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FROM THE THEOREMS OF MORSE, SARD, DUBOVITSKIĬ AND FEDERER TO THE LUZIN N-PROPERTY: THE STORY SO FAR

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Dedicated to the bright memory of Yurii Reshetnyak

Abstract. In this survey we demonstrate the universal synthesis of all the above mentioned analytic phenomena for continuous mappings of Hölder and Sobolev classes. This concludes the long time research started with our previous joint papers with Jean Bourgain (2013, 2015).

1. Introduction

The Morse–Sard theorem in its classical form states that the image of the set of critical points of a smooth mapping from an open subset $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^d$ has zero Lebesgue measure in $\mathbb{R}^d$. More precisely, assuming that the mapping $v\colon \Omega \to \mathbb{R}^d$ is C^k, the set of critical points for v is

$$Z_v = \{x \in \Omega \,:\, \operatorname{rank} \nabla v(x) < d\}, \tag{1.1}$$

and the conclusion is that

$$\mathscr{L}^d(v(Z_v)) = 0 \tag{1.2}$$

provided $k \geq \max\{n-d+1, 1\}$. The theorem was proved by Morse [64] in 1939 in the scalar case $d = 1$ and subsequently by Sard [78] in 1942 for the general vector–valued case $d > 1$. The celebrated results of Whitney [86] show that the C^{n-d+1} smoothness assumption on the mapping v is sharp: the conclusion (1.2) may fail if the map v is only C^j for $j < n-d+1$.

This result has a huge number of applications: it is one of the main technical tools in differential topology of manifolds (see for instance [62]), in real analysis (see for instance [28]), in mathematical economics (see for instance [9, 51]), and even in mathematical fluid mechanics (see in particular [49, 50]), etc. We are far from intending to indicate all possible applications of this remarkable theorem.

On the one hand, and at a superficial level, the statement of the Morse-Sard theorem (1.2) looks quite natural and is in agreement with intuition. On the other hand, and after closer inspection, there is something mysterious in the formulation

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of the theorem itself. Indeed, in order to make sense of its statement only the first order partial derivatives of the mapping v are required, but in the theorem we have to assume additional C^{n-d+1}-regularity!? It looks strange. From general intuition, the image of the set of critical points should be "small" even for C^1-mappings! Just recently an essential progress was achieved in the understanding of this mysterious phenomenon. Our paper is devoted to the survey of this recent progress. But before we turn to that we must recall another interesting phenomenon of real analysis: the Luzin N-property, namely that the image $v(E)$ has zero measure whenever E has zero measure. More explicitly, a continuous map $v\colon \Omega \to \mathbb{R}^n$ of an open subset $\Omega \subset \mathbb{R}^n$ is said to satisfy the Luzin N-condition if it preserves Lebesgue null sets: for subsets $E \subset \Omega$ we have that

$$\mathscr{L}^n(v(E)) = 0 \quad \text{whenever} \quad \mathscr{L}^n(E) = 0. \tag{1.3}$$

The Luzin N-property plays a fundamental role in various results from classical real analysis and differentiation theory. Early applications primarily concerning the case $n = 1$ can be found in [77]. Its crucial importance in dimension $n > 1$ is witnessed by its role for the validity of various formulas in geometric measure theory and geometric function theory – see e.g. [69, Th. 3, p. 364] and [59].

The formulation of the Morse–Sard theorem seems to be very different from the Luzin N-property. Nevertheless, evidently, the conclusions of both of them are very similar. They assert, roughly speaking, that the image $v(E)$ has small measure if the rank of differential of v on E is small (Morse–Sard), or if the set E itself is small (Luzin). Recently the natural synthesis of these two phenomena was obtained [11, 63, 33], which is another topic of the present paper. Moreover, a natural extension of the Luzin N condition, where the Lebesgue measure is replaced by lower dimensional Hausdorff measures, was decisive in works extending the Morse-Sard theorem and related results to classes of Sobolev maps which need neither be Lipschitz nor everywhere differentiable – see, in particular, [16, 17, 32].

2. From the Morse-Sard to the Dubovitskiĭ-Federer theorems: a bridge across the Hölder and Sobolev spaces

2.1. **Classical smoothness and Hölder regularity.** As we have already mentioned, by the elegant results of Whitney [86] (extended further by many authors, see, e.g., [65], [30, §3.4.3], see [36] for an elementary exposition), the additional smoothness assumptions on the mapping v in the Morse–Sard theorem are sharp, i.e., the conclusion (1.2) fails in general for C^k-smooth mappings with $k < n-d+1$. Only minor refinements and weakening of the smoothness assumptions turn out to be possible. For example, Bates proved the validity of (1.2) for $C^{n-d,1}$-smooth mappings, i.e., the higher $(n-d)$-derivatives satisfy the Lipschitz condition; for other attempts to relax 'a bit' the classical smoothness assumptions, see, for example [7, 65], etc. As it was mentioned by Norton [65, page 369], the absence of a Fubini-type theorem for Hausdorff measures is an obstacle for proofs of some new Morse–Sard type theorems. However, in 1957 Dubovitskiĭ obtained the following pioneering result that gives valuable "smallness" information also for less smooth mappings.

Theorem 2.1 (Dubovitskiĭ [26])**.** *Let $n, d, k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be an open subset and let $v\colon \Omega \to \mathbb{R}^d$ be a C^k–smooth mapping. Put $\nu = n - d - k + 1$. Then*

$$\mathscr{H}^\nu(Z_v \cap v^{-1}(y)) = 0 \qquad \text{for } \mathscr{L}^d \text{ a.a. } y \in \mathbb{R}^d, \tag{2.1}$$

where $\mathscr{H}^\nu$ denotes the ν–dimensional Hausdorff measure and Z_v is the set of critical points (see (1.1)).

Let us explicit note that there are no restrictions on the dimensions n, d and that both cases $d \le n$ and $d > n$ are included; we interpret $\mathscr{H}^\beta$ as the counting measure when $\beta \le 0$. Thus for $k \ge n - d + 1$ we have $\nu \le 0$, and $\mathscr{H}^\nu$ in (2.1) becomes simply the counting measure, so the Dubovitskiĭ theorem contains the Morse–Sard theorem as particular case[1].

A few years later and almost simultaneously, Dubovitskiĭ [27] in 1967 and Federer [30, Theorem 3.4.3] in 1969[2] published another important generalization of the Morse–Sard theorem.

Theorem 2.2 (Dubovitskiĭ–Federer)**.** *For $n, k, d \in \mathbb{N}$ let $m \in \{0, \dots, \min(n, d) - 1\}$, $\Omega \subset \mathbb{R}^n$ be an open subset and $v\colon \Omega \to \mathbb{R}^d$ be a C^k–smooth mapping. Put $q_\circ = m + \frac{n-m}{k}$. Then*

$$\mathscr{H}^{q_\circ}(v(Z_{v,m})) = 0, \tag{2.2}$$

where $Z_{v,m}$ denotes the set of m–critical points of v defined as

$$Z_{v,m} = \{x \in \Omega \,:\, \operatorname{rank} \nabla v(x) \le m\}. \tag{2.3}$$

In 2001 Moreira [63] extended the last result to the Hölder class $\mathrm{C}^{k,\alpha}$, i.e., he proved that for a mapping $v \in \mathrm{C}^{k,\alpha}(\Omega, \mathbb{R}^d)$ the equality (2.2) holds with $q_\circ = m + \frac{n-m}{k+\alpha}$. Recall, that a mapping $v\colon \Omega \to \mathbb{R}^d$ belongs to the class $\mathrm{C}^{k,\alpha}(\Omega, \mathbb{R}^d)$ for some positive integer k and $0 < \alpha \le 1$ if for each compact subset K of Ω there exists a constant $L = L_K \ge 0$ such that for multi-indices β of length $|\beta| \le k$ we have

$$|\partial^\beta v(x) - \partial^\beta v(y)| \le L\,|x - y|^\alpha \qquad \text{for all } x, y \in K.$$

Here $|\cdot|$ is the usual euclidean norm on $\mathbb{R}^d$ on the left-hand side and on $\mathbb{R}^n$ on the right-hand side. To simplify the notation, let us make the following agreement: for $\alpha = 0$ we identify $\mathrm{C}^{k,\alpha}$ with usual spaces of C^k-smooth mappings.

Now the very natural question arose. Theorem 2.1 asserts that $\mathscr{H}^m$-almost all preimages are small (with respect to $\mathscr{H}^\nu$-measure), and Theorem 2.2 claims that $\mathscr{H}^{q_\circ}$-almost all preimages are empty. Could we connect these results? More precisely, could we say something about $\mathscr{H}^q$–almost all preimages for other values of q, say, for $q \in [m - 1, q_\circ]$? The affirmative answer is contained in the next theorem, which is formulated for convenience in the general scale of Hölder spaces.

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[1] It is interesting to note that for a long time this first Dubovitskiĭ theorem remained almost unnoticed by mathematicians outside the Soviet Union; another proof was given in the recent paper [14] covering also some extensions to the case of Hölder spaces; see also [37] for the Sobolev case.

[2] Federer announced [29] his result in 1966, this announcement (without any proofs) was sent on 08.02.1966. For the historical details, Dubovitskiĭ sent his paper [27] (with complete proofs) a month earlier, on 10.01.1966.

Theorem 2.3 (Bridge D.-F. theorem [38, 32])**.** *Let $m \in \{0,\dots,n-1\}$, $k \geq 0$, $0 \leq \alpha \leq 1$, $k+\alpha \geq 1$, $d > m$, $\Omega \subset \mathbb{R}^n$ be an open subset, and $v \in C^{k,\alpha}(\Omega,\mathbb{R}^d)$. Then for any $q \in (m,\infty)$ the equality*

$$\mathscr{H}^{\mu_q}(Z_{v,m} \cap v^{-1}(y)) = 0 \qquad \text{for } \mathscr{H}^q\text{-a.a. } y \in \mathbb{R}^d$$

holds, where

$$\mu_q = n - m - (k+\alpha)(q-m),$$

and $Z_{v,m}$ denotes the set of m-critical points of v (see (2.3)).

Let us note that for the classical C^k-case, i.e., when $\alpha = 0$, the behaviour of the function μ_q is very natural:

$\mu_q = 0$ for $q = q_\circ = m + \frac{n-m}{k}$ (Dubovitskiĭ–Federer Theorem 2.2);

$\mu_q < 0$ for $q > q_\circ$ [ibid.];

$\mu_q = \nu$ for $q = m+1$ (Dubovitskiĭ Theorem 2.1);

$\mu_q = n - m$ for $q = m$.

The last value cannot be improved in view of the elementary example of a linear mapping $L\colon \mathbb{R}^n \to \mathbb{R}^d$ of rank m.

Thus, Theorem 2.3 contains all the previous theorems (Morse–Sard, 2.1–2.2, and even the Bates theorem for $C^{k,1}$-Lipschitz functions [10]) as particular cases.

Intuitively, the meaning of the Bridge Theorem 2.3 resembles the *Heisenberg's uncertainty principle in theoretical physics: the more precise information we have on the measure of the image of the critical set*, the less precisely the preimages are described, and vice versa.

Remark 2.4. As we mentioned before, for $q = q_\circ = m + \frac{n-m}{k+\alpha}$ and $\mu_q = 0$ (as in the Dubovitskiĭ-Federer Theorem 2.2) the assertion of Theorem 2.3 was proved in 2001 in the paper of Moreira [63]. For the minimal rank value $m = 0$ (i.e., when the gradient totally vanishes on the critical set) and $q = q_\circ = \frac{n}{k+\alpha}$, $\mu_q = 0$, the assertion of Theorem 2.3 was proved by Kucera [52] in 1972. Further, for the particular case $q = m+1 = d$ (as in the Dubovitskiĭ Theorem 2.1) and under the additional assumption that

$$|\partial^\beta v(x) - \partial^\beta v(y)| \leq \omega\big(|x-y|\big)\cdot|x-y|^\alpha \quad \text{with } \omega(r) \to 0 \text{ as } r \to 0, \tag{2.4}$$

holds for all multi-indices β of lenght $|\beta| \leq k$ and x, $y \in \Omega$the assertion of Theorem 2.3 was proved in the paper [14] by Bojarski et al. from 2005. Under the same asymptotic assumption (2.4) the above result by Moreira (i.e., when $q = q_\circ$, $\mu_q = 0$) was proved by Yomdin in the paper [87] in 1983. In fact, Yomdin obtained some interesting refinements of the previous Morse-Sard results, in that he obtained estimates in terms of Minkowski contents and also stability versions of these bounds.

The next natural step is to extend the above bridge theorem to the case of Sobolev spaces, but first we need some preparations and preliminary results.

2.2. **Some preliminaries.** For a subset S of $\mathbb{R}^n$ we write $\mathscr{L}^n(S)$ for its outer Lebesgue measure. The m–dimensional Hausdorff measure is denoted by $\mathscr{H}^m$ and the m–dimensional Hausdorff content by $\mathscr{H}^m_\infty$. Recall that for any subset S of $\mathbb{R}^n$ we have by definition

$$\mathscr{H}^m(S) = \lim_{\alpha \searrow 0} \mathscr{H}^m_\alpha(S) = \sup_{\alpha > 0} \mathscr{H}^m_\alpha(S),$$

where for each $0 < \alpha \leq \infty$,

$$\mathscr{H}^m_\alpha(S) = \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} S_i)^m \ : \ \operatorname{diam} S_i \leq \alpha, \quad S \subset \bigcup_{i=1}^{\infty} S_i \right\}.$$

It is well known that $\mathscr{H}^n(S) = \mathscr{H}^n_\infty(S) \sim \mathscr{L}^n(S)$ for sets $S \subset \mathbb{R}^n$.

To simplify the notation, we write $\|f\|_{\mathrm{L}_p}$ instead of $\|f\|_{\mathrm{L}_p(\mathbb{R}^n)}$, etc.

For an open subset Ω of $\mathbb{R}^n$ the Sobolev space $\mathrm{W}^k_p(\Omega, \mathbb{R}^d)$ is as usual defined as consisting of those $\mathbb{R}^d$-valued functions $f \in \mathrm{L}_p(\Omega)$ whose distributional partial derivatives of orders $m \leq k$ belong to $\mathrm{L}_p(\Omega)$. The simplified notation $\mathrm{W}^k_p(\Omega)$ will be used for real-valued functions. The corresponding local spaces are indicated with a subscript, such as $\mathrm{W}^k_{p,\mathrm{loc}}(\Omega, \mathbb{R}^d)$, etc. (for detailed definitions and differentiability properties of such functions see, e.g., [28], [60], [88], [24]). Denote by $\nabla^m f$ the vector-valued function consisting of all m-th order partial derivatives of f arranged in some fixed order. We use the norm

$$\|f\|_{\mathrm{W}^k_p} = \|f\|_{\mathrm{L}_p} + \|\nabla f\|_{\mathrm{L}_p} + \cdots + \|\nabla^k f\|_{\mathrm{L}_p},$$

where in each case $\big\|\nabla^m f\big\|_{\mathrm{L}_p} = \big\||\nabla^m f|\big\|_{\mathrm{L}_p}$ and $|\nabla^m f(x)|$ is the usual euclidean norm of $\nabla^m f(x)$. Working with locally integrable functions, we always assume that the precise representatives are chosen. If $w \in \mathrm{L}_{1,\mathrm{loc}}(\Omega)$, then the precise representative w^* is defined for *all* $x \in \Omega$ by

$$w^*(x) = \begin{cases} \lim\limits_{r \searrow 0} \fint_{B(x,r)} w(z)\,\mathrm{d}z, & \text{if the limit exists and is finite,} \\ 0 & \text{otherwise,} \end{cases} \tag{2.5}$$

where the dashed integral as usual denotes the integral mean,

$$\fint_{B(x,r)} w(z)\,\mathrm{d}z = \frac{1}{\mathscr{L}^n(B(x,r))} \int_{B(x,r)} w(z)\,\mathrm{d}z,$$

and $B(x,r) = \{y : |y-x| < r\}$ is the open ball of radius r centered at x. Henceforth we omit special notation for the precise representative writing simply $w^* = w$.

We will say that x is an L_p–Lebesgue point of w (and simply a Lebesgue point when $p = 1$), if

$$\fint_{B(x,r)} |w(z) - w(x)|^p\,\mathrm{d}z \to 0 \quad \text{as} \quad r \searrow 0.$$

If $k < n$, then it is well-known that functions that are locally on $\Omega \subset \mathbb{R}^n$ in the Sobolev space W^k_p admit continuous representatives when $p > \frac{n}{k}$ and that they could be essentially discontinuous when $p \leq p_\circ = \frac{n}{k}$ (see, e.g., [60, 88]). The Sobolev–Lorentz space $\mathrm{W}^k_{p_\circ,1}(\Omega) \subset \mathrm{W}^k_{p_\circ}(\Omega)$ is a refinement of the corresponding Sobolev space that for our purposes turns out to be convenient. Among other things functions that are locally in $\mathrm{W}^k_{p_\circ,1}$ on Ω have in particular continuous representatives. Here we shall mainly be concerned with the Lorentz space $\mathrm{L}_{p,1}$, and in this case one may rewrite the norm as (see for instance [57, Proposition 3.6])

$$\|f\|_{p,1,\Omega} = \int\limits_0^{+\infty} \left[\mathscr{L}^n(\{x \in \Omega : |f(x)| > t\})\right]^{\frac{1}{p}} \mathrm{d}t. \tag{2.6}$$

We refer the reader to [57] or [88] for more information about Lorentz spaces. Denote by $\mathrm{W}^k_{p,1}(\Omega)$ the space of all functions $v \in \mathrm{W}^k_p(\Omega)$ such that all its distributional partial derivatives of order at most k belong to $\mathrm{L}_{p,1}(\Omega)$.

2.3. **Extension of the Bridge D.-F.-theorem to mappings of Sobolev and fractional Sobolev spaces.** Sobolev spaces have numerous applications in the modern theory of PDEs (where they often replace the classical smoothness notion), so it is a natural and compelling problem to extend the above results to this case. Surprisingly, it can be successfully done in a very natural way, namely, just replacing the C^k-spaces by $\mathrm{W}^k_{p,\mathrm{loc}}(\Omega, \mathbb{R}^d)$ and suitably redefining the sets $Z_{\nu,m}$ (see (2.9) below) in the above Theorem 2.3 it remains true. We emphasize in particular that the integrability assumptions in Theorem 2.5 below are minimal and sharp, they are of the kind $kp \geq n$, so that they guarantee in general only *the continuity, but not the everywhere differentiability* of a mapping. However, the 'bad' set of nondifferentiability points of such Sobolev mappings is fortunately negligible in the above bridge theorem (see (2.10)) because of some Luzin type N–properties with respect to lower dimensional Hausdorff measures established in [17, 31, 48], see also Section 4. This ensures us that the nature of Sobolev spaces is in deep harmony with the Morse–Sard–type theorems. Because the results that follow are all of a local nature we henceforth assume that all maps are defined on full space $\mathbb{R}^n$ and whenever convenient we impose the regularity conditions globally on $\mathbb{R}^n$ in order to keep the presentation as non-technical as possible.

Let $k \in \mathbb{N}$, $1 < p < \infty$ and $0 \leq \alpha < 1$. One of the most natural types of fractional Sobolev spaces are the *(Bessel) potential spaces* $\mathscr{L}^{k+\alpha}_p$. They can be seen as the Sobolev analog of the classical Hölder classes $\mathrm{C}^{k,\alpha}$. Recall, that a map $v\colon \mathbb{R}^n \to \mathbb{R}^d$ belongs to the space $\mathscr{L}^{k+\alpha}_p$, if it can be expressed as a convolution of a function $g \in \mathrm{L}_p(\mathbb{R}^n)$ with the Bessel kernel $G_{k+\alpha}$, defined as the function with Fourier transform $\widehat{G_{k+\alpha}}(\xi) = (1 + 4\pi^2\xi^2)^{-(k+\alpha)/2}$. It is well known that for the integer exponents (i.e., when $\alpha = 0$) one has the identity

$$\mathscr{L}^k_p(\mathbb{R}^n) = \mathrm{W}^k_p(\mathbb{R}^n) \qquad \text{if} \qquad 1 < p < \infty, \tag{2.7}$$

where $\mathrm{W}^k_p(\mathbb{R}^n)$ is the classical Sobolev space consisting of functions whose generalised derivatives up to order $\leq k$ belong to the Lebesgue space $\mathrm{L}_p(\mathbb{R}^n)$.

As usual, if $(k + \alpha)p > n$, then functions from the potential space $\mathscr{L}^{k+\alpha}_p(\mathbb{R}^n)$ are continuous by Sobolev theorem. But if $(k + \alpha)p = n$, then functions from potential spaces $\mathscr{L}^{k+\alpha}_p(\mathbb{R}^n)$ are discontinuous in general. Thus for this limiting case we need to consider the Bessel–Lorentz potential space $\mathscr{L}^{k+\alpha}_{p,1}(\mathbb{R}^n)$ to have the continuity. Namely, $\mathscr{L}^{k+\alpha}_{p,1}(\mathbb{R}^n)$ denotes the space of functions that can be represented as a convolution of the Bessel potential $G_{k+\alpha}$ with a function g from the Lorentz space $\mathrm{L}_{p,1}$ (see the definition of these spaces in the section 2.2). Similarly to (2.7), for the integer exponents (i.e., when $\alpha = 0$) one has the identity

$$\mathscr{L}^k_{p,1}(\mathbb{R}^n) = \mathrm{W}^k_{p,1}(\mathbb{R}^n) \qquad \text{if} \qquad 1 < p < \infty, \tag{2.8}$$

where $\mathrm{W}^k_{p,1}(\mathbb{R}^n)$ consists of all functions $v \in \mathrm{W}^k_p(\mathbb{R}^n)$ whose partial derivatives of order $\le k$ belong to the Lorentz space $\mathrm{L}_{p,1}$ (see, e.g., [31]).

Theorem 2.5 (Bridge D.-F. theorem for Sobolev case [38, 32])**.** *Let $m \in \{0, \dots, n-1\}$, $k \ge 1$, $d > m$, $0 \le \alpha < 1$, $p \ge 1$ and let $v\colon \mathbb{R}^n \to \mathbb{R}^d$ be a mapping for which one of the following cases holds:*

(i) $\alpha = 0$, $kp > n$, *and* $v \in \mathrm{W}^k_p(\mathbb{R}^n, \mathbb{R}^d)$;
(ii) $\alpha = 0$, $kp = n$, *and* $v \in \mathrm{W}^k_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$;
(iii) $0 < \alpha < 1$, $p > 1$, $(k+\alpha)p > n$, *and* $v \in \mathscr{L}^{k+\alpha}_p(\mathbb{R}^n, \mathbb{R}^d)$;
(iv) $0 < \alpha < 1$, $p > 1$, $(k+\alpha)p = n$, *and* $v \in \mathscr{L}^{k+\alpha}_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$.

Then the mapping v is continuous and for any $q \in (m, \infty)$ the equality

$$\mathscr{H}^{\mu_q}(Z_{v,m} \cap v^{-1}(y)) = 0 \qquad \textit{for } \ \mathscr{H}^q\textit{-a.a. } y \in \mathbb{R}^d$$

holds, where again

$$\mu_q = n - m - (k+\alpha)(q-m),$$

and $Z_{v,m}$ denotes the set of m-critical points:

$$Z_{v,m} = A_v \cup \{x \in \mathbb{R}^n \setminus A_v : \operatorname{rank} \nabla v(x) \le m\}. \tag{2.9}$$

Here A_v means the set of 'bad' points at which either the function v is not differentiable or which are not the Lebesgue points for ∇v; this 'bad' set is "automatically" included by definition into the critical set. Recall, that by approximation results (see, e.g., [81] and [48]) under conditions of Theorem 2.5 the equalities

$$\mathscr{H}^\tau(A_v) = 0 \qquad \forall \tau > \tau_* := n - (k+\alpha-1)p \qquad \text{in cases (i), (iii);}$$
$$\mathscr{H}^{\tau_*}(A_v) = \mathscr{H}^p(A_v) = 0 \qquad \tau_* := n - (k+\alpha-1)p = p \qquad \text{in cases (ii), (iv)}$$

are valid (in particular, $A_v = \emptyset$ if $(k+\alpha-1)p > n$). However, it was proved in [31] that the impact of the "bad" set A_v is negligible in the Bridge D.-F. Theorem 2.5, i.e.,

$$\mathscr{H}^{\mu_q}(A_v \cap v^{-1}(y)) = 0 \qquad \text{for } \ \mathscr{H}^q\text{–a.a. } y \in \mathbb{R}^d \tag{2.10}$$

for any $q > m$.

Remark 2.6. Note that since $\mu_q \le 0$ for $q \ge q_\circ = m + \frac{n-m}{k+\alpha}$, the assertions of Theorem 2.5 is equivalent to the equality $0 = \mathscr{H}^q\big[v(Z_{v,m})\big]$ for $q \ge q_\circ$, so it is sufficient to check the assertions of Theorem 2.5 for $q \in (m, q_\circ]$ only.

Remark 2.7. Note that in the pioneering paper by De Pascale [23] the assertion of the initial Morse–Sard Theorem (1.2) (i.e., when $k = n-m$, $q = q_\circ = m+1 = d$, $\mu_q = 0$) was obtained for the Sobolev classes $\mathrm{W}^k_p(\mathbb{R}^n, \mathbb{R}^d)$ under the additional assumption $p > n$ (in this case the classical embedding $\mathrm{W}^k_p(\mathbb{R}^n, \mathbb{R}^d) \hookrightarrow \mathrm{C}^{k-1}$ holds, so there are no problems with nondifferentiability points). For the same Sobolev class $\mathrm{W}^k_p(\mathbb{R}^n,\mathbb{R}^d)$ with $p>n$ the assertion of the Dubovitskiĭ Theorem 2.1 was proved in the recent paper [37] by Hajłasz and Zimmermann.

In addition to the above mentioned papers there is a growing number of papers on the topic, including [7, 8, 9, 10, 18, 34, 66, 82, 83].

The last identity (2.10) demonstrates the necessity of studying the Luzin N-type properties in order to formulate the Morse–Sard type theorems for the Sobolev case. Really, it was the starting point for our research which recently was finished by finding some natural synthesis of these phenomena with some interesting and unexpected properties.

3. Luzin N-property for Sobolev mappings with respect to Hausdorff measures

Recall, that if a map $v\colon \Omega\subset\mathbb{R}^n\to\mathbb{R}^d$ is locally Lipschitz or everywhere differentiable, then it is easily seen that, for subsets $E\subset\Omega$,

$$\mathscr{H}^\alpha(v(E))=0 \qquad \text{whenever} \quad \mathscr{H}^\alpha(E)=0, \tag{3.1}$$

where $\alpha\in(0,n]$ and $\mathscr{H}^\alpha$ denotes the α-dimensional Hausdorff measure. In particular, the Luzin N-property (1.3) follows from (3.1), when $d=n$.

The intriguing issue is that in many important special cases it is required to establish the Luzin N-property for Sobolev mappings that are neither Lipschitz nor everywhere differentiable. In particular, the Lusin N-property is critical in the proof of central properties of quasiconformal maps and, more generally, of maps with bounded distortion, i.e., mappings $f\colon\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$ of Sobolev class $\mathrm{W}^1_n(\mathbb{R}^n)$ such that $\|\nabla f(x)\|^n\le K\det\nabla f(x)$ holds almost everywhere with some constant $K\in[1,+\infty)$, where $\|\cdot\|$ is the operator norm. These maps were introduced by Reshetnyak in [70]; we also emphasize his excellent books [71], [72], [35]. He proved that they satisfy the N-property and this was very helpful in his subsequent proofs of other basic topological properties of such mappings (openness, discreteness and etc.). Subsequently this theory was successfully developed by many mathematicians in both analytical and geometrical directions, and many interesting and deep results were obtained (see, e.g., monographs [74, 42]).

Note that the membership of a continuous mapping v to the Sobolev class $\mathrm{W}^1_{n,\mathrm{loc}}(\Omega,\mathbb{R}^n)$ is crucial for N-property. Indeed, every continuous mapping of class $\mathrm{W}^1_p(\Omega,\mathbb{R}^n)$ with $p>n$ has the N-property (it is a simple consequence of the Morrey inequality). But a continuous mapping of class $\mathrm{W}^1_{n,\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n)$ need not have the N-property [73]. On the other hand, the N-property holds for continuous mappings of the class $\mathrm{W}^1_{n,\mathrm{loc}}(\Omega,\mathbb{R}^n)$ under some additional assumptions on its topological features, namely, for homeomorphic and open mappings and for pseudo-monotone[3] mappings [85, 56].

The results above are very delicate and sharp: indeed, for any $p<n$ there are homeomorphisms $f\colon\mathbb{R}^n\to\mathbb{R}^n$ locally of class W^1_p without the N-property. This phenomenon was discovered by Ponomarev [68]. In recent years his construction has gone through a number of interesting refinements. For instance an example of

[3] Some of these results were generalised for the more delicate case of Carnot groups and manifolds, see, e.g., [84].

a Sobolev homeomorphism with zero Jacobian a.e. which belongs simultaneously to all the classes $\mathrm{W}^1_{p,\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n)$ with $p<n$ can be found in [39, 19] — of course, this exotic homeomorphism certainly fails to have the N-property[4].

In the positive direction, it was proved in [45] (see also [75]), that every mapping of the Sobolev–Lorentz class $\mathrm{W}^1_{n,1,\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n)$ satisfies the N-property. Note that this space $\mathrm{W}^1_{n,1,\mathrm{loc}}$ is limiting in a natural sense between classes $\mathrm{W}^1_{n,\mathrm{loc}}$ and $\mathrm{W}^1_{p,\mathrm{loc}}$ with $p>n$ (see section 2.2 for the exact definitions).

Another direction is to study the N-properties with respect to Hausdorff (instead of Lebesgue) measures. One of the most elegant results was achieved for the class of plane quasiconformal mappings. Namely, the famous area distortion theorem of Astala [6] implies the following dimension distortion result: if $f\colon \mathbb{C}\to\mathbb{C}$ is a K-quasiconformal mapping (i.e., it is a plane homeomorphic mapping with K-bounded distortion) and E is a compact set of Hausdorff dimension $t\in(0,2)$, then the image $f(E)$ has Hausdorff dimension at most $t'=\frac{2Kt}{2+(K-1)t}$. This estimate is sharp; however, it leaves open the endpoint case: does $\mathscr{H}^t(E)=0$ imply $\mathscr{H}^{t'}(f(E))=0$? The remarkable paper [54] gives an affirmative answer to Astala's conjecture.

To formulate the main result of the section, we use the following terminology: a continuous mapping $v\colon\mathbb{R}^n\to\mathbb{R}^d$ is said to satisfy (τ,σ)-*N-property*, if for subsets $E\subset\mathbb{R}^n$,

$$\mathscr{H}^\sigma(v(E))=0 \qquad \text{whenever} \quad \mathscr{H}^\tau(E)=0.$$

Similarly, $v\colon\mathbb{R}^n\to\mathbb{R}^d$ is said to satisfy *strict* (τ,σ)-*N-property*, if for subsets $E\subset\mathbb{R}^n$,

$$\mathscr{H}^\sigma(v(E))=0 \qquad \text{whenever} \quad \mathscr{H}^\tau(E)<\infty.$$

Define a continuous function $\sigma(\tau)$ by the following rule:

$$\sigma(\tau):=\begin{cases}\tau, & \text{if} \quad \tau\ge\tau_*:=n-(\alpha-1)p;\\[2ex] \frac{p\,\tau}{\alpha p-n+\tau}, & \text{if} \qquad\qquad 0<\tau<\tau_*.\end{cases} \tag{3.2}$$

Theorem 3.1 ([47, 31])**.** *Let $\alpha>0$, $1<p<\infty$, $\alpha p>n$, and $v\in\mathscr{L}^\alpha_p(\mathbb{R}^n,\mathbb{R}^d)$. Suppose that $0<\tau\le n$. Then the following assertions hold:*

(i) if $\tau\ne\tau_*=n-(\alpha-1)p$, then v has the (τ,σ)-N-property, where the values τ_* and $\sigma=\sigma(\tau)$ are defined in (3.2)*. This (τ,σ)-N-property is strict in case $\tau<\tau_*$.*

*(ii) if $\alpha>1$ and $\tau=\tau_*>0$ then $\sigma(\tau)=\tau_*$, but the mapping v in general has **no** (τ_*,τ_*)-N-property, i.e., it could be that $\mathscr{H}^{\tau_*}(v(E))>0$ for some $E\subset\mathbb{R}^n$ with $\mathscr{H}^{\tau_*}(E)=0$.*

Theorem 3.1 omits the limiting cases $\alpha p=n$ and $\tau=\tau_*$. As above, it is possible to cover these cases as well using the Lorentz norms.

[4]Moreover, even the examples of bi-Sobolev homeomorphisms of class $\mathrm{W}^1_{p,\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n)$, $p<n-1$, with zero Jacobian a.e. were constructed recently, see, e.g., [22], [20]. Such homeomorphisms are impossible in the Sobolev class $\mathrm{W}^1_{n-1,\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n)$. Furthermore, in [41] authors constructed an example of a Sobolev homeomorphism $f\in\mathrm{W}^1_{1,\mathrm{loc}}((0,1)^n,\mathbb{R}^n)$ such that the Jacobian $\det f'(x)$ changes its sign on the sets of positive measures.

Theorem 3.2 ([17, 48, 31])**.** *Let $v\colon \mathbb{R}^n \to \mathbb{R}^d$ be a mapping for which one of the following cases holds:*

(i) $v \in \mathrm{W}_1^k(\mathbb{R}^n, \mathbb{R}^d)$ *for some* $k \in \mathbb{N}$, $k \geq n$;

(ii) $v \in \mathscr{L}^\alpha_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$ *for some* $\alpha > 0$, $p \in (1, \infty)$ *with* $\alpha p \geq n$.

Suppose that $0 < \tau \leq n$. Then v is a continuous function satisfying the (τ, σ)-N-property, where again the value $\sigma = \sigma(\tau)$ is defined in (3.2) (with $\alpha = k$ and $p = 1$ for the (i) case).

So, in the last theorem the critical case $\tau = \tau_*$ is **included**.

Remark 3.3. The assertion of Theorem 3.1 for $\alpha = 1$ was proved by Kaufman [44]. The Hausdorff dimension distortion (with the same $\sigma(\tau)$) was established by Hencl and Honzík [40]. The assertion similar to Theorem 3.1 was announced in [5] (without complete proofs), see [31] for our commentaries and other historical remarks. The assertion of Theorem 3.2 for $\alpha = 1$ and $\tau = n$ was proved in [45], [75]. Finally, very recently the results were extended to the Orlicz–Sobolev case [21] (also for $\alpha = 1$).

In order to prove that the 'bad' set of nondifferentiability points is negligible in the Morse–Sard–Dubovitskiĭ type theorems (see identity (2.10)), we need the following result.

Theorem 3.4 ([38, 31], Sobolev case)**.** *Let $\alpha > 0$, $1 < p < \infty$, $\alpha p > n$, and $v \in \mathscr{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$. Suppose that $0 < \tau \leq n$ and $\tau \neq \tau_* = n - (\alpha - 1)p$. Then for every $q \in [0, \sigma]$ and for any set $E \subset \mathbb{R}^n$ with $H^\tau(E) = 0$ the equality*

$$\mathscr{H}^\mu(E \cap v^{-1}(y)) = 0 \qquad \text{for } \mathscr{H}^q\text{-a.a. } y \in \mathbb{R}^d \tag{3.3}$$

holds, where $\mu = \tau\bigl(1 - \frac{q}{\sigma}\bigr)$ and the value $\sigma = \sigma(\tau)$ is defined in (3.2).

The above Theorem 3.4 omits the limiting cases $\alpha p = n$ and $\tau = \tau_*$. As above, it is possible to cover these cases as well using the Lorentz norms.

Theorem 3.5 ([38, 31], Sobolev–Lorentz case)**.** *Let $v\colon \mathbb{R}^n \to \mathbb{R}^d$ be a mapping for which one of the following cases holds:*

(i) $v \in \mathrm{W}_1^k(\mathbb{R}^n, \mathbb{R}^d)$ *for some* $k \in \mathbb{N}$, $k \geq n$;

(ii) $v \in \mathscr{L}^\alpha_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$ *for some* $\alpha > 0$, $p \in (1, \infty)$ *with* $\alpha p \geq n$.

Suppose that $0 < \tau \leq n$. Then for every $q \in [0, \sigma]$ and for any set $E \subset \mathbb{R}^n$ with $H^\tau(E) = 0$ the equality (3.3) holds with the same μ and σ defined in (3.2) (with $\alpha = k$ and $p = 1$ for the case (i)).

4. On synthesis of Dubovitskiĭ–Federer Theorem 2.2 and Luzin type properties

The above mentioned properties assert, roughly speaking, that the image $v(E)$ has small measure if the rank of differential of v on E is small (Morse–Sard), or if the set E itself is small (Luzin). This leads us to the following natural generalizations. For a pair of positive parameters τ and σ and for an integer $m \in \mathbb{Z}_+$ we say that that a mapping $v\colon \mathbb{R}^n \to \mathbb{R}^d$ satisfies the $(m : \tau, \sigma)$-*N-property,* if

$$\mathscr{H}^\sigma(v(E)) = 0 \quad \text{whenever} \quad E \subset Z_{v,m} \quad \text{with} \quad \mathscr{H}^\tau(E) = 0.$$

Further, we say that that a mapping $v\colon \mathbb{R}^n \to \mathbb{R}^d$ satisfies the *strict* $(m:\tau,\sigma)$-N-*property,* if

$$\mathscr{H}^\sigma(v(E)) = 0 \quad \text{whenever} \quad E \subset Z_{v,m} \quad \text{with} \quad \mathscr{H}^\tau(E) < \infty.$$

Using this notation, the above classical Morse–Sard theorem means that every C^k-mapping $v\colon \mathbb{R}^n \to \mathbb{R}^d$ has the strict $(d-1:n,d)$-N-property if $k \geq n-d+1$. The starting point for this section is the following

Theorem 4.1 (Bates and Moreira, 2002 [11, 63])**.** *Let* $m \in \{0,\dots,n-1\}$*,* $k \geq 1$*,* $d \geq m$*,* $0 \leq \alpha \leq 1$*, and* $v \in \mathrm{C}^{k,\alpha}(\mathbb{R}^n,\mathbb{R}^d)$*. Then for any* $\tau \in [m,n]$ *the mapping* v *has the* $(m:\tau,\sigma)$-N*-property with*

$$\sigma = m + \frac{\tau - m}{k+\alpha}. \tag{4.1}$$

Moreover, this N*-property is strict if at least one of the following additional assumptions is fulfilled:*

1) $\tau = n$ *(in particular, it includes the case of the classical Morse–Sard theorem);*
2) $\tau > m$ *and* $\alpha = 0$ *(that means* $v \in \mathrm{C}^k$*);*
3) $\tau > m$ *and* $v \in \mathrm{C}^{k,\alpha+}(\mathbb{R}^n,\mathbb{R}^d)$*.*

Here we say that a mapping $v\colon \mathbb{R}^n \to \mathbb{R}^d$ belongs to the class $C^{k,\alpha+}$ for some positive integer k and $0 < \alpha \leq 1$, if there exists a function $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(r) \to 0$ as $r \to 0$ and

$$|\nabla^k v(x) - \nabla^k v(y)| \leq \omega(r)\cdot |x-y|^\alpha \qquad \text{whenever} \quad |x-y| < r. \tag{4.2}$$

See the recent paper [61] for the case of mappings into the infinite-dimensional normed space (and for a good historical survey). Note that the assertion of Theorem 4.1 is sharp: namely, if its additional conditions 1)–3) are not satisfied, then the corresponding $(m:\tau,\sigma)$-N-property is not strict. This follows from Whitney's counterexamples [86], see also [65, 63] for further comments.

Of course, the assertion of Theorem 4.1 includes the Morse–Sard theorem and many other results on this topic as particular cases. The next step is to extend this result to the case of Sobolev mappings. Recall, that in the Sobolev case we consider the set of 'bad' nonregular points (where ∇v is not well-defined) automatically as m-critical for any m (see Theorem 2.5 for the precise definition; of course, this only makes the corresponding $(m:\tau,\sigma)$-N-properties stronger).

Theorem 4.2. *Let* $m \in \{0,\dots,n-1\}$*,* $k \geq 1$*,* $d \geq m$*,* $0 \leq \alpha < 1$*,* $p > 1$*,* $(k+\alpha)p > n$*, and let* $v \in \mathscr{L}^{k+\alpha}_p(\mathbb{R}^n,\mathbb{R}^d)$*. Denote* $\tau_* = n - (k+\alpha-1)p$*. Suppose in addition that*

$$\tau > m \qquad \text{and} \qquad \tau > \tau_*,$$

then the mapping v *has* ***strict*** $(m:\tau,\sigma)$-N*-property with*

$$\sigma = m + \frac{p(\tau-m)}{\tau + (k+\alpha)p - n}. \tag{4.3}$$

Further, if $\tau = m > \tau_*$*, then* v *has nonstrict* $(m:\tau,m)$-N*-property.*

We make several remarks here.

- First of all, note that the value σ in Theorems 4.1–4.2 coincide for the borderline cases $\tau = m$ or $\tau = n$, but they are **different** for the intermediate range $m < \tau < n$ (of course, then σ is larger for the Sobolev case). Nevertheless, σ in Theorem 4.1 can be obtained by taking a limit in (4.3) as $p \to \infty$;
- The condition $\tau > \tau_*$ in Theorem 4.2 is essential and sharp: namely, in the paper [31] we constructed a counterexample of a mapping from $\mathscr{L}^{k+\alpha}_p(\mathbb{R}^n)$ that does not have the $(m : \tau, \sigma)$-N-property with $\tau = \tau_* = m = \sigma = 1$.

Theorem 4.2 omits the limiting cases $(k + \alpha)p = n$ and $\tau = \tau_*$. However, it is possible to cover these cases as well using the Lorentz norms:

Theorem 4.3. *Let $m \in \{0, \dots, n-1\}$, $k \geq 1$, $d \geq m$, $0 \leq \alpha < 1$, $p \geq 1$ and let $v \colon \mathbb{R}^n \to \mathbb{R}^d$ be a mapping for which one of the following cases holds:*

(i) $\alpha = 0$, $k \geq n$, *and* $v \in \mathrm{W}^k_1(\mathbb{R}^n, \mathbb{R}^d)$*;*

(ii) $0 \leq \alpha < 1$, $p > 1$, $(k + \alpha)p \geq n$, *and* $v \in \mathscr{L}^{k+\alpha}_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$.

Denote $\tau_* = n - (k + \alpha - 1)p$. Suppose in addition that

$$\tau > m \qquad \text{and} \qquad \tau \geq \tau_*,$$

then the mapping v has ***strict*** *$(m : \tau, \sigma)$-N-property with the same σ defined by (4.3). Further, if $\tau = m \geq \tau_*$, then v has the corresponding nonstrict $(m : \tau, m)$-N-property.*

So here the limiting case $\tau = \tau_*$ is included.

4.1. **Some additional historical remarks.** The above formulated results of §4 include many previous Morse–Sard type theorems as particular cases. For example, for the smooth case if $\alpha = 0$, $\tau = n$, then we have

$$\sigma = m + \frac{n-m}{k},$$

and the assertion of Theorem 4.1 coincides with the classical Federer–Dubovitskiĭ Theorem 2.2. Of course, it includes the original Morse–Sard theorem as a particular case (when $k = n - m, \sigma = m + 1$).

Note also that Theorem 4.1 was formulated as a conjecture by Norton in [65, page 369] and it includes as particular cases some related results of other mathematicians: Norton himself (who proved the assertion for the case $\sigma = d$, $\tau = (k+\alpha)(d-m)+m$), Yomdin [87] (case $\tau = n$, $v \in C^{k,\alpha+}$, see also [14]), Kucera [52] (case $\tau = n$, $m = 1$, i.e., when the gradient vanishes on the critical set), etc.

Finally, the assertions of Theorems 4.2 and 4.3 for the most important case $\tau = n$ follows from the bridge theorems 2.3, 2.5.

5. On universal synthesis of Bridge Dubovitskiĭ–Federer Theorem 2.3 and Luzin type N-properties

In this section we describe the universal synthesis of all the above phenomena: Morse–Sard theorem, Luzin N-property, Dubovitskiĭ–Federer Theorems 2.1–2.2, and Bates–Moreira Theorem 4.1.

In order to ease the presentation we introduce the following terminology. For parameters $\mu \geq 0$, $q \geq m$, $\tau > 0$ we say that that a mapping $v\colon \mathbb{R}^n \to \mathbb{R}^d$ has the $(m:\tau,\mu,q)$-*N-property* if for subsets $E \subset Z_{v,m}$ with $\mathscr{H}^\tau(E) = 0$ we have that

$$\mathscr{H}^\mu(E \cap v^{-1}(y)) = 0 \quad \text{for } \mathscr{H}^q\text{-almost all } y \in v(E). \tag{5.1}$$

Recall that here, as above, the set of m-critical points is defined by (2.3) for the classical smooth case or by (2.9) for the Sobolev case. Obviously,

$$\text{if } \mu \leq 0, \text{ then the } (m:\tau,\mu,q)\text{-}N\text{-property is equivalent to the } (m:\tau,q)\text{-}N\text{-property.} \tag{5.2}$$

Further, we say that a mapping $v\colon \mathbb{R}^n \to \mathbb{R}^d$ has the *strict* $(m:\tau,\mu,q)$-*N-property* if for subsets $E \subset Z_{v,m}$ with $\mathscr{H}^\tau(E) < \infty$ it holds that

$$\mathscr{H}^\mu(E \cap v^{-1}(y)) = 0 \quad \text{for } \mathscr{H}^q\text{-almost all } y \in v(E).$$

Theorem 5.1 (Smooth case $v \in \mathrm{C}^{k,\alpha}(\mathbb{R}^n,\mathbb{R}^d)$)**.** *Under the assumptions of Theorem 4.1 one can replace the assertion about $(m:\tau,\sigma)$-N-properties by the stronger assertion about $(m:\tau,\mu,q)$-N-property for any $\tau \in [m,n]$ and $q \in [m,\sigma]$ with*

$$\mu = \tau - m - (k+\alpha)(q-m). \tag{5.3}$$

Further, if $q > m$ and at least one of the corresponding conditions 1)–3) of Theorem 4.1 is fulfilled, then this $(m:\tau,\mu,q)$-N-property is strict.

Similar assertions hold for Sobolev and Sobolev–Lorentz cases.

Theorem 5.2 (Sobolev case $v \in \mathscr{L}^{k,\alpha}_p(\mathbb{R}^n,\mathbb{R}^d)$, $(k+\alpha)p > n$)**.** *Under assumptions of Theorem 4.2 one can replace the assertion about strict $(m:\tau,\sigma)$-N-properties by the stronger assertion about strict $(m:\tau,\mu,q)$-N-property for any $\tau > \max(\tau_*,m)$, $q \in (m,\sigma]$ with*

$$\mu = \tau - m - (k+\alpha - \frac{n}{p} + \frac{\tau}{p})(q-m). \tag{5.4}$$

Further, if $q = m$, $\tau > \tau_*$, and $\tau \geq m$, then v has nonstrict $(m:\tau,\mu,m)$-N-property with $\mu = \tau - m$.

Theorem 5.3 (Sobolev–Lorentz case $v \in \mathscr{L}^{k,\alpha}_{p,1}(\mathbb{R}^n,\mathbb{R}^d)$, $kp \geq n$)**.** *Under assumptions of Theorem 4.3 one can replace the assertion about strict $(m:\tau,\sigma)$-N-properties by the stronger assertion about strict $(m:\tau,\mu,q)$-N-property for any $\tau \geq \tau_*$, $\tau > m$, $q \in (m,\sigma]$, and with the same μ as in (5.4). Further, if $q = m$ and $\tau \geq \max(m,\tau_*)$, then v has nonstrict $(m:\tau,\mu,m)$-N-property with $\mu = \tau - m$.*

It is easy to see thatf we in the statement of Theorems 5.1–5.3 take $q = \sigma$, then $\mu = 0$, where σ is defined in the corresponding Theorems 4.1–4.3. It means (see (5.2)) that Theorems 5.1–5.3 include the previous Theorems 4.1–4.3 as particular cases.

6. Final remarks

The results of the last Section 5 are not merely about relaxing the smoothness assumptions in the classical theorems as can be seen when noticing that our results are new even for the classical case of C^k-smooth mappings! The aims of the research were quite different: first of all, to obtain the synthetic bridge theorem that links together all these remarkable phenomena of real analysis (Morse–Sard theorems, Dubovitskiĭ–Federer theorems, and Luzin N-property; an important step in this direction was taken earlier in the Bates–Moreira Theorem 4.1), and secondly, to extend these theorems to the very important function spaces in modern PDEs — Hölder, Sobolev, and fractional Sobolev (=Bessel potential) spaces. Success in achieving all these aims became the basis for writing this survey paper.

Many remarks and comments were already made, but let us here emphasize an important and perhaps surprising point. Namely that for for $\tau = n$ (which is the case in all Morse–Sard–Dubovitskiĭ–Federer type theorems) the conclusions for classically C^k-smooth mappings and for Sobolev W^k_p-mappings are exactly the same, despite the fact, that the integrability assumptions for the latter are very weak and sharp: they guarantee in general only continuity of the mapping itself, so the mappings under consideration are neither smooth, nor Lipschitz, nor even everywhere differentiable! The existence of points where the map fails to be differentiable does not cause any harm as the image of these point turns out to be negligible. This is by virtue of the Luzin type N-properties that we establish for such Sobolev mappings. In particular, for functions of the simplest Sobolev class $\mathrm{W}^n_1(\mathbb{R}^n, \mathbb{R})$ almost all levels sets turn out to be C^1-smooth manifolds despite the fact, that the function itself is not C^1-smooth, nor even Lipschitz (see [17]). This observation turned out to be crucial and very useful for the resolution of some important problems in fluid mechanics (see, e.g., [49]–[50]). Only starting from $\tau < n$ the integrability assumptions of Sobolev mappings begin to influence the distortion of the Hausdorff dimensions.

7. Appendix I: Some crucial ingredients for the proofs

In the research we develop ideas from our initial joint papers with Jean Bourgain [16, 17]. As in [17], we crucially use Yomdin's [87] entropy estimates of near critical values for polynomials (recalled in Theorem 7.1 below). These interesting results of Yomdin are very useful in this context, see, e.g., the recent paper [9], where the Morse-Sard theorems were proved for min-type functions and for Lipschitz selections. Another key ingredient are the elegant Adams–Maz'ya estimates of Choquet type integrals for Riesz potentials.

7.1. On Yomdin's entropy estimates for the near-critical values of polynomials.

For a subset A of $\mathbb{R}^d$ and $\varepsilon > 0$ the ε–entropy of A, denoted by $\mathrm{Ent}(\varepsilon, A)$, is the minimal number of closed balls of radius ε covering A. Further, for a linear map $L\colon \mathbb{R}^n \to \mathbb{R}^d$ we denote by $\lambda_j(L)$, $j = 1, \dots, d$, its singular values arranged in decreasing order: $\lambda_1(L) \geq \lambda_2(L) \geq \cdots \geq \lambda_d(L)$. Geometrically the singular values are the lengths of the semiaxes of the ellipsoid $L(\partial B(0, 1))$. We recall that the singular values of L coincide with the eigenvalues repeated according to multiplicity of the symmetric nonnegative linear map $\sqrt{LL^*}\colon \mathbb{R}^d \to \mathbb{R}^d$. Also for a mapping $f\colon \mathbb{R}^n \to \mathbb{R}^d$ that is approximately differentiable at $x \in \mathbb{R}^n$ put $\lambda_j(f, x) = \lambda_j(d_x f)$, where by $d_x f$ we denote the approximate differential of f at x. The next result is the basic ingredient of our proof.

Theorem 7.1 ([87]). *Let $m \in \{0, \dots, n-1\}$ and $m < d$. Then for any polynomial $P\colon \mathbb{R}^n \to \mathbb{R}^d$ of degree at most k, for each n-dimensional cube $Q \subset \mathbb{R}^n$ of size*

$\ell(Q)=r>0$, and for any number $\varepsilon>0$ we have that

$$\operatorname{Ent}\bigl(\varepsilon r,\{P(x):\,x\in Q,\ \lambda_1\le 1+\varepsilon,\dots,\lambda_m\le 1+\varepsilon,\lambda_{m+1}\le\varepsilon,\dots,\lambda_d\le\varepsilon\}\bigr)\le C_Y\bigl(1+\varepsilon^{-m}\bigr),$$

where the constant C_Y depends on n,d,k,m only and for brevity we wrote $\lambda_j=\lambda_j(P,x)$.

7.2. **On estimates of integrals of Sobolev functions with respect to measures satisfying a Morrey condition.** Let $\mathscr{M}^\beta$ be the space of all nonnegative Borel measures μ on $\mathbb{R}^n$ such that

$$\|\!|\mu|\!\|_\beta=\sup_{I\subset\mathbb{R}^n}\ell(I)^{-\beta}\mu(I)<\infty,$$

where the supremum is taken over all n–dimensional cubic intervals $I\subset\mathbb{R}^n$ and $\ell(I)$ denotes side–length of I.

Recall the following classical theorem proved by Adams.

Theorem 7.2 (§1.4.1 in [60] or [1])**.** *Let $\alpha>0,\ n-\alpha p>0,\ s>p>1$ and μ be a positive Borel measure on $\mathbb{R}^n$. Then for any $g\in\mathrm{L}_p(\mathbb{R}^n)$ the estimate*

$$\int\bigl|I_\alpha g\bigr|^s\,\mathrm{d}\mu\le C\|\!|\mu|\!\|_\beta\cdot\|g\|^s_{\mathrm{L}_p} \tag{7.1}$$

holds with $\beta=\frac{s}{p}(n-\alpha p)$, where C depends on $n,\ p,\ s,\ \alpha$ only, and

$$I_\alpha g(x):=\int_{\mathbb{R}^n}\frac{g(y)}{|y-x|^{n-\alpha}}\,\mathrm{d}y$$

is the Riesz potential of order α.

The above estimate (7.1) fails for the limiting case $s=p$. Namely, there exist functions $g\in\mathrm{L}_p(\mathbb{R}^n)$ such that $|I_\alpha g|(x)=+\infty$ on some set of positive $(n-\alpha p)$–Hausdorff measure. Nevertheless, there are two ways to cover this limiting case $s=p$. The first way is using the fractional maximal function M_α instead of the Riesz potential in the left hand side of (7.1).

Theorem 7.3 (Theorem 7 on page 28 in [3])**.** *Let $\alpha>0,\ n-\alpha p>0,\ s\ge p>1$ and μ be a positive Borel measure on $\mathbb{R}^n$. Then for any $g\in\mathrm{L}_p(\mathbb{R}^n)$ the estimate*

$$\int\bigl|M_\alpha g\bigr|^s\,\mathrm{d}\mu\le C\|\!|\mu|\!\|_\beta\cdot\|g\|^s_{\mathrm{L}_p} \tag{7.2}$$

holds with $\beta=\frac{s}{p}(n-\alpha p)$, where C depends on $n,\ p,\ s,\ \alpha$ only.

The second way is using the Lorentz norm instead of the Lebesgue norm in the right hand side of (7.1). Such a possibility was proved in [48].

Theorem 7.4 (Theorem 1.2 in [48])**.** *Let $\alpha>0,\ 1<p<\infty,\ n-\alpha p>0,$ and μ be a positive Borel measure on $\mathbb{R}^n$. Then for any $g\in\mathrm{L}_{p,1}(\mathbb{R}^n)$ the estimate*

$$\int\bigl|I_\alpha g\bigr|^p\,\mathrm{d}\mu\le C\|\!|\mu|\!\|_\beta\cdot\|g\|^p_{\mathrm{L}_{p,1}}$$

holds with $\beta=n-\alpha p$, where C depends on $n,\ p,\ \alpha$ only.

We are indebted to Igor Verbitsky for pointing out that Theorem 7.4 may also be proved using the results from [79].

The above theorems are not fulfilled in general for $p = 1$. However, Maz'ya [60, §1.4.3] proved the corresponding result for derivatives of Sobolev mappings:

Theorem 7.5 ([60]). *Let $k, l \in \{1, \ldots, n\}$, $l < k$, and μ be a positive Borel measure on $\mathbb{R}^n$. Then for any function f from the Sobolev space $\mathrm{W}_1^k(\mathbb{R}^n)$ the estimates*

$$\int \left|\nabla^l f\right| \mathrm{d}\mu \leq C \|\mu\|_\beta \cdot \|\nabla^k f\|_{\mathrm{L}_1} \tag{7.3}$$

hold, where $\beta = n - k + l$ and the constant C depends on n, k, l.

7.3. **On Choquet type integrals.** The estimates of previous section imply boundedness of certain Choquet type integrals of maximal functions with respect to the Hausdorff measure due to the following elegant result by Adams:

Theorem 7.6 (see Theorem A, Proposition 1 and its Corollary in [2]). *Let $\beta \in (0, n)$. Then for nonnegative functions $f \in \mathrm{C}_0(\mathbb{R}^n)$ the estimates*

$$\begin{aligned} \int_0^\infty \mathscr{H}_\infty^\beta(\{x \in \mathbb{R}^n : \mathcal{M}f(x) \geq t\})\,\mathrm{d}t &\leq C_1 \int_0^\infty \mathscr{H}_\infty^\beta(\{x \in \mathbb{R}^n : f(x) \geq t\})\,\mathrm{d}t \\ &\leq C_2 \sup\left\{\int f\,\mathrm{d}\mu : \mu \in \mathscr{M}^\beta,\ \|\mu\|_\beta \leq 1\right\}, \end{aligned}$$

hold, where the constants C_1, C_2 depend on β, n only, and $\mathcal{M}f$ denotes the usual Hardy-Littlewood maximal function of f.

In particular, from the Maz'ya Theorem 7.5 we get

Theorem 7.7. *Let $k, l \in \{1, \ldots, n\}$, $l < k$. Then for any function f from the Sobolev space $\mathrm{W}_1^k(\mathbb{R}^n)$ the estimates*

$$\int_0^\infty \mathscr{H}_\infty^\tau(\{x \in \mathbb{R}^n : \mathcal{M}\big(\nabla^l f\big)(x) \geq t\})\,\mathrm{d}t \leq C\|\nabla^k f\|_{\mathrm{L}_1} \tag{7.4}$$

hold, where $\tau = n - k + l$ and the constant C depends on n, k, l.

Similar estimates in terms of Choquet type integrals can be established for Riesz potentials by combination of Theorems 7.2, 7.4 with Theorem 7.6.

8. Appendix II. Proof sketch for the simplest case of $\mathrm{W}_1^2(\mathbb{R}^2)$ Sobolev functions

The classical Morse theorem guarantees for a domain Ω in $\mathbb{R}^2$ and for every C^2-smooth function $f \colon \Omega \to \mathbb{R}$ that

$$\mathscr{H}^1(f(Z_f)) = 0$$

holds, where Z_f is the set of critical points, $Z_f = \{x \in \Omega : \nabla f(x) = 0\}$. Let us recall some differentiability properties of Sobolev functions.

Lemma 8.1 (see Proposition 1 in [24]). *Let $\psi \in \mathrm{W}_1^2(\mathbb{R}^2)$. Then the function ψ is continuous and there exists a set A_ψ such that $\mathscr{H}^1(A_\psi) = 0$, and the function ψ is differentiable (in the classical sense) at each $x \in \mathbb{R}^2 \setminus A_\psi$. Furthermore, the* *classical derivative at such points x coincides with* $\nabla\psi(x) = \lim_{r\to 0} \fint_{B_r(x)} \nabla\psi(z)\,\mathrm{d}z$, *and* $\lim_{r\to 0} \fint_{B_r(x)} |\nabla\psi(z) - \nabla\psi(x)|^2\,\mathrm{d}z = 0$.

Now the Morse theorem for the Sobolev case can be formulated as follows:

Theorem 8.2 ([16, 17])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and $\psi \in \mathrm{W}^2_1(\Omega)$. Then*

(i) $\mathscr{H}^1(\{\psi(x) : x \in \bar{\Omega} \setminus A_\psi \ \&\ \nabla\psi(x) = 0\}) = 0$*;*

(ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $U \subset \bar{\Omega}$ with $\mathscr{H}^1_\infty(U) < \delta$ the inequality $\mathscr{H}^1(\psi(U)) < \varepsilon$ holds;

(iii) for $\mathscr{H}^1$–almost all $y \in \psi(\bar{\Omega}) \subset \mathbb{R}$ the preimage $\psi^{-1}(y)$ is a finite disjoint family of C^1–curves S_j, $j = 1, 2, \dots, N(y)$. Each S_j is either a cycle in Ω (i.e., $S_j \subset \Omega$ is homeomorphic to the unit circle $\mathbb{S}^1$) or it is a simple arc with endpoints on $\partial\Omega$ (in this case S_j is also transversal to $\partial\Omega$).

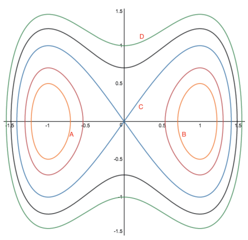

Figure 1. Typical level sets structure

We emphasize that this Morse–Sard theorem has a somewhat surprising consequence. Namely that **almost all level sets of a continuous map that is locally of Sobolev class $\mathrm{W}^{2,1}$ on the plane are C^1 smooth curves**. This is despite the fact, that such a function itself could be nondifferentiable. We present here the complete proof of Theorem 8.2 based on our initial joint article with Jean Bourgain [17]. Because the proofs for most of the other results in the present paper to a large extent are based on the same ideas, it is a reasonable illustration.

8.1. **Some preliminaries for the planar case.** By an *n-dimensional interval* we mean a closed cube $I = [a_1, a_1 + s] \times \cdots \times [a_n, a_n + s] \subset \mathbb{R}^n$ with sides parallel to the coordinate axes, where $a_1, \dots, a_n \in \mathbb{R}$ and $s > 0$. Furthermore we write $\ell(I) = s$ for its sidelength. The following well-known assertion follows immediately from the definition of Sobolev spaces.

Lemma 8.3. *Let $f \in \mathrm{W}^k_1(\mathbb{R}^n)$. Then for any $\varepsilon > 0$ there exist functions $f_0 \in \mathrm{C}^\infty_c(\mathbb{R}^n)$, $f_1 \in \mathrm{W}^{k,1}(\mathbb{R}^n)$, such that $f = f_0 + f_1$ and $\|f_1\|_{\mathrm{W}^{k,1}} < \varepsilon$.*

Definition 8.4. *Let μ be a positive measure on $\mathbb{R}^2$. We say that μ has $(*)$-property, if*

$$\mu(I) \le \ell(I) \tag{8.1}$$

for any 2-dimensional interval $I \subset \mathbb{R}^2$.

Now the Maz'ya Theorem 7.5 for the considered situation can be formulated as follows.

Theorem 8.5. *If $f \in \mathrm{W}_1^1(\mathbb{R}^2)$ and μ has $(*)$-property, then*

$$\int |f| d\mu \le C \|\nabla f\|_{\mathrm{L}^1}, \tag{8.2}$$

where C does not depend on μ, f.

Consequently, the Adams Theorem 7.7 gives us

Theorem 8.6. *If $f \in \mathrm{W}_1^1(\mathbb{R}^2)$, then*

$$\int_0^\infty \mathscr{H}^1_\infty(\{x \in \mathbb{R}^2 : \mathcal{M}f(x) \ge \lambda\})\,\mathrm{d}\lambda \le C \int_{\mathbb{R}^2} |\nabla f(y)|\,\mathrm{d}y,$$

where C is a universal constant.

We will also use the following simple technical assertion.

Lemma 8.7 (see, e.g., Remark 2 of §1.4.5 in [60])**.** *Suppose $v \in \mathrm{W}_1^2(\mathbb{R}^2)$. Then v has a continuous representative and for any 2-dimensional interval $I \subset \mathbb{R}^2$ the estimate*

$$\sup_{y,x \in I} |v(y) - v(x)| \le C\left(\frac{\|\nabla v\|_{\mathrm{L}^1(I)}}{\ell(I)} + \|\nabla^2 v\|_{\mathrm{L}^1(I)}\right) \tag{8.3}$$

holds, where C is some universal constant.

The last lemma has the following useful extension.

Lemma 8.8. *Suppose $v \in \mathrm{W}_1^2(\mathbb{R}^2)$. Then v is a continuous function and for any 2-dimensional interval $I \subset \mathbb{R}^2$ with a center point $x_0 \in I$ the function*

$$v_I(x) := v(x) - \fint_I v(y)\,dy - \left(\fint_I \nabla v(y)\,dy\right) \cdot (x - x_0)$$

can be extended from I to the whole of $\mathbb{R}^2$ such that $v_I \in \mathrm{W}_1^2(\mathbb{R}^2)$ and

$$\|\nabla^2 v_I\|_{L^1(\mathbb{R}^2)} \le C \|\nabla^2 v\|_{\mathrm{L}^1(I)}, \tag{8.4}$$

where C is also some universal constant.

Yomdin's Theorem 7.1 for the considered particular plane case is as follows.

Theorem 8.9. *For $A \subset \mathbb{R}^m$ and $\varepsilon > 0$ let $\mathrm{Ent}(\varepsilon, A)$ denote the minimal number of balls of radius ε covering A. Then for any polynomial $P\colon \mathbb{R}^n \to \mathbb{R}^m$ of degree at most k, for each ball $B \subset \mathbb{R}^n$ of radius $r > 0$, and for any number $\varepsilon > 0$ the estimate*

$$\mathrm{Ent}(\varepsilon r, \{P(x) : x \in B,\ |\nabla P(x)| \le \varepsilon\}) \le C_*$$

holds, where C_* depends on n, k, m only.

To apply Theorem 8.6, we need also the following simple estimate and its corollary.

Lemma 8.10 (see Lemma 2 in [24])**.** *Let $u \in \mathrm{W}_1^1(\mathbb{R}^n)$. Then for any ball $B(z,r) \subset \mathbb{R}^n$, $B(z,r) \ni x$, the estimate*

$$\left| u(x) - \fint_{B(z,r)} u(y)\,dy \right| \le Cr(\mathcal{M}\nabla u)(x)$$

holds, where C depends on n only.

Corollary 8.11. *Let $u \in \mathrm{W}_1^1(\mathbb{R}^n)$. Then for any ball $B \subset \mathbb{R}^n$ of a radius $r > 0$ and for any number $\varepsilon > 0$ the estimate*

$$\operatorname{diam}(\{u(x) : x \in B,\ (\mathcal{M}\nabla u)(x) \leq \varepsilon\}) \leq C_{**}\varepsilon r$$

holds, where C_{**} is a constant depending on n only.

An important role in the proof is also played by the following natural approximation result, which surprisingly has been proved only rather recently.

Theorem 8.12 ([17], see also [14])**.** *Let $k, l \in \{1, \dots, n\}$, $k \leq l$. Then for any $f \in \mathrm{W}_1^l(\mathbb{R}^n)$ and for each $\varepsilon > 0$ there exist an open set $U \subset \mathbb{R}^n$ and a function $g \in \mathrm{C}^k(\mathbb{R}^n)$ such that $\mathscr{H}_\infty^{n-l+k}(U) < \varepsilon$ and $f \equiv g$, $\nabla^m f \equiv \nabla^m g$ on $\mathbb{R}^n \setminus U$ for $m = 1, \dots, k$, and*

$$\|f - g\|_{\mathrm{W}_1^{k'}} < \varepsilon,$$

where $k' = \min(k + 1, l)$.

We record as a corollary the particular planar case that will be used in the proof below.

Corollary 8.13. *For any $f \in \mathrm{W}_1^2(\mathbb{R}^2)$ and for each $\varepsilon > 0$ there exist an open set $U \subset \mathbb{R}^2$ and a function $g \in \mathrm{C}^1(\mathbb{R}^2)$ such that $\mathscr{H}_\infty^1(U) < \varepsilon$ and $f \equiv g$, $\nabla f \equiv \nabla g$ on $\mathbb{R}^2 \setminus U$, and*

$$\|f - g\|_{\mathrm{W}_1^2} < \varepsilon.$$

8.2. **On images of sets of small Hausdorff 1-content.** The main result of this section is the following Luzin N–property for $\mathrm{W}_1^2(\mathbb{R}^2)$–functions:

Theorem 8.14. *Let $v \in \mathrm{W}_1^2(\mathbb{R}^2)$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $E \subset \mathbb{R}^2$ if $\mathscr{H}_\infty^1(E) < \delta$, then $\mathscr{H}^1(v(E)) < \varepsilon$. In particular, $\mathscr{H}^1(v(E)) = 0$ whenever $\mathscr{H}^1(E) = 0$.*

For the remainder of this section we fix a function $v \in \mathrm{W}_1^2(\mathbb{R}^2)$. To prove Theorem 8.14, we need some preliminary lemmas that we turn to next.
By a 2-dimensional *dyadic interval* we understand an interval of the form $[\frac{k_1}{2^m}, \frac{k_1+1}{2^m}] \times [\frac{k_2}{2^m}, \frac{k_2+1}{2^m}]$, where k_i, m are integers. The following assertion is straightforward, and hence we omit its proof here.

Lemma 8.15. *For any 2-dimensional interval $I \subset \mathbb{R}^2$ there exist dyadic intervals $Q_1, \dots, Q_4$ such that $I \subset Q_1 \cup \dots \cup Q_4$ and $\ell(Q_1) = \dots = \ell(Q_4) \leq 2\ell(I)$.*

Let $\{I_\alpha\}_{\alpha \in A}$ be a family of 2-dimensional dyadic intervals, where $A \subset \mathbb{N}$. We say that the family $\{I_\alpha\}_{\alpha \in A}$ is *regular*, if for any 2-dimensional dyadic interval Q the estimate

$$\ell(Q) \geq \sum_{\alpha : I_\alpha \subset Q} \ell(I_\alpha) \tag{8.5}$$

holds.

Lemma 8.16 ([16, 17])**.** *Let $\{I_\alpha\}_{\alpha \in A}$ be a family of 2-dimensional dyadic intervals. Then there exists a regular family $\{J_\beta\}_{\beta \in B}$ of 2-dimensional dyadic intervals such that $\cup_\alpha I_\alpha \subset \cup_\beta J_\beta$ and*

$$\sum_\beta \ell(J_\beta) \leq \sum_\alpha \ell(I_\alpha).$$

Proof. Define

$$\mathcal{F}=\left\{J\,:\,J\subset\mathbb{R}^2\text{ dyadic interval; }\sum_{I_\alpha\subset J}\ell(I_\alpha)\geq\ell(J)\right\}.$$

Thus $I_\alpha\in\mathcal{F}$ for each α. Denote by $\mathcal{F}^*=\{J_\beta\}$ the collection of maximal elements of $\mathcal{F}$. Clearly

$$\bigcup_\alpha I_\alpha\subset\bigcup_\beta J_\beta, \tag{8.6}$$

and since dyadic intervals are either disjoint or contained in one another, the $\{J_\beta\}$ are mutually disjoint[5]. It follows that

$$\sum_\beta\ell(J_\beta)\leq\sum_\beta\sum_{I_\alpha\subset J_\beta}\ell(I_\alpha)\leq\sum_\alpha\ell(I_\alpha). \tag{8.7}$$

Observe also that for any dyadic interval $Q\subset\mathbb{R}^2$,

$$\sum_{J_\beta\subset Q}\ell(J_\beta)\leq\ell(Q). \tag{8.8}$$

Indeed, if $J_\beta\subset Q$ for some β, then clearly either $J_\beta=Q$ or $J_\beta\neq Q$. In the first case the estimate is evident, and in the second case we deduce from maximality of J_β that $Q\notin\mathcal{F}$, and hence that

$$\sum_{J_\beta\subset Q}\ell(J_\beta)\leq\sum_{I_\alpha\subset Q}\ell(I_\alpha)<\ell(Q).$$

□

Lemma 8.17. *For each $\varepsilon>0$ there exists $\delta=\delta(\varepsilon,v)>0$ such that for any regular family $\{I_\alpha\}_{\alpha\in A}$ of 2-dimensional dyadic intervals we have*

$$\sum_\alpha\left(\frac{\|\nabla v\|_{\mathrm{L}^1(I_\alpha)}}{\ell(I_\alpha)}+\|\nabla^2v\|_{\mathrm{L}^1(I_\alpha)}\right)<\varepsilon\qquad\textit{whenever}\quad\sum_\alpha\ell(I_\alpha)<\delta.$$

Proof. Fix $\varepsilon>0$ and let $\{I_\alpha\}_{\alpha\in A}$ be a regular family of 2-dimensional dyadic intervals with $\sum_\alpha\ell(I_\alpha)<\delta$, where $\delta>0$ will be specified below. By virtue of Lemma 8.3 we can find a decomposition $v=v_0+v_1$, where $\|\nabla^jv_0\|_{\mathrm{L}^\infty}\leq K=K(\varepsilon,v)$ for all $j=0,1,2$ and

$$\|\nabla v_1\|_{\mathrm{L}^1}+\|\nabla^2v_1\|_{\mathrm{L}^1}<\varepsilon. \tag{8.9}$$

Assume that

$$\sum_\alpha\ell(I_\alpha)<\delta<\tfrac{1}{K+1}\varepsilon. \tag{8.10}$$

Define the measure μ by

$$\mu=\left(\sum_\alpha\frac{1}{\ell(I_\alpha)}\mathbf{1}_{I_\alpha}\right)\mathscr{L}^2, \tag{8.11}$$

[5] By *disjoint dyadic intervals* we mean intervals with disjoint interior.

where $\mathbf{1}_{I_\alpha}$ denotes the indicator function of the set I_α and $\mathscr{L}^2$ is the Lebesgue measure.

Claim. $\frac{1}{4}\mu$ has property $(*)$ from the Definition 8.4, i.e.,

$$\mu(I) \le 4\ell(I)$$

holds for any 2-dimensional interval $I \subset \mathbb{R}^2$.

Indeed, write for a dyadic interval Q

$$\mu(Q) = \sum_{I_\alpha \subset Q} \ell(I_\alpha) + \sum_{Q \subsetneq I_\alpha} \frac{\ell(Q)^2}{\ell(I_\alpha)} \le \ell(Q), \tag{8.12}$$

where we invoked (8.5) and the fact that $Q \subsetneq I_\alpha$ for at most one α (and in this case the first term in (8.12) is zero). Then for any interval I we deduce from Lemma 8.15 the estimate $\mu(I) \le 4\ell(I)$. This proves the claim.

In addition to (8.10) we now decrease $\delta > 0$ further such that

$$\sum_\alpha \|\nabla^2 v\|_{\mathrm{L}^1(I_\alpha)} < \varepsilon/2. \tag{8.13}$$

Then by the bounds (8.9), (8.2) (applied with $f = \nabla v_1$) we have

$$\begin{aligned} \sum_\alpha \left(\|\nabla^2 v\|_{\mathrm{L}^1(I_\alpha)} + \frac{1}{\ell(I_\alpha)} \int_{I_\alpha} |\nabla v| \right) &\le \varepsilon/2 + \tfrac{K}{K+1}\varepsilon + \sum_\alpha \frac{1}{\ell(I_\alpha)} \int_{I_\alpha} |\nabla v_1| \\ &= C'\varepsilon + \int |\nabla v_1|\, d\mu \le C''\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the proof of Lemma 8.17 is complete. □

Proof of Theorem 8.14. Denote

$$R(I) = \|\nabla^2 v\|_{\mathrm{L}^1(I)} + \frac{1}{\ell(I)} \int_I |\nabla v|.$$

First we record the estimate $\operatorname{diam} v(I) \le CR(I)$ that by virtue of Lemma 8.7 holds for any 2-dimensional interval $I \subset \mathbb{R}^2$. Fix $\varepsilon > 0$ and take $\delta = \delta(\varepsilon)$ from Lemma 8.17, i.e., for any regular family $\{I_\alpha\}_{\alpha \in A}$ of 2-dimensional dyadic intervals we have if $\sum_\alpha \ell(I_\alpha) < \delta$ that $\sum_\alpha R(I_\alpha) < \varepsilon$, and consequently, $\sum_\alpha \operatorname{diam} v(I_\alpha) < C\varepsilon$. Now the assertion of Theorem 8.14 follows from Lemmas 8.15–8.16. Indeed by these lemmas there exists $\delta_1 > 0$ such that if $\mathscr{H}^1_\infty(E) < \delta_1$, then E can be covered by a regular family $\{I_\alpha\}_{\alpha \in A}$ of 2-dimensional dyadic intervals with $\sum_\alpha \ell(I_\alpha) < \delta$. □

8.3. **The proof of the Morse–Sard theorem: the main part.** Recall that if $v \in \mathrm{W}^2_1(\mathbb{R}^2)$, then $\nabla v(x)$ is well-defined for $\mathcal{H}^1$-almost all $x \in \mathbb{R}^2$ (see Proposition 8.1). In particular, v is differentiable (in the classical Fréchet sense) and the classical derivative coincides with $\nabla v(x) = \lim_{r \to 0} ⨍_{B(x,r)} \nabla v(z)\, \mathrm{d}z$ and $\lim_{r\to 0} ⨍_{B_r(x)} |\nabla v(y) - \nabla v(x)|^2\, \mathrm{d}y < \infty$ at all points $x \in \mathbb{R}^2 \setminus A_v$, where $\mathscr{H}^1(A_v) = 0$. Consequently, in view of Theorem 8.14, $\mathscr{H}^1(v(A_v)) = 0$.

Denote $Z_v = \{x \in \mathbb{R}^2 \setminus A_v : \nabla v(x) = 0\}$. The main result of the subsection is as follows:

Theorem 8.18. *If $v \in \mathrm{W}^2_1(\mathbb{R}^2)$, then $\mathscr{H}^1(v(Z_v)) = 0$.*

For the remainder of the section we fix a function $v \in \mathrm{W}^2_1(\mathbb{R}^2)$.
The key point of the proof is contained in the following lemma.

Lemma 8.19. *For any 2-dimensional dyadic interval $I \subset \mathbb{R}^2$ the estimate*

$$\mathscr{H}^1(v(Z_v \cap I)) \le C\|\nabla^2 v\|_{\mathrm{L}^1(I)} \tag{8.14}$$

holds, where C is a universal constant.

Proof. Fix a 2-dimensional dyadic interval $I \subset \mathbb{R}^2$. Note that by formula (8.4) it is sufficient to prove the estimate

$$\mathscr{H}^1(v(Z_v \cap I)) \le C\|\nabla^2 v_I\|_{L^1(\mathbb{R}^2)}, \tag{8.15}$$

where the function $v_I(x) = v(x) - P_I(x)$ was defined in Lemma 8.8, and by P_I we denote the corresponding first-order polynomial:

$$P_I(x) := \fint_I v(y)\,dy + \left(\fint_I \nabla v(y)\,dy\right) \cdot (x - x_0),$$

where x_0 is the center point of I.

Denote $\sigma = \|\nabla^2 v_I\|_{\mathrm{L}^1(\mathbb{R}^2)}$ and

$$E_j = \{x \in \mathbb{R}^2 : (\mathcal{M}\nabla v_I)(x) \in (2^{j-1}, 2^j]\}, \quad \delta_j = \mathscr{H}^1_\infty(E_j), \quad j \in \mathbb{Z}.$$

Then applying Theorem 8.6 with $f = \nabla v_I$ we find

$$\sum_{j=-\infty}^{\infty} \delta_j 2^j \le C_1\sigma.$$

By construction, for each $j \in \mathbb{Z}$ there exists a family $\{B_{ij}\}_{i\in\mathbb{N}}$ of balls B_{ij} in $\mathbb{R}^2$ of radii r_{ij} such that

$$E_j \subset \bigcup_{i=1}^{\infty} B_{ij} \text{ and } \sum_{i=1}^{\infty} r_{ij} \le 3\delta_j.$$

Denote

$$Z_{ij} = Z_v \cap I \cap E_j \cap B_{ij}.$$

By construction each point $x \in Z_v$ is a Lebesgue point of ∇v and $\mathcal{M}\nabla v(x) < \infty$, therefore, $Z_v \cap I = \bigcup_{i,j} Z_{ij}$ and

$$\mathscr{H}^1(v(I \cap Z_v)) \le \sum_{j=-\infty}^{\infty} \sum_{i=1}^{\infty} \mathscr{H}^1(v(Z_{ij})).$$

Thus it remains to estimate $\mathscr{H}^1(v(Z_{ij}))$.

Since $\nabla P_I(x) = -\nabla v_I(x)$ at each point $x \in Z_v \cap I$, we have by construction for all i, j:

$$Z_{ij} \subset \{x \in B_{ij} : |\nabla P_I(x)| = |\nabla v_I(x)| \le (\mathcal{M}\nabla v_I)(x) \le 2^j\}.$$

Because of the identity $v = v_I + P_I$, applying Theorem 8.9 and Corollary 8.11 to functions P_I, v_I, respectively, with $B = B_{ij}$ and $\varepsilon = 2^j$, we find a finite family of intervals $T_k \subset \mathbb{R}$ each of length $2(1 + C_{**})2^j r_{ij}$, $k = 1, \dots, C_*$, such that

$$\bigcup_{k=1}^{C_*} T_k \supset v(Z_{ij}).$$

Therefore

$$\mathscr{H}^1(v(Z_{ij})) \le 2C_*(1+C_{**})2^j r_{ij},$$

and consequently,

$$\mathscr{H}^1(v(Z_v \cap I)) \le \sum_{j=-\infty}^{\infty} \sum_{i=1}^{\infty} 2C_*(1+C_{**})2^j r_{ij} \le 6C_*(1+C_{**}) \sum_{j=-\infty}^{\infty} 2^j \delta_j \le C'\sigma.$$

The last estimate finishes the proof of the lemma. □

From the last result and the absolute continuity of the Lebesgue integral we infer

Corollary 8.20. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $E \subset \mathbb{R}^2$ if $\mathscr{H}^2_\infty(E) \le \delta$, then $\mathscr{H}^1(v(Z_v \cap E)) \le \varepsilon$. In particular, $\mathscr{H}^1(v(Z_v \cap E)) = 0$ for any $E \subset \mathbb{R}^2$ with $\mathscr{H}^2_\infty(E) = 0$.*

Because of the classical Morse-Sard theorem for $g \in \mathrm{C}^2(\mathbb{R}^2)$, Theorem 8.12 (applied to the case $k = n = 2$) implies

Corollary 8.21. *There exists a set $Z_{0,v}$ of 2-dimensional Lebesgue measure zero such that $\mathscr{H}^1(v(Z_v \setminus Z_{0,v})) = 0$. In particular, $\mathscr{H}^1(v(Z_v)) = \mathscr{H}^1(v(Z_{0,v}))$.*

From Corollaries 8.21, 8.20 we conclude the proof of Theorem 8.18.

Proof of Theorem 8.2. The first two assertions (i)–(ii) follow immediately from the Theorems 8.14 and 8.18. The last assertion (iii) concerning the level sets follows from the same Theorems 8.14, 8.18 and from approximation Theorem 8.13 by some elementary arguments, see [17] or [16] for the details. □

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