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CURVELINEAR FUNCTIONALS OF TANGENT ABELIAN DISKS IN UNIVERSAL TEICHMÜLLER SPACE

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Abstract. We investigate the analytic, geometric and potential features arising from an old Vekua problem and the related Belinskii theorem, giving essential strengthening of this theorem, and provide its generalizations to holomorphic disks in the universal Teichmüller space. This is applied to explicit estimating the curvelinear functionals associated with univalent functions and quasicircles (the Teichmüller and Grunsky norms, reflection coefficient, the first Fredholm eigenvalue).

1. Background and preliminaries

The aim of this paper is to investigate the analytic, geometric and potential features arising from an old Vekua problem and the related Belinskii theorem. We essentially strengthen this theorem and provide its generalizations to holomorphic disks in the universal Teichmüller space **T**. All this has interesting applications related to geometric analysis, complex geometry of the universal Teichmüller space and potential theory.

1.1. **The Belinskii theorem and Vekua problem.** The following theorem of P. Belinskii is a deep underlying result in the variational calculus for quasiconformal maps.

Theorem A ([5]). *Let a function $\mu(\zeta)$ be defined on the plane $\mathbb{C}$ and C^1-smooth, up to jumps on a finite number of closed smooth curves. Let*

$$|\mu(\zeta)| < \varepsilon, \ |\partial_\zeta \mu| < \varepsilon, \ |\partial_{\overline{\zeta}} \mu| < \varepsilon,$$

and let either $\mu(1/\zeta)$ or $(\zeta/\overline{\zeta})^2 \mu(1/\overline{\zeta})$ satisfy in a neighborhood of the point $\zeta = 0$ the same assumptions, as the function $\mu(\zeta)$ in the finite points. Then, for sufficiently small $\varepsilon > 0$, the function

$$w(z) = z - \frac{z(z-1)}{\pi} \iint\limits_{|\zeta|<\infty} \frac{\mu(\zeta) d\xi d\eta}{\zeta(\zeta-1)(\zeta-z)}$$

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provides a quasiconformal homeomorphism of the whole plane $\widehat{\mathbb{C}}$ whose Beltrami coefficient is $\widetilde{\mu} = \mu + O(\|\mu\|_\infty^2)$, and this map differs from the map with Beltrami coefficient $\mu(z)$ and the same normalization up to a quantity of order ε^2 uniformly in any bounded domain.

The original proof of Theorem A is complicated and relates on the deep results from geometric function theory and from the potential theory.

This theorem relates to the **problem of I.N. Vekua** posed in 1961. Consider in the space $L_p(\mathbb{C})$ with $p > 2$ the well-known integral operators

$$T\rho(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{\rho(\zeta) d\xi d\eta}{\zeta - z}, \quad \Pi\rho(\zeta) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{\rho(\zeta) d\xi d\eta}{(\zeta - z)^2} = \partial_z T\rho(z)$$

assuming for simplicity that ρ has a compact support in $\mathbb{C}$. Then the second integral exists as a Cauchy principal value, and the derivative $\partial_z T$ generically is understanding as distributional.

Each quasiconformal automorphism w^μ with $\|\mu\|_\infty = k < 1$ of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ with $\|\mu\|_\infty = k < 1$ is represented in the form $w^\mu(z) = z + T\rho(z)$, where ρ is the solution in L_p (for $2 < p < p_0(k)$) of the integral equation $\rho = \mu + \mu\Pi\rho$, given by the series

$$\rho = \mu + \mu\Pi\mu + \mu\Pi\mu(\Pi\mu) + \dots. \tag{1.1}$$

Denote by μ_n be the n-th partial sum of the series (1), and set

$$f_n(z) = z - \frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{\mu_n(\zeta) d\xi d\eta}{\zeta - z}.$$

The question of Vekua was, whether all f_n also are homeomorphisms.

Much later, T. Iwaniec constructed a counterexample which shows that the smoothness and smallness assumptions in the Belinskii theorem cannot be dropped completely. A simple modification of his construction allows us to define $\varepsilon \in (0, 1)$ and a Beltrami coefficient μ, so that the second iteration

$$f_2(z) = z + T\mu(z) + T(\mu\Pi\mu)(z)$$

is not injective in $\mathbb{D}$. The details are exposed in survey [14].

Decomposing any quasiconformal automorphism w^μ of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ via $w^\mu = w^{\mu_2} \circ w^{\mu_1}$ with the Beltrami coefficients μ_1 and μ_2 supported, respectively, in the unit disk $\mathbb{D} = \{|z| < 1\}$ and in the domain $D_2 = \widehat{\mathbb{C}} \setminus w^{\mu_1}(\mathbb{D})$, one arrives to univalent holomorphic functions with quasiconformal extension. Such functions play a crucial role in geometric complex analysis and in Teichmüller space theory.

1.2. We strengthen the theorems of Belinskii type for univalent functions by estimating the associated curvelinear functionals. This provides new rather broad classes of univalent functions with equal Grunsky and Teichmüller norms, which is important for applications. Together with the classical Kühnau-Schiffer theorem on reciprocity of the Grunsky norm to the least positive Fredholm eigenvalue of the corresponding quasicircle, this essentially increase the known collections of curves whose eigenvalues are given explicitly.

In addition, we establish that the features created by this theorem naturally appear on holomorphic disks in the universal Teichmüller space whose tangent disks are abelian.

1.3. The Grunsky inequalities for univalent functions. In 1939, Grunsky discovered the necessary and sufficient conditions for univalence of a holomorphic function in a finitely connected domain on the extended complex plane $\widehat{\mathbb{C}}$ in terms of an infinite system of the coefficient inequalities. In particular, his theorem for the canonical disk

$$\mathbb{D}^* = \{z \in \widehat{\mathbb{C}} : \ |z| > 1\}$$

yields that a holomorphic function $f(z) = z + \text{const} + O(z^{-1})$ in a neighborhood of $z = \infty$ can be extended to a univalent holomorphic function on the $\mathbb{D}^*$ if and only

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \le 1,$$

where α_{mn}, called the Grunsky coeffcients of f, are defined from the series

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\mathbb{D}^*)^2, \tag{1.2}$$

the sequence $\mathbf{x} = (x_n)$ runs over the unit sphere $S(l^2)$ of the Hilbert space l^2 with norm $\|\mathbf{x}\|^2 = \sum\limits_1^\infty |x_n|^2$, and the principal branch of the logarithmic function is chosen (cf. [10]). The quantity

$$\varkappa \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| : \ \mathbf{x} = (x_n) \in S(l^2) \Big\} \le 1 \tag{1.3}$$

is called the **Grunsky norm** of f.

For the functions with k-quasiconformal extensions ($k < 1$), we have a stronger bound

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \le k \quad \text{for any } \mathbf{x} = (x_n) \in S(l^2) \tag{1.4}$$

established first in [21] (see also [18]). Then $\varkappa(f) \le k(f)$, where $k(f)$ denotes the **Teichmüller norm** of f which is equal to the infimum of dilatations $k(w^\mu) = \|\mu\|_\infty$ of quasiconformal extensions of f to $\widehat{\mathbb{C}}$. Here w^μ denotes a homeomorphic solution to the Beltrami equation $\partial_{\overline{z}} w = \mu \partial_z w$ on $\mathbb{C}$ extending f; accordingly, μ is called the **Beltrami coefficient** (or complex dilatation) of w.

Note that the Grunsky (matrix) operator $\mathcal{G}(f) = (\sqrt{mn}\ \alpha_{mn}(f))_{m,n=1}^\infty$ acts as a linear operator $l^2 \to l^2$ contracting the norms of elements $\mathbf{x} \in l^2$; the norm of this operator equals $\varkappa(f)$. For most functions f, we have the strong inequality $\varkappa(f) < k(f)$ (moreover, the functions satisfying this inequality form a dense subset of Σ_Q see, [17], [20]), while the functions with the equal norms play a crucial role in many applications

The method of Grunsky inequalities was generalized in several directions, even to bordered Riemann surfaces X with a finite number of boundary components. We shall deal here only with unbounded simply connected domains $X = D^* \ni \infty$ with quasiconformal boundaries (quasidisks). For any such domain, one must use instead of (2) the expansion

$$-\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} \frac{\beta_{mn}}{\sqrt{mn}\ \chi(z)^m\ \chi(\zeta)^n}, \tag{1.5}$$

where χ denotes a conformal map of D^* onto the disk $\mathbb{D}^*$ so that $\chi(\infty) = \infty,\ \chi'(\infty) > 0$.

Each coefficient $\beta_{mn}(f)$ in (1.6) is represented as a polynomial of a finite number of the initial coefficients $b_1, b_2, \dots, b_s$ of f; hence it depends holomorphically on Beltrami coefficients of quasiconformal extensions of f as well as on the **Schwarzian derivatives**

$$S_f(z) = \Big(\frac{f''(z)}{f'(z)}\Big)' - \frac{1}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2, \quad z \in D^* \tag{1.6}$$

provided that $f(z)$ (and their quasiconformal extensions) have a full normalization, for example,

$$f(\infty) = \infty, \quad f'(\infty) = 1, \quad f(0) = 0. \tag{1.7}$$

The univalent functions in $\mathbb{D}^*$ normalized by (1.7) form the class Σ^0. All these functions are zero-free in $\mathbb{D}^*$ and their inversions $F_f(z) = 1/f(1/z) = z + a_2 z^2 + \dots$ are univalent in the unit disk and have the same Grunsky coefficients.

The Schwarzian derivatives (1.6) of univalent functions in D^* with quasiconformal extensions range over a bounded domain in the complex Banach space $\mathbf{B}(D^*)$ of hyperbolically bounded holomorphic functions φ in D^* with norm

$$\|\varphi\|_{\mathbf{B}(D^*)} = \sup_{D^*} \lambda_{D^*}(z)^{-2}(z)|\varphi(z)|,$$

where $\lambda_{D^*}(z)|dz|$ denotes the hyperbolic metric of D^* of Gaussian curvature -4. This domain models the **universal Teichmüller space T** with the base point $\chi'(\infty)^{-1}D^*$ (in holomorphic Bers' embedding of $\mathbf{T}$; see, e.g., [8], [9]).

A theorem of Milin [25] extending the Grunsky univalence criterion for the disk $\mathbb{D}^*$ to multiply connected domains D^* states that a holomorphic function $f(z) = z + \text{const} + O(z^{-1})$ in a neighborhood of $z = \infty$ can be continued to a univalent function in the whole domain D^* if and only if the coefficients α_{mn} in (1.5) satisfy, similar to the case of the disk $\mathbb{D}^*$, the inequality

$$\Big|\sum_{m,n=1}^{\infty} \beta_{mn}\ x_m x_n\Big| \le 1 \tag{1.8}$$

for any point $\mathbf{x} = (x_n) \in S(l^2)$. Now we consider the class $\Sigma_{D^*}(0))$ of univalent functions in D^* with hydrodynamical normalization

$$f(z) = z + b_0 + b_1 z^{-1} + \dots \quad \text{near } z = \infty,$$

added by $f(0) = 0$ (provided that $0 \in D$), and call the quantity

$$\varkappa_{D^*}(f) = \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \beta_{mn}\, x_m x_n \Big| : \ \mathbf{x} = (x_n) \in S(l^2) \Big\} \le 1 \tag{1.9}$$

the **generalized Grunsky norm** of f. Note that in the case $D^* = \mathbb{D}^*$, $\beta_{mn} = \sqrt{mn}\, \alpha_{mn}$; for this disk, we shall use the notations Σ and $\varkappa(f)$.

The quasiconformal theory of generic Grunsky coefficients (for arbitrary quasidiska) was created in [18].

Consider the subspace $A_1(D)$ of $L_1(D)$ formed by integrable holomorphic quadratic differentials $\psi(z)dz^2$ on D, and its subset

$$A_1^2(D) = \{\psi \in A_1(D) : \ \psi = \omega^2, \ \omega \text{ holomorphic}\}$$

which consists of quadratic differentials having in D only zeros of even order. Put

$$\langle \mu, \psi \rangle_D = \iint_D \mu(z)\psi(z) dxdy, \quad \mu \in L_\infty(D), \ \psi \in L_1(D) \ \ (z = x + iy).$$

Given function $f \in \Sigma_{D^*}(0)$, take its extremal quasiconformal extension f^{μ_0} to D with Beltrami coefficient $\mu_0 \in L_\infty(D)$ (hence, $k(f) = \|\mu_0\|_\infty$) and assign to this function the quantity

$$\alpha_D = \sup \left\{ \Big| \iint_D \mu_0(z)\psi(z) dxdy \Big| : \ \psi \in A_1^2(D), \ \|\psi\|_{A_1(D)} = 1 \right\}, \tag{1.10}$$

Lemma 1.1 ([13, 18]). *The Grunsky norm $\varkappa_{D^*}(f)$ of every function $f \in \Sigma(D^*)$ is estimated by its Teichmüller norm $k = k(f)$ and the quantity (1.10) via*

$$\varkappa_{D^*}(f) \le k \frac{k + \alpha_D(f)}{1 + \alpha_D(f)k},$$

and $\varkappa_D^*(f) < k$ unless $\alpha_D(f) = \|\mu_0\|_\infty$. The last equality occurs if and only if $\varkappa(f) = k(f)$.

In addition, if $\varkappa(f) = k(f)$ and the equivalence class of f (the collection of maps equal to f on ∂D) is a Strebel point (which means that this class contains the Teichmüller extremal map w^{μ_0}), then the extremal coefficient μ_0 is necessarily of the form

$$\mu_0(z) = \|\mu_0\|_\infty |\psi_0(z)|/\psi_0(z) \quad \text{with} \ \ \psi_0 \in A_1^2(D).$$

The point is that, due to the Hamilton-Krushkal-Reich-Strebel theorem, the norm of any extremal Beltrami coefficient minimizing the dilatation is attained on the unit sphere of the space $A_1(D)$, while the value of the Grunsky norm $\varkappa_{D^*}(f)$ is attained on the abelian quadratic differentials $\omega^2 dz^2$ with L_1-norm 1.

1.4. Quasireflections. The **quasiconformal relections** (or quasireflections) are the orientation reversing quasiconformal homeomorphisms of the sphere $\widehat{\mathbb{C}}$ which preserve point-wise some (oriented) quasicircle $L \subset \widehat{\mathbb{C}}$ and interchange its interior and exterior domains. In other words, quasireflections are the topological involutions of the sphere $\widehat{\mathbb{C}}$ whose fixed Jordan curve is a quasicircle.

We denote the indicated domains determined by this curve by D_L and D_L^*, respectively.

One defines for this curve L its **reflection coefficient**

$$q_L = \inf k(f) = \inf \ \|\partial_z f/\partial_{\overline{z}} f\|_\infty,$$

taking the infimum over all quasireflections across L, and **quasiconformal dilatation**

$$Q_L = (1 + q_L)/(1 - q_L) \geq 1.$$

Lemma 1.2 ([4, 24])**.** *For any qusicircle $L \subset \widehat{\mathbb{C}}$, its dilatation $Q_L = K_L^2$, where*

$$K_L = (1 + k_L)/(1 - k_L)$$

and k_L is the minimal dilatation among all orientation preserving quasiconformal automorphisms f_* of $\widehat{\mathbb{C}}$ carrying the unit circle onto L, with $k(f_*) = \|\partial_{\overline{z}} f_*/\partial_z f_*\|_\infty$.

On the properties of quasireflections and obtained results see, e.g., [3], [16], [19], [24].

1.4. Connection with Fredholm eigenvalues. Recall that the Fredholm eigenvalues ρ_n of an oriented smooth closed Jordan curve L on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ are the eigenvalues of its double-layer potential, or equivalently, of the integral equation

$$u(z) + \frac{\rho}{\pi} \int\limits_L u(\zeta) \frac{\partial}{\partial n_\zeta} \log \frac{1}{|\zeta - z|} ds_\zeta = h(z),$$

which often appears in applications (here n_ζ is the outer normal and ds_ζ is the length element at $\zeta \in L$).

The least positive eigenvalue $\rho_L = \rho_1$ plays a crucial role and is naturally connected with conformal and quasiconformal maps . It can be defined for any oriented closed Jordan curve L by

$$\frac{1}{\rho_L} = \sup \ \frac{|\mathcal{D}_G(u) - \mathcal{D}_{G^*}(u)|}{\mathcal{D}_G(u) + \mathcal{D}_{G^*}(u)},$$

where G and G^* are, respectively, the interior and exterior of L; $\mathcal{D}$ denotes the Dirichlet integral, and the supremum is taken over all functions u continuous on $\widehat{\mathbb{C}}$ and harmonic on $G \cup G^*$. In particular, $\rho_L = \infty$ only for the circle.

An upper bound for ρ_L is given by Ahlfors' inequality [3]

$$\frac{1}{\rho_L} \leq q_L, \tag{1.11}$$

where q_L denotes the minimal dilatation of quasireflections across L .

In view of the invariance of all quantities in (1.11) under the action of the Möbius group $PSL(2, \widehat{\mathbb{C}})/\pm\mathbf{1}$, it suffices to consider the quasiconformal homeomorphisms of the sphere carrying $\mathbf{S}^1$ onto L whose Beltrami coefficients $\mu_f(z) = \partial_{\overline{z}} f/\partial_z f$ have support in the unit disk $\mathbb{D} = \{|z| < 1\}$, and $f|\mathbb{D}^*(z) = z + b_0 + b_1 z^{-1} + ...$, where $\mathbb{D}^* = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ (or in the upper half-plane $U = \{\Im z > 0\}$). Then q_L is equal to the minimum $k_0(f)$ of dilatations $k(f) = \|\mu\|_\infty$ of quasiconformal extensions of the function $f^* = f|\mathbb{D}^*$ into $\mathbb{D}$.

The inequality (11) and the indicated above Kühnau-Schiffer theorem (stating that $\varkappa(f) = 1/q_{f(\mathbf{S}^1)}$ serve as a background for defining the value ρ_L. Another useful tool here is the Kühnau jump inequality [24], which asserts that if a closed curve $L \subset \widehat{\mathbb{C}}$ contains two analytic arcs with the interior intersection angle $\pi\alpha$, then

$$\frac{1}{\rho_L} \geq |1 - |\alpha||;$$

the restriction of analiticity can be essentially weakened. The results obtained in these ways can be found in surveys [14], [18], [22], [24] and the references cited there.

1.5. Holomorphic functions generated by Grunsky coefficients. The Grunsky-Milin coefficients $\beta_{mn}(f^\mu)$ of the functions $f^\mu \in \Sigma_{D^*}(0)$ generate for each $\mathbf{x} = (x_n) \in l^2$ with $\|\mathbf{x}\| = 1$ the holomorphic maps

$$h_{\mathbf{x}}(\mu) = \sum_{m,n=1}^{\infty} \beta_{mn}(f^\mu) x_m x_n : \ \mathrm{Belt}(D)_1 \to \mathbb{D}, \tag{1.12}$$

so that

$$\sup_{\mathbf{x}} |h_{\mathbf{x}}(f^\mu)| = \varkappa_{D^*}(f^\mu).$$

The holomorphy of these functions follows from the holomorphy of coefficients β_{mn} with respect to Beltrami coefficients $\mu \in \mathrm{Belt}(D)_1$ and to Schrazians mentioned above using the estimate

$$\Big| \sum_{m=j}^{M} \sum_{n=l}^{N} \beta_{mn} x_m x_n \Big|^2 \leq \sum_{m=j}^{M} |x_m|^2 \sum_{n=l}^{N} |x_n|^2$$

which holds for any finite M, N and $1 \leq j \leq M, \ 1 \leq l \leq N$. This estimate is a simple corollary of the Milin univalence theorem (cf. [25], p. 193; [28], p. 61).

The functions (1.12) imply a powerful tool in applications of the Grunsky operator to Teichmüller spaces.

2. Main theorems

2.1. A direct strengthening of the Belinskii theorem. Let $L \subset \mathbb{C}$ be an oriented closed $C^{1+\sigma}$-smooth Jordan curve (hence, a quasicircle), separating the points 0 and ∞ ($\sigma > 0$). Denote its interior and exterior domains by D_L and D_L^*, respectively, and consider the corresponding spaces $A_1(D_L)$ and $A_1^2(D_L)$.

Theorem 2.1. *Let a function $\psi = \varphi^2 \in A_1^2(D_L)$ be $C^{1+\sigma}$-smooth on the closed domain $\overline{D_L}$, with exception of a finite number of (simple) poles on L, and let ψ have on the boundary L only a finite number of zeros. Then, for sufficiently small $|t| > 0$, the map*

$$w_\varphi(z;t) = z - \frac{t}{\pi} \iint\limits_{D_L} \frac{|\psi(\zeta)|/\psi(\zeta)}{\zeta - z} d\xi d\eta = z - \frac{t}{\pi} \iint\limits_{D_L} \frac{\overline{\varphi(\zeta)}/\varphi(\zeta)}{\zeta - z} d\xi d\eta \tag{2.1}$$

provides a quasiconformal automorphism of the sphere $\widehat{\mathbb{C}}$ with the complex dilatation

$$\mu(z,t) = t|\psi(z)|/\psi(z) + O(t^2) \quad \textit{for} \ \ z \in D_L$$

and conformal on D_L^**, and this map has equal Teichmüller and Grunsky norms satisfying*

$$k(w_\varphi(\cdot;t)) = \varkappa_{D_L^*}(w_\varphi(\cdot;t)) = |t| + O(|t|^2), \tag{2.2}$$

with uniform estimate of the remainder.

In the case of the disk, this theorem simultaneously implies the approximate values of the reflection coefficients and Fredholm eigenvalues of quasicircles $L_t = w_\varphi(\mathbf{S}^1; t)$.

Corollary 2.2. *If the curve* L *is the unit circle* $\mathbf{S}^1$ *(hence,* $D_L = \mathbb{D}$*), then the equalities (2.2) imply the following sharp approximate representation of the Fredholm eigenvalues and quasireflection coefficients of quasicircles* $L_t = w_\varphi(\mathbf{S}^1; t)$*:*

$$\begin{aligned} k(w_\varphi(\cdot;t)) = \varkappa(w_\varphi(\cdot;t)) &= \frac{1}{\rho_{L_t}} \\ &= \sup_{\psi \in A_1^2(\mathbb{D}), \|\psi\|_{A_1}=1} |t| \Big| \iint_{\mathbb{D}} \mu(z,t)\psi(z)dxdy \Big| + O(|t|^2) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} Q_{L_t} &= \Big(\frac{1 + k(w_\varphi(\cdot;t))}{1 - k(w_\varphi(\cdot;t))}\Big)^2 \\ &= 1 + 4|t| \sup_{\psi \in A_1^2(\mathbb{D}), \|\psi\|_{A_1}=1} \Big| \iint_{\mathbb{D}} \mu(z,t)\psi(z)dxdy \Big| + O(|t|^2). \end{aligned} \tag{2.4}$$

2.2. Extension to holomorphic disks in universal Teichmüller space T. Theorem 1 can be generalized to some holomorphic disks in the space $\mathbf{T}$ whose tangent disks at the origin are geodesic in the Teichmüller-Kobayashi metric of this space.

First recall that the points of $\mathbf{T}$ representing the univalent functions in U^*, which admit the extremal extensions of Teichmüller type, are called **Strebel points**. Such points are dense in $\mathbf{T}$.

Theorem 2.3. *Let* $h: \ t \to \varphi(\cdot, t) = S_f(z,t)$ *be a holomorphic map from a disk* $\mathbb{D}_\varepsilon = \{|t| < \varepsilon\}$ *into the universal Teichmüller space* $\mathbf{T}$ *such that for small* $|t| > 0$*,*

$$\varphi(z,t) = t\varphi_0(z) + t^2\varphi_1(z) + \dots$$

with $\varphi_0 \in A_1^2(\mathbb{D})$ *and* $\|\varphi_0\|_{A_1} = 1$*. Then:*

(a) *for sufficiently small* $|t| > 0$*, the corresponding Schwarzians* $S_f(z,t)$ *form a holomorphic disk* $S_f(\mathbb{D})$ *in the space* $\mathbf{T}$*, and for sufficiently small* $|t| > 0$ *the extremal Beltrami coefficients*

$$\mu_t(z) = \mu(z,t) = \partial_{\bar z} f(z,t)/\partial_z f(z,t)$$

are of the form

$$\mu(z,t) = t\frac{|\varphi_0(z)|}{\varphi_0(z)} + O(|t|^2), \tag{2.5}$$

and this estimate is sharp and uniform for $|t| < t_0$ *in* L_∞*-norm;*

(b) *the corresponding integrals (2.1) for* f^μ *with* μ *given by (2.5) represent* $(|t| + O(|t|^2))$*-quasiconformal automorphisms of the complex plane* $\widehat{\mathbb{C}}$ *with equal Teichmüller and Grunsky norms;*

(c) *for small* $|t| > 0$*, the corresponding Fredholm eigenvalues* ρ_{L_t} *and quasireflection coefficients* Q_{L_t} *are represented similar to (2.3), (2.4).*

It can be shown by applying the methods, which are far from the framework of this paper, that generically the points $S_f(\cdot, t)$ are not Strebel.

2.3. Two examples. We illustrate Theorem 2.1 on the cases, when L is the unit circle $\mathbf{S}^1$ or an ellipse. These cases are of special interest, because one can find explicitly an orthonormal basis in $A_1^2(D_L)$, which provides a lot of new sets of univalent functions f with explicitly given Teichmüller and Grunsky norms and Fredholm eigenvalues of quasicircles $f(\mathbf{S}^1)$.

Example 1. Let D be the unit disk $\mathbb{D}$. It is shown in [13] that every holomorphic quadratic differential $\psi \in A_1^2(\mathbb{D})$, i.e., with zeros of even order in $\mathbb{D}$ has the form

$$\psi(z) = \frac{1}{\pi} \sum_{m+n=0}^{\infty} \sqrt{mn}\, x_m x_n z^{m+n} = \frac{1}{\pi} \Big(\sum_{0}^{\infty} \sqrt{n} x_n z^n \Big)^2,$$

and $\|\psi\|_{A_1(\mathbb{D})} = \|\mathbf{x}\|_{l^2} = 1,\ \mathbf{x} = (x_n)$. For any such ψ obeying the assumptions of Theorem 2.1, i.e., sufficiently smooth up to the boundary circle $\mathbf{S}^1$, the equalities (2.2) are valid.

In particular, all this is valid for any polynomial

$$P_N(z) = \sum_{0}^{N} \sqrt{n} x_n z^n \quad \text{with} \quad \sum_{0}^{N} |x_n|^2 = 1$$

and $\psi(z) = P_N^2(z)$. In this case, the extremal $\mu_0(z) = t\overline{P_N(z)}/P_N(z)$, and the representations (2.3), (2.4) for the Fredholm eigenvalues of quasicircles L_t and of their quasireflection dilatations assume the form

$$\frac{1}{\rho_{L_t}} = |t| + O(|t|^2), \quad Q_{L_t} = \Big(\frac{1 + |t| + O(|t|^2)}{1 - |t| + O(|t|^2)} \Big)^2 = 1 + 4|t| + O(|t|^2);$$

so, up to the quantities of order 2, these values do not depend on N.

Example 2. Let D^* be the exterior $D_{\mathcal{E}}^*$ of the ellipse $\mathcal{E}$ with the foci at $-1, 1$ and semiaxes a, b $(a > b)$. The branch of the function

$$\chi(z) = (z + \sqrt{z^2 - 1})/(a + b)$$

positive for real $z > 1$ maps this exterior onto $\mathbb{D}^*$. A conformal map of the interior of this ellipse $D_{\mathcal{E}}$ onto the disk involves an elliptic function.

As is well known (see [27]), an orthonormal basis in the space

$$A_2(D_{\mathcal{E}}) = \{\omega \in L_2(D_{\mathcal{E}}) :\ \omega \ \text{ holomorphic in } \ D_{\mathcal{E}}\}$$

is formed by the polynomials

$$P_n(z) = 2\sqrt{\frac{n+1}{\pi}}\, (r^{n+1} - r^{-n-1})\, U_n(z),$$

where $r = (a+b)^2$ and $U_n(z)$ are the Chebyshev polynomials of the second kind,

$$U_n(z) = \frac{1}{\sqrt{1-z^2}} \sin[(n+1)\arccos z], \quad n = 0, 1, \dots \ .$$

Using the Riesz-Fisher theorem, one obtains that each function $\psi \in A_2(D_{\mathcal{E}})$ is of the form (cf. [13])

$$\psi(z) = \sum_0^\infty x_n P_n(z), \quad \mathbf{x} = (x_n) \in l^2,$$

with $\|\psi\|_{A_2} = \|\mathbf{x}\|_{l^2}$.

Now we have that a function $f \in \Sigma^0(D_{\mathcal{E}}^*)$ with Teichmüller extension f^μ to $D_{\mathcal{E}}$ satisfies

$$\varkappa_{D_{\mathcal{E}}^*}(f) = k(f) = \kappa$$

if and only if any its extremal Beltrami coefficient $\mu \in \mathrm{Belt}(D_{\mathcal{E}})$ satisfies

$$\sup \Big| \Big\langle \mu, \sum_{m,n\ge 0}^\infty x_m x_n P_m P_n \Big\rangle_{D_{\mathcal{E}}} \Big| = \|\mu\|_\infty = \kappa,$$

taking the supremum over all $\mathbf{x} = (x_n) \in l^2$ with $\|\mathbf{x}\| = 1$. In particular, this holds for

$$\mu(z) = \kappa \overline{\sum_0^\infty x_n^0 P_n(z)} \Big/ \sum_0^\infty x_n^0 P_n(z)$$

with some $\mathbf{x}^0 = (x_n^0) \in S(l^2)$.

Note also that for every $f \in \Sigma^0(D_{\mathcal{E}}^*)$, its constant $\alpha_{D_{\mathcal{E}}}(f)$ is given explicitly by

$$\alpha_{D_{\mathcal{E}}}(f) = \sup_{\mathbf{x}=(x_n)\in S(l^2)} \Big| \iint_{D_{\mathcal{E}}} \frac{\mu(z)}{\|\mu\|_\infty} \sum_{m,n\ge 0}^\infty x_m x_n P_m(z) P_n(z) dx dy \Big|,$$

taking any extremal μ in the equivalence class of f. Similar to the above case, the equalities (2.2) are valid for all integrals (2.1) with $\mu(z) = |S_N(z)|/S_N(z)$ defined by the finite sums

$$S_N(z) = 2 \sum_0^N \sqrt{\frac{n+1}{\pi}} \, (r^{n+1} - r^{-n-1}) \, U_n(z).$$

3. Proof of Theorem 1

Under the assumptions on the Beltrami coefficient $\mu(z) = t|\psi(z)|/\psi(z) = t\overline{\varphi(z)}/\varphi(z)|$, the quasiconformality of the map $w_\varphi(z;t)$ follows from Theorem A, and by this theorem, the Beltrami coefficient $\widetilde{\mu}(z)$ of $w_\varphi(z;t)$ equals $\mu(z) + O(|\mu(z)|^2)$ (equivalently, $k(w_\varphi(\cdot;t)) = |t| + O(|t|^2)$). Indeed,

$$\widetilde{\mu}(z) = \mu/(1 + \Pi\mu(z))$$

and the indicated estimates follow from the well-known properties of operators T and Π in the Hölder spaces $C^{m+\alpha}$ (also from the lambda-lemma for holomorphic motions).

Hence, the crucial step in the proof is to establish that the map (13) has equal Teichmüller and Grunsky norms.

We first establish how uniform is the bound (2.2). The arguments from [5] indicated above provide that there exists $r_0 > 0$ such that the integrals in (2.1) determine quasiconformal (homeomorphic) map for any fixed t with $|t| < r_0$. Let $\mathbb{D}_{r_0} = \{|t| < r_0\}$.

For such t, the integrals(2.1) define a holomorphic motion $\widehat{\mathbb{C}} \times \mathbb{D}_{r_0} \to \widehat{\mathbb{C}}$, that means, $w_\varphi(z;t)$ is injective in $z \in \widehat{\mathbb{C}}$ for any fixed t, holomorphic in t for a fixed z, and $w_\varphi(z;0) \equiv z$ (in fact, $w_\varphi(z;t)$ can be defined in a broader domain containing the disk $\mathbb{D}_{r_0}$). By the lambda-lemma for holomorphic motions, the Beltrami coefficient

$$\mu_t(z) := \mu(z;t) = \partial_{\overline{z}} w_\varphi(z;t)/\partial_z w_\varphi(z;t)$$

defines a holomorphic map from the disk $\mathbb{D}_{r_0}$ into the unit ball of the space $L_\infty(\widehat{\mathbb{C}})$, and this map, in view of Theorem A, must have near $t = 0$ the expansion

$$\mu(z;t) = t\mu_0(z) + t^2\mu_1(z) + \dots, \tag{3.1}$$

where μ_0 is of Teichmüller form; moreover,

$$\mu_0 = |\psi|/\psi \quad \text{with given} \quad \psi \in A_1^2(D_L^*).$$

This implies

$$\|\mu(z;t)\|_\infty = |t| + O(|t|^2),$$

with uniform bound in L_∞-norm.

Now, using, if needed, the similarity map $z \mapsto rz$, $r > 0$, we may assume that the Riemann mapping function $\chi: \mathbb{D}^* \to D_L^*$ has the expansion $\chi(z) = z + b_0 + b_1 z^{-1} + \dots$ (i.e. $\chi'(\infty) = 1$). Then the domain D_L^* can be chosen to be the base point of the universal Teichmüller space $\mathbf{T}$.

To establish the equality of Teichmüller and Grunsky norms claimed above, we apply the following important result from [18] which implies that the Grunsky norm is lower semicontinuous in the weak topology (of locally uniform convergence) on the set $\Sigma^0(D_L^*)$ and locally Lipschitz continuious with respect to Teichmüller metric.

Lemma 3.1. .

(i) *If a sequence $\{f_n\} \subset \Sigma^0(D^*)$ is convergent locally uniformly on D^* to f_0, then*

$$\varkappa_{D^*}(f_0) \le \liminf_{n\to\infty} \varkappa_{D^*}(f_n).$$

(ii) *The functional $\varkappa_{D^*}(\varphi)$ regarded as a function of points $\varphi = S_f$ from the universal Teichmüller space $\mathbf{T}$ is locally Lipschitz continuous and logarithmically plurisubharmonic on $\mathbf{T}$.*

The Teichmüller norm has similar properties. Its continuity and plurisubharmonicity is a consequence, for example, of the following result strengthening of the fundamental Royden-Gardiner theorem.

Lemma 3.2 ([14])**.** *The differential (infinitesimal)Kobayashi metric $\mathcal{K}_{\mathbf{T}}(\varphi, v)$ on the tangent bundle $\mathcal{T}(\mathbf{T})$ of the universal Teichmüller space $\mathbf{T}$ is logarithmically plurisubharmonic in $\varphi \in \mathbf{T}$, equals the canonical Finsler structure $F_{\mathbf{T}}(\varphi, v)$ on $\mathcal{T}(\mathbf{T})$*

generating the Teichmüller metric of $\mathbf{T}$ *and has constant holomorphic sectional curvature* $\kappa_{\mathcal{K}\mathbf{T}}(\varphi, v) = -4$ *on* $\mathcal{T}(\mathbf{T})$.

The proof of these lemmas essentially involves the holomorphy of functions (1.12) generated by the Grunsky coefficients.

Subharmonicity allows one to apply the maximum principle for estimating the distortion of functionals depending on Teichmüller and Grunsky norms.

The proof of local Lipschitz continuity of Teichmüller metric on the space $\mathbf{T}$ is elementary. Indeed, for any two points φ_1, $\varphi_2 \in \mathbf{T}$, there exists an extremal (geodesic in Teichmüller metric) disk $\mathbb{D}_{12}$ passing trough these points, and the Teichmüller distance on this disk is equal to the hyperbolic distance on the unit disk (with differential form of Gaussian curvature -4). This last distance is locally Lipschitz on $\mathbb{D}_{12}$.

This continuity can be derived also from some other results in quasiconformal theory.

Now the desired equalities (2.2) follow from (3.1) and indicated local Lipschitz continuity of the Teichmüller and Grunsky norms. This completes the proof of Theorem 2.1.

The assertions of Corollary 2.1 follow from Theorem 2.1 and above lemmas.

4. Additional remarks

1. Proof of Theorem 2.2. The arguments applied above in the proof of Theorem 2.1 are extended straightforwardly to more general holomorphic disks in the space $\mathbf{T}$ presented in Theorem 2.2. Its proof follows the same lines (in fact, Theorem 2.1 is a special case of the second theorem).

2. To Theorem 2.1. There is also another more complicated proof of this theorem based on comparison of the infinitesimal Kobayashi-Teichmüller metric with a metric generated by functions (1.12) (the infinitesimal form of the Grunsky structure) and their curvature properties. It follows the lines of [17] and involves the generalized curvatures of subharmonic metrics.

The **generalized Gaussian curvature** κ_λ of an upper semicontinuous Finsler metric $ds = \lambda(t)|dt|$ in a domain $\Omega \subset \mathbb{C}$ is defined by

$$\kappa_\lambda(t) = -\frac{\Delta \log \lambda(t)}{\lambda(t)^2}, \tag{4.1}$$

where Δ is the **generalized Laplacian**

$$\Delta\lambda(t) = 4 \liminf_{r\to 0} \frac{1}{r^2}\Big\{\frac{1}{2\pi}\int_0^{2\pi} \lambda(t + re^{i\theta})d\theta - \lambda(t)\Big\}$$

(provided that $-\infty \le \lambda(t) < \infty$). Similar to C^2 functions, for which Δ coincides with the usual Laplacian, one obtains that λ is subhatrmonic on Ω if and only if $\Delta\lambda(t) \ge 0$; hence, at the points t_0 of local maximuma of λ with $\lambda(t_0) > -\infty$, we have $\Delta\lambda(t_0) \le 0$.

The sectional **holomorphic curvature** of a Finsler metric on a complex Banach manifold X is defined in a similar way as the supremum of the curvatures (4.1) over appropriate collections of holomorphic maps from the disk into X for a given tangent direction in the image. The holomorphic curvature of the Kobayashi metric $\mathcal{K}_X(x, v)$ of any complete hyperbolic manifold X satisfies $\kappa_{\mathcal{K}} \geq -4$ at all points (x, v) of the tangent bundle $\mathcal{T}(X)$ of X, while for the metric $\lambda_{\varkappa}(S_f, v)$ generated on $\mathbf{T}$ by the Grunsky coefficients we have $\kappa_{\lambda_{\varkappa}}(x, v) \leq -4$ (cf., e.g., [1], [6], [1], [14]). All this is essentially applied in the proof of Lemma 3.2 in [14].

3. Open question. It would be very interesting to find the extent in which the equalities (2.2) in the above theorems and their consequences remain valid for Beltrami coefficients of non-Teichmüller type.

4. One must keep in the mind, that if a curve L and the boundary values of $f \in \Sigma_{D_L^*}(0)$ are sufficiently regular, then f necessarily admits extremal quasiconformal extension of Teichmüller type (see [9], [30]). In view of this, the assumptions of Theorems 2.1 and 2.2 can be regarded to be enough general.

5. For $\mu \in L_\infty(\mathbb{D}^*)$ satisfying $\mu(z) \to 0$ as $|z| \to 1$ (for example, $|\mu(z)| = O(|z| - 1)^{1+\sigma})$), one can strengthen the assertion of Theorem A (accordingly, taking more special holomorphic disks in Theorems 2.1 and 2.2), getting an explicit upper bound for admissible ε. This will be given in a separate paper.

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