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WEAK QUASICONFORMAL MAPPINGS AND WEIGHTED POINCARÉ-SOBOLEV INEQUALITIES

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Abstract. In the article, we prove weighted Sobolev inequalities with weights which are Jacobians of mappings inverse to weak quasiconformal mappings. On this base, we obtain estimates of the first non-trivial eigenvalue of the two-dimensional Neumann-Laplacian in Hölder singular domains.

1. Introduction

The quasiconformal mapping theory originates as a generalization of the conformal mappings theory [2] and is the important part of the modern geometric functions theory [25]. In the two-dimensional case, the theory of quasiconformal mappings has numerous applications in the quasilinear elliptic equations [3] and in the continuum mechanics problems [4]. Unfortunately, in the space $\mathbb{R}^n$, $n \geq 3$, the quasiconformal mapping theory has no significant applications in the theory of PDEs. Because of it, were considered various generalizations of quasiconformal mappings, see for example, [21, 33].

The natural generalizations of quasiconformal mappings which arise in the Sobolev embedding theory [11] are weak (p,q)-quasiconformal mappings, $1 \leq q \leq p \leq \infty$. In the case $p = q = n$ they are usual quasiconformal mappings and are solutions of the Reshentyak Problem (1968). This problem connects composition operators on Sobolev spaces and quasiconformal mappings [36]. Recall that the Reshentyak Problem was initiated by previous works [24, 29]. The generalized Reshetnyak Problem was solved in [35]. In this work [35] were given necessary and sufficient conditions on homeomorphic mappings, which generate bounded embedding operators on Sobolev spaces $L^1_p(\widetilde{\Omega})$ and $L^1_q(\Omega)$. In the series of subsequent works the theory of weak (p,q)-quasiconformal mappings, $1 \leq q \leq p \leq \infty$, was founded, see, for example, [41, 42].

Recall that a homeomorphic mapping $\varphi : \Omega \to \widetilde{\Omega}$ is called the weak (p,q)-quasiconformal mapping [7, 35, 39], if $\varphi \in W^1_{q,\mathrm{loc}}(\Omega)$, has finite distortion and

$$K^p_p(\varphi;\Omega) = K^p_{p,p}(\varphi;\Omega) = \operatorname*{ess\,sup}_{\Omega} \frac{|D\varphi(x)|^p}{|J(x,\varphi)|} < \infty, \text{ if } 1 \leq q = p < \infty, \tag{1.1}$$

$$K^{\frac{pq}{p-q}}_{p,q}(\varphi;\Omega) = \int\limits_{\Omega} \left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}\right)^{\frac{q}{p-q}} dx < \infty, \text{ if } 1 \leq q < p < \infty. \tag{1.2}$$

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The special cases $1 \leq q < p = \infty$ and $q = n-1 < p = n$ were studied in details in [13].

Remark 1.1. In the recent work [1], the weak $(p, n-1)$-quasiconformal mappings were considered, but the article [1] repeats some results from works [13, 35, 41, 42]. In the our present work, we recall the basic properties of the weak quasiconformal mappings and we consider its applications to the weighted Poincaré-Sobolev inequalities.

In the frameworks of the geometric function theory generalizations of quasiconformal mappings base on (weighted) moduli inequalities and are called Q-mappings [25]. In the recent articles [10, 28], it was proved that the weak $(p, n-1)$-quasiconformal mappings are Q-homeomorphisms with $Q \in L_1(\Omega)$.

The weighted Poincaré-Sobolev inequalities represent the significant part of the geometric analysis of PDEs, see, for example, [26]. The applications of the weak (p,q)-quasiconformal mappings to the weighted Sobolev embedding theory base on weights which are generated by the weak quasiconformal geometry of domains. The weights which are Jacobians of (quasi)conformal mappings represent natural weights of the weighted Poincaré-Sobolev inequalities and were considered in [14, 15]. The weighted Poincaré-Sobolev inequalities with quasiconformal weights were considered in [20]. In the present paper we prove the weighted Poincaré-Sobolev inequalities with weights which are (inverse) Jacobians of weak (p,q)-quasiconformal mappings:

Let a bounded domain $\widetilde{\Omega} \subset \mathbb{R}^n$ be such that there exists a weak (p,q)-quasiconformal mapping $\varphi : \Omega \to \widetilde{\Omega}$, $1 \leq q \leq p < \infty$, where $\Omega \subset \mathbb{R}^n$ is a bounded (s,q)-Poincaré-Sobolev domain, and φ has the Luzin N-property if $p > n$. Then the weighted (s,p)–Poincaré-Sobolev inequality

$$\inf_{c \in \mathbb{R}} \left(\int\limits_{\widetilde{\Omega}} |f(y) - c|^s w(y)\, dy \right)^{\frac{1}{s}} \leq B^w_{s,p}(\widetilde{\Omega}) \left(\int\limits_{\widetilde{\Omega}} |\nabla f(y)|^p \, dy \right)^{\frac{1}{p}}$$

holds for any function $f \in W^1_p(\widetilde{\Omega})$ with the weight $w(y) = J_{\varphi^{-1}}(y)$, where $J_{\varphi^{-1}}(y)$ is a volume derivative of the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$.

In the second part of the article we consider applications of two-dimensional weak (p,q)-quasiconformal mappings, in the limit case $q = 1 < p = 2$, to the theory of elliptic operators. Spectral estimates of Neumann eigenvalues of the Laplace operator in non-convex domains are the long-standing complicated problem [30, 31]. The approach, which allows to obtain estimates in non-convex domains, is based on the geometric theory of composition operators on Sobolev spaces [41, 42] and was suggested in [16, 17]. In the present article, we give estimates of the first non-trivial eigenvalue of the Neumann-Laplacian in two-dimensional Hölder cusp domains:

Let $Q_\alpha \subset \mathbb{R}^2$ be the Hölder cusp domain. Then the first non-trivial Neumann eigenvalue of the Laplace operator satisfies

$$\mu_1(Q_\alpha) \geq \frac{4(\alpha+1)}{9\pi^3 \alpha}.$$

The applications of the weak (p,q)-quasiconformal mappings to the Sobolev extension theory can be found in the recent works [22, 43].

2. Sobolev spaces and weighted embedding theorems

2.1. **Sobolev spaces.** Let us recall the basic notions of Sobolev spaces and the change of variables formula. Let Ω be an open subset of the Euclidean space $\mathbb{R}^n$. The Sobolev space $W^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined [26] as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^1_p(\Omega)\| = \|f \mid L_p(\Omega)\| + \|\nabla f \mid L_p(\Omega)\|,$$

where ∇f is the weak gradient of the function f, i. e. $\nabla f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$. In accordance with the non-linear potential theory [27] we consider elements of Sobolev spaces $W^{1,p}(\Omega)$ as equivalence classes up to a set of p-capacity zero [26]. The Sobolev space $W^1_{p,\mathrm{loc}}(\Omega)$ is defined as a space of functions $f \in W^1_p(U)$ for every open and bounded set $U \subset \Omega$ such that $\overline{U} \subset \Omega$.

The homogeneous seminormed Sobolev space $L^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined as a space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L^1_p(\Omega)\| = \|\nabla f \mid L_p(\Omega)\|.$$

Let Ω and $\widetilde{\Omega}$ be domains in the Euclidean space $\mathbb{R}^n$. Then a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ belongs to the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega)$, if its coordinate functions belong to $W^1_{p,\mathrm{loc}}(\Omega)$. In this case, the formal Jacobi matrix $D\varphi(x)$ and its determinant (Jacobian) $J(x,\varphi)$ are well defined at almost all points $x \in \Omega$. We use $|D\varphi(x)|$ to denote the operator norm of $D\varphi(x)$.

Let us recall the change of variables formula for homeomorphisms $\varphi : \Omega \to \widetilde{\Omega}$. It can be found, for example, in [19]. Suppose that $\varphi : \Omega \to \widetilde{\Omega}$ is a homeomorphism. We define a volume derivative of the inverse mapping by

$$J_{\varphi^{-1}}(y) := \lim_{r\to 0} \frac{|\varphi^{-1}(B(y,r))|}{|B(y,r)|},$$

where $B(y,r)$ is a ball with a center at a point x and with a radius r. This function $J_{\varphi^{-1}}$ belongs to the space $L_{1,\mathrm{loc}}(\widetilde{\Omega})$ and, if $\varphi^{-1} \in W^1_{n,\mathrm{loc}}(\widetilde{\Omega})$, then this volume derivative coincides with the Jacobian a.e. in $\widetilde{\Omega}$: $J_{\varphi^{-1}}(y) = J(y,\varphi^{-1})$ for almost all $y \in \widetilde{\Omega}$ (see, for example, [18, 38]).

Let, in addition, the homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ possess the Luzin N^{-1}-property (the preimage of a set of measure zero has measure zero), then the following change of variables formula

$$\int\limits_{\Omega} f \circ \varphi(x)dx = \int\limits_{\widetilde{\Omega}} f(y)J_{\varphi^{-1}}(y)dy, \tag{2.1}$$

holds for every nonnegative measurable function $f : \widetilde{\Omega} \to \mathbb{R}$.

2.2. **Composition operators on Sobolev spaces.** Let Ω and $\widetilde{\Omega}$ be domains in the Euclidean space $\mathbb{R}^n$. Then a homeomorphism $\varphi:\Omega\to\widetilde{\Omega}$ generates a bounded composition operator

$$\varphi^{*}: L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega),\ \ 1\leq q\leq p\leq\infty,$$

by the composition rule $\varphi^*(f)=f\circ\varphi$, if for any function $f\in L^1_p(\widetilde{\Omega})$, the composition $\varphi^*(f)\in L^1_q(\Omega)$ is defined quasi-everywhere in Ω and there exists a constant $K_{p,q}(\varphi;\Omega)<\infty$ such that

$$\|\varphi^*(f)\mid L^1_q(\Omega)\|\leq K_{p,q}(\varphi;\Omega)\|f\mid L^1_p(\widetilde{\Omega})\|.$$

The analytic description of composition operators

$$\varphi^{*}: L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega),$$

was obtained in [35, 40] and in the limit case $p=\infty$ in [13]. This description is given in terms of integral characteristics of mappings of finite distortion and represents the solution of the generalized Reshetnyak Problem. Recall that a homeomorphism $\varphi:\Omega\to\widetilde{\Omega}$ of the class $W^1_{1,\mathrm{loc}}(\Omega)$ is a mapping of finite distortion if $D\varphi(x)=0$ for almost all x in $Z=\{x\in\Omega : J(x,\varphi)=0\}$ [38].

In the case of homeomorphisms $\varphi:\Omega\to\widetilde{\Omega}$ of the class $W^1_{1,\mathrm{loc}}(\Omega)$, one can define the p-dilatation [5] at a point $x\in\Omega$ as

$$K_p(x)=\inf\{k(x) : |D\varphi(x)|\leq k(x)|J(x,\varphi)|^{\frac{1}{p}}\}.$$

Theorem 2.1. *A homeomorphism $\varphi:\Omega\to\widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ generates a bounded composition operator*

$$\varphi^{*}: L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega),\ \ 1\leq q\leq p\leq\infty,$$

if and only if $\varphi\in W^1_{q,\mathrm{loc}}(\Omega)$, φ has finite distortion, and

$$K_{p,q}(\varphi;\Omega):=\|K_p\mid L_\kappa(\Omega)\|<\infty,\ 1/\kappa=1/q-1/p.$$

The norm of the operator φ^* has the upper bound $\|\varphi^*\|\leq K_{p,q}(\varphi;\Omega)$.

Composition operators on Sobolev spaces have the capacitary description also [35]. This description, in particular, allows us to establish a connection of the composition operators theory with the theory of Q-homeomorphisms, as it was mentioned in the introduction.

First of all, we recall the notion of a variational p-capacity [38]. A condenser in a domain $\Omega\subset\mathbb{R}^n$ is the pair (F_0,F_1) of connected disjoint closed relatively to Ω sets $F_0,F_1\subset\Omega$. Then a continuous function $f\in L^1_p(\Omega)$ is called an admissible function for the condenser (F_0,F_1), if the set $F_i\cap\Omega$ is contained in some connected component of the set $\mathrm{Int}\{x|f(x)=i\}$, $i=0,1$.

The p-capacity of the condenser (F_0,F_1) relatively to domain Ω is the following quantity:

$$\mathrm{cap}_p(F_0,F_1;\Omega)=\inf\|f|L^1_p(\Omega)\|^p.$$

Here the greatest lower bound is taken over all admissible functions for the condenser $(F_0,F_1)\subset\Omega$. If the condenser has no admissible functions we put the capacity equal to infinity.

The next two theorems give the capacitary description of composition operators on Sobolev spaces. The first theorem was not formulated, but proved in [35] by using the approximation by extremal functions [37].

Theorem 2.2. *Let $1 < p < \infty$. A homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ generates a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_p(\Omega)$$

if and only if for every condenser $(F_0, F_1) \subset \widetilde{\Omega}$ the inequality

$$\operatorname{cap}_p^{1/p}(\varphi^{-1}(F_0), \varphi^{-1}(F_1); \Omega) \leq K_{p,p}(\varphi; \Omega) \operatorname{cap}_p^{1/p}(F_0, F_1; \widetilde{\Omega})$$

holds.

In the case $q < p$ the capacitory description requires set functions introduced in [35].

Theorem 2.3. [35] *Let $1 < q < p < \infty$. A homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ generates a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega)$$

if and only if there exists a bounded monotone countable-additive set function $\widetilde{\Phi}_{p,q}$ defined on open subsets of $\widetilde{\Omega}$ such that for every condenser $(F_0, F_1) \subset \widetilde{\Omega}$ the inequality

$$\operatorname{cap}_q^{1/q}(\varphi^{-1}(F_0), \varphi^{-1}(F_1); \Omega) \leq \widetilde{\Phi}_{p,q}(\widetilde{\Omega} \setminus (F_0 \cup F_1))^{\frac{p-q}{pq}} \operatorname{cap}_p^{1/p}(F_0, F_1; \widetilde{\Omega})$$

holds.

Recall that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ of domains $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$ is called a (p, Q)-*homeomorphism* (see, for example, [25, 32]), with a non-negative measurable function Q, if

$$M_p(\varphi\Gamma) \leqslant \int\limits_{\Omega} Q(x) \cdot \rho^p(x) dx$$

for every family Γ of rectifiable paths in Ω and every admissible function ρ for Γ. The following connection between composition operators on Sobolev spaces and (p, Q)-homeomorphisms was proved in [28] in the case $p = n$ and it was generalized for the case $n - 1 < p < \infty$ in [10].

Theorem 2.4. *Let Ω and $\widetilde{\Omega}$ be domains in $\mathbb{R}^n$. Suppose that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ is a weak $(p, n-1)$-quasiconformal mapping. Then φ is a (p', Q)-homeomorphism with $p' = p/(p - n + 1)$ and $Q \in L_1(\Omega)$.*

The weak (p, q)-quasiconformal mappings, $1 \leq q \leq p \leq n$, have the following measure distortion properties [39, 40]:

Proposition 2.5. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a weak (p, q)-quasiconformal mapping, $1 \leq q \leq p \leq n$. Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ has the Luzin N-property (an image of a set of measure zero has measure zero) and $J_{\varphi^{-1}} \in L_t(\widetilde{\Omega})$, $t = \frac{p}{p-q} \frac{n-q}{n}$.*

In the case $n - 1 \leq q \leq p \leq \infty$ we have the following differential properties of the inverse mapping [35, 13].

Proposition 2.6. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a weak (p,q)-quasiconformal mapping, $n-1 \leq q \leq p \leq \infty$, and has the Luzin N-property if $q = n-1$. Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ belongs to the Sobolev space $W^1_{p',\mathrm{loc}}(\widetilde{\Omega})$, has finite distortion and differentiable a.e. in $\widetilde{\Omega}$ if $p \leq (n-1)^2/(n-2)$.*

Remark 2.7. In the case $p = (n-1)^2/(n-2)$ we have $p' = n-1$ and differentiability a.e. in $\widetilde{\Omega}$ follows from [34].

2.3. **Weighted Poincaré-Sobolev inequalities.** In this section we prove the weighted Poincaré-Sobolev inequality with weights, which are volume derivatives of the mappings which inverse to the corresponding weak (p,q)-quasiconformal homeomorphisms. Note that by Proposition 2.5, this weight $w = J_{\varphi^{-1}}$ is locally integrable function.

Let us recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is said to be an (s,q)-Poincaré-Sobolev domain (see, for example, [12, 16]), $1 \leq q \leq s \leq \infty$, if the following Poincaré-Sobolev inequality

$$\inf_{c\in\mathbb{R}} \|g - c \mid L_s(\Omega)\| \leq B_{s,q}(\Omega)\|\nabla g \mid L_q(\Omega)\|$$

holds for any function $g \in W^1_q(\Omega)$ with the best constant $B_{s,q}(\Omega) < \infty$.

Theorem 2.8. *Let a bounded domain $\widetilde{\Omega} \subset \mathbb{R}^n$ be such that there exists a weak (p,q)-quasiconformal mapping $\varphi : \Omega \to \widetilde{\Omega}$, $1 \leq q \leq p < \infty$, where $\Omega \subset \mathbb{R}^n$ is a bounded (s,q)-Poincaré-Sobolev domain, and φ has the Luzin N-property if $p > n$. Then the weighted (s,p)–Poincaré-Sobolev inequality*

$$\inf_{c\in\mathbb{R}} \left(\int\limits_{\widetilde{\Omega}} |f(y) - c|^s w(y)\, dy \right)^{\frac{1}{s}} \leq B^w_{s,p}(\widetilde{\Omega}) \left(\int\limits_{\widetilde{\Omega}} |\nabla f(y)|^p\, dy \right)^{\frac{1}{p}}$$

holds for any function $f \in W^1_p(\widetilde{\Omega})$ with the weight $w(y) = J_{\varphi^{-1}}(y)$, where $J_{\varphi^{-1}}(y)$ is a volume derivative of the mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$.

Proof. Let a function $f \in W^1_p(\widetilde{\Omega})$. Since there exists a weak (p,q)-quasiconformal mapping $\varphi : \Omega \to \widetilde{\Omega}$, then the composition $\varphi^*(f)$ belongs to $L^1_q(\Omega)$ and

$$\|\varphi^*(f) \mid L^1_q(\Omega)\| \leq K_{p,q}(\varphi;\Omega)\|f \mid L^1_p(\widetilde{\Omega})\|,\ f \in W^1_p(\widetilde{\Omega}). \tag{2.2}$$

Because Ω is the bounded (s,q)-Poincaré-Sobolev domain, $s \geq q$, then by the embedding theorem [26] the composition $\varphi^*(f)$ belongs to $L_q(\Omega)$ and so the function $\varphi^*(f) \in W^1_q(\Omega)$.

By [39, 40] it is known that weak (p,q)-quasiconformal mappings possess the Luzin N^{-1}-property, if $1 \leq q \leq p \leq n$. Hence by the change of variable formula

(2.1) and by using that Ω is a bounded (s,q)-Sobolev–Poincaré domain, we have

$$\inf_{c\in\mathbb{R}}\left(\int\limits_{\widetilde{\Omega}} |f(y)-c|^s J_{\varphi^{-1}}(y)\,dy\right)^{\frac{1}{s}} = \inf_{c\in\mathbb{R}}\left(\int\limits_{\Omega} |f(\varphi(x))-c|^s\,dx\right)^{\frac{1}{s}}$$
$$\leq B_{s,q}(\Omega)\left(\int\limits_{\Omega} |\nabla f(\varphi(x))|^q\,dx\right)^{\frac{1}{q}},$$

where $B_{s,q}(\Omega)$ is a best constant in the (s,q)-Sobolev–Poincaré inequality.

By the inequality (2.2) we obtain

$$\inf_{c\in\mathbb{R}}\left(\int\limits_{\widetilde{\Omega}} |f(y)-c|^s J_{\varphi^{-1}}(y)\,dy\right)^{\frac{1}{s}} \leq B_{s,q}(\Omega)\left(\int\limits_{\Omega} |\nabla f(\varphi(x))|^q\,dx\right)^{\frac{1}{q}}$$
$$\leq B_{s,q}(\Omega)K_{p,q}(\Omega;\varphi)\left(\int\limits_{\widetilde{\Omega}} |\nabla f(y)|^p\,dy\right)^{\frac{1}{p}} = B^w_{s,p}(\widetilde{\Omega})\left(\int\limits_{\widetilde{\Omega}} |\nabla f(y)|^p\,dy\right)^{\frac{1}{p}},$$

for any function $f\in W^1_p(\widetilde{\Omega})$.

The constant $B^w_{s,p}(\widetilde{\Omega})$ has the following estimate:

$$B^w_{s,p}(\widetilde{\Omega}) \leq B_{s,q}(\Omega)K_{p,q}(\Omega;\varphi).$$

□

3. Spectral estimates of elliptic operators

In this section we give spectral estimates of the two-dimensional Laplace operator in Hölder cusp domains. The detailed description of applications of the geometric theory of composition operators on Sobolev spaces to the spectral theory of non-linear elliptic operators, see for example [8, 16, 17].

In this section we consider the two-dimensional weak (p,q)-quasiconformal mappings in the limit case $1=q<p=2$. The two-dimensional weak $(2,1)$-quasiconformal mappings we will call as weak quasiconformal mappings. Namely, a homeomorphism $\varphi:\Omega\to\widetilde{\Omega}$ we call a weak quasiconformal mapping, if $\varphi\in W^1_{1,\mathrm{loc}}(\Omega)$, has finite distortion, and

$$K(\varphi;\Omega)=\left(\int\limits_{\Omega}\frac{|D\varphi(x)|^2}{|J(x,\varphi)|}\,dx\right)^{\frac{1}{2}}<\infty.$$

Recall that weak quasiconformal mappings coincide with Q-homeomorphisms, $Q\in L_1(\Omega)$.

In the case of bounded domains $\Omega,\widetilde{\Omega}\subset\mathbb{R}^2$, by the measurable Riemann mapping theorem [2], there exists a K-quasiconformal mapping $\varphi:\Omega\to\widetilde{\Omega}$, which will be a weak quasiconformal mapping, since

$$K(\varphi;\Omega)=\left(\int\limits_{\Omega}\frac{|D\varphi(x)|^2}{|J(x,\varphi)|}\,dx\right)^{\frac{1}{2}}\leq\left(\int\limits_{\Omega}K\,dx\right)^{\frac{1}{2}}=(K|\Omega|)^{\frac{1}{2}}<\infty.$$

Note, that the class of weak quasiconformal mappings wider than the class of usual quasiconformal mappings and such mappings can be easily constructed explicitly.

Now we consider the Neumann spectral problem for the Laplace operator

$$\begin{cases}-\Delta u=\mu u \text{ in } Q_\alpha,\\ \frac{\partial u}{\partial\nu}=0 \quad \text{ on } \partial Q_\alpha,\end{cases}$$

in the Hölder cusp domains $Q_\alpha\subset\mathbb{R}^2$, $1<\alpha<\infty$, which are the images of the square $Q=\{(x,y):|x|+|y|\leq 1\}$ under locally Lipschitz homeomorphisms $\varphi:Q\to Q_\alpha$ of the form

$$\varphi(x,y)=\begin{cases}(x,y^\alpha), & y\geq 0,\\ (x,-y^\alpha), & y<0.\end{cases}$$

By the Min-Max Principle the first non-trivial Neumann eigenvalue in the domains Ω which satisfy the quasihyperbolic boundary condition [23] can be characterized as

$$\mu_1(\Omega)=\left(B_{2,2}(\Omega)\right)^{-2},$$

where $B_{2,2}(\Omega)$ is the best constant in the corresponding Poincaré-Sobolev inequality.

The Hölder cusp domains $Q_\alpha\subset\mathbb{R}^2$ satisfy this quasihyperbolic boundary condition. Therefore, we need estimate $B_{2,2}(Q_\alpha)$ in the Hölder cusp domain Q_α. To do this we first provide the estimate for the constants $B_{2,2}(\Omega_L)$ in the Lipschitz domain $\Omega_L\subset\mathbb{R}^2$. The following version of the critical Poincaré -Sobolev inequality on disks with the upper estimates of the sharp constant was proven in [9].

Theorem 3.1 ([9])**.** *Let $f\in W^1_1(\Omega)$, $\Omega\subset\mathbb{R}^2$. Then for any $r>0$ and any $z_0\in\Omega$, such that $D(z_0,r)\subset\Omega$, the following inequality holds:*

$$\left(\iint\limits_{D(z_0,r)}|f(z)-f_{D(z_0,r)}|^2\,dxdy\right)^{\frac{1}{2}}\leq\frac{3\sqrt{\pi^3}}{4}\iint\limits_{D(z_0,r)}|\nabla f(z)|\,dxdy. \tag{3.1}$$

Remark 3.2. Theorem 3.1 in [9] contains an additional assumption $\operatorname{dist}(z_0,\partial\Omega)>2r$ instead of $D(z_0,r)\subset\Omega$. In fact, authors of [9] proved the theorem without this additional assumption. Indeed, in the proof authors considered an extension by reflection of a function f defined on a disk $D(z_0,r)\subset\Omega$ to a disk $D(z_0,2r)$. But it is irrelevant whether the initial function f is well defined on the disk $D(z_0,2r)$ or not, because an extension $E(f)$ of f is well defined on the disk of double radii by its definition. Therefore, we need not additionally assume, that f is defined on $D(z_0,2r)$ and it is enough to assume that $D(z_0,r)\subset\Omega$ only.

By using the bi-Lipschitz change of variables, we can extend this result from a disk to a general domain with the Lipschitz boundary and obtain an estimate for the constant $B_{2,2}(\Omega_L)$.

Theorem 3.3. *Suppose that there exists a bi-Lipschitz mapping* $\psi : D(0,r) \to \Omega_L$ *and that* $f \in W^1_1(\Omega_L)$*. Then the following inequality holds:*

$$\left(\iint\limits_{\Omega_L} |f(x,y) - f_{\Omega_L}|^2\, dxdy\right)^{\frac{1}{2}} \leq \frac{3L^2\sqrt{\pi^3}}{4} \iint\limits_{\Omega_L} |\nabla f(x,y)|\, dxdy, \tag{3.2}$$

where L *is the bi-Lipschitz constant of the mapping* ψ*.*

Proof. Without loss of generality we can assume, that $f_{\Omega_L} = 0$. Note, that in this case by the Min-Max Principle we have that

$$\left(\inf_{c\in\mathbb{R}} \iint\limits_{\Omega_L} |f(u,v) - c|^2\, dudv\right)^{\frac{1}{2}} = \left(\iint\limits_{\Omega_L} |f(u,v)|^2\, dudv\right)^{\frac{1}{2}},$$

where

$$\iint\limits_{\Omega_L} f(u,v)\, dudv = 0.$$

Since the function f belongs to the space $W^1_1(\Omega_L)$, the composition $f \circ \psi$ of the function f with a bi-Lipschitz mapping $\psi : D(0,r) \to \Omega_L$ belongs to $W^1_1(D(0,r))$, and we can apply the change of variables formula.

$$\begin{aligned}\left(\iint\limits_{\Omega_L} |f(u,v)|^2\, dudv\right)^{\frac{1}{2}} &= \left(\inf_{c\in\mathbb{R}} \iint\limits_{\Omega_L} |f(u,v) - c|^2\, dudv\right)^{\frac{1}{2}} \\ &= \left(\inf_{c\in\mathbb{R}} \iint\limits_{D(0,r)} |f(\psi(x,y)) - c|^2 |J((x,y),\psi)|\, dxdy\right)^{\frac{1}{2}} \\ &\leq \operatorname*{esssup}_{(x,y)\in D(0,r)} |J((x,y),\psi)|^{\frac{1}{2}} \left(\iint\limits_{D(0,r)} |f(\psi(x,y)) - f_{D(0,r)}|^2\, dxdy\right)^{\frac{1}{2}}\end{aligned}$$

By Theorem 3.1, we obtain the following estimate:

$$\left(\iint\limits_{\Omega_L} |f(u,v)|^2\, dudv\right)^{\frac{1}{2}} \leq \frac{3\sqrt{\pi^3}}{4} \operatorname*{esssup}_{(x,y)\in D(0,r)} |J((x,y),\psi)|^{\frac{1}{2}} \iint\limits_{D(0,r)} |\nabla f(\psi(x,y))|\, dxdy.$$

Using the chain rule and the change of variables formula a second time, we infer that

$$
\left(\iint\limits_{\Omega_L} |f(u,v)|^2\, dudv\right)^{\frac{1}{2}}
\leq \frac{3\sqrt{\pi^3}}{4} \operatorname*{esssup}_{(x,y)\in D(0,r)} |J((x,y),\psi)|^{\frac{1}{2}} \operatorname*{esssup}_{(x,y)\in D(0,r)} \frac{|D\psi(x,y)|}{|J((x,y),\psi)|} \iint\limits_{\Omega} |\nabla f(u,v)|\, dudv.
$$

By the Hadamard inequality, we can choose a constant L such that

$$
0 < L^{-1} \leq |J((x,y),\psi)|^{1/2} \leq |D\psi(x,y)| \leq L.
$$

Then

$$
\left(\iint\limits_{\Omega_L} |f(u,v)|^2\, dudv\right)^{\frac{1}{2}} \leq \frac{3L^2\sqrt{\pi^3}}{4} \iint\limits_{\Omega_L} |\nabla f(u,v)|\, dudv.
$$

□

Now we are ready to formulate and proof the lower estimates for the first non-trivial Neumann eigenvalue of the Laplace operator in Hölder cusp domains Q_α.

Theorem 3.4. *Let $Q_\alpha \subset \mathbb{R}^2$ be the Hölder cusp domain. Then the first non-trivial Neumann eigenvalue of the Laplace operator satisfies*

$$
\mu_1(Q_\alpha) \geq \frac{4(\alpha+1)}{9\pi^3\alpha}.
$$

Proof. We need estimate the constant $B_{2,2}(Q_\alpha)$ in the Poincaré-Sobolev inequality. The estimate is based on the following anti-commutative diagram [6, 12]:

$$
\begin{array}{ccc}
W^1_2(Q_\alpha) & \xrightarrow{\varphi^*} & W^1_1(Q) \\
\downarrow & & \downarrow \\
L_2(Q_\alpha) & \xleftarrow{(\varphi^{-1})^*} & L_2(Q)
\end{array}
$$

First note, that the weak quasiconformal mapping $\varphi : Q \to Q_\alpha$ generates a bounded composition operator

$$
\varphi^* : L^1_2(Q_\alpha) \to L^1_1(Q).
$$

Indeed, $\varphi \in W^1_{1,\mathrm{loc}}(Q)$ and we have the following calculations:

$$
|D\varphi(x,y)| = \alpha y^{\alpha-1}, \quad J((x,y),\varphi) = \alpha y^{\alpha-1}.
$$

Hence

$$
K(\varphi;Q) = \left(4\int\limits_0^1 dx \int\limits_0^x \alpha y^{\alpha-1}\, dy\right)^{\frac{1}{2}} = \left(\frac{4}{\alpha+1}\right)^{\frac{1}{2}}
$$

and

$$
\sup_{(x,y)\in Q} |J((x,y),\varphi)|^{\frac{1}{2}} = \sqrt{\alpha}.
$$

Therefore, φ is a weak quasiconformal mapping and, by Theorem 2.1, φ^* is the bounded composition operator with $\|\varphi^*\| \leq K(\varphi; Q)$. Moreover, as we know that

$$\inf_{(x,y)\in Q} |J((x,y),\varphi^{-1})| = \sup_{(x,y)\in Q} |J((x,y),\varphi)| < \infty,$$

The Poincaré-Sobolev inequality (3.2) implies

$$\inf_{c\in\mathbb{R}} \left(\iint\limits_{Q_\alpha} |f(x,y) - c|^2 \, dxdy \right)^{\frac{1}{2}}$$
$$\leq B^w_{2,2}(Q_\alpha) \frac{1}{\inf\limits_{(x,y)\in Q} |J((x,y),\varphi^{-1})|} \left(\iint\limits_{Q_\alpha} |\nabla f(x,y)|^2 \, dxdy \right)^{\frac{1}{2}},$$

and we conclude

$$B_{2,2}(Q_\alpha) \leq B^w_{2,2}(Q_\alpha) \frac{1}{\inf\limits_{(x,y)\in Q} |J(x,y,\varphi^{-1})|} = B_{2,1}(Q) K_{2,1}(\varphi; Q) \sup_{(x,y)\in Q} |J(x,y,\varphi)|.$$

It remains to calculate the constant $B_{2,1}(Q)$ in inequality (3.2) for the square $Q = \{(x,y) : |x| + |y| \leq 1\}$. As the bi-Lipschitz mapping, we consider the radial transformation $\psi : D(0,1) \to Q$, defined by the rule

$$\psi(x,y) = (l(x,y)x, l(x,y)y), \quad l(x,y) = \frac{\sqrt{x^2+y^2}}{|x|+|y|}.$$

By calculations, we obtain

$$|D\psi(x,y)| = l(x,y), \quad J(x,y,\psi) = l^2(x,y).$$

Then, for $(x,y) \in D(0,1)$,

$$\sup_{(x,y)} |J((x,y),\psi)|^{\frac{1}{2}} \sup_{(x,y)} \frac{|D\psi(x,y)|}{J((x,y),\psi)} = 1.$$

Substituting these values to inequality (3.2) with the bi-Lipschitz mapping ψ, we obtain

$$B_{2,1}(Q) = \frac{3\sqrt{\pi^3}}{4}$$

By combining all the estimates, we infer that

$$\mu_1(Q_\alpha) \geq \frac{4(\alpha+1)}{9\pi^3\alpha}.$$

□

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