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DIFFERENTIABILITY ALMOST EVERYWHERE OF WEAK LIMITS OF BI-SOBOLEV HOMEOMORPHISMS

ANNA DOLEŽALOVÁ AND ANASTASIA MOLCHANOVA

Abstract. This paper investigates the differentiability of weak limits of bi-Sobolev homeomorphisms. Given $p > n-1$, consider a sequence of homeomorphisms f_k with positive Jacobians $J_{f_k} > 0$ almost everywhere and $\sup_k(\|f_k\|_{W^{1,n-1}} + \|f_k^{-1}\|_{W^{1,p}}) < \infty$. We prove that if f and h are weak limits of f_k and f_k^{-1}, respectively, with positive Jacobians $J_f > 0$ and $J_h > 0$ a.e., then $h(f(x)) = x$ and $f(h(y)) = y$ both hold a.e. and f and h are differentiable almost everywhere.

[Reshetnyak's] synthesis of classical function theory and Sobolev function classes was so fruitful that it was given a special name: quasiconformal analysis.

A. D. Aleksandrov, 1999 Russ. Math. Surv. 54 1069

1. Introduction

Let Ω and Ω' be domains, i.e. non-empty connected open sets, in $\mathbb{R}^n$ and $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ be a mapping from Ω to Ω'. According to classic results of Geometric Analysis, if $p > n$, the mapping f is differentiable almost everywhere. This result was established in 1941 for $n = 2$ by Cesari [4] and later generalized to arbitrary n by Calderón [2]. The a.e.-differentiability of continuous and monotone mappings was studied from a geometrical perspective by Väisälä [29] and Reshetnyak [24, 25, 26]. This includes mappings with bounded distortion, also known as quasiregular mappings, and mappings with finite distortion (even for $p = n$). Further details on these results can be found in [25, 29]. The results also extend to $W^{1,1}$-homeomorphisms in dimension $n = 2$, as shown by Gehring and Lehto [10], and $W^{1,p}$-homeomorphisms with $p > n-1$ if $n \geq 3$, see Väisälä [29] (also Onninen [22, Theorem 1.2 and Example 1.3]).

For $W^{1,n-1}$-Sobolev homeomorphisms with $n \geq 3$, the a.e.-differentiability was established by considering the integrability of the inner distortion $K_I \in L^1(\Omega)$,

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where $J_f(x) := \det Df(x)$ is the Jacobian, $\operatorname{adj} Df$ is the adjugate matrix of Df and

$$K_I := \frac{|\operatorname{adj} Df|^n}{J_f(x)^{n-1}},$$

if f has finite inner distortion, e.g. $\operatorname{adj} Df(x) = 0$ almost everywhere in the zero set of the Jacobian $\{x \in \Omega : J_f(x) = 0\}$, see [28]. This condition on integrability of distortion is sharp, meaning that for any $\delta \in (0,1)$ and $n \geq 3$ there exists a homeomorphism f which belongs to $W^{1,n-1}((-1,1)^n, \mathbb{R}^n)$ such that $K_I \in L^\delta((-1,1)^n)$ and f is not classically differentiable on a set of positive measure [14]. The a.e.-differentiability of $W^{1,n-1}$-Sobolev maps also holds for continuous, open, and discrete mappings of finite distortion with nonnegative Jacobian if a particular weighted distortion function is integrable [31]. The condition $K_I \in L^1(\Omega)$ essentially means that $f^{-1} \in W^{1,n}(f(\Omega), \mathbb{R}^n)$ [30, Theorem 3], [20, Theorem 1.1], see also [23]. The regularity of the inverse, together with the oscillation estimate [23, Lemma 2.1], gives

$$\limsup_{r \to 0+} \frac{\operatorname{osc}_{B(x,r)} f}{r} < \infty$$

for almost all $x \in \Omega$, and hence f is differentiable in x by the Stepanov theorem. Thus, instead of assumptions on distortion, we can directly consider bi-Sobolev homeomorphisms. Indeed, if $f \in W^{1,n-1}$, $J_f > 0$ a.e., and $f^{-1} \in W^{1,p}$ with $p > n-1$, then f^{-1} has finite distortion, e.g. [12, Lemma 5.2], and hence both f and f^{-1} are differentiable almost everywhere [32, Theorem 27]. However, no similar result holds for $W^{1,n-1}$-bi-Sobolev homeomorphisms and we need the inverse to be in $W^{1,p}$ for $p > n-1$. In fact, Csörnyei, Hencl, and Malý constructed in Example 5.2 in [5] a homeomorphism $f \in W^{1,n-1}((-1,1)^n, \mathbb{R}^n)$, $n \geq 3$, with $J_f > 0$ a.e. that is nowhere differentiable and its inverse $f^{-1} \in W^{1,n-1}((-1,1)^n, \mathbb{R}^n)$ is also nowhere differentiable.

In this work, we examine the a.e.-differentiability of a class of *weak limits of homeomorphisms*. This class of mappings is well suited for the Calculus of Variations approach and may serve as deformations in Continuum Mechanics models. For further information, refer to [15, 17, 19]. Weak limits of Sobolev homeomorphisms have received significant attention in recent years, with various studies conducted, including [1, 3, 6, 7, 8, 9, 13, 16].

Here we consider the energy functional

$$\mathcal{E}(f) := \int_\Omega |Df(x)|^{n-1}\, dx + \int_{\Omega'} |Df^{-1}(y)|^p\, dy$$

for bi-Sobolev mappings $f\colon \Omega \to \Omega'$ such that f is invertible almost everywhere, $f \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, and $f^{-1} \in W^{1,p}(\Omega', \mathbb{R}^n)$ for some $p > n-1$.

The main result, which is proven in Section 1.2, reads as follows.

Theorem 1.1. *Let $n \geq 2$, $p > n-1$, Ω, $\Omega' \subset \mathbb{R}^n$ be bounded domains and $f_k \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, $k = 0, 1, 2 \dots$, be homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega'}$ with $J_{f_k} > 0$ a.e. and*

$$\sup_k \mathcal{E}(f_k) < \infty.$$

Assume that $f\colon \Omega \to \mathbb{R}^n$ is a weak limit of $\{f_k\}_{k\in\mathbb{N}}$ in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ with $J_f > 0$ a.e. and $h\colon \Omega' \to \mathbb{R}^n$ is a weak limit of $\{f_k^{-1}\}_{k\in\mathbb{N}}$ in $W^{1,p}(\Omega', \mathbb{R}^n)$ with $J_h > 0$ a.e. Then for a.e. $x \in \Omega$ we have $h(f(x)) = x$ and for a.e. $y \in \Omega'$ we have $f(h(y)) = y$, and both f and h are differentiable almost everywhere.

Let us note the following result, which better suits the Calculus of Variations approach since it formulates the assumptions only for f_k.

Corollary 1.2. *Let $n \geq 2$, $p > n-1$, Ω, $\Omega' \subset \mathbb{R}^n$ be bounded domains and φ be a positive convex function on $(0, \infty)$ with*

$$\lim_{t\to 0^+} \varphi(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty. \tag{1.1}$$

Let $f_k \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, $k = 0, 1, 2 \dots$, be homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega'}$ with $J_{f_k} > 0$ a.e. such that $\sup_k \mathcal{F}(f_k) < \infty$, where

$$\mathcal{F}(f) := \int_\Omega |Df(x)|^{n-1} + \frac{|\operatorname{adj} Df(x)|^p}{J_f^{p-1}(x)} + \varphi(J_f(x))\, dx.$$

Assume that $f\colon \Omega \to \mathbb{R}^n$ is a weak limit of $\{f_k\}_{k\in\mathbb{N}}$ in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ and $h\colon \Omega' \to \mathbb{R}^n$ is a weak limit of $\{f_k^{-1}\}_{k\in\mathbb{N}}$ in $W^{1,p}(\Omega', \mathbb{R}^n)$. Then for a.e. $x \in \Omega$ we have $h(f(x)) = x$ and for a.e. $y \in \Omega'$ we have $f(h(y)) = y$, and both f and h are differentiable almost everywhere.

2. Preliminaries

By $B(c, r)$, we denote the open euclidean ball with centre $c \in \mathbb{R}^n$ and radius $r > 0$, and by $S(c, r)$ the corresponding sphere.

2.1. **Topological image and** (INV) **condition.** Although a weak limit of homeomorphisms may not be a homeomorphism, it may possess an invertibility property known as the (INV) condition. The (INV) condition states, informally, that a ball $B(x, r)$ is mapped inside the image of the sphere $f(S(x, r))$ and the complement $\Omega \setminus \overline{B(x, r)}$ is mapped outside $f(S(x, r))$. This concept was introduced for $W^{1,p}$-mappings, where $p > n-1$, by Müller and Spector [21], although the fact that a ball $B(x, r)$ is mapped inside the image of a sphere $f(S(a, r))$ was known in literature before as *monotonicity*, see [25] and [33, §2]. Suppose that $f\colon S(y, r) \to \mathbb{R}^n$ is continuous, we define the *topological image* of $B(x, r)$ as

$$f^T(B(x, r)) := \{z \in \mathbb{R}^n \setminus f(S(x, r)) : \deg(f, S(x, r), z) \neq 0\} \tag{2.1}$$

and the *topological image* of x as

$$f^T(x) := \bigcap_{r>0,\, r\notin N_x} {f^*}^T(B(x, r)) \cup f^*(S(x, r)),$$

where a representative f^* and a set N_x will be defined further along.

Definition 2.1. A mapping $f\colon \Omega \to \mathbb{R}^n$ satisfies the (INV) *condition*, provided that for every $x \in \Omega$ there exist a constant $r_x > 0$ and an $\mathcal{L}^1$-null set N_x such that for all $r \in (0, r_x) \setminus N_x$, the restriction $f|_{S(x,r)}$ is continuous and

(i) $f(z) \in f^T(B(x,r)) \cup f(S(x,r))$ for a.e. $z \in \overline{B(x,r)}$,
(ii) $f(z) \in \mathbb{R}^n \setminus f^T(B(x,r))$ for a.e. $z \in \Omega \setminus B(x,r)$.

Let us note that for a particular representative of a Sobolev mapping, Definition 2.1 allows for some points to escape their destiny, e.g. a null-set inside the ball may be mapped outside the image of this ball. Thus, we also consider a stronger version of the (INV) condition.

Definition 2.2. A mapping $f\colon \Omega \to \mathbb{R}^n$ satisfies the *strong* (INV) *condition*, provided that for every $x \in \Omega$ there exist a constant $r_x > 0$ and an $\mathcal{L}^1$-null set N_x such that for all $r \in (0, r_x) \setminus N_x$ the restriction $f|_{S(x,r)}$ is continuous and

(i) $f(z) \in f^T(B(x,r)) \cup f(S(x,r))$ for every $z \in \overline{B(x,r)}$,
(ii) $f(z) \in \mathbb{R}^n \setminus f^T(B(x,r))$ for every $z \in \Omega \setminus B(x,r)$.

2.2. **Precise, super-precise, and hyper-precise representative of a Sobolev mapping.** Let $1 \le p \le n$ and $f \in W^{1,p}(\mathbb{R}^n)$, then the *precise representative* of f is given by

$$f^*(a) := \begin{cases} \lim\limits_{r\to 0^+} \dfrac{1}{|B(a,r)|} \displaystyle\int_{B(a,r)} f(x)\,dx & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Note that the representative f^* is p-quasicontinuous (see remarks after [21, Proposition 2.8]).

Let now $f\colon \Omega \to \mathbb{R}^n$ be a $W^{1,p}$-weak limit of homeomorphisms $f_k\colon \Omega \to \mathbb{R}^n$ with $p \in (n-1, n]$ for $n > 2$ or $p \in [1,2]$ for $n = 2$. Then by [1, Theorem 5.2] there exists an $\mathcal{H}^{n-p}$-null set $NC \subset \Omega$ and a representative f^{**} of f such that f^{**} is continuous at every $x \in \Omega \backslash NC$, a set-valued image $f^T(x)$ is a singleton for every $y \in \Omega \backslash NC, f^{**} = f^*\,\mathrm{cap}_p$-a.e., and f^{**} can be chosen so that $f^{**}(x) \in f^T(x)$ for every $x \in \Omega$. We will call f^{**} the *super-precise representative* of f.

The *hyper-precise* representative $\tilde{f}$ is defined as

$$\tilde{f}(a) := \limsup_{r\to 0^+} \frac{1}{|B(a,r)|} \int_{B(a,r)} f(x)\,dx. \tag{2.3}$$

We need the following monotonicity property of mappings satisfying the strong (INV) condition.

Lemma 2.3. *Let $n \ge 2$ and $\Omega' \subset \mathbb{R}^n$ be a bounded domain. If $h\colon \Omega' \to \mathbb{R}^n$ satisfies the strong* (INV) *condition, then h is monotone for almost all radii, i.e., for $y \in \Omega'$ there exists an $\mathcal{L}^1$-null set N_y such that for all $r \in (0, r_y) \setminus N_y$ it holds that $\operatorname{osc}_{B(y,r)} h \le \operatorname{osc}_{S(y,r)} h$.*
If, moreover, $h \in W^{1,p}(\Omega', \mathbb{R}^n)$ with $p > n-1$, then for any $r \in \left(0, \frac{r_y}{2}\right)$ the following estimate holds

$$\operatorname{osc}_{B(y,r)} h \le Cr \left(r^{-n} \int_{B(y,2r)} |Dh|^p \right)^{1/p}.$$

Proof. Let N_y be a set from Definition 2.2. Then for $y \in \Omega'$ and $r \in (0, r_y) \setminus N_y$ it holds that h is continuous on the sphere $S(y,r)$ and $h(z) \in h^T(B(y,r)) \cup h(S(y,r))$ for every $z \in \overline{B(y,r)}$. In this case, $h(S(y,r))$ is a compact set and $h^T(B(y,r)) \subseteq \mathbb{R}^n \setminus A$, where A is the unbounded component of $\mathbb{R}^n \setminus h(S(y,r))$ (since by the basic properties of the topological degree [12, p. 48(d)] we have $\deg(h, S(y,r), \xi) = 0$ for all $\xi \in A$), and therefore $\operatorname{osc}_{B(y,r)} h \leq \operatorname{osc}_{S(y,r)} h$.

Further, for $y \in \Omega'$ and $r > 0$, and for a.e. $t \in [r, 2r)$, it holds that

$$\operatorname{osc}_{B(y,r)} h \leq \operatorname{osc}_{B(y,t)} h \leq \operatorname{osc}_{S(y,t)} h.$$

Then by the Sobolev embedding theorem on spheres [12, Lemma 2.19], following the proof of [12, Theorem 2.24], we obtain that

$$\operatorname{osc}_{B(y,r)} h \leq \operatorname{osc}_{S(y,t)} h \leq Ct \left(t^{-n+1} \int_{S(y,t)} |Dh|^p \right)^{1/p}$$
$$\leq Cr \left(r^{-n} \int_{B(y,2r)} |Dh|^p \right)^{1/p}.$$

□

Remark 2.4. In case $p > n$, $h^* = h^{**} = \tilde{h}$ is the continuous representative of h and h^* is differentiable almost everywhere [2] and satisfies the Lusin (N) condition in Ω [18]. Moreover, due to compact embedding of $W^{1,p}$ into the Hölder space $C^{0,\alpha}$, weak convergence in $W^{1,p}$ implies uniform convergence on compact sets. With these properties, the subsequent analysis becomes simplified, and the details are left to the reader.

3. A.e.-invertibility of f

Since a limit of homeomorphisms may not be a homeomorphism, we need to define a weaker notion of an inverse mapping. First recall that a mapping $f \colon \Omega \to \Omega'$ is called *injective a.e. in domain* if there exists a null set $\Sigma \subset \Omega$, $|\Sigma| = 0$, such that the restriction $f|_{\Omega \setminus \Sigma} \colon \Omega \setminus \Sigma \to f(\Omega \setminus \Sigma)$ is injective. A mapping $f \colon \Omega \to \Omega'$ is called *injective a.e. in image* if there exists a null set $\Sigma' \subset \Omega'$, $|\Sigma'| = 0$, such that for any $y \in f(\Omega) \setminus \Sigma'$ the preimage $f^{-1}(y) := \{x \in \Omega : f(x) = y\}$ consists of precisely one point. Note that if f is injective a.e. in image and satisfies the $(\mathrm{N})^{-1}$ condition, then f is injective a.e. in domain. If instead f is injective a.e. in domain, f satisfies the (N) condition, and $|\Omega'| = |f(\Omega)|$ then f is injective a.e. in image. We say that $h \colon \Omega' \to \Omega$ is the a.e.-inverse to $f \colon \Omega \to \Omega'$ if for a.e. $x \in \Omega$ we have $h(f(x)) = x$ and for a.e. $y \in \Omega'$ we have $f(h(y)) = y$. Note that if f satisfies the $(\mathrm{N})^{-1}$ condition, then f is injective a.e. in image if and only if there exists the a.e.-inverse to f.

The following lemma provides some additional conditions that guarantee the a.e.-invertibility of f in our setting.

Lemma 3.1. *Let $n \geq 2$, Ω and Ω' be bounded domains in $\mathbb{R}^n$, $p > n-1$, and let $f_k \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ be homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega'}$ with $J_{f_k} > 0$. Let also $f \colon \Omega \to \mathbb{R}^n$ be a weak limit of $\{f_k\}_{k\in\mathbb{N}}$ in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ with $J_f > 0$ a.e. Assume* *also that the sequence* $\{f_k^{-1}\}_{k\in\mathbb{N}}$ *converges* $W^{1,p}$*-weakly to* $h\colon \Omega' \to \mathbb{R}^n$ *with* $J_h > 0$ *a.e. Then* $h^{**}(f(x)) = x$ *a.e. in* Ω *and* $f(h^{**}(y)) = y$ *a.e. in* Ω'.

Proof. Let $p > n-1$, and fix a representative of f, which we denote by the same symbol. If needed, we pass to a subsequence so that $f_k \to f$ and $f_k^{-1} \to h$ pointwise a.e. Since h is a $W^{1,p}$-weak limit of Sobolev homeomorphisms with $p > n-1$, the super-precise representative h^{**} satisfies the strong (INV) condition [1, Theorem 5.2 and Lemma 5.3]. Then there exists a set $G_1' \subset \Omega'$ of full measure $|G_1'| = |\Omega'|$: $J_{h^{**}}(y) > 0$ for all $y \in G_1'$, h^{**} is injective in G_1' (see [21, Lemma 3.4] and [1, Theorem 1.2]) and $f_k^{-1}(y) \to h^{**}(y)$ for all $y \in G_1'$.

Step 1. $h^{**}(f(x)) = x$ a.e.: By Lemma 2.3, we know that $\operatorname{osc}_{B(y,r)} h^{**} \underset{r\to 0}{\longrightarrow} 0$ for a.e. $y \in \Omega'$. Since $J_f > 0$ a.e. (and therefore f satisfies the $(\mathrm{N})^{-1}$ condition), $\operatorname{osc}_{B(f(x),r)} h^{**} \underset{r\to 0}{\longrightarrow} 0$ for a.e. $x \in \Omega$.

Let $G_1 \subset f^{-1}(G_1')$ be a set such that $|G_1| = |\Omega|$ and for all $x \in G_1$ it holds that $f_k(x) \to f(x)$ and $\operatorname{osc}_{B(f(x),r)} h^{**} \underset{r\to 0}{\longrightarrow} 0$.

For $x \in G_1$ and $r > 0$, by the pointwise convergence of f_k in $x \in G_1$ and f_k^{-1} in $f(x) \in G_1'$, we can find $k_0 \in \mathbb{N}$ big enough such that

$$f_k(x) \in B(f(x), r) \quad \text{and} \quad f_k^{-1}(f(x)) \in B(h^{**}(f(x)), r)$$

for all $k \geq k_0$. Moreover, by [21, Lemma 2.9] (though it is formulated for the precise representative h^*, it holds also for the super-precise representative h^{**} with an analogous proof), there exists a subsequence $\{f_{k_j}\}_{j\in\mathbb{N}}$ (that depends on r) and a number $j_0 \in \mathbb{N}$ big enough such that for all $j \geq j_0$

$$\operatorname{osc}_{S(f(x),r)} f_{k_j}^{-1} \leq \operatorname{osc}_{S(f(x),r)} h^{**} + r.$$

Then we have

$$\begin{aligned}
|f_{k_j}^{-1}(f_{k_j}(x)) - &h^{**}(f(x))| \\
&\leq |f_{k_j}^{-1}(f_{k_j}(x)) - f_{k_j}^{-1}(f(x))| + |f_{k_j}^{-1}(f(x)) - h^{**}(f(x))| \\
&\leq \operatorname{osc}_{B(f(x),r)} f_{k_j}^{-1} + r \leq \operatorname{osc}_{S(f(x),r)} f_{k_j}^{-1} + r \\
&\leq \operatorname{osc}_{S(f(x),r)} h^{**} + r + r \leq \operatorname{osc}_{B(f(x),2r)} h^{**} + 2r.
\end{aligned}$$

Therefore, by definition of G_1,

$$|x - h^{**}(f(x))| = |f_{k_j}^{-1}(f_{k_j}(x)) - h^{**}(f(x))| \leq \lim_{r\to 0}(\operatorname{osc}_{B(f(x),2r)} h^{**} + 2r) = 0$$

for all $x \in G_1$, which concludes Step 1.

Step 2. $f(h^{**}(y)) = y$ a.e.: We know that h^{**} is injective a.e. on G_1' and both f and h^{**} satisfies the $(\mathrm{N})^{-1}$ condition, so when we set

$$G_2' := \big(G_1' \cap (h^{**})^{-1}(G_1)\big) \setminus (h^{**})^{-1}(f^{-1}(\Omega' \setminus G_1')),$$

we know it is a set of full measure. Let us take $y \in G_2'$. Since f_k^{-1} is a homeomorphism onto Ω, we can find $y_k \in \Omega'$ such that $f_k^{-1}(y_k) = h^{**}(y)$. Therefore,

$$y_k = f_k(f_k^{-1}(y_k)) = f_k(h^{**}(y)) \to f(h^{**}(y)),$$

so y_k converges to some $\tilde{y} = f(h^{**}(y))$. We apply h^{**} to both sides to get $h^{**}(\tilde{y}) = h^{**}(f(h^{**}(y)))$. From $y \in G_2'$ we have that $h^{**}(y) \in G_1$. Since $h^{**}(f(x)) = x$ on G_1 we get $h^{**}(\tilde{y}) = h^{**}(f(h^{**}(y))) = h^{**}(y)$. Now we can have either $\tilde{y} \in G_1'$ or $\tilde{y} \notin G_1'$. In the first case, $\tilde{y} = y$ as h^{**} is injective on G_1', so $f(h^{**}(y)) = y$. In the other case, $f(h^{**}(y)) \in \Omega' \setminus G_1'$, which is a contradiction to $y \in G_2'$.

□

Remark 3.2. If $p > n$, equality $h^{**}(f(x)) = x$ can be derived easily from

$$|x - h^{**}(f(x))| \leq |f_k^{-1}(f_k(x)) - f_k^{-1}(f(x))| + |f_k^{-1}(f(x)) - h^{**}(f(x))|,$$

using uniform convergence $f_k^{-1} \rightrightarrows h^{**}$ (up to subsequence) and the Morrey inequality for f_k^{-1}. The other relation $f(h^{**}(y)) = y$ follows the same way as above.

Remark 3.3. Since both f and h satisfy the $(\mathrm{N})^{-1}$ condition, the identities $h(f(x)) = x$ a.e. in Ω and $f(h(y)) = y$ a.e. in Ω' hold for arbitrary representatives.

4. Differentiability

First, let us notice the following well-known fact.

Lemma 4.1. *Let $n \geq 2$, $p > n-1$ and Ω' be a bounded domain in $\mathbb{R}^n$. If $h \in W^{1,p}_{\mathrm{loc}}(\Omega', \mathbb{R}^n)$ satisfies the strong* (INV) *condition, then h is differentiable a.e. in Ω'.*

Proof. By Lemma 2.3 we have

$$\operatorname{osc}_{B(y,r)} h \leq Cr \left(r^{-n} \int_{B(y,2r)} |Dh|^p \right)^{1/p},$$

which implies by setting $r = |z-y|$ that

$$\limsup_{z \to y} \frac{|h(z) - h(y)|}{|z-y|} \leq C|Dh(y)| < \infty$$

for any Lebesgue point y of $|Dh|^p$ and, therefore, h is differentiable a.e. by the Stepanov theorem [27], see also [12, Theorem 2.23].

□

We also need the following modification of [12, Lemma A.29], which gives us the a.e.-differentiability of mapping f from Theorem 1.1 – but the derivative is only with respect to a set of full measure.

Lemma 4.2. *Let $n \geq 2$ and Ω, Ω' be bounded domains in $\mathbb{R}^n$. Let $\Lambda \subset \Omega$, $\Lambda' \subset \Omega'$ be sets of full measure and $h\colon \Omega' \to \Omega$ such that $h\colon \Lambda' \to \Lambda = h(\Lambda')$ is differentiable with respect to the relative topology in Λ', i.e., induced by the topology in $\mathbb{R}^n$, and $J_h(y) > 0$ for all $y \in \Lambda'$. Assume also that $h|_{\Lambda'}$ is injective, and the inverse mapping $f := h^{-1}$ is continuous in Λ with respect to the relative topology in Λ. Then f is differentiable on Λ with respect to the relative topology in Λ and $Df(x) = (Dh(f(x)))^{-1}$ for all $x \in \Lambda$.*

Proof. Since $h\colon \Lambda' \to \Lambda$ is a homeomorphism, the proof of this lemma follows the lines of the proof of [12, Lemma A.29]. We present it here for the convenience of the reader.

By the differentiability of h we know that for $y \in \Lambda'$

$$\lim_{\bar{y}\to y,\, \bar{y}\in\Lambda'} \frac{h(\bar{y}) - h(y) - Dh(y)(\bar{y}-y)}{|\bar{y}-y|} = 0. \tag{4.1}$$

For $\bar{x}$, $x \in \Lambda$ denote $\bar{y} = f(\bar{x})$, $y = f(x) \in \Lambda'$, then

$$h(\bar{y}) - h(y) = h(f(\bar{x})) - h(f(x)) = \bar{x} - x.$$

Since $J_h(y) > 0$ we obtain for $\bar{y}$ close to y enough that

$$|\bar{x} - x| = |h(\bar{y}) - h(y)| \approx |Dh(y)(\bar{y}-y)| \approx |\bar{y} - y|.$$

Then from (4.1) it follows

$$\begin{aligned} 0 &= \lim_{\bar{y}\to y,\, \bar{y}\in\Lambda'} \frac{(Dh(y))^{-1}\left(h(\bar{y}) - h(y) - Dh(y)(\bar{y}-y)\right)}{|y'-y|} \\ &= \lim_{\bar{y}\to y,\, \bar{y}\in\Lambda'} \frac{(Dh(y))^{-1}\left(h(\bar{y}) - h(y)\right) - (\bar{y}-y)}{|y'-y|} \\ &\approx \lim_{\bar{x}\to x,\, \bar{x}\in\Lambda} \frac{(Dh(f(x)))^{-1}(\bar{x}-x) - (f(\bar{x}) - f(x))}{|\bar{x}-x|}, \end{aligned}$$

which concludes the proof. □

The following proposition is a version of an inverse function theorem.

Proposition 4.3. *Let $n \geq 2$, $p > n-1$, Ω and Ω' be bounded domains in $\mathbb{R}^n$, $\Lambda \subset \Omega$ and $\Lambda' \subset \Omega'$ be sets of full measure and $h \in W^{1,p}(\Omega', \Omega)$ satisfy the strong* (INV) *condition and be differentiable with $J_h(y) > 0$ for any $y \in \Lambda'$. Assume also that the restriction $h|_{\Lambda'}\colon \Lambda' \to \Lambda$ is one-to-one. Then for any $y_0 \in \Lambda'$ there exists a sequence $\{r_m\}_{m\in\mathbb{N}} \searrow 0$ such that the topological image $h^T(B(y_0, r_m))$ contains $B\left(h(y_0), \frac{r_m}{3}\right)$.*

Proof. Without loss of generality, by a translation and a linear change of variables, we may assume that $y_0 = 0$, $h(y_0) = 0$, and $Dh(y_0) = Id$. Since h is differentiable at 0, it holds that $h(y) = y + o(|y|)$ if $y \to 0$. That means that there exists $r_0 > 0$ such that

$$|h(y) - y| \leq \frac{|y|}{2} \quad \text{for all } y \in B(0, r_0) \subset \Omega'. \tag{4.2}$$

Consider a sequence $\{r_m\}_{m\in\mathbb{N}} \searrow 0$ such that h is continuous on $S(0, r_m)$ and Definition 2.2 (i)–(ii) is fulfilled. Let now $z \in B\left(0, \frac{r_m}{3}\right) \subset \Omega$, the inequality (4.2) implies $z \notin h(S(0, r_m))$. Since $\operatorname{dist}(z, S(0, r_m)) > r_m/2$, from (4.2) we know that $1 = \deg(z, Id, S(0, r_m)) = \deg(z, h, S(0, r_m))$. Therefore, $B\left(0, \frac{r_m}{3}\right) \subset h^T(B(0, r_m))$, see Figure 1 for illustration. □

The closing theorem of this section concludes the differentiability part of Theorem 1.1.

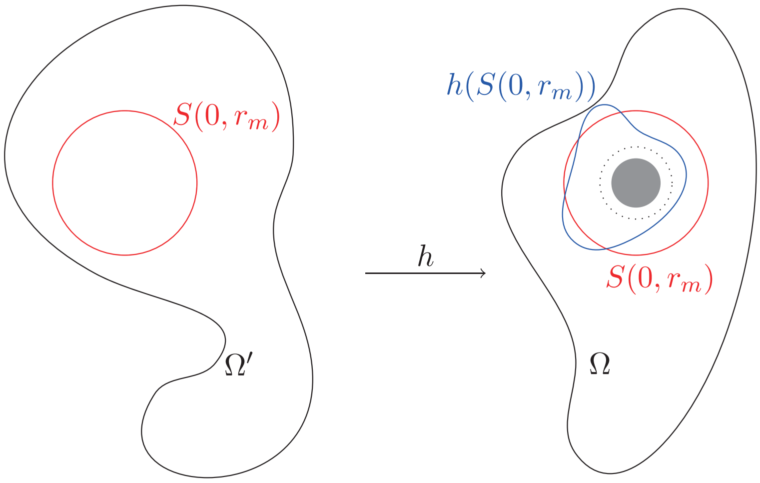

FIGURE 1. Mapping h maps the red sphere $S(0,r_m)$ to $h(S(0,r_m))$ (blue); the grey ball $B(0,r_m/3)$ does not intersect $h(S(0,r_m))$, since its distance from 0 is at least $r_m/2$ (denoted by the dotted sphere).

Theorem 4.4. *Let $n \geq 2$, $p > n-1$, Ω and Ω' be bounded domains in $\mathbb{R}^n$ and $f_k \in W^{1,n-1}(\Omega,\mathbb{R}^n)$ be homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega'}$ with $J_{f_k} > 0$. Let $f\colon \Omega \to \mathbb{R}^n$ be a weak limit of $\{f_k\}_{k\in\mathbb{N}}$ in $W^{1,n-1}(\Omega,\mathbb{R}^n)$ with $J_f > 0$ a.e. Assume also that the sequence $\{f_k^{-1}\}_{k\in\mathbb{N}}$ converges $W^{1,p}$-weakly to $h\colon \Omega' \to \mathbb{R}^n$ with $J_h > 0$ a.e. Then h^{**} is differentiable a.e. in Ω' and $\tilde{f}$ is differentiable a.e. in Ω.*

Proof. We again pass to a subsequence (if needed) so that $f_k \to f$ and $f_k^{-1} \to h$ pointwise a.e. Since h is a $W^{1,p}$-weak limit of Sobolev homeomorphisms with $p > n-1$, the super-precise representative h^{**} is continuous on almost all spheres [11, Lemma 2.19], satisfies the strong (INV) condition [1, Theorem 5.2 and Lemma 5.3], and is injective a.e. (see [21, Lemma 3.4] and [1, Theorem 1.2]). By Lemma 4.1, h is differentiable a.e. in Ω'. Moreover, since $J_h(y) > 0$ a.e. in Ω', by the change-of-variables formula we conclude that h satisfies the $(\mathrm{N})^{-1}$ condition.

Step 1. Finding sets Λ, Λ': Let f be an arbitrarily fixed representative, and let us introduce *good* sets $G \subset \Omega$, $G' \subset \Omega'$ as

$$G := \{x \in \Omega \,:\, h^{**}(f(x)) = x\} \subset \Omega,$$
$$G' := \{y \in \Omega' \,:\, f(h^{**}(y)) = y\} \subset \Omega'.$$

It is easy to check that $f(G) = G'$, $h^{**}(G') = G$, and by Lemma 3.1, $|G| = |\Omega|$, $|G'| = |\Omega'|$. And we define *bad* sets $\Sigma \subset G$, $\Sigma' \subset G'$ as

$$\Sigma := G \setminus \{x \in \Omega \,:\, J_f(x) > 0,\ f_k(x) \to f(x)\},$$
$$\Sigma' := G' \setminus \{y \in \Omega' \,:\, h^{**} \text{ is differentiable in } y,\ J_{h^{**}}(y) > 0,$$
$$f_k^{-1}(y) \to h^{**}(y)\}.$$

Clearly $|\Sigma| = |\Sigma'| = 0$. Then *very good* sets $\Lambda \subset G$, $\Lambda' \subset G'$ are defined by

$$\Lambda' := G' \setminus (\Sigma' \cup f^{-1}(\Sigma)) \qquad \text{and} \qquad \Lambda := h^{**}(\Lambda').$$

By Lemma 3.1 and the $(\mathrm{N})^{-1}$ condition for f and h^{**}, it is not difficult to see that $|\Lambda'| = |G'| = |\Omega'|$, $|\Lambda| = |G| = |\Omega|$ and $f(\Lambda) = \Lambda'$.

Step 2. $f|_\Lambda$ is continuous: The restriction $f|_\Lambda \colon \Lambda \to \Lambda'$ is continuous with respect to the relative topology in Λ. Indeed, let $f|_\Lambda$ be not continuous in some point $x_0 \in \Lambda$, then there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \Lambda$, $x_k \to x_0$, but $f(x_k) \nrightarrow f(x_0)$. We set $y_k := f(x_k) \in \Lambda'$ and $y_0 := (h^{**})^{-1}|_\Lambda(x_0) = f(x_0)$. Since $h^{**}|_{\Lambda'} = (f|_\Lambda)^{-1}$, we have $h^{**}(y_k) \to h^{**}(y_0)$, but $y_k \nrightarrow y_0$.

By Proposition 4.3 there exists a sequence $\{r_m\}_{m\in\mathbb{N}} \searrow 0$ such that

$$B\left(h^{**}(y_0), \frac{r_m}{3}\right) \subset (h^{**})^T(B(y_0, r_m)).$$

Let m and $k_0 \in \mathbb{N}$ be big enough so that infinitely many y_k are outside of $B(y_0, r_m)$ for $k \geq k_0$ and $h^{**}(y_{k_0}) \in B\left(h^{**}(y_0), \frac{r_m}{6}\right)$. Passing to a subsequence, we can, for now, assume that $y_k \notin B(y_0, r_m)$ for all k. Then we can find $r > 0$ such that

$$B(y_{k_0}, r) \cap B(y_0, r_m) = \varnothing$$

and, since $h^{**}|_{\Lambda'}$ is continuous,

$$h^{**}(B(y_{k_0}, r) \cap \Lambda') \subset B\left(h^{**}(y_{k_0}), \frac{r_m}{6}\right).$$

Summarizing the above, we obtain

$$\begin{aligned} h^{**}(B(y_{k_0}, r) \cap \Lambda') \subset B\left(h^{**}(y_{k_0}), \frac{r_m}{6}\right) &\subset B\left(h^{**}(y_0), \frac{r_m}{3}\right) \\ &\subset (h^{**})^T(B(y_0, r_m)). \end{aligned}$$

Thus, for every

$$z \in (B(y_{k_0}, r) \cap \Lambda') \subset (\Omega' \setminus B(y_0, r_m))$$

it holds that $h^{**}(z) \in (h^{**})^T(B(y_0, r_m))$, the latter contradicts the strong (INV) condition for h^{**}, since a set of positive measure $B(y_{k_0}, r) \cap \Lambda'$ from outside of the ball $B(y_0, r_m)$ is mapped inside the topological image of this ball.

Therefore, f is continuous on Λ with respect to the relative topology, and by Lemma 4.2, we conclude that f is differentiable on Λ with respect to the relative topology.

Step 3. $\tilde{f}$ is differentiable a.e.: It is left to show that a hyper-precise representative $\tilde{f}$, given by (2.3), is differentiable at $x_0 \in \Lambda$ with respect to Ω. Since Λ is a set of full measure and f is continuous on Λ with respect to the relative topology, any point $x \in \Lambda$ is a Lebesgue point of f, and therefore $\tilde{f} = f$ on Λ.

Fix $x_0 \in \Lambda$ and $\varepsilon > 0$. By differentiability of f on Λ with respect to the relative topology, there exists $s > 0$ such that for any $x \in B(x_0, s) \cap \Lambda$ it holds that

$$\begin{aligned} &\frac{|f(x) - f(x_0) - Df(x_0)(x - x_0)|}{|x - x_0|} \\ &\qquad = \frac{|\tilde{f}(x) - \tilde{f}(x_0) - Df(x_0)(x - x_0)|}{|x - x_0|} < \frac{\varepsilon}{2}, \end{aligned} \tag{4.3}$$

where $Df(x_0)$ denotes the derivative $Df|_\Lambda(x_0)$ with respect to the relative topology. To prove differentiability of $\tilde{f}$, we need to show that for an arbitrary x' close to x_0

it holds that

$$\frac{|\tilde f(x') - \tilde f(x_0) - Df(x_0)(x'-x_0)|}{|x'-x_0|} < \varepsilon. \tag{4.4}$$

If $x' \in \Lambda$, (4.4) follows immediately from (4.3). In the other case, roughly speaking, we want to find a point $z \in \Lambda$ such that $\frac{|\tilde f(x') - \tilde f(z)|}{|x'-x_0|}$ and $\frac{|x'-z|}{|x'-x_0|}$ are small, and so we can estimate

$$\begin{aligned}&\frac{|\tilde f(x') - \tilde f(x_0) - Df(x_0)(x'-x_0)|}{|x'-x_0|}\\ &\quad\le \frac{|\tilde f(x') - \tilde f(z)| + |Df(x_0)(x'-z)|}{|x'-x_0|} + \frac{|\tilde f(z) - \tilde f(x_0) - Df(x_0)(z-x_0)|}{|x'-x_0|}\\ &\quad< \varepsilon.\end{aligned}$$

Now we prove the above paragraph rigorously. Let $x' \in B\left(x_0, \frac{s}{2}\right)$. By (2.3), there exists a sequence $\{r_k\}_{k\in\mathbb{N}} \searrow 0$ such that $r_k < 2^{-k}|x'-x_0|$ and

$$\left|\tilde f(x') - \frac{1}{|B(x',r_k)|}\int_{B(x',r_k)\cap\Lambda} \tilde f(x)\,dx\right| < 2^{-k}|x'-x_0|. \tag{4.5}$$

In the following, we proceed coordination-wise for $i \in \{1,\dots,n\}$. Denote by a_k^i and b_k^i points in $B(x',r_k)\cap\Lambda$ such that

$$\tilde f_i(a_k^i) \ge \frac{1}{|B(x',r_k)|}\int_{B(x',r_k)\cap\Lambda} f_i(x)\,dx - 2^{-k}|x'-x_0|, \tag{4.6}$$

$$\tilde f_i(b_k^i) \le \frac{1}{|B(x',r_k)|}\int_{B(x',r_k)\cap\Lambda} f_i(x)\,dx + 2^{-k}|x'-x_0|. \tag{4.7}$$

If there is an equality in (4.6) or (4.7), we define x_k^i as a_k^i or b_k^i, correspondingly. Otherwise, by continuity of $\tilde f_i$ on Λ, there exist two balls $B(a_k^i, \rho(a_k^i))$ and $B(b_k^i, \rho(b_k^i))$, contained in $B(x',r_k)$, such that (4.6) holds for any $a \in B(a_k^i,\rho(a_k^i))\cap\Lambda$ and (4.7) holds for any $b \in B(b_k^i,\rho(b_k^i))\cap\Lambda$. Without loss of generality, we may assume $a_k^i = (0,\dots,0)$ and $b_k^i = (b_1, 0, \dots, 0)$. Let us now consider the lines $l_d := (t, d_2, \dots, d_n)$ connecting $B(a_k^i,\rho(a_k^i))$ and $B(b_k^i,\rho(b_k^i))$. Since Λ is of full measure, for $\mathcal{L}^{n-1}$-a.e. $d := (d_2,\dots,d_n)$ a line l_d contains $x_a \in B(a_k^i,\rho(a_k^i))\cap\Lambda$ and $x_b \in B(b_k^i,\rho(b_k^i))\cap\Lambda$, and $\mathcal{L}^1(l_d\setminus\Lambda) = 0$. Moreover, $\tilde f_i \in W^{1,n-1}$ and hence $\tilde f_i$ is absolutely continuous on $\mathcal{L}^{n-1}$-a.e. l_d. Therefore, by the intermediate value property, there is a point $c_k^i \in l_d$ such that

$$\left|\tilde f_i(c_k^i) - \frac{1}{|B(x',r_k)|}\int_{B(x',r_k)\cap\Lambda} \tilde f_i(x)\,dx\right| \le 2^{-k}|x'-x_0|. \tag{4.8}$$

Moreover, there exists $x_k^i \in l_d\cap\Lambda \subset B(x',r_k)$ such that

$$|\tilde f(c_k^i) - \tilde f(x_k^i)| \le 2^{-k}|x'-x_0|. \tag{4.9}$$

Then, by (4.5), (4.8), and (4.9),

$$|\tilde{f}_i(x_k^i) - \tilde{f}_i(x')| \le |\tilde{f}_i(x_k^i) - \tilde{f}_i(c_k^i)| + |\tilde{f}_i(c_k^i) - \tilde{f}_i(x')| < 2^{-k+2}|x' - x_0|. \tag{4.10}$$

Further,

$$\begin{aligned} \frac{|\tilde{f}_i(x') - \tilde{f}_i(x_0) - Df_i(x_0)(x' - x_0)|}{|x' - x_0|} &\le \frac{|\tilde{f}_i(x') - \tilde{f}_i(x_k^i)| + |Df_i(x_0)(x' - x_k^i)|}{|x' - x_0|} \\ &\quad + \frac{|\tilde{f}_i(x_k^i) - \tilde{f}_i(x_0) - Df_i(x_0)(x_k^i - x_0)|}{|x' - x_0|}. \end{aligned} \tag{4.11}$$

Since $x_k^i \in B(x', r_k)$ and (4.10) holds, the first term in (4.11) can be estimated as

$$\frac{|\tilde{f}_i(x') - \tilde{f}_i(x_k^i)| + |Df_i(x_0)(x' - x_k^i)|}{|x' - x_0|} \le 2^{-k+2} + 2^{-k}|Df(x_0)|.$$

While to estimate the second term in (4.11), we note that

$$|x_k^i - x_0| \le |x_k^i - x'| + |x' - x_0| \le (1 + 2^{-k})|x' - x_0| \le 2|x' - x_0| \le s,$$

since $x_k^i \in B(x', r_k)$. And hence, by (4.3), we conclude

$$\begin{aligned} &\frac{|\tilde{f}_i(x_k^i) - \tilde{f}_i(x_0) - Df_i(x_0)(x_k^i - x_0)|}{|x' - x_0|} \\ &\qquad \le \frac{2|\tilde{f}_i(x_k^i) - \tilde{f}_i(x_0) - Df_i(x_0)(x_k^i - x_0)|}{|x_k^i - x_0|} \le \varepsilon. \end{aligned}$$

Summarizing the above, we obtain that for $x_0 \in \Lambda$ and any $\varepsilon > 0$ there exists $s > 0$ such that for any $x' \in B\left(x_0, \frac{s}{2}\right)$ it holds

$$\frac{|\tilde{f}_i(x') - \tilde{f}_i(x_0) - Df_i(x_0)(x' - x_0)|}{|x' - x_0|} \le \liminf_{k\to\infty} (2^{-k}(4 + |Df_i(x_0)|) + \varepsilon) = \varepsilon.$$

Therefore, $\tilde{f}_i$ is differentiable in any $x_0 \in \Lambda$ with respect to Ω and, moreover, $D\tilde{f}_i(x_0) = Df_i|_\Lambda(x_0)$.

□

5. Proofs of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1. Theorem 1.1 immediately follows from Lemma 3.1 and Theorem 4.4. □

Proof of Corollary 1.2. Let us first note that following the proof of [20, Theorem 1.1] with substituting n by p, we obtain

$$\int_{\Omega'} |Df_k^{-1}|^p(y)\,dy \le \int_\Omega \frac{|\operatorname{adj} Df_k|^p(x)}{(J_{f_k}(x))^{p-1}}\,dx.$$

Hence, $\mathcal{E}(f_k) \leq \mathcal{F}(f_k)$ and the sequence $\{f_k^{-1}\}_{k\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega', \mathbb{R}^n)$ and, passing to a subsequence if needed, there exists a weak limit h. Moreover, by [8, Lemma 2.3] and (1.1), the inequality

$$\int_\Omega \varphi(J_f(x))\, dx \leq C$$

guarantees that $J_f > 0$ a.e. in Ω and $J_h > 0$ a.e. in Ω'. To finish the proof, we apply Theorem 1.1. □

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A. Doležalová
Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic
E-mail address: dolezalova@karlin.mff.cuni.cz

A. Molchanova
Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria
E-mail address: anastasia.molchanova@univie.ac.at