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CAPACITY AND MODULUS MEASURES IN METRIC MEASURE SPACES

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Abstract. The new variational $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$–capacity, introduced in [15], is directly connected to the M_p–modulus and thus fits well to metric measure spaces where path families play a more important role than in $\mathbb{R}^n$. Let $\Gamma(E)$ and Γ^E be the path families whose paths meet a set $E \subset X$ and lie in E, respectively. Since the Poincaré inequality is not used, it is shown that the M_p–modulus of $\Gamma(E)$ and that of Γ^E are intimately connected to the sets of zero $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$–capacity. The differences between the M_1–modulus and the M_p–modulus, $p > 1$, are also considered.

1. Introduction

In a good metric measure space (X, d, μ) the standard assumptions are: X is proper, the measure μ is doubling and X supports the p– Poincaré inequality. These assumptions imply that X is complete and quasiconvex, see the excellent books [3] and [10]. The variational $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$–capacity is based on the M_p–modulus and uses Lipschitz functions and quasiconvexity, see [15], but does not employ the doubling property or the p–Poincaré inequality. Hence the sets E of zero $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$–capacity are characterized in terms of the M_p–modulus and it turns out that $\mu(E) = 0$ can be replaced by $M_p(\Gamma^E) = 0$ where Γ^E is the family of all paths in E. This approach avoids the use of Suslin sets in the Choquet capacity theory.

Sections 2 and 3 are devoted to the modulus theory and to the properties of the $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$–capacity. In Section 4 modulus measures are considered and in Section 5 sets of zero $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$–capacity are compared to the sets of zero modulus measures. Section 6 is devoted to the $\mathrm{Cap}_{1}^{\mathrm{M}}$–capacity.

2. M_p– and AM_p–modulus

Let (X, d) be a metric space equipped by a Borel measure μ. A continuous mapping $\gamma\colon [a, b] \to X$ is called a *path* if it has finite and non–zero total length; in this case we parametrize γ by its arclength. It is essential that a constant curve is not a path. The *locus* of γ is defined as $\gamma([0, \ell(\gamma)])$ and denoted by $\langle\gamma\rangle$ and the length of γ by $\ell(\gamma)$.

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Let Γ be a family of paths in X. A non–negative Borel function ρ is *M–admissible*, or simply admissible, for Γ if

$$\int_\gamma \rho \, ds \geq 1$$

for every $\gamma \in \Gamma$. For $p \in [1, \infty)$ the *M_p–modulus* of Γ is defined as

$$M_p(\Gamma) = \inf \int_X \rho^p \, d\mu$$

where the infimum is taken over all admissible functions ρ.

A sequence of non–negative Borel functions ρ_i, $i = 1, 2, \ldots$, is *AM–admissible*, or simply admissible, for Γ if

$$\liminf_{i \to \infty} \int_\gamma \rho_i \, ds \geq 1 \tag{2.1}$$

for every $\gamma \in \Gamma$. The *approximation modulus*, AM_p–modulus for short, of Γ is defined as

$$AM_p(\Gamma) = \inf_{(\rho_i)} \Big\{ \liminf_{i \to \infty} \int_X \rho_i^p \, d\mu \Big\} \tag{2.2}$$

where the infimum is taken over all AM–admissible sequences (ρ_i) for Γ. We use the phrase "almost every path", a.e. for short, to mean every path except a family of M_p– or AM_p–modulus zero.

If the space X is proper (closed bounded sets are compact), instead of admissible Borel functions it is possible to use lower semicontinuous non–negative functions as admissible for the M_p– and AM_p–modulus, see [6, Proposition 7.14].

The following lemma contains the most important properties of the AM_p– and M_p–modulus. See [14] and [13] for the properties of the AM_p–modulus, [1] for (f) and [3] and [7] for those of the M_p–modulus.

Lemma 2.1. *Suppose that X is a metric space equipped with a Borel measure μ and $p \in [1, \infty)$ unless otherwise stated. The AM_p– and M_p–modulus are outer measures in the set of path families in X, i.e.*

(a) $AM_p(\emptyset) = 0$,
(b) $\Gamma_1 \subset \Gamma_2 \implies AM_p(\Gamma_1) \leq AM_p(\Gamma_2)$,
(c) $\Gamma = \bigcup_{j=1}^\infty \Gamma_j \implies AM_p(\Gamma) \leq \sum_{i=j}^\infty AM_p(\Gamma_j)$.
(d) *If every $\gamma \in \Gamma$ has a subpath $\tilde{\gamma}$, then $AM_p(\Gamma) \leq AM_p(\tilde{\Gamma})$ where $\tilde{\Gamma}$ is the family of these subpaths.*

The properties (a)–(d) also hold for the M_p–modulus, $p \geq 1$.

(e) $AM_1(\Gamma) \leq M_1(\Gamma)$ *and* $AM_p(\Gamma) = M_p(\Gamma)$, $p > 1$, *for every path family* Γ.
(f) $p > 1$ *and* $\Gamma_1 \subset \Gamma_2 \subset \ldots \implies \lim_{i \to \infty} M_p(\Gamma_i) = M_p(\cup_i \Gamma_i)$.
(g) $M_p(\Gamma) = 0 \iff$ *there is a Borel function* $\rho \in L^p(X)$ *such that*

$$\int_\gamma \rho \, ds = \infty \text{ for every } \gamma \in \Gamma.$$

(h) $AM_1(\Gamma) = 0 \iff$ *there is a sequence* (ρ_i) *of non–negative Borel functions such that*

$$\lim_{i\to\infty}\int_\gamma \rho_i\, ds = \infty \text{ for every } \gamma\in\Gamma \text{ and } \liminf_{i\to\infty}\int_X \rho_i\, d\mu < \infty.$$

3. Modulus measures

In this section we assume that X is a metric space and μ is a Borel measure in X.

We employ the following notation for path families associated with arbitrary sets $E, F \subset X$: $\Gamma(E)$ denotes the family of all paths in X which meet E, Γ^E is the family of all paths in E and $\Gamma(E,F)$ is the family of paths which meet both E and F. We mostly use the path family $\Gamma(E, X\setminus E)$ which consists of paths meeting both E and its complement.

To the M_p–modulus and AM_p–modulus, $p \geq 1$, we associate the corresponding *modulus measures* $E \mapsto M_p(\Gamma(E))$ and $E \mapsto AM_p(\Gamma(E))$ where E is an arbitrary subset of X. By the properties of the M_p– and AM_p–modulus in Lemma 2.1 it is easy to see that the modulus measures are outer measures in X and, moreover, metric outer measures, see Theorem 1 in [11]. Hence all Borel sets are measurable. Since for $p > 1$, $M_p(\Gamma(E)) = AM_p(\Gamma(E))$ for every set E, only the AM_1–modulus measure is of interest. Note that $AM_1(\Gamma(E)) \leq M_1(\Gamma(E))$ for $E \subset X$.

Remark 3.1. (a) If $\mu(E) = 0$ or E is totally disconnected or, more generally, does not contain rectifiable paths, then $M_p(\Gamma^E) = AM_p(\Gamma^E) = 0$ for $p \in [1,\infty)$. Note also that $M_p(\Gamma^E) = 0$ does not imply $\mu(E) = 0$.
(b) Let $co\mathcal{H}^p$ denote the p–codimensional Hausdorff meassure, $p \geq 1$, defined as

$$co\mathcal{H}^p(E) = \sup_{\delta>0}\, co\mathcal{H}^p_\delta(E)$$

where for $\delta > 0$

$$co\mathcal{H}^p_\delta(E) = \inf\Big\{\sum_{j=1}^\infty \frac{\mu(B(x_j,r_j))}{r_j^p} : E \subset \bigcup_{j=1}^\infty B(x_j,r_j),\ \sup_j r_j < \delta\Big\}.$$

If μ is a doubling measure, i.e.

$$0 < \mu(B(x,2r)) \leq C_\mu\, \mu(B(x,r)) < \infty \text{ for all balls } B(x,r),$$

then

$$M_p(\Gamma(E)) \leq C\, co\mathcal{H}^p(E) \tag{3.1}$$

where E is an arbitrarary set and C depends only on the doubling constant C_μ, see Theorem 2.1 in [13]. The standard $(n-p)$–dimensional Hausdorff measure $\mathcal{H}^{n-p}$ in $\mathbb{R}^n$ satisfies $\mathcal{H}^{n-p} \approx co\mathcal{H}^p$.
(c) If X is a good metric space, then $M_p(\Gamma(E)) = 0$ implies $\mu(E) = 0$ and this yields $M_p(\Gamma^E) = 0$, see Proposition 4.9 in [3] and for the measurability question of E see Proposition 1.5 in [2].

Next we show that the M_p–modulus measure, $p \geq 1$, degenerates to either 0 or ∞ on every set E and thus all sets E are M_p–measurable.

Lemma 3.2. *For $p \geq 1$ either $M_p(\Gamma^E) = 0$ or $M_p(\Gamma^E) = \infty$ and the same is true for $AM_p(\Gamma^E)$.*

Proof. We consider the AM_p–modulus measure. The proof for the M_p–modulus measure is simpler.

Suppose that $AM_p(\Gamma^E) \in (0, \infty)$. Let (ρ_i) be an admissible sequence for Γ^E with

$$\liminf_{i\to\infty} \int_X \rho_i^p \, d\mu < \infty.$$

Then there is a path $\gamma \in \Gamma^E$ such that

$$\liminf_{i\to\infty} \int_\gamma \rho_i \, ds = M \in [1, \infty) \tag{3.2}$$

since otherwise $AM_p(\Gamma^E) = 0$, see Lemma 2.1 (h). Choose an integer $m > M$ and then disjoint intervals $I_j \subset [0, \ell(\gamma)]$, $j = 1, 2, \dots m$. Let $\gamma_j = \gamma | I_j$. Since each $\gamma_j \in \Gamma^E$

$$\liminf_{i\to\infty} \int_{\gamma_j} \rho_i \, ds \geq 1$$

and so

$$\liminf_{i\to\infty} \int_\gamma \rho_i \, ds \geq \liminf_{i\to\infty} \sum_{j=1}^m \int_{\gamma_j} \rho_i \, ds \geq \sum_{j=1}^m \liminf_{i\to\infty} \int_{\gamma_j} \rho_i \, ds \geq m > M$$

which contradicts (3.2). □

Theorem 3.3. *If $p \geq 1$ and $E \subset X$ is an arbitrary set, then $M_p(\Gamma(E)) \in \{0, \infty\}$.*

Proof. Note first that $\Gamma(E) = \Gamma(E, X \setminus E) \cup \Gamma^E$.

If $M_p(\Gamma^E) = \infty$, then $M_p(\Gamma(E)) = \infty$ and thus by Lemma 3.2 we can assume that $M_p(\Gamma^E) = 0$ and it suffices to show that $M_p(\Gamma(E, X \setminus E)) \in (0, \infty)$ leads to contradiction. Let ρ be an admissible function for $\Gamma(E, X \setminus E)$ with

$$\int_X \rho^p \, d\mu < \infty.$$

Now for a.e. path $\gamma \in \Gamma(E, X \setminus E)$

$$\int_\gamma \rho \, ds < \infty \tag{3.3}$$

because the family of those paths in $\Gamma(E, X \setminus E)$ for which the above integral is $= \infty$ has zero M_p–modulus., see Lemma 2.1 (g).

Let $\gamma \in \Gamma(E, X \setminus E)$ satisfy (3.3). If γ has a subpath which belongs to Γ^E, then by Lemma 2.1 (d) the family of these paths γ has zero M_p–modulus because $M_p(\Gamma^E) = 0$. If γ does not belong to this subfamily, then there is $t_0 \in [0, \ell(\gamma)]$ such that $\gamma(t_0) \in E$ and also a sequence of points $t_i \in [0, \ell(\gamma)]$ with $\gamma(t_i) \in X \setminus E$ and $t_i \to t_0$. The paths γ_i obtained by restricting γ to the intervals generated by the points t_i and t_0 belong to $\Gamma(E, X \setminus E)$ and hence

$$\int_{\gamma_i} \rho \, ds \geq 1$$

and this contradicts (3.3) by the absolute continuity of an integral. □

Theorem 3.3 is not true for the AM_1–modulus measure. In [12] it is shown that in a good metric space $AM_1(\Gamma(E)) \approx co\mathcal{H}^1(E)$ for each Suslin set E.

4. $\mathrm{Cap}_\mathrm{p}^\mathrm{M}$–capacity

In this section we assume that X is a proper metric space with a Borel regular measure μ which is finite on compact sets. In a proper space closed bounded sets are compact. Since the concept of the $\mathrm{Cap}_\mathrm{p}^\mathrm{M}$–capacity is based on the modulus theory, it is essential that X contains plenitude of paths and we assume that X is qusiconvex. This means that there is $c \in [1,\infty)$ such that for all x and y in X, $x \neq y$, there is a path γ joining x to y with $\ell(\gamma) \le c\, d(x,y)$. We do not assume that X supports the p–Poincaré inequality nor that the measure μ is doubling.

We first recall the basic properties of the $\mathrm{Cap}_\mathrm{p}^\mathrm{M}$–capacity from [15].

For a Lipschitz function u a non-negative Borel function g is an upper gradient of u if for every path γ in X

$$|u(\gamma(\ell)) - u(\gamma(0))| \le \int_\gamma g\, ds,$$

see [3, Chapters 1–2] for the properties of functions and their upper gradients. The lower pointwise dilatation

$$|\nabla u(x)| = \liminf_{r\to 0} \sup_{y\in B(x,r)} \frac{|u(y)-u(x)|}{r}$$

is an upper gradient of u, see [3, Proposition 1.14]. In $\mathbb{R}^n$, $|\nabla u(x)|$ is a unique minimal upper gradient for a Lipschitz function u, see [3, Examples A1].

Let G be a fixed bounded open set in X and E an arbitrary subset of G. An increasing sequence (u_i) of non–negative Lipschitz functions in X is called *admissible*, $(u_i) \in Ad(E,G)$, for the condenser (E,G) if $u_i = 0$ in $X \setminus G$ and $\lim_{i\to\infty} u_i(x) \ge 1$ for $x \in E$. For $p \ge 1$ we define

$$\mathrm{Cap}_\mathrm{p}^\mathrm{M}(E,G) = \inf\Big\{ \liminf_{i\to\infty} \int_G g_i^p\, d\mu :\ (u_i) \in Ad(E,G) \text{ and } g_i \text{ is an upper gradient of } u_i \Big\}.$$

The next theorem contains the basic properties of the $\mathrm{Cap}_\mathrm{p}^\mathrm{M}(E,G)$-capacity from [15] and for (d) see Lemma 3.3 in [13].

Theorem 4.1. *Let E and E_i, $i = 1, 2, \ldots$ be subsets of G and $p \ge 1$.*

(a) $E_1 \subset E_2 \subset G \implies \mathrm{Cap}_\mathrm{p}^\mathrm{M}(E_1,G) \le \mathrm{Cap}_\mathrm{p}^\mathrm{M}(E_2,G)$ *(monotonicity).*

(b) $K_1 \supset K_2 \supset \ldots$ *compact sets in* $G \implies$

$$\lim_{i\to\infty} \mathrm{Cap}_\mathrm{p}^\mathrm{M}(K_i,G) = \mathrm{Cap}_\mathrm{p}^\mathrm{M}(\bigcap_i K_i, G).$$

(c) *There is a Borel set E' such that $E \subset E' \subset G$ and*

$$\mathrm{Cap}_\mathrm{p}^\mathrm{M}(E',G) = \mathrm{Cap}_\mathrm{p}^\mathrm{M}(E,G).$$

(d) $\mathrm{Cap}_\mathrm{p}^\mathrm{M}(K,G) = M_p(\Gamma(K, X\setminus G))$ *if $K \subset G$ is compact.*

(e) $AM_p(\Gamma(E, X \setminus G)) \le \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E,G) \le M_p(\Gamma(E, X \setminus G))$ *and for* $p > 1$ *the inequalities are equalities.*

Since for $p > 1$, $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E,G) = M_p(\Gamma(E, X \setminus G))$ for all sets $E \subset G$, the properties of the M_p–modulus imply the following.

Theorem 4.2. *If* $p > 1$ *then:*

(a) $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$ *is subadditive. i.e. if* $E_i \subset G$, $i = 1, 2, \ldots$, *then*

$$\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(\bigcup_i E_i, G) \le \sum_i \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E_i, G).$$

(b) $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$ *is a Choquet capacity, i.e. for a Suslin set* $E \subset G$,

$$\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E,G) = \sup\big\{ \mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(K,G) : K \subset E \text{ compact}\big\}.$$

Proof. The subadditivity of the M_p–modulus, see Lemma 2.1 (c), implies (a). By the Choquet capacitibility theorem, see [5] and [4], for (b) it suffices subadditivity, monotonicity and (b) in Theorem 4.1. □

5. Sets of zero M_p–modulus measure and zero $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$–capacity

In this section we assume that X is proper, quasiconvex and the measure μ supports the Vitali covering theorem. Note that if μ is doubling, then this comes for free, see Theorem 1.6 and Remark 1.13 in [9].

A set $E \subset X$ has *zero p–capacity* if for all bounded open sets G, $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E \cap G, G) = 0$. It is clear that a subset of a set of zero capacity has also zero $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$–capacity.

Theorem 5.1. *Suppose that* $p > 1$ *and* $E \subset X$ *is an arbitrary set. Then* E *has zero* $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$*–capacity if and only if* $M_p(\Gamma(E)) < \infty$ *or, in order words,* E *has zero* M_p*–modulus measure.*

Proof. Suppose that E has zero $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}$–capacity. We first show that $M_p(\Gamma^E) = 0$. For each $x \in E$ and for $r > 0$ the set $S(x,r) = \{z : |z - x| = r\}$ has zero μ–measure for at most countably many values of $r > 0$. Hence by the Vitali covering theorem for each $j = 1, 2, \ldots$ there is a collection of closed balls $\overline{B}(x_i^j, r_i^j)$, $i = 1, 2, \ldots$ such that $r_i^j < 1/j$, $\mu(S(x_i^j, r_i^j)) = 0$ and

$$\mu(F_j) = 0 \text{ where } F_j = E \setminus \bigcup_i B(x_i^j, r_i^j).$$

Let Γ_j be the family of all paths γ in Γ^E with $diam\langle\gamma\rangle > 2/j$. Now $\Gamma_j = \Gamma_j^1 \cup \Gamma_j^2$ where $\gamma \in \Gamma_j^1$ meets $B(x_i^j, r_i^j) \cap E$ for some i and $\gamma \in \Gamma_j^2$ lies in the set $F_j \cap E$. Now $M_p(\Gamma_j^2) = 0$ because $\mu(F_j) = 0$. On the other hand each $\gamma \in \Gamma_j^1$ meets both $S(x_i^j, r_i^j)$ and $B(x_i^j, r_i^j) \cap E$ for some i and hence $\gamma \in \Gamma(E \cap B(x_i^j, r_i^j), X \setminus B(x_i^j, r_i^j))$. Now

$$M_p(\Gamma_j^1) = M_p(\bigcup_i \Gamma(E \cap B(x_i^j, r_i^j), X \setminus B(x_i^j, r_i^j))) = 0$$

because by Theorem 4.1 (e) for each i

$$M_p(\Gamma(E \cap B(x_i^j, r_i^j), X \setminus B(x_i^j, r_i^j))) = 0$$

and hence $M_p(\Gamma_j) = 0$ for each j. By the subadditivity of the M_p–modulus $M_p(\Gamma^E) = 0$ because $\Gamma^E = \cup_j \Gamma_j$.

Now it suffices to show that $M_p(\Gamma(E, X \setminus E) = 0$. Since X is proper for each integer j we can choose a covering of E by balls $B(x_i^j, r_i^j)$ such that $r_i^j < 1/j$ for every i. Let Γ_i^j be the family of all paths in $\Gamma(E, X \setminus E)$ which belong to $\Gamma(E \cap B(x_i^j, r_i^j), X \setminus B(x_i^j, r_i^j))$. Since E has zero $\mathrm{Cap}_\mathrm{p}^\mathrm{M}$–capacity, $M_p(\Gamma(E \cap B(x_i^j, r_i^j), X \setminus B(x_i^j, r_i^j)) = 0$ and thus by Theorem 4.1(e), $M_p(\Gamma_i^j) = 0$ for each i. By the subadditivity of the M_p–modulus $M_p(\Gamma^j) = 0$ where $\Gamma^j = \cup_i \Gamma_i^j$. Let $\tilde{\Gamma}^j$ be the family of all paths $\gamma \in \Gamma(E, X \setminus E)$ with $diam(\langle\gamma\rangle) > 2/j$. Since each path $\gamma \in \tilde{\Gamma}^j$ belongs to Γ^j, $M_p(\tilde{\Gamma}_j) = 0$. Now we obtain

$$M_p(\Gamma(E, X \setminus E)) = M_p(\bigcup_j \tilde{\Gamma}_j)) = 0$$

and hence $M_p(\Gamma(E)) = 0$.

If $M_p(\Gamma(E)) < \infty$, then from Theorem 3.3 it follows that $M_p(\Gamma(E)) = 0$ and then by Theorem 4.1 (e), $\mathrm{Cap}_\mathrm{p}^\mathrm{M}(E \cap G, X \setminus G) = 0$ for each bounded open set G and consequently E has zero $\mathrm{Cap}_\mathrm{p}^\mathrm{M}$–capacity. □

Lemma 5.2. *If $K \subset X$ is compact and has zero $\mathrm{Cap}_1^\mathrm{M}$–capacity, then $M_1(\Gamma^K) = 0$.*

Proof. We first show that

$$M_1(\Gamma(K \cap B(x, r), X \setminus B(x, r))) = 0 \tag{5.1}$$

for every ball $B(x, r)$. Choose $0 < t_1 < t_2 \dots < r$ with $\lim_i t_i = r$. Now $K \cap \overline{B}(x, t_i)$ is compact and by Theorem 4.1 (d)

$$M_1(\Gamma(K \cap \overline{B}(x, t_i), X \setminus B(x, r))) = \mathrm{Cap}_1^\mathrm{M}(K \cap \overline{B}(x, t_i), B(x, r)) = 0$$

because each subset of a set of zero $\mathrm{Cap}_1^\mathrm{M}$–capacity has zero $\mathrm{Cap}_1^\mathrm{M}$–capacity. On the other hand

$$\bigcup_i \Gamma(K \cap \overline{B}(x, t_i), X \setminus B(x, r))) = \Gamma(K \cap B(x, r), X \setminus B(x, r))$$

and by the subadditivity of the M_1–modulus we obtain (5.1).

Now we can proceed as in the proof of Theorem 5.1 to conclude $M_1(\Gamma^K) = 0$. □

The next theorem gives a sufficient condition for zero $\mathrm{Cap}_\mathrm{p}^\mathrm{M}$–capacity in the case of compact sets. This result is well known in $\mathbb{R}^n$, see Lemma 2.34 in [8].

Theorem 5.3. *If $K \subset X$ is compact, $M_p(\Gamma^K) = 0$, $p > 1$ and for some sequence of bounded open sets*

$$G_1 \supset G_2 \supset \dots \supset K, \bigcap_i G_i = K \text{ and } \mathrm{Cap}_\mathrm{p}^\mathrm{M}(K, G_i) \le M < \infty$$

for all i, then K has zero $\mathrm{Cap}_\mathrm{p}^\mathrm{M}$–capacity and $M_p(\Gamma(K)) = 0$.

Proof. Since

$$\Gamma(K, X \setminus G_i) \subset \Gamma(K, X \setminus G_{i+1}) \text{ and } \bigcup_i \Gamma(K, X \setminus G_i) = \Gamma(K, X \setminus K),$$

by Lemma 2.1 (f)

$$M \geq \lim_{i\to\infty} M_p(\Gamma(K, X \setminus G_i)) = M_p(\Gamma(K, X \setminus K)),$$

and because $M_p(\Gamma^K) = 0$ we have $M_p(\Gamma(K)) = 0$ by Theorem 3.3. Now Theorem 5.1 implies that K has zero Cap_p^M–capacity. □

Remark 5.4. (a) Theorem 5.3 is not true for $p = 1$. For the simplest example take $X = \mathbb{R}$, $K = \{0\}$, and $G_i = (-1/i, 1/i)$. Then $M_1(\Gamma^K) = 0$ because K does not contain paths and it easily follows $\mathrm{Cap}_1^M(K, G_i) = 2$ for all i but

$$M_1(\Gamma(K, \mathbb{R} \setminus K)) \geq \lim_i M_1(\Gamma(K, \mathbb{R} \setminus G_i)) = 2$$

and hence $M_1(\Gamma(K)) > 0$.
(b) The subspace $X = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq |x| \leq 1\}$ of $\mathbb{R}^2$ equipped with the Lebesgue measure m_2 is a proper, 2–quasiconvex metric space and m_2 is a doubling measure in X. Set $G_i = \{(x, y) \in X : x > -1/i\}$ and $K = \{(x, y) \in X : x \geq 0\}$. Now for each i and $1 \leq p \leq 2$, $\mathrm{Cap}_p^M(K, G_i) = 0$, since $\Gamma(K, X \setminus G_i) \subset \Gamma(\{(0, 0)\})$ and $M_p(\Gamma(\{(0, 0)\})) = 0$, see e.g. (3.1). In this case $M_p(\Gamma^K) = \infty$ and thus $M_p(\Gamma^K) = 0$ is a necessary condition in Theorem 5.3. Note that $m_2(K) > 0$ and X does not support the p–Poincaré inequality.

6. Sets of zero M_1–modulus measure and zero Cap_1^M–capacity

Relations between the Cap_1^M–capacity and the M_1–modulus measure are more complicated than those in the case $p > 1$. However, Theorem 4.1 (d) makes it possible to extend Theorem 5.3 to the M_1–modulus measure with slightly stronger assumptions. We assume that X is a proper quasiconvex metric space with a Borel regular measure μ finite on compact sets.

Theorem 6.1. *Suppose that a compact set $K \subset X$ satisfies $M_1(\Gamma^K) = 0$. Then K has zero Cap_1^M–capacity if and only if $M_1(\Gamma(K, X \setminus K)) = 0$ or $M_1(\Gamma(K)) = 0$, i.e. K has zero M_1–modulus measure.*

Proof. Suppose first that K has zero Cap_1^M–capacity. Since $M_1(\Gamma^K) = 0$ it suffices to show that $M_1(\Gamma(K, X \setminus K)) = 0$. Choose bounded open sets $G_1 \supset G_2 \supset \ldots \supset K$ such that $\cap_i G_i = K$. If now $\gamma \in \Gamma(K, X \setminus K)$ then γ meets some $X \setminus G_i$ and hence $\gamma \in \Gamma(K, X \setminus G_i)$. By Theorem 4.1 (d), $M_1(\Gamma(K, X \setminus G_i)) = 0$ for each i and since

$$\Gamma(K, X \setminus K) = \bigcup_i \Gamma(K, X \setminus G_i),$$

the subadditivity of the M_1–modulus yields $M_1(\Gamma(K, X \setminus K)) = 0$.

Suppose that $M_1(\Gamma(K, X \setminus K)) = 0$. Since also $M_1(\Gamma^K) = 0$ we have $M_1(\Gamma(K)) = 0$ and for an arbitrary bounded open set G, $\Gamma(K \cap G, X \setminus G) \subset \Gamma(K)$ and thus $M_1(\Gamma(K \cap G, X \setminus G)) = 0$. This implies that K has zero Cap_1^M–capacity. □

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