said to be inscribed in a curve $\mathbf{c} : [a,b] \to \mathcal{M}$ if P is obtained by choosing a partition $a \leq t_0 < t_1 < \cdots < t_m \leq b$ and connecting with geodesic segments the consecutive points $\mathbf{c}(t_i)$ of the curve. For a general curve $\mathbf{c}$ supported in $\mathcal{M}$, we shall denote by $\mathcal{P}_{\mathcal{M}}(\mathbf{c})$ the class of polygonals in $\mathcal{M}$ which are inscribed in $\mathbf{c}$. Also, if $\mathbf{c}$ is rectifiable (and parameterized in arc-length) the mesh of a polygonal P in $\mathcal{P}_{\mathcal{M}}(\mathbf{c})$ is equivalently given by the maximum of the length of the arcs of $\mathbf{c}$ bounded by the consecutive vertexes of P. Notice that one clearly has $\mathbf{k}_{\mathcal{M}}(P) \le \mathrm{TC}(P)$, and that the difference $\mathrm{TC}(P) - \mathbf{k}_{\mathcal{M}}(P)$ is equal to the sum of the integrals of the modulus of the normal curvature $\mathfrak{K}_n$ of the geodesic arcs of P.

If e.g. $\mathcal{M} = \mathcal{S}^N$, the unit hyper-sphere in $\mathbb{R}^{N+1}$, then $\mathfrak{K}_n \equiv -1$ and hence $\mathrm{TC}(P) = \mathbf{k}_{\mathcal{S}^2}(P) + \mathcal{L}(P)$. In general, by smoothness and compactness of $\mathcal{M}$, the normal curvature of geodesic arcs of $\mathcal{M}$ is uniformly bounded, and hence there exists a constant $c_{\mathcal{M}} > 0$ depending on $\mathcal{M}$ such that for each polygonal P in $\mathcal{M}$

$$\mathrm{TC}(P) \le \mathbf{k}_{\mathcal{M}}(P) + c_{\mathcal{M}} \cdot \mathcal{L}(P)\,.$$

The following property has been proved in [7].

Theorem 5.1. ([7, Thm. 3.4]) *Let $\mathbf{c}$ be a regular curve in $\mathcal{M}$ of class C^2, parameterized by arc-length. Then, for any sequence $\{P_n\} \subset \mathcal{P}_{\mathcal{M}}(\mathbf{c})$ such that* $\operatorname{mesh} P_n \to 0$*, one has*

$$\lim_{n\to\infty} \mathbf{k}_{\mathcal{M}}(P_n) = \int_{\mathbf{c}} |\mathfrak{K}_g|\, ds = \int_0^L |\mathfrak{K}_g(s)|\, ds\,.$$

As a consequence, for a curve $\mathbf{c}$ in $\mathcal{M}$, one is tempted to define its total intrinsic curvature as in the Euclidean case, i.e., as the supremum of the intrinsic rotation $\mathbf{k}_{\mathcal{M}}(P)$ computed among all the polygonals P in $\mathcal{P}_{\mathcal{M}}(\mathbf{c})$. However, as observed in [7], if $\mathcal{M}$ has positive sectional (Gauss) curvature, as e.g. $\mathcal{M} = \mathcal{S}^2$, the latter definition does not work. In fact, if $P, P' \in \mathcal{P}_{\mathcal{M}}(\mathbf{c})$, and P' is obtained by adding a vertex in $\mathbf{c}$ to the vertices of P, then the monotonicity inequality $\mathbf{k}_{\mathcal{M}}(P) \le \mathbf{k}_{\mathcal{M}}(P')$ holds true in general provided that $\mathcal{M}$ has non-positive sectional curvature. In fact, it relies on the fact that in this case the sum of the interior angles of a geodesic triangle of $\mathcal{M}$ is not greater than π, see [7, Lemma 4.1].

Example 5.2. If e.g. $\mathcal{M} = \mathcal{S}^2$, and $\mathbf{c}$ is a parallel which is not a great circle, then the opposite inequality $\mathbf{k}_{\mathcal{S}^2}(P) \ge \mathbf{k}_{\mathcal{S}^2}(P')$ holds, and for any $P \in \mathcal{P}_{\mathcal{S}^2}(\mathbf{c})$ one has $\mathbf{k}_{\mathcal{S}^2}(P) > \int_{\mathbf{c}} |\mathfrak{K}_g|\, ds$, see Example 2.9.

In order to overcome this drawback, the good intrinsic notion turns out to be the one proposed by S. B. Alexander and R. L. Bishop [2], that goes back to the one considered by Alexandrov-Reshetnyak [3]. For this purpose, compare e.g. [18], we recall that the *modulus* $\mu_{\mathbf{c}}(P)$ of a polygonal P in $\mathcal{P}_{\mathcal{M}}(\mathbf{c})$ is the maximum of the geodesic diameter of the arcs of $\mathbf{c}$ determined by the consecutive vertexes in P. For $\varepsilon > 0$, we also let

$$\Sigma_\varepsilon(\mathbf{c}) := \{P \in \mathcal{P}_{\mathcal{M}}(\mathbf{c}) \mid \mu_{\mathbf{c}}(P) < \varepsilon\}\,.$$

Definition 5.3. The *total intrinsic curvature* of a curve $\mathbf{c}$ in $\mathcal{M}$ is

$$\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) := \lim_{\varepsilon\to 0^+} \sup\{\mathbf{k}_{\mathcal{M}}(P) \mid P \in \Sigma_\varepsilon(\mathbf{c})\}\,.$$

Clearly, the above limit is equal to the infimum as $\varepsilon > 0$ of $\sup\{\mathbf{k}_{\mathcal{M}}(P) \mid P \in \Sigma_\varepsilon(\mathbf{c})\}$. Moreover, arguing as in [18, Prop. 2.1], for a polygonal P in $\mathcal{M}$ we always have $\mathrm{TC}_{\mathcal{M}}(P) = \mathbf{k}_{\mathcal{M}}(P)$. Also, since $\mathcal{M}$ is compact, a curve with finite total curvature $\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$ is rectifiable, too (cf. [18, Prop. 2.4]). Most importantly, making use of a result by Dekster [8], as a consequence of [18, Prop. 2.4] one obtains:

Proposition 5.4. *The total curvature* $\mathrm{TC}_{\mathcal{M}}(\mathbf{c})$ *of any curve* $\mathbf{c}$ *in* $\mathcal{M}$ *is equal to the limit of the rotation* $\mathbf{k}_{\mathcal{M}}(P_h)$ *of* any *sequence of polygonals* $\{P_h\} \subset \mathcal{P}_{\mathcal{M}}(\mathbf{c})$ *such that* $\mu_{\mathbf{c}}(P_h) \to 0$.

Proposition 5.4 is proved in [3, Thm. 6.3.2], when $\mathcal{M} = \mathcal{S}^2$, and in [7, Prop. 4.3], when $\mathcal{M}$ has non-positive Gauss curvature. The proof for general smooth surfaces $\mathcal{M}$ is obtained by arguing as in [18, Prop. 2.4], where it is firstly proved for curves in CAT(K) spaces. It suffices to observe that the Gauss curvature of $\mathcal{M}$ is bounded, provided that $\mathcal{M}$ is smooth and compact. A crucial step is the following result (cf. [3, Thm. 2.1.3]): if $\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$, for each $\varepsilon > 0$ there exists $\delta > 0$ such that if γ is an arc of $\mathbf{c}$ with geodesic diameter lower than δ, the length of γ is smaller than ε. As a consequence, if $\{P_n\} \subset \mathcal{P}_{\mathcal{M}}(\mathbf{c})$ is such that the modulus $\mu_{\mathbf{c}}(P_n) \to 0$, then also $\operatorname{mesh} P_n \to 0$, the converse implication being trivial.

Proposition 5.4 fills the gap given by the lack of monotonicity observed e.g. in Example 5.2, yielding to the conclusion that Definition 5.3 involves a control on the modulus and not on the mesh, at least when the sectional curvature of $\mathcal{M}$ fails to be non-negative.

As a consequence, by Theorem 5.1 one infers that for smooth curves $\mathbf{c}$ in $\mathcal{M}$ one has $\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) = \int_{\mathbf{c}} |\mathfrak{K}_g| \, ds$. By [7, Cor. 3.6], for piecewise smooth curves $\mathbf{c}$ in $\mathcal{M}$ one similarly obtains that

$$\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) = \int_0^L |\mathfrak{K}_g(s)| \, ds + \sum_i |\alpha_i| \, . \tag{5.1}$$

In this formula, the integral is computed separately outside the corner points of $\mathbf{c}$, where the geodesic curvature $\mathfrak{K}_g$ is well-defined, and the second addendum denotes the finite sum of the absolute value of the oriented turning angles α_i between the incoming and outcoming unit tangent vectors at each corner point of $\mathbf{c}$. Therefore, for piecewise smooth curves we can rewrite formula (5.1) as

$$\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) = \int_0^L |\dot{\mathbf{t}} \bullet \mathbf{u}| \, ds + \sum_{s \in J_{\mathbf{t}}} d_{\mathbb{S}^N}(\mathbf{t}(s+), \mathbf{t}(s-)) \, .$$

For a curve $\mathbf{c}$ in $\mathcal{M}$, we clearly have $\mathrm{TC}_{\mathcal{M}}(c) \leq \mathrm{TC}(\mathbf{c})$, but it is false in general that if $\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$, then also $\mathrm{TC}(\mathbf{c}) < \infty$. If one e.g. takes a curve in $\mathcal{S}^2$ that winds around an equator infinitely many times, its total intrinsic curvature is zero but its length and total curvature are both infinite.

To this purpose, we recently found a flaw in [3, Thm. 6.3.1], where the authors erroneously stated that if the geodesic turn of a spherical curve is finite, then its spatial turn is also finite. This is true if the spherical diameter of the curve is smaller than a dimensional constant δ_0. In this case, in fact, for polygonal curves in $\mathcal{S}^2$ they obtain the inequality $\mathbf{k}^*(P) \leq \pi + 2\mathbf{k}_{\mathcal{S}^2}(P)$ between Euciledan and geodesic rotation. Therefore, their statement holds true provided that the curve can be divided in a finite number of arcs with spherical diameter smaller than δ_0. However, the latter property is false, in general, if the curve fails to be rectifiable, as the previous example shows.

Dealing with rectifiable curves $\mathbf{c}$ in $\mathcal{M}$, one instead has

$$\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) < \infty \iff \mathrm{TC}(\mathbf{c}) < \infty \, .$$

In fact, the normal curvature of geodesic arcs of $\mathcal{M}$ being uniformly bounded, we recover the nontrivial implication $\Rightarrow$ in the previous equivalence by arguing as in the model case $\mathcal{M}=\mathcal{S}^2$ considered in [3].

As a consequence, in the sequel we shall always assume that $\mathbf{c}:\overline{I}_L\to\mathcal{M}\subset\mathbb{R}^{N+1}$ is a rectifiable curve parameterized in arc-length, where $I_L=(0,L)$ and $L=\mathcal{L}(\mathbf{c})$.

If in addition $\mathrm{TC}_{\mathcal{M}}(\mathbf{c})<\infty$, moreover, the curve is *one-sidedly smooth* in the sense of [3, Sec. 3.1], i.e., it has a left and a right tangent $\mathbf{T}_\pm(s)$ at all the points $\mathbf{c}(s)$ in the so called "strong sense". This implies that for each $s\in[0,L[$ and $\delta>0$ we can find $\varepsilon>0$ such that any secant inscribed in the arc $\mathbf{c}_{|\,[s,s+\varepsilon]}$ forms with the straight line $\mathbf{T}_+(s)$ an angle less than δ, and similarly for the left tangent.

In addition, recalling that the tantrix $\mathbf{t}$ is a function of bounded variation, the weak conormal $\mathbf{u}\in\mathrm{BV}(I_L,\mathbb{S}^N)$ is well defined, and $\mathbf{u}(s)\in T_{\mathbf{c}(s)}\mathcal{M}$ for a.e. $s\in I_L$, and one has

$$D^C\mathbf{t}=\mathbf{u}\,(\mathbf{u}\bullet D^C\mathbf{t})$$

i.e., the Cantor component $D^C\mathbf{t}$ of the distributional derivative of the tantrix is tangential to $\mathcal{M}$.

5.2. **Weak parallel transport.** The following compactness property holds:

Theorem 5.5. *Let $\mathbf{c}$ be a rectifiable curve in $\mathcal{M}$ with finite total intrinsic curvature, parameterized by arc-length. Let $\{P_n\}\subset\mathcal{P}_{\mathcal{M}}(\mathbf{c})$ be such that the modulus $\mu_{\mathbf{c}}(P_n)\to 0$. For each n, let $P_n:[0,L]\to\mathcal{M}$ be parameterized with constant velocity, and let $X_n:I_L\to\mathbb{R}^3$ be the parallel transport along P_n, with constant initial condition $X_n(0)=\mathbf{t}(0)\in\mathbb{S}^N$. Then, possibly passing to a subsequence, the sequence $\{X_n\}$ strongly converges in $W^{1,1}$ to some function $X\in W^{1,1}(I_L,\mathbb{R}^{N+1})$ satisfying*

$$X(s)=\cos\Theta(s)\,\mathbf{t}(s)-\sin\Theta(s)\,\mathbf{u}(s)$$

for $\mathcal{L}^1$-a.e. $s\in I_L$, where $\mathbf{t}=\dot{\mathbf{c}}$ is the unit tangent vector, and the conormal $\mathbf{u}$ agrees with the weak-* BV-limit of the sequence $\{\mathbf{u}_n\}$ of conormals to a subsequence of $\{P_n\}$. Furthermore, $\mathbf{t}$ and $\mathbf{u}$ are functions in $\mathrm{BV}(I_L,\mathbb{S}^2)$, and the angle function Θ has bounded variation in $\mathrm{BV}(I_L)$.

In principle, the angle function Θ depends on the subsequence corresponding to the approximating sequence $\{P_n\}$. In order to overcome this drawback, we introduce the *energy functional*:

$$\mathcal{F}(\mathbf{t}):=\int_0^L|\dot{\mathbf{t}}\bullet\mathbf{u}|\,ds+|D^C\mathbf{t}|(I_L)+\sum_{s\in J_{\mathbf{t}}}d_{\mathbb{S}^N}(\mathbf{t}(s+),\mathbf{t}(s-)) \tag{5.2}$$

where, we recall, $\dot{\mathbf{t}}\bullet\mathbf{u}$ is the tangential component of the differential of the tantrix $\mathbf{t}:=\dot{\mathbf{c}}$, so that $|\dot{\mathbf{t}}|\geq|\dot{\mathbf{t}}\bullet\mathbf{u}|$, as $\mathcal{M}$ is "curved". Therefore, by (2.1) we clearly have $\mathcal{F}(\mathbf{t})\leq\mathrm{Var}_{\mathbb{S}^N}(\mathbf{t})$, where strict inequality holds in general.

Remark 5.6. In Theorem 5.5, we may and do assume that at each Jump point $s\in J_\Theta$, the Jump

$$[\Theta]_s:=\Theta(s+)-\Theta(s-)$$

is bounded by π, i.e., $|[\Theta]_s|\leq\pi$.

The *optimal* angle function Θ this way obtained is essentially unique, and hence the parallel transport X along irregular curves $\mathbf{c}$ with finite total curvature is well-defined in the $W^{1,1}$ setting. In fact, we have

Theorem 5.7. *Under the hypotheses of Theorem* 5.5*, and on account of Remark* 5.6*, we have*

$$|D\Theta|(I_L) = \mathcal{F}(\mathbf{t}) .$$

More precisely, by decomposing $|D\Theta|(I_L)$ *we have:*

$$|D^a\Theta|(I_L) = \int_0^L |\dot{\mathbf{t}} \bullet \mathbf{u}| \, ds \,, \quad |D^C\Theta|(I_L) = |D^C\mathbf{t}|(I_L) \,,$$
$$|D^J\Theta|(I_L) = \sum_{s \in J_{\mathbf{t}}} d_{\mathbb{S}^N}(\mathbf{t}(s+), \mathbf{t}(s-)) .$$

5.3. **Gauss-Bonnet theorem.** Gauss-Bonnet formula holds true in the setting of domains in $\mathcal{M}$ bounded by simple and closed curves with finite total curvature:

Theorem 5.8. *Let* $\mathcal{M}$ *be a smooth, closed, compact, and immersed surface in* $\mathbb{R}^{N+1}$*, where* $N \geq 3$*. Let* $\mathbf{c} : [0, L] \to \mathcal{M}$ *be a simple and closed rectifiable curve with finite total curvature,* $\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$*. Let* $k(s)\, ds := D\Theta[0, s)$*, where* Θ *is the left-continuous representative of the optimal angle function of the parallel transport along* $\mathbf{c}$*, see Theorems* 5.5 *and* 5.7*, so that*

$$\int_0^L k(s)\, ds = \Theta(L) - \Theta(0) .$$

Let U *be the open set in* $\mathcal{M}$ *enclosed by the oriented curve* $\mathbf{c}$*. Moreover, assume that* U *is simply connected, and that for a.e.* $s \in I_L$ *the tangent vector* $\mathbf{t}(s)$ *is positively oriented w.r.t. the natural orientation on the boundary of* U *at* $\mathbf{c}(s)$*. Finally, let* $\mathbf{K}$ *denote the Gauss curvature of* $\mathcal{M}$*, and* α *the oriented angle from* $\mathbf{t}(L-)$ *to* $\mathbf{t}(0+)$ *at the junction point* $\mathbf{c}(0) = \mathbf{c}(L)$*. Then we have:*

$$\int_U \mathbf{K}\, dA = 2\pi - \int_0^L k(s)\, ds - \alpha .$$

Notice that if $\mathbf{c}$ is smooth, we know that $D\Theta = \dot{\Theta}\, \mathcal{L}^1$, with $\dot{\Theta}(s) = \mathfrak{K}_g(s)$ for each s, so that we recover the classical formula, as $\int_0^L k(s)\, ds = \int_{\mathbf{c}} \mathfrak{K}_g(s)\, ds$. In a similar way one may proceed in the case of piecewise smooth curves, this time obtaining an extra term given by the sum of the oriented turning angles at the corner points of $\mathbf{c}$, in correspondence to the Jump points of the angle function Θ in I_L, plus a possible extra term at the junction point $\mathbf{c}(0) = \mathbf{c}(L)$. Therefore, our Theorem 5.8 extends the classical Gauss-Bonnet theorem to the wider class of curves with finite total curvature.

If $\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) = \infty$, in fact, we expect that there is no way to find a finite measure that contains the information (given by the derivative $D\Theta$ of the angle function of the parallel transport along the curve) on the "signed geodesic curvature" of the curve $\mathbf{c}$.

Finally, a more general result could be obtained if U fails to be simply-connected, assuming $\mathcal{M}$ oriented. This time, the term $2\pi \cdot \chi(U)$ appears, $\chi(U)$ being the Euler-Poincaré characteristic of U.

5.4. **Representation formula.** By the sequential lower-semicontinuity of the total variation w.r.t. the weak-* convergence, in Theorem 5.5 (that holds true for curves contained in surfaces $\mathcal{M}$ of $\mathbb{R}^{N+1}$) we only have

$$|D\Theta|(I_L) \leq \lim_{h\to\infty} |D\Theta_h|(I_L) = \lim_{h\to\infty} \mathbf{k}_{\mathcal{M}}(P_h) = \mathrm{TC}_{\mathcal{M}}(\mathbf{c})$$

where the last equality follows from Proposition 5.4. As a consequence, by Theorem 5.7 we obtain:

$$\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) \geq \mathcal{F}(\mathbf{t}) \tag{5.3}$$

where $\mathcal{F}(\mathbf{t})$ is the energy functional given by (5.2), and we expect that equality holds in (5.3) in full generality.

In fact, for piecewise smooth and regular curves $\mathbf{c}$ in $\mathcal{M}$, one has:

$$\mathcal{F}(\mathbf{t}) = \int_0^L |\mathfrak{K}_g(s)|\, ds + \sum_i |\alpha_i|$$

so that it suffices to apply Theorem 5.1 and (5.1).

Remark 5.9. It is readily checked that *equality holds in* (5.3) *for convex or concave curves* with finite total intrinsic curvature, i.e., for simple and closed curves $\mathbf{c}$ such that the right-hand (or left-end) side region with boundary the trace of $\mathbf{c}$ is a geodesically-convex subset of $\mathcal{M}$. For non-closed curves, this means that all the length minimizing arcs connecting two points of the curve lie on the same side w.r.t. the tantrix of the curve. In this case, in fact, for any polygonal P_n in $\mathcal{M}$ inscribed in $\mathbf{c}$, the angle Θ_n of the parallel transport along P_n is a monotone function. Therefore, for each $(a,b)\subset I_L$ we have $|D\Theta_h|(a,b) = |\Theta_h(b-)-\Theta_h(a+)|$. The a.e. convergence of Θ_n to Θ, that holds true for a subsequence, yields that the angle Θ is a monotone function, too, whence $|D\Theta|(a,b) = |\Theta(b-)-\Theta(a+)|$. As a consequence, we obtain the strict convergence $|D\Theta_h|(I) \to |D\Theta|(I)$, which implies the equality sign in (5.3), on account of Theorem 5.7.

By exploiting (in Proposition 5.11) the generalized Gauss-Bonnet theorem 5.8, we are able to prove that equality holds in (5.3), even in the non trivial case of surfaces $\mathcal{M}$ with positive Gauss curvature.

Theorem 5.10. *Let $\mathcal{M}$ a smooth (at least of class C^3), closed, and compact (not necessarily oriented) immersed surface in $\mathbb{R}^{N+1}$. For every rectifiable curve $\mathbf{c}$ in $\mathcal{M}$ with finite total curvature,* $\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$,

$$\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) = \mathcal{F}(\mathbf{t})$$

where $\mathcal{F}(\mathbf{t})$ is given by (5.2) *and $\mathbf{t} = \dot{\mathbf{c}}$ is the tantrix of the curve.*

Theorem 5.10 holds true as a consequence of the following

Proposition 5.11. *Let $\mathbf{c} : [0,L] \to \mathcal{M}$ be a rectifiable curve with finite total curvature (parameterized by arc-length), and let Θ denote the left-continuous representative of the optimal angle of the parallel transport X along $\mathbf{c}$, with initial condition $X(0) = \mathbf{t}(0)$. Let $\{P_n\} \subset \mathcal{P}_{\mathcal{M}}(\mathbf{c})$ with modulus $\mu_{\mathbf{c}}(P_n) \to 0$. Assume that P_n is generated by the consecutive vertexes $\mathbf{c}(s_i)$, where $0 = s_0 < s_1 < \cdots < s_m = L$ (with $\{s_i\}$ and m depending on n), and that every s_i is not a Jump point of the* *angle function* Θ. *Also, let* Θ_n *denote the angle of the parallel transport* X_n *along* P_n, *with initial condition* $X_n(0) = \mathbf{t}(0)$. *Then, for* n *sufficiently large there exists a piecewise constant function* $\widetilde{\Theta}_n : I_L \to \mathbb{R}$ *such that:*

(*a*) *for each* $i = 1, \ldots, m$, *there exists a parameter* $\widetilde{s}_i \in [s_{i-1}, s_i[$ *such that*

$$\widetilde{\Theta}_h(s) = t_i \,\Theta(\widetilde{s}_i+) + (1 - t_i)\Theta(\widetilde{s}_i-)$$

for any $s \in]s_{i-1}, s_i[$, *where* $t_i \in [0, 1]$ *;*

(*b*) $\mathrm{Var}(\Theta_n) \le \mathrm{Var}(\widetilde{\Theta}_n) + \varepsilon_n$, *where* $\varepsilon_n \to 0^+$ *as* $n \to \infty$.

Proposition 5.11 is proved by exploiting Theorem 5.8, and it is based on the following *localization* result, which is illustrated in Figure 4.

Lemma 5.12. *Given any one-sidedly smooth curve* $\gamma : [0, L] \to \mathcal{M}$, *parameterized in arc length, there is* $\varepsilon_0 > 0$ *such that for any* $[a, b] \subset [0, L]$ *satisfying* $b - a < \varepsilon_0$ *we can find a simply-connected closed set* $\Omega \subset \mathcal{M}$ *for which* $\gamma([a, b]) \subset \Omega$ *and* $\gamma(a), \gamma(b) \in \partial\Omega$, *in such a way that the minimal geodesic arcs connecting any couple of points in the curve* $\gamma([a, b])$ *are contained in* Ω. *In particular, the geodesic arc connecting* $\gamma(a)$ *and* $\gamma(b)$ *divides* Ω *in two connected components.*

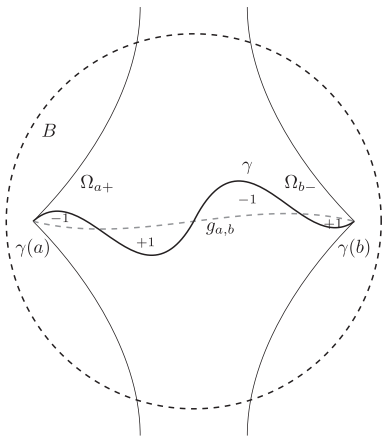

FIGURE 4. The simply-connected closed set $\Omega = B \cap \Omega_{a+} \cap \Omega_{b-}$ of Lemma 5.12. The arc γ is drawn with a continuous line, and the geodesic arc connecting $\gamma(a)$ and $\gamma(b)$ with a dashed line.

The argument outlined in Remark 2.2 is the starting point to takle the proof of Proposition 5.11 where, moreover, we have to consider the angle of the parallel transport, and to deal with the extra term given by the integral of the Gauss curvature.

In order to illustrate our strategy, for a planar curve $\mathbf{c}$ in $\mathbb{R}^2$, we thus denote by $\omega(s)$ the oriented angle from $\mathbf{t}(s)$ to the fixed direction $\mathbf{t}(0)$, where we choose $\mathbf{t}$ equal to the left-continuous representative of the BV-function $\dot{\mathbf{c}}$. We assume moreover that $P_n : [0, L] \to \mathbb{R}^2$ is parameterized with constant velocity on each interval $]s_{i-1}, s_i[$, in such a way that $P_n(s_i) = \mathbf{c}(s_i)$ for each i, and that every s_i is not a Jump point of $\mathbf{t}$.

If $\omega_h(s)$ is the oriented angle from $\mathbf{t}_n(s)$ to $\mathbf{t}(0)$, then $\omega_n(s)$ is constant on each interval $]s_{i-1}, s_i[$. In order to show that $\mathrm{Var}(\omega_h) \to \mathrm{Var}(\omega)$, by [27, Lemma 1] we may and do assume that $\mathbf{c}$ is a simple arc. Also, by Lemma 5.12 we can reduce to the following situation, for h large enough.

Denote by $\angle \mathbf{t}(s)\mathbf{v}_i$ the oriented angle from $\mathbf{t}(s)$ to $\mathbf{v}_i$, where $s \in [s_{i-1}, s_i[$, and $\mathbf{v}_i$ is the oriented segment of P_n from $\mathbf{c}(s_{i-1})$ to $\mathbf{c}(s_i)$. For $i = 1, \ldots, m$, letting $\alpha_i := \angle \mathbf{t}(s_{i-1})\mathbf{v}_i$, if $\alpha_i \neq 0$, we choose the *first* parameter $\overline{s}_i$ in the interval $]s_{i-1}, s_i]$ such that $\mathbf{c}(\overline{s}_i) \in \mathbf{v}_i$. Then, by Lemma 5.12, *the angle* $\overline{\beta}_i := \angle \mathbf{t}(\overline{s}_i)\mathbf{v}_i$ *cannot have the same sign as* α_i, i.e., $\alpha_i \cdot \overline{\beta}_i \leq 0$. Moreover, denoting by γ_i the oriented closed curve given by the join of the arc $\mathbf{c}_i := \mathbf{c}_{|[s_{i-1}, \overline{s}_i]}$ plus the segment of P_n from $\mathbf{c}(\overline{s}_i)$ to $\mathbf{c}(s_{i-1})$, *the index of* γ_i *on the open set* U_i *enclosed by* γ_i *is equal to the sign of* α_i, see Figure 4. Whence:

$$\omega(\overline{s}_i) - \omega(s_{i-1}) = \alpha_i - \overline{\beta}_i\,, \qquad \alpha_i \neq 0\,, \quad \alpha_i \cdot \overline{\beta}_i \leq 0\,.$$

Letting now $f_i(s) := \omega(s) - \omega(s_{i-1})$, we get $f_i(s_{i-1}) < \alpha_i$ and $f_i(\overline{s}_i) \geq \alpha_i$, when $\alpha_i > 0$ and $\overline{\beta}_i \leq 0$, whereas $f_i(s_{i-1}) > \alpha_i$ and $f_i(\overline{s}_i) \leq \alpha_i$, when $\alpha_i < 0$ and $\overline{\beta}_i \geq 0$. Therefore, using that ω is a function of bounded variation, we find $\widetilde{s}_i \in]s_{i-1}, \overline{s}_i[$ such that either $\alpha_i = t_i\, f_i(\widetilde{s}_i+) + (1-t_i) f_i(\widetilde{s}_i-)$ for some $t_i \in [0,1]$, if $\widetilde{s}_i$ is a Jump point of f_i, or $\alpha_i = f_i(\widetilde{s}_i)$, otherwise. When $\alpha_i = 0$, we clearly have $\alpha_i = f_i(0)$.

Recall that $\omega(s_0) = 0$ and $\alpha_i := \angle \mathbf{t}(s_{i-1})\mathbf{v}_i$. Setting $\beta_i := \angle \mathbf{t}(s_i)\mathbf{v}_i$, by the previous discussion based on Lemma 5.12, we also get:

$$\omega(s_j) - \omega(s_{j-1}) = \alpha_j - \beta_j \qquad \forall\, j = 1, \ldots, m\,.$$

Moreover, for $j = 1, \ldots, m-1$, the oriented turning angle of the polygonal P_n at the corner point $\mathbf{c}(s_j)$ is equal to $\alpha_{j+1} - \beta_j$. We thus have $\omega_n(s) = \alpha_1$ if $s \in]s_0, s_1[$, whereas if $s \in]s_{i-1}, s_i[$, and $i = 2, \ldots, m$, then

$$\begin{aligned} \omega_h(s) = &\quad \alpha_1 + \sum_{j=1}^{i-1} (\alpha_{j+1} - \beta_j) = \alpha_i + \sum_{j=1}^{i-1} (\alpha_j - \beta_j) \\ = &\quad \alpha_i + \sum_{j=1}^{i-1} (\omega(s_j) - \omega(s_{j-1})) = \alpha_i + \omega(s_{i-1})\,. \end{aligned}$$

We thus conclude that for each $i = 1, \ldots, m$ there exists $\widetilde{s}_i \in [s_{i-1}, s_i[$ and $t_i \in [0,1]$ such that

$$\omega_h(s) = t_i\, \omega(\widetilde{s}_i+) + (1-t_i)\, \omega(\widetilde{s}_i-) \qquad \forall\, s \in]s_{i-1}, s_i[\,.$$

The above property, that actually expresses the parallelism condition in terms of angle functions, implies that ω_m is a competitor to the computation of the essential variation of ω, whence $\mathrm{Var}(\omega_n) \leq \mathrm{Var}(\omega)$. By the weak-* BV convergence of ω_n to ω, which ensures that $\mathrm{Var}(\omega) \leq \liminf_n \mathrm{Var}(\omega_n)$, we obtain the strict convergence $\mathrm{Var}(\omega_n) \to \mathrm{Var}(\omega)$.

5.5. **Curves into Riemannian surfaces.** Our previous results extend the more general case of curves into Riemannian surfaces, i.e., 2-dimensional Riemannian manifolds $(\widetilde{\mathcal{M}}, g)$. We assume that $\widetilde{\mathcal{M}}$ is smooth (at least of class C^3), closed, and compact. Recall that we can always find a smooth isometric embedding $F : \widetilde{\mathcal{M}} \hookrightarrow \mathbb{R}^{N+1}$ of $\widetilde{\mathcal{M}}$ into a surface $\mathcal{M} = F(\widetilde{\mathcal{M}})$ immersed in the $(N+1)$-dimensional

Euclidean space, for some $N \geq 3$. Since the total intrinsic curvature of piecewise smooth curves involves the geodesic curvature and the turning angles at corner points, we do not need $\widetilde{\mathcal{M}}$ to be oriented.

We first extend Definition 5.3, by saying that the *total intrinsic curvature* of any curve γ in $\widetilde{\mathcal{M}}$ is

$$\mathrm{TC}_{\widetilde{\mathcal{M}}}(\gamma) := \lim_{\varepsilon \to 0^+} \sup\{\mathbf{k}_{\widetilde{\mathcal{M}}}(\widetilde{P}) \mid \widetilde{P} \in \Sigma_\varepsilon(\gamma)\}$$

where $\Sigma_\varepsilon(\gamma)$ is the class of polygonals $\widetilde{P}$ in $\widetilde{\mathcal{M}}$ inscribed in γ and with modulus $\mu_\gamma(\widetilde{P}) < \varepsilon$, and $\mathbf{k}_{\widetilde{\mathcal{M}}}(\widetilde{P})$ is the rotation of $\widetilde{P}$, both modulus and rotation being defined as in the case of surfaces $\mathcal{M}$ in $\mathbb{R}^{N+1}$.

Theorem 5.13. *For every curve γ in $\widetilde{\mathcal{M}}$ with finite total intrinsic curvature, we have*

$$\mathrm{TC}_{\widetilde{\mathcal{M}}}(\gamma) = \mathcal{F}(\mathbf{t})$$

where the energy functional $\mathcal{F}(\mathbf{t})$ is defined by (5.2) *in correspondence to the tangent indicatrix $\mathbf{t} = \dot{\mathbf{c}}$ of $\mathbf{c} = F \circ \gamma$, and F is any isometric embedding of $\widetilde{\mathcal{M}}$ as above.*

In order to prove Theorem 5.13, following e.g. [10, Sec. 4.12], on small open domains U of $\widetilde{\mathcal{M}}$ homeomorphic to a disk, we introduce the *geodesic polar coordinates* $ds^2 = dr^2 + g(r,\phi)\, d\phi^2$, where g is a non-negative smooth function on U. We shall denote by $f_{,r}$, $f_{,\phi}$, $f_{,rr}$, $f_{,r\phi}$, and $f_{,\phi\phi}$ the partial first and second derivatives of a function $f(r,\phi)$ on U. The coefficient g of the Riemannian metric satisfies

$$\lim_{r\to 0} g = 0\,, \quad \lim_{r\to 0}(\sqrt{g})_{,r} = 1 \qquad \forall\, \phi$$

compare [9, Sec. 4.6]. Also, in coordinates the non-trivial Christoffel coefficients of the Levi-Civita connection ∇_g of the Riemannian metric are

$$\Gamma^1_{22} = -\frac{1}{2}\, g_{,r}\,, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2g}\, g_{,r}\,, \quad \Gamma^2_{22} = \frac{1}{2g}\, g_{,\phi}\,.$$

Let $\gamma : I \to \widetilde{\mathcal{M}}$ be a smooth and regular curve parameterized by arc-length. Assume that $\gamma(\widetilde{I}) \subset U$ for some open interval $\widetilde{I} \subset I$. Also, we choose the pole of the coordinates not lying on the trace $\gamma(\widetilde{I})$ of the curve. Therefore, there exists a positive real constant c such that $g(r,\phi) \geq c > 0$ for every $(r,\phi) \in \gamma(\widetilde{I})$.

In coordinates, we thus have $\gamma(s) = (r(s), \phi(s))$ for some smooth functions $r(s)$ and $\phi(s)$ satisfying $\langle \dot\gamma(s), \dot\gamma(s)\rangle_g = \dot r^2 + g(r,\phi)\,\dot\phi^2 = 1$ for every $s \in \widetilde{I}$. Therefore, the unit tangent vector and unit conormal are

$$\dot\gamma = (\dot r, \dot\phi)\,, \qquad \dot\gamma^\perp := (-g^{1/2}\dot\phi, g^{-1/2}\dot r)\,.$$

The acceleration vector $\nabla_{\dot\gamma}\dot\gamma$ can be written in components as $(\nabla_{\dot\gamma}\dot\gamma)^k = \ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j$, for $k = 1, 2$, so that in the previous local coordinates we get

$$(\nabla_{\dot\gamma}\dot\gamma)^1 = \ddot r - \frac{1}{2}\, g_{,r}\, \dot\phi^2\,, \qquad (\nabla_{\dot\gamma}\dot\gamma)^2 = \ddot\phi + \frac{1}{g}\, g_{,r}\, \dot r\, \dot\phi + \frac{1}{2g}\, g_{,\phi}\, \dot\phi^2\,. \tag{5.4}$$

We have $\langle \nabla_{\dot\gamma}\dot\gamma, \dot\gamma\rangle_g = 0$, whence $\nabla_{\dot\gamma}\dot\gamma = \mathfrak{K}_g\,\dot\gamma^\perp$, where $\mathfrak{K}_g := \langle \nabla_{\dot\gamma}\dot\gamma, \dot\gamma^\perp\rangle_g$ is the geodesic curvature of γ, so that $|\mathfrak{K}_g| = |\nabla_{\dot\gamma}\dot\gamma|_g$. This yields to the local expression:

$$\begin{aligned}\mathfrak{K}_g = {} & \sqrt{g}\left[-\dot\phi\,(\nabla_{\dot\gamma}\dot\gamma)^1 + \dot r\,(\nabla_{\dot\gamma}\dot\gamma)^2\right] \\ = {} & \sqrt{g}\left[(\dot r\,\ddot\phi - \dot\phi\,\ddot r) + \frac{1}{2}\left(g_{,r}\,\dot\phi^3 + 2\,\frac{g_{,r}}{g}\,\dot r^2\,\dot\phi + \frac{g_{,\phi}}{g}\,\dot r\,\dot\phi^2\right)\right].\end{aligned} \tag{5.5}$$

Example 5.14. If e.g. $\widetilde{\mathcal{M}} = \mathcal{M} = \mathcal{S}^2$ and $g(r,\phi) = \sin^2 r$, with $r = \theta$ and $\phi = \varphi$, using that

$$\Gamma^1_{22} = -\sin\theta\cos\theta\,, \quad \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta\,, \quad \Gamma^2_{22} = 0$$

we recover the formula (2.13) for $\mathfrak{K}_g$.

Remark 5.15. We also recall that if ω denotes the angle between $\dot\gamma$ and the fixed direction $(1,0)$, we find

$$\tan\omega = \sqrt{g}\,\frac{\dot\phi}{\dot r}\,, \qquad \dot\omega = \mathfrak{K}_g - (\sqrt{g})_{,r}\,\dot\phi\,.$$

Therefore, if the curve γ parameterizes the positively oriented boundary of the smooth domain U, by Stokes theorem, compare [10, Sec. 4.12], one has

$$\oint_{\partial U} (\sqrt{g})_{,r}\,\dot\phi\,ds = -\int_U \mathbf{K}\,dA\,, \qquad \mathbf{K} = -\frac{1}{\sqrt{g}}\,(\sqrt{g})_{,rr}$$

where $\mathbf{K}$ is the Gauss curvature of $(\widetilde{\mathcal{M}}, g)$, yielding to the local formula of Gauss-Bonnet theorem:

$$\int_U \mathbf{K}\,dA = 2\pi - \oint_{\partial U} \mathfrak{K}_g\,ds\,.$$

Now, given an isometric embedding $F : \widetilde{\mathcal{M}} \hookrightarrow \mathcal{M} \subset \mathbb{R}^{N+1}$, we let $\overline{g}$ and $\overline{\nabla}$ denote the (Gaussian) metric and (Levi-Civita) connection induced by the Euclidean metric of $\mathbb{R}^{N+1}$ on $\mathcal{M}$. The pull-back of $\overline{g}$ and of $\overline{\nabla}$ through F agree with the metric g and Levi-Civita connection ∇_g on $\mathcal{M}$, respectively. Therefore, in local coordinates as above, writing $F = F(r,\phi) : U \to \mathbb{R}^{N+1}$, we have

$$F_{,r} \bullet F_{,r} = 1\,, \quad F_{,r} \bullet F_{,\phi} = 0\,, \quad F_{,\phi} \bullet F_{,\phi} = g\,. \tag{5.6}$$

By computing the partial second derivatives, we thus obtain the six formulas for the scalar products in $\mathbb{R}^{N+1}$

$$\begin{aligned} &F_{,r} \bullet F_{,rr} = 0\,, \quad F_{,r} \bullet F_{,r\phi} = 0\,, \qquad F_{,r} \bullet F_{,\phi\phi} = -\frac{1}{2}\,g_{,r}\,, \\ &F_{,\phi} \bullet F_{,rr} = 0\,, \quad F_{,\phi} \bullet F_{,r\phi} = \frac{1}{2}\,g_{,r}\,, \quad F_{,\phi} \bullet F_{,\phi\phi} = \frac{1}{2}\,g_{,\phi}\,. \end{aligned} \tag{5.7}$$

Letting $\mathbf{c}(s) := F \circ \gamma(s)$, where $s \in \widetilde{I}$, the unit tangent vector and conormal corresponding to $\dot\gamma$ and $\dot\gamma^\perp$ take the expression

$$\mathbf{t} = \dot r\,F_{,r} + \dot\phi\,F_{,\phi}\,, \qquad \mathbf{u} = -g^{1/2}\dot\phi\,F_{,r} + g^{-1/2}\dot r\,F_{,\phi}\,.$$

The curvature vector of the curve $\mathbf{c}$ in $\mathbb{R}^{N+1}$ then becomes

$$\mathbf{k} = \dot{\mathbf{t}} = \ddot r\,F_{,r} + \ddot\phi\,F_{,\phi} + \dot r^2\,F_{,rr} + 2\,\dot r\,\dot\phi\,F_{,r\phi} + \dot\phi^2\,F_{,\phi\phi}\,. \tag{5.8}$$

Computing the geodesic curvature of $\mathbf{c}$ in $\mathcal{M}$ through the formula $\mathfrak{K}_g := \dot{\mathbf{t}} \bullet \mathbf{u}$, by (5.6) and (5.7) we obtain:

$$\begin{aligned}\mathfrak{K}_g &= -g^{1/2}\,\dot\phi\Big(\ddot r + \dot\phi^2\Big(-\frac{1}{2}\,g_{,r}\Big)\Big) + g^{-1/2}\dot r\,\Big(g\,\ddot\phi + 2\,\dot r\,\dot\phi\,\Big(\frac{1}{2}\,g_{,r}\Big) + \dot\phi^2\Big(\frac{1}{2}\,g_{,\phi}\Big)\Big)\\ &= \sqrt{g}\,\Big[(\dot r\,\ddot\phi - \dot\phi\,\ddot r) + \frac{1}{2}\,\Big(g_{,r}\,\dot\phi^3 + 2\,\frac{g_{,r}}{g}\,\dot r^2\,\dot\phi + \frac{g_{,\phi}}{g}\,\dot r\,\dot\phi^2\Big)\Big]\end{aligned}$$

which agrees with the local expression (5.5) for the geodesic curvature of γ in $\widetilde{\mathcal{M}}$.

Remark 5.16. If γ is a geodesic in $\widetilde{\mathcal{M}}$, the curve $\mathbf{c} = F\circ\gamma$ is a geodesic in $\mathcal{M}$, whence the curvature vector $\dot{\mathbf{t}}$ is orthogonal to both $F_{,r}$ and $F_{,\phi}$. By (5.8), (5.6) and (5.7) we have

$$0 = \dot{\mathbf{t}} \bullet F_{,r} = \ddot r - \frac{1}{2}\,g_{,r}\,\dot\phi^2\,, \qquad 0 = \dot{\mathbf{t}} \bullet F_{,\phi} = g\,\ddot\phi + g_{,r}\,\dot r\,\dot\phi + \frac{1}{2}\,g_{,\phi}\,\dot\phi^2$$

and hence for a geodesic $\mathbf{c}$ one recovers the local expressions of the equations $\nabla_{\dot\gamma}\dot\gamma = 0$ from (5.4):

$$\ddot r = \frac{1}{2}\,g_{,r}\,\dot\phi^2\,, \qquad \ddot\phi = -\frac{1}{2g}\,\big(2g_{,r}\,\dot r\,\dot\phi + g_{,\phi}\,\dot\phi^2\big)\,. \tag{5.9}$$

Summing up, length, angles and geodesics are preserved by isometries, and the intrinsic local expression (5.5) does not depend on the choice of isometric embedding. In a similar way, one checks that the rotation of a polygonal $\widetilde{P}$ in $\widetilde{\mathcal{M}}$ is an intrinsic notion. As a consequence, we obtain:

Proposition 5.17. *For any piecewise smooth curve γ in $\widetilde{\mathcal{M}}$, we have*

$$\mathrm{TC}_{\widetilde{\mathcal{M}}}(\gamma) = \mathrm{TC}_{\mathcal{M}}(\mathbf{c}) \qquad \text{if} \qquad \mathbf{c} := F\circ\gamma$$

independently of the chosen isometric embedding F.

Moreover, all the previous results obtained for curves $\mathbf{c}$ in surfaces $\mathcal{M}$ of $\mathbb{R}^{N+1}$ extend to curves γ in a Riemannian surface $(\widetilde{\mathcal{M}}, g)$. In fact, it suffices to work with $\mathbf{c} = F\circ\gamma$ for any isometric embedding F, and to use standard arguments based on local geodesic coordinates and partition of unity.

We point out that a bit of care is needed when checking the validity of the compactness theorem 5.5. In fact, by a quick inspection it turns out that its proof is the unique point of the previous theory where we used non-intrinsic quantities. On account of Proposition 5.17 and Theorem 5.10, we finally conclude with the validity of Theorem 5.13.

5.6. **Development of curves.** The original idea of parallel transport by Tullio Levi-Civita involves the concept of *development* of a curve on a surface. If e.g. $\mathcal{M} = \mathcal{S}^2$, it corresponds to drawing in a plane the points of the trace of the oriented curve in $\mathcal{S}^2$ as the 2-sphere rolls without slipping or spinning in the plane, while staying tangent to the plane at the points of the curve. The above construction implies that the scalar curvature of the developed curve on $\mathbb{R}^2$ is equal to the modulus of the geodesic curvature of the given curve in $\mathcal{S}^2$, see Example 5.18.

We now wish to analyze the relationship between the definition of total intrinsic curvature and the notion of development of a smooth curve. We point out that similar arguments, based on considering iterations of the development of the "complete tangent indicatrix", are proposed by Reshetnyak [27] as a way to treat the "curvatures" of an irregular curve in $\mathbb{R}^{N+1}$.

Following e.g. [9], if $\gamma : I \to \mathcal{M}$ is a regular, smooth, and simple curve on a surface $\mathcal{M} \subset \mathbb{R}^3$, and $\dot{\mathbf{n}}(s) \neq 0$, where, we recall, $\mathbf{n}(s)$ is the unit normal $\mathbf{n}(s) := \dot\gamma(s)/\|\dot\gamma(s)\|$, then the *envelope of the tangent planes* is the ruled surface Σ parameterized by

$$X(s,v) := \gamma(s) + v\,\frac{\mathbf{n}(s) \times \dot{\mathbf{n}}(s)}{|\dot{\mathbf{n}}(s)|}$$

that in case $\mathcal{M} = \mathcal{S}^2$ clearly becomes $X(s,v) := \gamma(s) + v\,\mathbf{u}(s)$. Around the trace of the curve, the ruled surface Σ has zero Gauss curvature, and hence, by Minding's theorem, it is locally isometric to a planar domain. The parallel transport of tangent fields $X(s)$ along the curve is the same, when considering γ either as a curve on $\mathcal{M}$ or as a curve on Σ. In particular, when $X(s) = \mathbf{t}(s)$, one can use either local coordinates on $\mathcal{M}$ or on Σ in order to obtain the geodesic curvature $\mathfrak{K}_g$ of the curve γ. As a consequence, the parallel transport can be computed locally by pulling back the parallel transport along the development of the curve on the plane $\mathbb{R}^2$.

Moreover, we can define a tubular neighborhood (a strip) Σ of the envelope of the tangent planes to $\mathcal{M}$ along γ, in such a way that Σ is a surface with Gauss curvature equal to zero. As a consequence, the total curvature $\mathrm{TC}_\Sigma(\gamma)$ of γ as a curve in Σ is well-defined, according to Definition 5.3, by taking inscribed polygonals $\widetilde{P}$ in Σ with modulus sufficiently small (according to the width of the strip Σ, which actually depends on the maximum of the modulus of the geodesic curvature of the curve).

By means of the same vertexes as for $\widetilde{P}$, we may correspondingly consider the polygonal P in $\mathcal{M}$ inscribed in γ. However, in general the rotation of P in $\mathcal{M}$ is different from the rotation of $\widetilde{P}$ in Σ, i.e.,

$$\mathbf{k}_{\mathcal{M}}(P) \neq \mathbf{k}_\Sigma(\widetilde{P})\,.$$

In fact, if e.g. γ is a parallel of the 2-sphere $\mathcal{M} = \mathcal{S}^2$, and the vertexes of P are taken at equidistant points along γ, then the angles between $\widetilde{P}$ and γ are equal to the angles between the developed curve in $\mathbb{R}^2$ and the corresponding polygonal, whence they are smaller than the angles between P and γ.

Example 5.18. Following Example 2.9, if $\mathcal{M} = \mathcal{S}^2$ and $\gamma = \mathbf{c}_{\theta_0}$ is the parallel with constant co-latitude $\theta_0 \in]0,\pi/2]$, the geodesic polar coordinates on $\mathcal{S}^2$ give $g = \sin^2 r$, so that $r(s) \equiv \theta_0$ and $\phi(s) = s/\sin\theta_0$, where $s \in [0, 2\pi\,\sin\theta_0]$. The geodesic polar coordinates on Σ give instead $g = r^2$, whence $r(s) \equiv \tan\theta_0$ and $\phi(s) = \cot\theta_0 \cdot s$, where again $s \in [0, 2\pi\,\sin\theta_0]$. Therefore, the corresponding developed curve $\widetilde{\gamma}$ in $\mathbb{R}^2$ is the arc of a circle of radius $\tan\theta_0$ and length $2\pi\,\sin\theta_0$, i.e.,

$$\widetilde{\gamma}(s) = \tan\theta_0\,\bigl(\cos(\cot\theta_0\cdot s), \sin(\cot\theta_0\cdot s)\bigr)\,, \qquad s \in [0, 2\pi\,\sin\theta_0]\,.$$

The pointwise scalar curvature of $\widetilde{\gamma}$ is the reciprocal of the curvature radius of $\widetilde{\gamma}$, and hence it is equal to the pointwise geodesic curvature $\mathfrak{K}_g \equiv \cot\theta_0$ of the parallel $\mathbf{c} = \mathbf{c}_{\theta_0}$, whereas the total curvature of $\widetilde{\gamma}$ is equal to $2\pi\,\cos\theta_0$, i.e., to the total curvature $\mathrm{TC}_{\mathcal{S}^2}(\mathbf{c}_{\theta_0})$ of the parallel.

Notwithstanding, the total curvature $\mathrm{TC}_\Sigma(\gamma)$ of γ in the strip Σ can be computed by means of its development:

Proposition 5.19. *Let γ be a regular, smooth, and simple curve on a smooth surface $\mathcal{M} \subset \mathbb{R}^3$, with $\dot{\mathbf{n}} \neq 0$ everywhere. We have:*

$$\mathrm{TC}_\Sigma(\gamma) = \int_\gamma |\mathfrak{K}_g| \, ds \,.$$

Now, for any smooth curve γ as in Proposition 5.19, Theorem 5.1 says that the total curvature $\mathrm{TC}_\mathcal{M}(\gamma)$ agrees with the integral on the right-hand side of the previous formula, whence we get:

$$\mathrm{TC}_\mathcal{M}(\gamma) = \mathrm{TC}_\Sigma(\gamma) \,.$$

In particular, if $\{P_n\} \subset \mathcal{P}_\mathcal{M}(\gamma)$ satisfies $\mu_\gamma(P_n) \to 0$, and $\{\widetilde{P}_n\}$ is (for n large enough) the corresponding sequence of inscribed polygonals in Σ, even if in general one has $\mathbf{k}_\mathcal{M}(P_n) \neq \mathbf{k}_\Sigma(\widetilde{P}_n)$, we conclude that

$$\lim_{n\to\infty} \mathbf{k}_\mathcal{M}(P_n) = \lim_{n\to\infty} \mathbf{k}_\Sigma(\widetilde{P}_n) = \int_\gamma |\mathfrak{K}_g| \, ds \,.$$

References

[1] E. Acerbi and D. Mucci, *Curvature-dependent energies: a geometric and analytical approach*, Proc. Roy. Soc. Edinburgh **147A** (2017), 449–503.

[2] S. B. Alexander and R. L. Bishop, *The Fary-Milnor theorem in Hadamard manifolds*, Proc. Amer. Math. Soc. **126** (1998), 3427–3436.

[3] A. D. Alexandrov and Yu. G. Reshetnyak, *General theory of irregular curves*, Mathematics and its Applications, Soviet series, Kluwer Academic Publishers, Dordrecht, 1989.

[4] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variaton and free discontinuity problems*, Oxford University Press, Oxford, 2000.

[5] T. F. Banchoff, *Global geometry of polygons. III. Frenet frames and theorems of Jacobi and Milnor for space polygons*, Rad Jugoslav. Akad. Znan. Umjet., **396** (1982), 101–108.

[6] J. Cantarella, J. H. G. Fu, R. Kusner, J. M. Sullivan and N. C. Wrinkle, *Criticality for the Gehring link problem* Geom. Topol. **10** (2006), 2055–2116.

[7] M. Castrillón Lopez, V. Fernandéz Mateos and J. Muñoz Masqué, *Total curvatures of curves in Riemannian manifolds*, Diff. Geom. Appl. **28** (2010), 140–147.

[8] B. V. Dekster, *Upper estimates of the length of a curve in a Riemannian manifold with boundary*, J. Diff. Geom. **14** (1979), 149–166.

[9] M. P. do Carmo, *Differential geometry of curves & surfaces*, Revised and updated second edition, Dover Publications, Inc., Mineola, NY, 2016.

[10] B. Dubrovin, *Differential geometry*, Notes of the Lecture Course held at SISSA, Trieste. Web page: https://people.sissa.it/ dubrovin/

[11] K. Enomoto and J.-I. Itoh, *The total mixed curvature of open curves in* $\mathbb{E}^3$, Geom. Dedicata **194** (2018), 131–140.

[12] K. Enomoto, J.-I. Itoh and R. Sinclair, *The total absolute curvature of open curves in* $\mathbb{E}^3$, Illinois J. Math. **52** (2008), no. 1, 47–76.

[13] I. Fáry, *Sur la courbure totale d'une courbe gauche faisant un nœud*, Bull. Soc. Math. France **77** (1949), 128–138.

[14] W. Fenchel, *On the differential geometry of closed space curves*, Bull. Amer. Math. Soc. **57** (1951), 44–54.

[15] H. Gluck, *Higher curvatures of curves in Eucidean space*, Amer. Math. Monthly **73** (1966), 699–704.

[16] E. Gutkin, *Curvatures, volumes and norms of derivatives for curves in Riemannian manifolds*, J. Geom. Phys. **61** (2011), 2147–2161.
[17] C. Jordan, *Sur la théorie des courbes dans l'espace à n dimensions*, C. R. Acad. Sci. Paris **79** (1874), 795–797.
[18] C. Maneesawarng and Y. Lenbury, *Total curvature and length estimate for curves in CAT(K) spaces*, Diff. Geom. Appl. **19** (2003), 211–222.
[19] A. S. McRae, *The Milnor-Totaro theorem for space polygons*, Geom. Dedicata **84** (2001), 321–330.
[20] J. W. Milnor, *On the total curvature of knots,* Ann. of Math. **52** (1950), 248–257.
[21] D. Mucci and A. Saracco, *Bounded variation and relaxed curvature of surfaces*, Milan J. Math. **88** (2020), 191–223.
[22] D. Mucci and A. Saracco, *The weak Frenet frame of non-smooth curves with finite total curvature and absolute torsion*, Ann. Mat. Pura Appl. (4) **199** (2020), 2459–2488.
[23] D. Mucci and A. Saracco, *Weak curvatures of irregular curves in high dimension Euclidean spaces*, Annals of Global Analysis and Geometry **60** (2021), 181–216.
[24] D. Mucci and A. Saracco, *The total intrinsic curvature of curves in Riemannian surfaces* Rend. Circ. Mat. Palermo, II. Ser. **70** (2021), 521–557. *Erratum to:* Ibid. **70** (2021), 1137–1138.
[25] M. A. Penna, *Total torsion Amer. Math. Monthly* **87** (1980), 452–461.
[26] A. N. Pressley, *Elementary differential geometry*, Springer Verlag, London, 2001.
[27] Yu. G. Reshetnyak, *The theory of curves in differential geometry from the point of view of the theory of functions of a real variable*, Russian Math. Surveys **60** (2005), 1165–1181.
[28] J. M. Sullivan, *Curves of finite total curvature,* In: Discrete Differential Geometry (Bobenko, Schröder, Sullivan, and Ziegler, eds.), Oberwolfach Seminars, vol. 38, Birkäuser, 2008.
[29] B. Totaro, *Space curves with nonzero torsion*, Internat. J. Math. **1** (1990), no. 1, 109–117.

Manuscript received January 21 2022
revised December 2 2022

D. Mucci
Dipartimento di Scienze Matematiche, Fisiche ed Informatiche, Università di Parma, Parco Area delle Scienze 53/A, I-43124 Parma, Italy
E-mail address: *domenico.mucci@unipr.it*

A. Saracco
Dipartimento di Scienze Matematiche, Fisiche ed Informatiche, Università di Parma, Parco Area delle Scienze 53/A, I-43124 Parma, Italy
E-mail address: *alberto.saracco@unipr.it*