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THE GAUSS EQUATION ON SURFACES OF BOUNDED INTEGRAL CURVATURE

DANIELE BARTOLUCCI

Abstract. Based on some results of A. Huber and Y. G. Reshetnyak concerning Surfaces of Bounded Integral Curvature, we show that even in such a weak setting one can write down the "Gauss" equation (which in the classical case reads $-\Delta\rho = 2Ke^{\rho}$, where Δ is the Euclidean Laplace operator), locally relating the Gaussian curvature K to the metric and its weak Laplacian. This approach naturally leads to the analysis of the regularity theory of singular Liouville type equations as developed by H. Brezis and F. Merle. Several examples are provided to illustrate the local regularity properties of this sort of singular surfaces.

1. Introduction

The geometry of surfaces carrying a wide variety of singularities, thereby generalizing classical Riemannian theory, can be described through the concept of Surface of Bounded Integral Curvature (*SBC* for short), as introduced by A.D. Alexandrov [2], see [3], [19] for a complete account about the subject and [1] for a concise discussion about some of the main results. Due to a series of subtle contributions by Huber and Reshetnyak (see [19] §7) an *SBC* without cusps (see (1.3) and Remark 1 below) can be equivalently defined as a Riemann surface $\mathcal{M}$ equipped with a metric $\mathfrak{g}$, which admits an atlas of local charts $\mathcal{U} = \{U_j, \Phi_j\}_{j\in J}$, such that each Φ_j is an isometry of U_j on $\Omega_j = \Phi_j(U_j)$, with $\Omega_j \subset \mathbb{R}^2(\simeq \mathbb{C})$, a smooth, open and bounded set, such that $\mathfrak{g}$ in local coordinates takes the form of a quadratic differential, $\overset{*}{\Phi}_j(\mathfrak{g}) = e^{\rho_j(z)}|dz|^2$, $z = x + iy \in \mathbb{C}$. Here $*$ denotes the standard pull-back, $|dz|^2$ is the Euclidean metric and

$$\rho_j \equiv \rho = h + g_+ - g_- \tag{1.1}$$

where h is harmonic in Ω_j and $g_\pm$ are two superharmonic functions defined by,

$$g_\pm(z) = \int_{\Omega_j} \Gamma(z,y)d\omega_\pm(y), \quad \Gamma(z,y) = \frac{1}{2\pi}\log\left(\frac{1}{|z-y|}\right). \tag{1.2}$$

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The singular behavior of the surface is locally encoded in $\omega_\pm$, which are mutually orthogonal, regular, non negative and bounded measures on Ω, $\omega_\pm(\Omega) \in (0, +\infty)$. Equivalently one could have started with a signed measure ω of bounded total variation, to come up then with $\omega_\pm$ by the Jordan decomposition of ω on Ω_j, $\omega = \omega_+ - \omega_-$.
Any such system of coordinates is said to be isothermal and any metric taking the form $e^{\rho(z)}|dz|^2$ with ρ as in (1.2) is said to be subharmonic. We will focus on the following local model of an *SBC*.

Definition 1.1. *An Abstract Surface of Bounded Integral Curvature without Cusps (ASBC for short) is a pair $\mathcal{S} = \{\Omega, e^{\rho(z)}|dz|^2\}$, where Ω is an open, smooth and bounded domain and $\rho = h + g_+ - g_-$, h harmonic and bounded in Ω and $g_\pm$ defined as in* (1.2), *that satisfies the no cusp condition,*

$$\forall\, z \in \Omega,\ \omega_+(z) < 4\pi. \tag{1.3}$$

So if $\mathcal{S} = \{\Omega, e^{\rho(z)}|dz|^2\}$ is an *ASBC*, then according to Reshetnyak (see [19] §8), the total curvature on $\mathcal{S}$, denoted by $\mathcal{K}$, is the measure of bounded total variation defined as follows,

Definition 1.2. *Let $\mathcal{S} = \{\Omega, e^{\rho(z)}|dz|^2\}$ be an ASBC. The total curvature $\mathcal{K}(E)$ of any Borel set $E \subseteq \Omega$ is defined by:*

$$2\mathcal{K}(E) := \omega(E) = \omega_+(E) - \omega_-(E).$$

Remark 1. *If for some $z_0 \in \Omega$ we had $\omega_+(z_0) \geq 4\pi$, then the lengths and areas of sets containing z_0, defined via the metric $g = e^{\rho(z)}|dz|^2$, would not be well defined in general. Any point $z_0 \in \Omega$ which satisfies $\omega_+(z_0) \geq 4\pi$ is said to be a cusp.*

From now on we will assume that $\mathcal{S} = \{\Omega, e^{\rho(z)}|dz|^2\}$ is an *ASBC*. Surfaces with conical singularities ([5], [22]) are classical examples of this sort, see also Example 1 below. The local regularity of subharmonic metrics has been also recently discussed in [1], showing among other things that for any $p \in [1, 2)$, there exists a discrete set $\mathfrak{S}_p$ such that the metric is, locally far away from $\mathfrak{S}_p$, of class $W^{1,p}_{\text{loc}}$ with respect to the intrinsic volume measure. Although our results are related to those in [1], we are interested in a different problem. Indeed, as far as $\rho \in C^2(\Omega)$, it is well known that the Gaussian curvature can be defined in Ω by the "Gauss" equation, $K(z) = -\frac{1}{2}\frac{\Delta\rho}{e^\rho}$, which has a long history in mathematics and has been widely used to analyze the local geometry of surfaces, see [4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21] and references quoted therein.

Our aim is to show that in fact we can write the Gauss equation even in the very weak ASBC setting. The underlying idea is, starting just from a subharmonic metric and its total curvature, to come up with a potential u and a "regular" Gaussian curvature function K satisfying the Gauss equation in a suitably defined weak sense. Our argument relies in a crucial way on the regularity theory of solutions of Liouville-type equations by Brezis and Merle in ([7]).
Here and in the rest of this paper $\mathcal{H}^\gamma$, $\gamma > 0$, denotes the γ-dimensional Hausdorff measure and $S_{2\pi} = \{x \in \Omega \ :\ \omega_+(z) \geq 2\pi\}$. Also $L^p(\Omega)$, $p \in [1, +\infty)$, denotes the usual space of measurable functions whose p-th power of the modulus is integrable

in Ω, (where two functions are identified if they agree a.e. in Ω) while $W^{k,p}(\Omega)$, $p \in [1,+\infty)$, denotes the Sobolev space of functions weakly differentiable k times in Ω, whose derivatives up to the order k are in $L^p(\Omega)$. Finally $L^\infty(\Omega)$ denotes the space of measurable functions bounded almost everywhere (where again two functions are identified if they agree a.e. in Ω).

Theorem 1.3. *Let $\mathcal{S} = \left\{\Omega, e^{\rho(z)}|dz|^2\right\}$ be an ASBC. There exists $q_1 > 1$ such that the following holds:*

(a) *$e^\rho \in L^q(\Omega)$ for any $q < q_1$ and for any $r > 0$ small enough there exists $p_r > 2$ such that $e^\rho \in L^{p_r}(\Omega \setminus B_r(S_{2\pi}))$;*

(b) *Let $\mathcal{K} = Ke^\rho\mathcal{H}^2 + \omega_s$ be the Lebesgue decomposition of $\mathcal{K}$ w.r.t. $e^\rho\mathcal{H}^2$ and let $\omega_s = \omega_{s,+} - \omega_{s,-}$ be the Jordan decomposition of the singular part ω_s. Define $f = f_+ - f_-$, $f_\pm$ satisfying (1.2) with $\omega_\pm = 2\omega_{s,\pm}$. Then*

$$e^f \in L^{q_1}(\Omega),$$

and $u = \rho - f \in L^1(\Omega) \cap W^{1,s}_{\mathrm{loc}}(\Omega)$ for any $s \in (1,2)$ is a solution of,

$$-\Delta u = 2Ke^{f+u} \quad \text{in} \quad \Omega, \tag{1.4}$$

in the sense of distributions, $e^{t|u|} \in L^1(\Omega)$ for any $t \geq 1$ and $e^{u+f} \in L^q(\Omega)$ for any $q < q_1$.

Moreover, if $\rho = u + f$ for a pair $\{u, f\}$ as above, then, for any fixed h harmonic and bounded in Ω, the pair $\{u_h, f_h\} := \{u - h, f + h\}$ satisfies the same properties with $\rho = u_h + f_h$.

Needless to say, by definition $K \in L^1(\Omega; e^\rho\mathcal{H}^2)$ and for any Borel set $E \subseteq \Omega$ we have,

$$\mathcal{K}(E) = \int\limits_E Ke^{f+u} + \omega_s(E).$$

Actually it is well known that $\rho - h \in W^{1,s}(\Omega)$ for any $s \in (1,2)$ which, together with Theorem 1.3(a), implies that for any r small enough $e^{\rho-h} \in W^{1,t}(\Omega \setminus B_r(S_{2\pi}))$ for any $1 \leq t < \frac{2p_r}{p_r+2}\}$.

Remark that it is possible to construct an ASBC where the above decomposition yields $f \equiv 0$, $Ke^u \in L^1(\Omega)$, $e^{t|u|} \in L^1(\Omega)$ for any $t \geq 1$, but u is unbounded and there is no $s > 1$ such that $K^s e^u \in L^1(\Omega)$, see Examples 6 and 7 in section 3. In particular (see Example 7) it is not true in general that $e^\rho \in L^{q_1}(\Omega)$. Better estimates can be obtained via stronger assumptions on K.

Theorem 1.4. *Let $\mathcal{S} = \left\{\Omega, e^{\rho(z)}|dz|^2\right\}$ be an ASBC and let $\rho = u + f$, q_1, p_r and K as determined by Theorem 1.3. Assume that*

$$K \in L^s(\Omega) \text{ for some } s > \frac{q_1}{q_1 - 1}, \tag{1.5}$$

then u is a strong solution of (1.4), $u \in W^{2,t}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, $\forall\ 1 \leq t \leq \frac{sq_1}{s+q_1}$ and $e^\rho \in L^{q_1}_{\mathrm{loc}}(\Omega)$. Moreover:

- *if $K \in L^\infty(\Omega)$, then $u \in W^{2,q_1}_{\mathrm{loc}}(\Omega)$;*

- *if* $q_1 = \infty$ *and* (1.5) *holds with some* $s > 1$, *then* $u \in W^{2,s}_{\mathrm{loc}}(\Omega)$;
- *if* (1.5) *holds and for any* r *small enough there exists* $n_r > \frac{2p_r}{p_r-2}$ *such that* $K \in L^{n_r}(E \setminus B_r(S_{2\pi}))$, $\forall\, E \Subset \omega$, *then* $u \in W^{2,k}(E \setminus B_r(S_{2\pi}))$, *for any* $\forall\, E \Subset \Omega$ *and for any* $2 \le k \le \frac{n_r p_r}{n_r + p_r}$

The proof of Theorems 1.3, 1.4 are discussed in section 2. Several examples are provided in section 3 relevant to Theorems 1.3, 1.4.

2. The proof of Theorems 1.3 and 1.4

We first discuss the summability properties of $e^{g_\pm}$. Remark that the estimates about the local exponential integrability of e^{g_+} are essentially the same as those in [23], [7], the underlying idea being the following:

Lemma 2.1. ([23], [7]) *Let* g_+ *take the form* (1.2) *and assume that* $\omega_+(\Omega) \in (0, \frac{4\pi}{p})$. *Then*

$$\int\limits_{\Omega} e^{pg_+} \le \frac{2\pi}{2-\sigma}(d_\Omega)^{2\sigma},$$

where d_Ω *is the diameter of* Ω *and* $\sigma = \omega_+(\Omega)\dfrac{p}{2\pi} \in (0,2)$.

However we provide a detailed proof of the Proposition 2.2 below since our statement is slightly different from those in [7], [23]. Let $g_\pm$ be two superharmonic functions satisfying the assumptions of Definition 1.1 and let us recall that $S_{2\pi} = \{x \in \Omega \,:\, \omega_+(x) \ge 2\pi\}$. Obviously $S_{2\pi}$ is finite.

Proposition 2.2.

(j) $e^{-g_-} \in L^\infty(\Omega)$ *and for any* r *small enough there exists* $p_r > 2$ *such that* $e^{g_+} \in L^{p_r}(\Omega \setminus B_r(S_{2\pi}))$.

(jj) *Assume that the no cusp condition* (1.3) *holds. Then* $e^{g_+} \in L^q(\Omega)$ *for some* $q > 1$.

(jjj) *In particular let* $\omega_+ = \mu_0 + \mu_1$ *be the Lebesgue decomposition of* ω_+ *w.r.t.* $\mathcal{H}^2$, *where* μ_1 *denotes the singular part of* ω_+ *and accordingly define* $g_+ = g_{+,0} + g_{+,1}$. *Then* $e^{g_{+,0}} \in L^q(\Omega)$ *for any* $q \ge 1$.

Proof. (j) Clearly for $z \in \Omega$ we have

$$g_-(z) = \int\limits_{\Omega} \Gamma(z,y)d\omega_-(y) \ge \min\left\{-\frac{\omega_-(\Omega)}{2\pi}\log(d_\Omega), 0\right\},$$

whence $e^{-g_-} \in L^\infty(\Omega)$.
Concerning g_+, by definition we can write $g_+ = g_{+,0} + g_{+,1}$, where $g_{+,0}$(resp. $g_{+,1}$) takes the form (1.2) in Ω with ω replaced by μ_0(resp. μ_1). First of all, since $\mu_0 \in L^1(\Omega)$, for any $q \ge 1$ we can write $\mu_0 = \mu_{0,\varepsilon} + \mu_{0,\infty}$, where $\|\mu_{0,\varepsilon}\|_{L^1(\Omega)} \le \varepsilon < \frac{4\pi}{q}$, $\mu_{0,\infty} \in L^\infty(\Omega)$, and correspondingly define $g_{+,0,\varepsilon}$ and $g_{+,0,\infty}$. Clearly $g_{+,0,\infty} \in L^\infty(\Omega)$ while, by Lemma 2.1, $g_{+,0,\varepsilon}$ satisfies $\|e^{g_{+,0,\varepsilon}}\|_{L^q(\Omega)} \le C(\Omega,\varepsilon)$, which, in view of $g_{+,0} = g_{+,0,\varepsilon} + g_{+,0,\infty}$, proves (jjj).

Next let us set $\Omega_d = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < d\}$ and observe that, by the inner regularity of ω_+, for any fixed $q \geq 1$ we can find $d_q > 0$ such that $\omega_+(\Omega_{4d_q}) \leq \omega_+(\overline{\Omega_{4d_q}}) < \frac{4\pi}{q}$. Assume w.l.o.g. that $4d_q < 1$, then we have

$$g_{+,1}(x) - \frac{\omega_+(\Omega)}{2\pi}\log(\frac{1}{2d_q}) \leq w(x) := \frac{1}{2\pi}\int_{\Omega_{4d_q}} \log\left(\frac{1}{|x-y|}\right) d\omega_+, \ \forall\, x \in \Omega_{2d_q}.$$

Assuming w.l.o.g. that $\omega_+(2d_q) > 0$ and putting $\sigma = \omega_+(2d_q)\frac{q}{2\pi} \in (0,2)$, by the Jensen inequality and Fubini-Tonelli we can estimate,

$$\begin{aligned}
\int_{\Omega_{2d_q}} e^{qw} &\leq \int_{\Omega_{2d_q}} dx \int_{\Omega_{4d_q}} \left(\frac{1}{|x-y|}\right)^{\sigma} \frac{d\omega_+(y)}{\omega_+(\Omega_{4d_q})} \\
&\leq \int_{\Omega_{4d_q}} \frac{d\omega_+(y)}{\omega_+(\Omega_{4d_q})} \int_{\Omega_{2d_q}} \left(\frac{1}{|x-y|}\right)^{\sigma} dx \\
&\leq \int_{\Omega_{4d_q}} \frac{d\omega_+(y)}{\omega_+(\Omega_{4d_q})} \int_{B_{d_\Omega}(y)} \left(\frac{1}{|x-y|}\right)^{\sigma} dx \\
&= \frac{2\pi}{2-\sigma}(d_\Omega)^{2-\sigma}.
\end{aligned}$$

Therefore $e^{g_{+,1}} \in L^q(\Omega_{2d_q})$ for any such q. Fix any $q \geq 6$ and pick any $r \in (0, d_q)$ small enough such that $\Omega_{4d_q} \cap B_r(S_{2\pi}) = \emptyset$, which of course can always be done since $S_{2\pi}$ is a finite set. Next let $\widetilde{\Omega} = \Omega \setminus \{\Omega_{d_q} \cup B_r(S_{2\pi})\}$, which is by construction compact, and let us fix $x_0 \in \widetilde{\Omega}$. Since by assumption $\omega_+(x_0) < 2\pi$, then by outer regularity there exists $p > 2$ such that there exists $R > 0$ depending on x_0 and p, such that the ball centered at x_0, $B_{2R} := B_{2R}(x_0)$, satisfies $B_{2R} \Subset \Omega \setminus S_{2\pi}$ and $\omega_+(B_{2R}) < \frac{4\pi}{p}$. Assume w.l.o.g. that $R < 1$, then we have,

$$g_{+,1}(x) - \frac{\omega_+(\Omega)}{2\pi}\log(\frac{1}{R}) \leq v(x) := \frac{1}{2\pi}\int_{B_{2R}} \log\left(\frac{4R}{|x-y|}\right) d\omega_+, \ \forall\, x \in B_R(x_0).$$

Assuming w.l.o.g. that $\omega_+(B_{2R}) > 0$ and putting $\sigma_0 = \omega_+(B_{2R})\frac{p}{2\pi} \in (0,2)$, by the Jensen inequality and Fubini-Tonelli we can estimate as above

$$\int_{B_R(x_0)} e^{pv} \leq \frac{2\pi}{2-\sigma_0}(d_\Omega)^{2-\sigma_0}.$$

Therefore $e^{g_{+,1}} \in L^p(B_R(x_0))$ for some $p > 2$ and $R > 0$ depending only by x_0 and p.

At this point, since $\widetilde{\Omega}$ is compact, we deduce by a covering argument that $e^{g_{+,1}} \in L^{q_r}(\widetilde{\Omega})$, for some $q_r > 2$. Thus, by defining $p_r = \min\{q_r, q\}$, we have $e^{g_{+,1}} \in L^{p_r}(\Omega \setminus B_r(S_{2\pi}))$ which completes the proof of (j).

At last (jj) follows by a covering argument as above, where the no cusp condition (1.3) replaces $\omega_+(x_0) < 2\pi$ in the compact set $\Omega \setminus \Omega_{2d_q}$. The proof is exactly the same as above and we omit the details to avoid repetitions. $\square$

Next we prove Theorems 1.3, 1.4. Our arguments rely in a crucial way on the regularity theory of solutions of Liouville-type equations as developed in [7].

The proof of Theorem 1.3. **STEP 1.** Recall that $\rho = h + g_+ - g_-$ where h is harmonic and bounded in Ω. Therefore Proposition 2.2-(jj) implies that $e^\rho \in L^{q_0}(\Omega)$ for some $q_0 > 1$ and consequently it is well defined the Lebesgue decomposition of $\mathcal{K}$ with respect to $e^\rho \mathcal{H}^2$,

$$\mathcal{K} = K e^\rho \mathcal{H}^2 + \omega_s, \quad K \in L^1\left(\Omega; e^\rho \mathcal{H}^2\right), \quad \omega_s \perp e^\rho \mathcal{H}^2, \tag{2.1}$$

where ω_s is a bounded Radon measure. Next observe that, since $\rho \in L^1(\Omega)$, then $-\Delta \rho = \omega_+ - \omega_-$ holds in the sense of distributions in Ω, whence, by (2.1) and the definition of $\mathcal{K}$, we see that the following equality,

$$-\Delta \rho = 2 K e^\rho + 2\omega_s,$$

holds as well in the sense of distributions in Ω. Let $\omega_s = \omega_{s,+} - \omega_{s,-}$ be the Jordan decomposition of ω_s and $f_\pm$ be defined by (1.2) with $\omega_\pm = 2\omega_{s,\pm}$. Then let us set $f = f_+ - f_-$ and

$$u := \rho - f.$$

Since $u \in L^1(\Omega)$ and since $-\Delta f = 2\omega_s$ in the sense of distributions, then we deduce that,

$$-\Delta u = 2 K e^{f+u} + 2\omega_s + \Delta f = 2 K e^{f+u},$$

that is, u satisfies (1.4) in the sense of distributions in Ω.
Obviously $2\omega_s$ satisfies the no cusp condition (1.3) as well. At this point, since ω_s is bounded, then we can apply Proposition 2.2 to $2\omega_s$, thereby deducing that $e^f \in L^{q_1}(\Omega)$ for some $q_1 > 1$, which completes the proof of (b).

STEP 2. We conclude the proof of (a) and (c). Since $K \in L^1(\Omega; e^{f+u}\mathcal{H}^2)$ and $u \in L^1(\Omega)$ is a solution of (1.4) in the sense of distributions, then by Remark 2 in [7] we have $e^{t|u|} \in L^1_{\rm loc}(\Omega)$ for any $t \geq 1$. Therefore by the Holder inequality $e^\rho = e^f e^u \in L^s$, for any $s < q_1$. This fact shows a posteriori that in fact q_0 in STEP 1 could be any $q < q_1$.
Next, let $\Omega_1 \Subset \Omega$ be any open, smooth and relatively compact subset. Set $u_2 = u - u_1$, where u_1 is the unique weak solution (in the sense of Stampacchia [20]) of the Dirichlet problem,

$$\begin{cases} -\Delta u_1 = 2 K e^{f+u} & \text{in } \Omega_1, \\ u_1 = 0 \quad \text{on } \Omega_1. \end{cases}$$

By the Weyl Lemma u_2 coincides a.e. with an harmonic function in Ω_1, and since it is well known ([20]) that $u_1 \in W^{1,s}_0(\Omega_1)$ for any $s \in (1,2)$, then we deduce that $u \in W^{1,s}_{\rm loc}(\Omega)$. In view of STEP 1, these facts prove (c).

Finally, it is obvious that the representation $\rho = u + f$ with all the properties established above still holds for $\{u_h, f_h\}$ where h is any harmonic and bounded function in Ω. $\square$

Next we prove Theorem 1.4.

The proof of Theorem 1.4. We assume first that $K \in L^s(\Omega)$, for some $s > \frac{q_1}{q_1-1}$. Recall that $e^f \in L^{q_1}(\Omega)$. Then by the Hölder inequality we have $Ke^f \in L^t(\Omega)$, $\forall\, 1 < t \leq \frac{sq_1}{s+q_1}$. On the other hand, since by Theorem 1.3(*b*) we have $e^u \in L^p_{\rm loc}(\Omega)$ for any $p \geq 1$, then, for fixed $1 < t \leq \frac{sq_1}{s+q_1}$, in particular we have $e^u \in L^{\frac{t}{t-1}}_{\rm loc}(\Omega)$ and then we can apply Remark 5 in [7] which yields $u \in L^\infty_{\rm loc}(\Omega)$. So we find $Ke^{f+u} \in L^t_{\rm loc}(\Omega)$ for any $1 < t \leq \frac{sq_1}{s+q_1}$ and by standard elliptic estimates we conclude that $u \in W^{2,t}_{\rm loc}(\Omega)$ for any $1 < t \leq \frac{sq_1}{s+q_1}$ and in particular that u is a strong solution of (1.4). Therefore, in particular if $K \in L^\infty_{\rm loc}(\Omega)$ then $Ke^{f+u} \in L^{q_1}_{\rm loc}(\Omega)$ and $u \in W^{2,q_1}_{\rm loc}(\Omega)$ by standard elliptic theory. On the other side, it is readily seen by the same argument that $u \in L^\infty_{\rm loc}(\Omega)$ whenever $q_1 = +\infty$ and $s > 1$. Therefore, in this case, $Ke^{f+u} \in L^s_{\rm loc}(\Omega)$ and $u \in W^{2,s}_{\rm loc}(\Omega)$ by standard elliptic theory.
At last, recall that p_r is given by (*a*) in Theorem 1.3 and assume that for any r small enough there exists $n_r > \frac{2p_r}{p_r-2}$ such that $K \in L^{n_r}(E \setminus B_r(S_{2\pi}))$ for any $E \Subset \Omega$. Since $\frac{n_r p_r}{n_r+p_r} > 2$ then $Ke^{f+u} \in L^k(E \setminus B_r(S_{2\pi}))$, $\forall\, 2 \leq k \leq \frac{np_r}{n+p_r}$. As a consequence, by standard elliptic estimates, we conclude also that $u \in W^{2,k}(E \setminus B_r(S_{2\pi}))$ for any $E \Subset \Omega$. $\square$

3. Examples

We discuss some examples relevant to Theorems 1.3, 1.4. Recall that an ASBC is said to have a conical singularity at point p if in local subharmonic coordinates (z, Ω) around p, such that $z(p) = 0$, the metric takes the form,

$$|z|^{2\alpha} e^u |dz|^2, \quad z \in \Omega \setminus \{0\},$$

for some $0 \neq \alpha \in (-1, +\infty)$ (α is said to be the order of the conical singularity) and some $u \in C^0(\Omega) \cap C^2(\Omega \setminus \{0\})$. The well known geometric interpretation is that the surface admits a tangent cone at p whose opening angle is $2\arcsin(1+\alpha)$.

Example 1. We illustrate Theorems 1.3, 1.4 for a metric with constant Gaussian curvature and one conical singularity. Let us consider the ASBC $\{B_1, e^\rho |dz|^2\}$ where ρ satisfies (1.1)-(1.2) with $h \equiv \log(1-\alpha)^2$, $\omega_- \equiv 0$ and

$$\omega_+ = 2V_\alpha \mathcal{H}^2 + 4\pi\alpha\delta_{z=0}, \quad V_\alpha = \frac{4(1-\alpha)^2 |z|^{-2\alpha}}{(1+|z|^{2(1-\alpha)})^2},\ \alpha \in (0,1).$$

Clearly

$$\omega_+ = 2\mathcal{K} = 2Ke^\rho \mathcal{H}^2 + 2\omega_s, \quad 2\omega_s = 4\pi\alpha\delta_{z=0},$$

where $K = V_\alpha e^{-\rho} \in L^1(B_1, e^\rho \mathcal{H}^2)$. Therefore, setting $f = f_+$ as in (1.2) with $\omega_+ = 2\omega_s$ and $u_\alpha = \rho - f$ we have $e^f = |z|^{-2\alpha} \in L^q(B_1)$ for any $q < \frac{1}{\alpha}$, and

$$-\Delta u_\alpha = 2Ke^\rho = 2Ke^{f+u_\alpha} = 2K|z|^{-2\alpha} e^{u_\alpha} \text{ in } B_1, \tag{3.1}$$

in the sense of distributions. On the other side, since by definition $u_\alpha = \log(1-\alpha)^2 + \int\limits_{B_1} \Gamma(x,y) 2V_\alpha(y)dy$, we have that u_α is radial and continuous in $\overline{B_1}$. Thus

$$K = V_\alpha |z|^{2\alpha} e^{-u_\alpha} = \frac{4(1-\alpha)^2}{(1+|z|^{2(1-\alpha)})^2}\, e^{-u_\alpha}, \tag{3.2}$$

and K is radial and positive in $\overline{B_1}$ and in particular constant on ∂B_1. Since $-\Delta u_\alpha = 2V_\alpha$ in B_1, then by a straightforward evaluation we find $\Delta \log(K) = 0$ in B_1, whence K is constant in B_1. Therefore, since

$$u_\alpha(0) = \log{(1-\alpha)^2} - \frac{1}{2\pi}\int\limits_{B_1} \log{(|y|)} 2V_\alpha(|y|)dy = \log 4(1-\alpha)^2,$$

we deduce from (3.2) that $K \equiv K(0) = 1$ which yields at once the explicit expression of u_α, $e^{u_\alpha} = V_\alpha |z|^{2\alpha}$. Clearly $u_\alpha \in W^{2,q}(B_1)$ for any $q < \frac{1}{\alpha}$.

It is readily seen that, modulo minor changes, the same construction works for any $\alpha < 0$ as well, yielding the classical example of a conical singularity where the surface (of constant Gaussian curvature) "winds" around the singular point more than once.

Example 2. We provide an example of an ASBC where the decomposition of Theorems 1.3, 1.4 yields the same equation of Example 1 (see (3.1)) but without a conical singularity. Of course in this case K is not constant.

Let us consider the ASBC $\left\{B_1, e^\rho |dz|^2\right\}$ where ρ satisfies (1.1)-(1.2) with $h \equiv \log(1-\alpha)^2$, $\omega_- \equiv 0$ and

$$\omega_+ = 2V_\alpha \mathcal{H}^2, \quad V_\alpha = \frac{4(1-\alpha)^2|z|^{-2\alpha}}{(1+|z|^{2(1-\alpha)})^2},\ \alpha \in (0,1).$$

Clearly

$$\omega_+ = 2\mathcal{K} = 2Ke^\rho \mathcal{H}^2,$$

where $K = V_\alpha e^{-\rho} \in L^1(B_1, e^\rho \mathcal{H}^2)$. In this case we just set $u_\alpha = \rho$ and then we have

$$-\Delta u_\alpha = 2Ke^\rho = 2Ke^{u_\alpha} \text{ in } B_1,$$

in the sense of distributions. As above, since u_α is radial and continuous, then

$$K = V_\alpha e^{-u_\alpha} = \frac{4(1-\alpha)^2|z|^{-2\alpha}}{(1+|z|^{2(1-\alpha)})^2} e^{-u_\alpha}, \tag{3.3}$$

is radial and constant on ∂B_1. In particular $|z|^{2\alpha}K$ is radial, positive in $\overline{B_1}$ and constant on ∂B_1. Therefore, since $-\Delta u_\alpha = 2V_\alpha$ in B_1, then as above we find $\Delta \log(|z|^{2\alpha}K) = 0$ in B_1, whence $|z|^{2\alpha}K$ is constant in B_1 and since once more we have $u_\alpha(0) = \log 4(1-\alpha)^2$, then we deduce from (3.3) that $|z|^{2\alpha}K \equiv |z|^{2\alpha}K(z)\big|_{z=0} = 1$ which yields at once $K(z) = |z|^{-2\alpha}$. Clearly we have that $e^{u_\alpha} = V_\alpha |z|^{2\alpha}$, $u_\alpha \in W^{2,s}(B_1)$ for any $s < \frac{1}{\alpha}$ in this case as well, but this time K is just in $L^s(B_1)$, for any $s < \frac{1}{\alpha}$.

Remark. We see from Examples 1 and 2 that the "Gauss" equation

$$-\Delta u = |z|^{-2\alpha} e^u$$

need not be associated to a conical singularity. In Example 1 the metric has a conical singularity (i.e. $|z|^{-2\alpha}$ is part of the conformal factor e^ρ) and K is constant while in Example 2 the singular part of ω (whence of ρ) vanishes and $K = |z|^{-2\alpha}$.

Example 3. We provide an example relevant to Theorems 1.3, 1.4, showing a possible mechanism to trigger the dependence of the exponent p_r by r. Let us consider the ASBC $\{B_1, e^{\rho}|dz|^2\}$ where ρ satisfies (1.1)-(1.2) with $h \equiv 0$, $\omega_- \equiv 0$ and

$$\omega_+ = 2V\mathcal{H}^2 + 4\pi\alpha_0\delta_{z=0} + 4\pi\sum_{n=1}^{+\infty}\alpha_n\delta_{z=z_n}, \quad V = \frac{1}{|z|^{2\alpha_0}}\prod_{n=1}^{+\infty}\frac{1}{|z-z_n|^{2\alpha_n}},$$

where $\alpha_0 = \frac{1}{2}$, $\alpha_n \in (0,1)$ is any sequence such that $\alpha := \sum\limits_n \alpha_n < \frac{1}{2}$, and $|z_n|$ is any strictly decreasing sequence such that $\frac{1}{2} \leq |z_n| \to 0$, as $n \to +\infty$. Clearly

$$\omega_+ = 2\mathcal{K} = 2Ke^{\rho}\mathcal{H}^2 + 2\omega_s, \quad 2\omega_s = 4\pi\alpha_0\delta_{z=0} + 4\pi\sum_{n=1}^{+\infty}\alpha_n\delta_{z=z_n},$$

where $K = Ve^{-\rho} \in L^1(B_1, e^{\rho}\mathcal{H}^2)$. Putting $f = f_+$ as in (1.2) with $\omega = 2\omega_s$ and $u = \rho - f$ we have $e^f = V$ and

$$K = Ve^{-\rho} = Ve^{-f-u} = e^{-u}.$$

Therefore we have,

$$-\Delta u = 2Ke^{\rho} = 2Ke^{f+u} \text{ in } B_1$$

in the sense of distributions. Since $\alpha_0 = \frac{1}{2}$ then $\omega(\{0\}) = 2\pi$ but for any $r \in (0,1)$ we have $\omega(B_r) > 2\pi$. On the other side it is not difficult to see that $e^f = V \in L^q(B_1)$ for any $q < 2$ and that for any $0 < r \leq \frac{1}{2}$, $V \in L^{p_r}(B_1 \backslash B_r)$ for some $p_r > 2$ and then, since $u = \int\limits_{B_1} \Gamma(x,y)2V(y)dy$, by standard elliptic estimates we have $u \in W^{2,q}(B_1)$ for any $q < 2$ and $u \in W^{2,p_r}(B_1 \setminus B_r)$ for any $0 < r \leq \frac{1}{2}$. We deduce in particular by the Sobolev embedding that $u \in L^{\infty}(B_1)$ and then we have that

$$e^{-c_-} \leq K(z) \leq e^{c_+}, \quad c_{\pm} = \|u_{\mp}\|_{\infty}.$$

This ASBC has a sequence of conical singularities converging to the origin of orders α_n. As a consequence u is not of class $C^2(B_1 \setminus \{0\})$ and in particular the origin cannot be said to be a conical singularity, although it looks like a conical singularity of order α_0.

Example 4. Similar to Example 3, we provide an ASBC relevant to Theorems 1.3,1.4, still with a non trivial dependence of the exponent p_r by r, but without conical singularities.
Let us consider the ASBC $\{B_1, e^{\rho}|dz|^2\}$ where ρ satisfies (1.1)-(1.2) with $h \equiv 0$, $\omega_- \equiv 0$ and

$$\omega_+ = 2V\mathcal{H}^2, \quad V = \frac{1}{|z|^{2\alpha_0}}\prod_{n=1}^{+\infty}\frac{1}{|z-z_n|^{2\alpha_n}},$$

where $\alpha_0 = \frac{1}{2}$, $\alpha_n \in (0,1)$ is any sequence such that $\alpha := \sum\limits_n \alpha_n < \frac{1}{2}$, and $|z_n|$ is any strictly decreasing sequence such that $\frac{1}{2} \leq |z_n| \to 0$, as $n \to +\infty$. Clearly

$$\omega_+ = 2\mathcal{K} = 2Ke^{\rho}\mathcal{H}^2,$$

where $K = Ve^{-\rho} \in L^1(B_1, e^{\rho}\mathcal{H}^2)$. Putting $u = \rho$ we have $K = Ve^{-\rho} = Ve^{-u}$ and

$$-\Delta u = 2Ke^{\rho} = 2Ke^{u} \text{ in } B_1$$

in the sense of distributions. Since $\alpha_0 = \frac{1}{2}$ then $\omega(\{0\}) = 2\pi$ but for any $r \in (0,1)$ we have $\omega(B_r) > 2\pi$. On the other side it is not difficult to see that $V \in L^q(B_1)$ for any $q < 2$ and that for any $0 < r \leq \frac{1}{2}$, $V \in L^{p_r}(B_1 \setminus B_r)$ for some $p_r > 2$ and then, since $u = \int\limits_{B_1} \Gamma(x,y)2V(y)dy$, by standard elliptic estimates we have $u \in W^{2,q}(B_1)$ for any $q < 2$ and $u \in W^{2,p_r}(B_1 \setminus B_r)$ for any $0 < r \leq \frac{1}{2}$. We deduce in particular by the Sobolev embedding that $u \in L^\infty(B_1)$ and then we have that

$$e^{-c_-}V(z) \leq K(z) \leq e^{c_+}V(z), \quad c_\pm = \|u_\mp\|_\infty,$$

so that this time there are no conical singularities, while K has the same regularity properties as V does.

Remark. The same arguments of Examples 1 and 2 with minor changes can be used to produce examples of an ASBC with or without conical singularities and negative Gaussian curvature, which is done essentially by setting $\omega_+ \equiv 0$ and $\omega_- = 2W_\alpha\mathcal{H}^2$ where

$$W_\alpha = \frac{4(1-\alpha)^2|z|^{-2\alpha}}{(1-|z|^{2(1-\alpha)})^2}, \quad |z| < \frac{1}{2}, \ \alpha \in (0,1).$$

Analogously, slight changes to Examples 3 and 4 yield an ASBC with the same sort of singularities and negative Gaussian curvature.

Example 5. We provide an example of an ASBC where the Gaussian curvature is in $L^\infty(\Omega)$ with a jump discontinuity along a circle. Let us set $\Omega = \{\frac{1}{2} < |z| < 2\}$ and consider the ASBC $\{\Omega, e^{\rho}|dz|^2\}$ where ρ satisfies (1.1)-(1.2) with $\omega_- \equiv 0$, $\omega_+ = 2\omega_{\underline{\alpha}}\mathcal{H}^2$, where $\underline{\alpha} = (\alpha_1, \alpha_2)$ and for fixed $0 \leq \alpha_2 < \alpha_1 < 1$ we define,

$$\omega_{\underline{\alpha}}(z) = \begin{cases} \omega_1(z) = \dfrac{4(1-\alpha_1)^2|z|^{-2\alpha_1}}{\left(1+|z|^{2(1-\alpha_1)}\right)^2}, & \frac{1}{2} \leq |z| < 1, \\ \\ \omega_2(z) = \dfrac{4(1-\alpha_2)^2|z|^{-2\alpha_2}}{\left(1+|z|^{2(1-\alpha_2)}\right)^2}, & 1 \leq |z| < 2. \end{cases}$$

Clearly

$$\omega_+ = 2\mathcal{K} = 2Ke^{\rho}\mathcal{H}^2,$$

where $K = \omega_{\underline{\alpha}}e^{-\rho} \in L^1(\Omega, e^{\rho}\mathcal{H}^2)$. Let us set

$$u(z) = \begin{cases} \log\left(\dfrac{4(1-\alpha_1)^2|z|^{-2\alpha_1}}{\left(1+|z|^{2(1-\alpha_1)}\right)^2}\right), & \frac{1}{2} \leq |z| < 1, \\ \\ \log\left(\dfrac{4(1-\alpha_1)^2|z|^{-2\alpha_2}}{\left(1+|z|^{2(1-\alpha_2)}\right)^2}\right), & 1 \leq |z| \leq 2. \end{cases}$$

which is radial, of class C^1 in $\overline{\Omega}$ and is a solution of $-\Delta u = 2\omega_{\underline{\alpha}}$ in Ω in the sense of distributions. Remark that, by standard elliptic theory, since $\omega_{\underline{\alpha}} \in L^\infty(\Omega)$, then

ρ is C^1 in $\overline{\Omega}$. By the Weyl lemma $h = \rho - u$ is harmonic in Ω. In particular h is continuous in $\overline{\Omega}$ and

$$-\Delta u = 2Ke^{\rho} = 2Ke^{h}e^{u} \text{ in } \Omega, \tag{3.4}$$

in the sense of distributions. At this point, since ρ is radial, by setting $c_1 = \rho(z)\mid_{|z|=1}$, we have that

$$K = \omega_{\underline{\alpha}}e^{-\rho},$$

is radial, continuous in $\frac{1}{2} \leq |z| < 1$ and in $1 < |z| \leq 2$ and satisfies

$$\lim_{r\to 1^-} K(z) = (1-\alpha_2)^2 e^{-c_1}, \qquad \lim_{r\to 1^+} K(z) = (1-\alpha_1)^2 e^{-c_1}.$$

In other words K has a jump discontinuity at the circle $|z| = 1$ while, by standard elliptic theory, $u \in W^{2,p}(\Omega)$ is a strong solution of (3.4).

Example 6. We use Example 1 in [7] to construct an ASBC $\left\{B_1, e^{\rho}|dz|^2\right\}$ such that $\{u, f, K, \omega_s\} = \{u_a, 0, K, 0\}$ as obtained in Theorem 1.3 have the following properties:
- $e^{\rho} \in L^{\infty}(B_1)$;
- $K \in L^1(B_1)$ but there is no $s > 1$ such that $K \in L^s(B_1)$;
- u is not locally bounded;
- u has all the properties claimed in Theorem 1.3.

Let ρ satisfy (1.1)-(1.2) with $\omega_+ \equiv 0$,

$$\omega_- = 2V_a\mathcal{H}^2, \quad V_a = \frac{a}{2}|z|^{-2}\left(\log\left(\frac{e}{|z|}\right)\right)^{-2}, \ a \in (0,1),$$

and

$$h(z) = -\int\limits_{B_1} H(z,y)d\omega_-(y), \quad H(z,y) = \frac{1}{2\pi}\log\left(|z|\left|\frac{z}{|z|^2} - y\right|\right), \quad z \in \overline{B_1}, y \in B_1.$$

Since $H(z,y)$ is the regular part of the Green's function on B_1, and since V_a is bounded far away from the origin, then h is harmonic and bounded in B_1. Clearly $V_a \in L^1(B_1)$ and

$$\omega = 2\mathcal{K} = 2Ke^{\rho}\mathcal{H}^2,$$

where $K = -V_a e^{-\rho} \in L^1(B_1, e^{\rho}\mathcal{H}^2)$. Therefore, setting $u_a = \rho$ we have

$$-\Delta u_a = 2Ke^{\rho} = 2Ke^{u_a} \text{ in } B_1,$$

in the sense of distributions. Actually, since for $z \in \partial B_1$ we have $\Gamma(z,y) + H(z,y) = 0$, $\forall y \in B_1$, and since V_a is bounded far away from the origin, then $u_a(z) = 0$ for $z \in \partial B_1$. Therefore u_a is the unique (radial) function that satisfies $\Delta u_a = 2V_a$ in B_1, $u_a = 0$ on ∂B_1, which is $u_a = -a\log\left(\log\left(\frac{e}{|z|}\right)\right)$, $z \in B_1\setminus\{0\}$. As a consequence we have

$$K(z) = -\frac{a}{2}|z|^{-2}\left(\log\left(\frac{e}{|z|}\right)\right)^{-(2-a)},$$

and we see that $K \in L^1(B_1)$, $e^{u_a} \in L^{\infty}(B_1)$ and $u_a(z) \to -\infty$ as $z \to 0$. On the other side there is no $s > 1$ such that $K \in L^s(B_1)$, so there is no chance that K satisfies the assumption (1.5) of Theorem 1.4. On the other side, it is easy to check that u_a has all the properties claimed in Theorem 1.3.

Example 7. We use again Example 1 in [7] to construct an ASBC $\{B_1, e^{\rho}|dz|^2\}$ such that $\{u, f, K, \omega_s\} = \{u_a, 0, K, 0\}$ as obtained in Theorem 1.3 have the following properties:
- $e^{\rho} \notin L^{\infty}(B_1)$, $e^{\rho} \in L^q(B_1)$, for any $q \geq 1$;
- $K \in L^1(e^{\rho}\mathcal{H}^2, B_1) \cap L^1(B_1)$ but there is no $s > 1$ such that $K \in L^s(B_1)$;
- u is not locally bounded;
- u has all the properties claimed in Theorem 1.3.

Let ρ satisfy (1.1)-(1.2) with $\omega_- \equiv 0$,

$$\omega_+ = 2V_a\mathcal{H}^2, \quad V_a = \frac{a}{2}|z|^{-2}\left(\log\left(\frac{e}{|z|}\right)\right)^{-2}, \; a > 0,$$

and

$$h(z) = \int\limits_{B_1} H(z,y)d\omega_+(y), \quad H(z,y) = \frac{1}{2\pi}\log\left(|z|\left|\frac{z}{|z|^2} - y\right|\right), \qquad z \in \overline{B_1}, y \in B_1.$$

Since $H(z,y)$ is the regular part of the Green's function on B_1, and since V_a is bounded far away from the origin, then h is harmonic and bounded in B_1.

Clearly $V_a \in L^1(B_1)$ and

$$\omega = 2\mathcal{K} = 2Ke^{\rho}\mathcal{H}^2,$$

where $K = V_a e^{-\rho} \in L^1(B_1, e^{\rho}\mathcal{H}^2)$. Therefore, setting $u_a = \rho$ we have

$$-\Delta u_a = 2Ke^{\rho} = 2Ke^{u_a} \text{ in } B_1,$$

in the sense of distributions. Actually, since for $z \in \partial B_1$ we have $\Gamma(z,y) + H(z,y) = 0$, $\forall y \in B_1$, and since V_a is bounded far away from the origin, then $u_a(z) = 0$ for $z \in \partial B_1$. Therefore u_a is the unique (radial) function that satisfies $-\Delta u_a = 2V_a$ in B_1, $u_a = 0$ on ∂B_1, which is $u_a = a\log\left(\log\left(\frac{e}{|z|}\right)\right)$, $z \in B_1 \setminus \{0\}$. Thus we also have

$$K(z) = \frac{a}{2}|z|^{-2}\left(\log\left(\frac{e}{|z|}\right)\right)^{-(2+a)},$$

and we see that $K \in L^1(B_1)$, $e^{u_a} \in L^q(B_1)$ for any $q \geq 1$ and $u_a(z) \to +\infty$ as $z \to 0$. However, as in Example 2, there is no $s > 1$ such that $K \in L^s(B_1)$ while it is easy to check that u_a has all the properties claimed in Theorem 1.3. Remark that, with the notations of Theorem 1.3, since $f \equiv 0$ we have $q_1 = \infty$ but in particular it is not true that $e^{\rho} = e^{u_a + f} \in L^{q_1}(B_1)$.

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Daniele Bartolucci
Department of Mathematics, University of Rome *"Tor Vergata"*, Via della ricerca scientifica n.1, 00133 Roma, Italy
E-mail address: `bartoluc@mat.uniroma2.it`