Pure and Applied Functional Analysis
Volume 9, Number 1, 2024, 93–109

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

COMPOSITION OPERATORS ON SOBOLEV SPACES AND BALL'S CLASSES

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Abstract. In this paper we study composition operators on Sobolev spaces in the context of nonlinear elasticity problems. On this base we obtain the measure distortion estimates and the capacity distortion estimates of mappings of Ball's classes.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, and $\varphi : \Omega \to \widetilde{\Omega}$ be a measurable mapping. In fundamental papers [1, 2] J. M. Ball demonstrated that solutions of typical boundary value nonlinear elasticity problems are minimizers of complicated energy integrals of deformations

$$I(\varphi, \Omega) := \int_{\Omega} W(x, \varphi(x), D\varphi(x))\, dx,$$

where $\varphi : \Omega \to \widetilde{\Omega}$ is a continuous weak differentiable mapping (deformation) that is a homeomorphism of boundaries $\partial\Omega$ and $\partial\widetilde{\Omega}$.

In the spatial case ($n = 3$) J. M. Ball [1, 2] proved that appropriate classes of minimizers (Ball's classes) are

$$A^{+}_{q,r}(\Omega; \widetilde{\Omega}) = \left\{ \varphi \in L^1_q(\Omega; \widetilde{\Omega}) : \operatorname{adj} D\varphi \in L_r(\Omega),\ J(x, \varphi) > 0 \text{ for a.e. } x \in \Omega \right\},$$

where $q > n - 1$ and $r \geq q/(q - 1)$. In the case $q > n$ J. M. Ball proved [1, 2] (with some regularity boundary conditions of the variational problem) existence of the inverse mappings and its weak regularity, namely φ^{-1} belongs to the Sobolev class $L^1_1(\widetilde{\Omega})$ where $\widetilde{\Omega} = \varphi(\Omega)$.

The case $n-1 < q \leq n$ was studied in subsequent works [12, 26] (see, also [21, 22]) where it was proved that outside of a "singular" set S of a variational q-capacity zero the mapping $\varphi : \Omega \setminus S \to \widetilde{\Omega}$ is a homeomorphism. Its inverse homeomorphism φ^{-1} belongs to the space $L^1_1(\widetilde{\Omega})$ and maps the set $\varphi(\Omega \setminus S)$ onto $\Omega \setminus S$. It was proved also ([12]) that the "singular" set $H := \widetilde{\Omega} \setminus \varphi(\Omega \setminus S)$ of φ^{-1} has $(n - 1)$-Hausdorff measure zero.

2020 *Mathematics Subject Classification.* 46E35, 30C65, 74H30.
Key words and phrases. Sobolev spaces, nonlinear elasticity, quasiconformal mappings.
The first author was supported by the United States-Israel Binational Science Foundation (BSF Grant No. 2014055).

In the present work, using the geometric theory of composition operators on Sobolev spaces L^1_q [27, 33, 35] we introduce a characterization of Ball's classes in terms of the composition operators. On this base we obtain accurate volume distortion estimates of topological mappings (homeomorphisms) of Ball's classes possessing the Luzin N-property. The absolute continuity of the of Ball's classes with respect to the Sobolev capacity is proved also. The capacity considered as an outer measure associated with corresponding Sobolev spaces L^1_q [3]. This approach allows us to refine in the capacitary terms results of [12, 26] about "singular" sets.

Remark 1.1. In [26] the change of variables formulas in Theorem 2 of [26] and Theorem 3 of [26] are valid if the topological mapping φ possesses the Luzin N-property, since these formulas imply that for any set of measure zero ($|A| = 0$) we obtain

$$|\varphi(A)| = \int\limits_A |J(x,\varphi)|\ dx = 0.$$

Thereby, in the case $n-1 < q < n$, the assumption about the Luzin N-property was missed in [26].

In the present work we study Ball's classes $A^+_{q,r}(\Omega;\widetilde{\Omega})$ in the case of the minimal possible second index r, namely $r = q/(q-1)$ or $1/q + 1/r = 1$ and so $r = q'$.

Remark 1.2. In the case $q = n$ ($q' = n/(n-1)$) by the duality composition theorem (Theorem 3.6) the Ball's class $A^+_{n,n'}(\Omega;\widetilde{\Omega})$ coincides with the Sobolev space $L^1_n(\Omega;\widetilde{\Omega})$ of mappings with $J(x,\varphi) > 0$ for almost all $x \in \Omega$.

The class of Sobolev homeomorphisms $L^1_n(\Omega;\widetilde{\Omega})$ was studied in details in [25] under a name BL-homeomorphisms.

Remark 1.3. In [32] it was proved that Sobolev homeomorphisms $\varphi \in L^1_{n,\mathrm{loc}}(\Omega;\widetilde{\Omega})$ possess the Luzin N-property. Therefore Ball's classes $A^+_{q,q'}(\Omega;\widetilde{\Omega})$, $n \leq q$ have the Luzin N-property.

We use the terminology Sobolev homeomorphism because a possible existence of "singular sets" in Ω [26] for the case $n-1 < q < n$. It will be demonstrated later that the "singular" sets are sets of capacity zero.

The proposed approach to the geometric properties of the Ball classes $A^+_{q,q'}(\Omega;\widetilde{\Omega})$ is based on the geometric theory of composition operators on Sobolev spaces [7, 10, 27, 33–36]. We show that a Sobolev topological mapping (homeomorphism) $\varphi : \Omega \to \widetilde{\Omega}$ such that $J(x,\varphi) > 0$ for almost all $x \in \Omega$, belongs to $A^+_{q,q'}(\Omega;\widetilde{\Omega})$ if and only if for any function $f \in L^1_\infty(\widetilde{\Omega})$ (i.e. for bounded gradients of the stress tensor after the deformation), the composition $\varphi^*(f) = f \circ \varphi \in L^1_q(\Omega)$ and the following inequality is correct

$$\|f \mid L^1_1(\widetilde{\Omega})\| \leq \|\operatorname{adj} D\varphi | L_{q'}(\Omega)\| \cdot \|\varphi^*(f) \mid L^1_q(\Omega)\|,$$

where $1/q + 1/q' = 1$. This inequality states that the second Ball's condition $\operatorname{adj} D\varphi \in L_{q'}(\Omega)$ is equivalent (for mappings with positive a. e. Jacobians) to the boundedness of the composition operator

$$\left(\varphi^{-1}\right)^* : L^1_q(\Omega) \to L^1_1(\widetilde{\Omega}),\ n-1 < q < \infty, \tag{1.1}$$

generated by the inverse mapping (inverse deformation) $\varphi^{-1}:\widetilde{\Omega}\to\Omega$.

Using this composition operators property (1.1) and results of papers [33–35] we obtain the following volume distortion estimates for mappings of Ball's classes (Theorem 4.2): *Let $\varphi:\Omega\to\widetilde{\Omega}$ belongs to $A^{+}_{q,q'}(\Omega;\widetilde{\Omega})$, then the following inequality*

$$|\varphi(A)|^{1-\frac{1}{n}} \leq \left(\frac{1}{|A|}\int\limits_{A} |\operatorname{adj} D\varphi(x)|^{\frac{q}{q-1}}\ dx\right)^{1-\frac{1}{q}} |A|^{1-\frac{1}{n}}, \quad n-1<q\leq n,$$

holds for any measurable set $A\subset\Omega$.

Using capacitary properties of composition operators on Sobolev spaces we obtain corresponding estimates for capacity distortion of mappings of the Ball's classes $A^{+}_{q,r}(\Omega;\widetilde{\Omega})$ (Theorem 4.4): *Let $\varphi:\Omega\to\widetilde{\Omega}$ belongs to $A^{+}_{q,q'}(\Omega;\widetilde{\Omega})$, then the following capacity inequality*

$$\operatorname{cap}_1(\varphi(E);\widetilde{\Omega}) \leq \|\operatorname{adj} D\varphi | L_{q/(q-1)}(\Omega)\| \operatorname{cap}_q^{\frac{1}{q}}(E;\Omega), \quad n-1<q<\infty,$$

holds for any Borel set $E\subset\Omega$. Corresponding local and point wise estimates are also correct.

This means that topological mappings of Ball's classes are absolutely continuous with respect to capacity and an image of cavitation of q-capacity zero ("singular" set) in a body Ω has 1-capacity zero (also "singular" set) in $\widetilde{\Omega}$ and can not lead to the body breaking upon these deformations. So, the cavitation can be characterized in capacitary terms. As a consequence we obtain characterization of cavitations in terms of the Hausdorff measure: for every set $E\subset\Omega$ such that $\operatorname{cap}_q(E;\Omega)=0$, $1\leq q\leq n$, the Hausdorff measure $H^{n-1}(\varphi(E))=0$.

Weak regularity properties of mappings inverse to Sobolev homeomorphisms arise to [24, 37] and were intensively studied in the last decades. In frameworks of the geometric theory of composition operators on Sobolev spaces in [28] it was proved that mappings inverse to Sobolev homeomorphisms of the class $L^1_q(\Omega)$, $q>n-1$, belong to the class $BV(\widetilde{\Omega})$ and in [10] it was proved that mappings inverse to Sobolev homeomorphisms of finite distortions of the class $L^1_{n-1}(\Omega)$ possess the Luzin N-property, belong to the class $L^1_1(\widetilde{\Omega})$ and generate a bounded composition operator

$$\left(\varphi^{-1}\right)^{*}: L^1_{\infty}(\Omega)\to L^1_1(\widetilde{\Omega}).$$

The weak differentiability of mappings which are inverse to Sobolev homeomorphisms of finite distortions of the class $W^1_{n-1,\mathrm{loc}}(\Omega)$ was studied in [4] also. Unfortunately the Luzin N-property was missed in [4] because of the weak inverse differential was defined in [4] as a composition with the inverse mapping and measurability had to be proven. The weak differentiability of mappings inverse to Sobolev homeomorphisms of finite distortion with $|D\varphi|\in L^{n-1,1}(\Omega)$ was proved in [15].

In the paper [17] were considered bi-Sobolev mappings $\varphi:\Omega\to\widetilde{\Omega}$ such that $\varphi\in L^1_n(\Omega)$ and $\varphi^{-1}\in L^1_n(\widetilde{\Omega})$ in connections with the the nonlinear elasticity problems. This class of mapping was introduced and studied in [23]. Note that these classes coincide with classes of weak $(n,1)$-quasiconformal mappings [27, 34].

2. Composition operators on Sobolev spaces

In this section we give necessary elements of the Sobolev spaces theory and the geometric theory of composition operators on Sobolev spaces.

2.1. **Sobolev spaces.** Let E be a measurable subset of $\mathbb{R}^n$, $n \geq 2$. The Lebesgue space $L_p(E)$, $1 \leq p \leq \infty$, is defined as a Banach space of p-summable functions $f : E \to \mathbb{R}$ equipped with the standard norm.

If Ω is an open subset of $\mathbb{R}^n$, the Sobolev space $W^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^1_p(\Omega)\| = \|f \mid L_p(\Omega)\| + \|\nabla f \mid L_p(\Omega)\|.$$

The homogeneous seminormed Sobolev space $L^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined as a space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L^1_p(\Omega)\| = \|\nabla f \mid L_p(\Omega)\|.$$

Recall that in Lipschitz domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, Sobolev spaces $W^1_p(\Omega)$ and $L^1_p(\Omega)$ are coincide (see, for example, [19]).

Sobolev spaces are Banach spaces of equivalence classes [19]. To clarify the notion of equivalence classes we use the nonlinear p-capacity associated with Sobolev spaces. With the help of the p-capacity the Lebesgue differentiation theorem was refined for Sobolev spaces [19].

Recall the definition of the p-capacity [8, 13, 19]. Suppose Ω is an open set in $\mathbb{R}^n$ and $F \subset \Omega$ is a compact set. The p-capacity of F with respect to Ω is defined by

$$\operatorname{cap}_p(F; \Omega) = \inf\{\|\nabla f | L_p(\Omega)\|^p\},$$

where the infimum is taken over all functions $f \in C_0(\Omega) \cap L^1_p(\Omega)$ such that $f \geq 1$ on F. Such functions are called admissible functions for the compact set $F \subset \Omega$. If $U \subset \Omega$ is an open set, we define

$$\operatorname{cap}_p(U; \Omega) = \sup_F \{\operatorname{cap}_p(F; \Omega) \, : \, F \subset U, \; F \text{ is compact}\}.$$

In the case of an arbitrary set $E \subset \Omega$ we define the inner p-capacity

$$\underline{\operatorname{cap}}_p(E; \Omega) = \sup_F \{\operatorname{cap}_p(F; \Omega) \, : \, F \subset E \subset \Omega, \; F \text{ is compact}\},$$

and the outer p-capacity

$$\overline{\operatorname{cap}}_p(E; \Omega) = \inf_E \{\operatorname{cap}_p(U; \Omega) \, : \, E \subset U \subset \Omega, \; U \text{ is open}\}.$$

A set $E \subset \Omega$ is called p-capacity measurable, if $\underline{\operatorname{cap}}_p(E; \Omega) = \overline{\operatorname{cap}}_p(E; \Omega)$. Let $E \subset \Omega$ be a p-capacity measurable set. The value

$$\operatorname{cap}_p(E; \Omega) = \underline{\operatorname{cap}}_p(E; \Omega) = \overline{\operatorname{cap}}_p(E; \Omega)$$

is called the p-capacity measure of the set $E \subset \Omega$. By [3] analytical sets (in particular Borel sets) are p-capacity measurable sets.

The notion of the p-capacity measure permits us to refine the notion of Sobolev functions. Let a function $f \in L^1_p(\Omega)$. Then the refined function

$$\tilde{f}(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)\ dy$$

is defined quasieverywhere i. e. up to a set of p-capacity zero and it is absolutely continuous on almost all lines [19]. This refined function $\tilde{f} \in L^1_p(\Omega)$ is called the unique quasicontinuous representation (*a canonical representation*) of the function $f \in L^1_p(\Omega)$. Recall that a function $\tilde{f}$ is termed quasicontinuous if for any $\varepsilon > 0$ there is an open set U_ε such that the p-capacity of U_ε is less than ε and on the set $\Omega \setminus U_\varepsilon$ the function $\tilde{f}$ is continuous (see, for example [13, 19]). In what follows we will use the quasicontinuous (refined) functions only.

Note that the first weak derivatives of the function f coincide almost everywhere with the usual partial derivatives (see, e.g., [19]).

2.2. **Composition operators on Sobolev spaces.** Let us recall basic notations and results of the geometric theory of composition operators on Sobolev spaces. Suppose $\varphi : \Omega \to \mathbb{R}^n$ is a mapping of the Sobolev class $W^1_{1,\mathrm{loc}}(\Omega; \mathbb{R}^n)$. Then the formal Jacobi matrix $D\varphi(x)$ and its determinant (Jacobian) $J(x, \varphi)$ are well defined at almost all points $x \in \Omega$. The norm $|D\varphi(x)|$ is the operator norm of $D\varphi(x)$, i. e., $|D\varphi(x)| = \max\{|D\varphi(x) \cdot h| : h \in \mathbb{R}^n, |h| = 1\}$. We also set $l(D\varphi(x)) = \min\{|D\varphi(x) \cdot h| : h \in \mathbb{R}^n, |h| = 1\}$.

The Sobolev mapping $\varphi : \Omega \to \mathbb{R}^n$ is a mapping of finite distortion if $D\varphi(x) = 0$ for almost all x from $Z = \{x \in \Omega : J(x, \varphi) = 0\}$ [32]. Of course, the condition $J(x, \varphi) > 0$ for almost all $x \in \Omega$ of Ball's classes is stronger than condition of finite distortion, i.e. any mapping of such classes has the finite distortion. It means that all general results of this paper are correct for Ball's classes $A^+_{q,q'}(\Omega)$.

Let us reproduce here the change of variable formula for the Lebesgue integral [5, 11]. Let a mapping $\varphi : \Omega \to \mathbb{R}^n$ be such that there exists a collection of closed sets $\{A_k\}_1^\infty$, $A_k \subset A_{k+1} \subset \Omega$ for which restrictions $\varphi|_{A_k}$ are Lipschitz mappings on the sets A_k and

$$\left|\Omega \setminus \bigcup_{k=1}^{\infty} A_k\right| = 0.$$

Then there exists a measurable set $S \subset \Omega$, $|S| = 0$ such that the mapping $\varphi : \Omega \setminus S \to \mathbb{R}^n$ has the Luzin N-property and the change of variable formula

$$\int\limits_E f \circ \varphi(x)|J(x, \varphi)|\ dx = \int\limits_{\mathbb{R}^n \setminus \varphi(S)} f(y) N_f(E, y)\ dy \tag{2.1}$$

holds for every measurable set $E \subset \Omega$ and every non-negative measurable function $f : \mathbb{R}^n \to \mathbb{R}$. Here $N_f(y, E)$ is the multiplicity function defined as the number of preimages of y under f in E.

Sobolev mappings of the class $W^1_{1,\mathrm{loc}}(\Omega; \mathbb{R}^n)$ satisfy the conditions of the change of variable formula [11] and so for Sobolev mappings the change of variable formula (2.1) holds.

If the mapping φ possesses the Luzin N-property (the image of a set of measure zero has measure zero), then $|\varphi(S)| = 0$ and the second integral can be rewritten as the integral on $\mathbb{R}^n$. Note, that Sobolev homeomorphisms of the class $W^1_{n,\mathrm{loc}}(\Omega;\widetilde{\Omega})$ possess the Luzin N-property [32].

Let Ω and $\widetilde{\Omega}$ be bounded domains in $\mathbb{R}^n$, $n \geq 2$. We say that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ induces a bounded composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \ \ 1 \leq q \leq p \leq \infty,$$

by the composition rule $\varphi^*(f) = f \circ \varphi$, if for any function $f \in L^1_p(\widetilde{\Omega})$, the composition $\varphi^*(f) \in L^1_q(\Omega)$ is defined quasi-everywhere in Ω and there exists a constant $K_{p,q}(\Omega) < \infty$ such that

$$\|\varphi^*(f) \mid L^1_q(\Omega)\| \leq K_{p,q}(\Omega) \|f \mid L^1_p(\widetilde{\Omega})\|.$$

The problem of composition operators on Sobolev spaces arises in [9] for Sobolev extension operators in cusp domains and is connected with the Reshennyak's problem (1969) [31]. Recall that in connection with the geometric function theory the p-distortion of a mapping φ at a point $x \in \Omega$ is defined as

$$K_p(x) = \inf\{k(x) : |D\varphi(x)| \leq k(x)|J(x,\varphi)|^{\frac{1}{p}}, \ x \in \Omega\}.$$

If $p = n$ we have the usual conformal dilatation and in the case $p \neq n$ the p-dilatation arises in [6] (see, also, [29]).

The following theorem gives characterizations of mappings which generate bounded composition operators on Sobolev spaces in the terms of the p-distortion K_p.

Theorem 2.1. *A homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ induces a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \ \ 1 \leq q \leq p \leq \infty,$$

if and only if φ is a Sobolev mapping of the class $L^1_q(\Omega;\widetilde{\Omega})$, has finite distortion and

$$K_{p,q}(\varphi;\Omega) = \|K_p \mid L_\kappa(\Omega)\| < \infty,$$

where $1/q - 1/p = 1/\kappa$ ($\kappa = \infty$, if $p = q$).

This theorem was proved in [27] (see, also, [34]), case $p = \infty$ was considered in [10]. Mappings that satisfy conditions of Theorem 2.1 are called weak (p,q)-quasiconformal mappings [7,33] and are a natural generalization of quasiconformal mappings ($p = q = n$). In [18], where Sobolev spaces $W^1_p(\Omega)$ were considered as spaces of locally-integrable functions defined up to a set of measure zero, another proof of sufficiency of conditions of Theorem 2.1 was given. Unfortunately, methods of [18] do not allow to prove necessity conditions of Theorem 2.1. In [20] the theory of multipliers was applied to the composition operators on Sobolev spaces.

Remark 2.2. The historical survey on the theory of composition operators on Sobolev spaces can be found in [30]. Unfortunately, this useful work [30] doesn't contain essential new results and contains some non-correct citations of previous original papers [27, 33, 34]. Let us remark also that some proofs are not complete: for example, the main result of Section 4 of [30] was formulated for general type of mappings, but the proof was given for homeomorphisms only (and so it repeats results of the work [10]) and even in this case contains gaps.

2.3. **Capacity estimates of composition operators.** Now we give the capacitary distortion estimates of Borel sets under homeomorphisms generating composition operators on Sobolev spaces.

Theorem 2.3. *Let a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ generates a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_p(\Omega),\ 1 \leq p < \infty.$$

Then the inequality

$$\operatorname{cap}_p^{1/p}(\varphi^{-1}(\widetilde{E});\Omega) \leq K_{p,p}(\varphi;\Omega)\operatorname{cap}_p^{1/p}(\widetilde{E};\widetilde{\Omega})$$

holds for every Borel set $\widetilde{E} \subset \widetilde{\Omega}$.

Proof. Let $F \subset E = \varphi^{-1}(\widetilde{E})$ be a compact set. Because φ is a homeomorphism $\widetilde{F} = \varphi(F) \subset \widetilde{E}$ is also a compact set. Let $f \in C_0(\widetilde{\Omega}) \cap L^1_p(\widetilde{\Omega})$ be an arbitrary function such that $f \geq 1$ on $\widetilde{F}$. Then the composition $g = \varphi^*(f)$ belongs to $C_0(\Omega) \cap L^1_p(\Omega)$, $g \geq 1$ on F and

$$\|\varphi^*(f) \mid L^1_p(\Omega)\| \leq K_{p,p}(\varphi;\Omega)\|f \mid L^1_p(\Omega)\|.$$

Since the function $g = \varphi^*(f) \in C_0(\Omega) \cap L^1_p(\Omega)$ is an admissible function for the compact $F \subset E$, then

$$\operatorname{cap}_p^{1/p}(\varphi^{-1}(\widetilde{F});\Omega) \leq \|\varphi^*(f) \mid L^1_p(\Omega)\| \leq K_{p,p}(\varphi;\Omega)\|f \mid L^1_p(\Omega)\|.$$

Taking infimum over all functions $f \in C_0(\widetilde{\Omega}) \cap L^1_p(\widetilde{\Omega})$ such that $f \geq 1$ on $\widetilde{F}$ we have

$$\operatorname{cap}_p^{1/p}(\varphi^{-1}(\widetilde{F});\Omega) \leq K_{p,p}(\varphi;\Omega)\operatorname{cap}_p^{1/p}(\widetilde{F};\widetilde{\Omega})$$

for any compact set $\widetilde{F} \subset \widetilde{E} \subset \widetilde{\Omega}$.

Now for the Borel set $\widetilde{E} \subset \widetilde{\Omega}$ we have (by the definition of the p-capacity of Borel sets)

$$\operatorname{cap}_p^{1/p}(\widetilde{F};\widetilde{\Omega}) \leq \underline{\operatorname{cap}}_p^{1/p}(\widetilde{E};\widetilde{\Omega}) = \operatorname{cap}_p^{1/p}(\widetilde{E};\widetilde{\Omega}).$$

Hence

$$\operatorname{cap}_p^{1/p}(\varphi^{-1}(\widetilde{F});\Omega) \leq K_{p,p}(\varphi;\Omega)\operatorname{cap}_p^{1/p}(\widetilde{E};\widetilde{\Omega}).$$

Since $F = \varphi^{-1}(\widetilde{F})$ is an arbitrary compact set, $F \subset E$, E is a Borel set as a preimage of the Borel set $\widetilde{E}$ under the homeomorphism φ, then

$$\operatorname{cap}_p^{1/p}(\varphi^{-1}(\widetilde{E});\Omega) = \underline{\operatorname{cap}}_p^{1/p}(\varphi^{-1}(\widetilde{E});\Omega) = \sup_{F \subset E}\operatorname{cap}_p^{1/p}(F;\Omega)$$
$$\leq K_{p,p}(\varphi;\Omega)\operatorname{cap}_p^{1/p}(\widetilde{E};\widetilde{\Omega}).$$

□

Theorem 2.4. *Let a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ generates a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega),\ 1 \leq q < p < \infty.$$

Then the inequality

$$\operatorname{cap}_q^{1/q}(\varphi^{-1}(\widetilde{E});\Omega) \leq K_{p,q}(\varphi;\Omega)\operatorname{cap}_p^{1/p}(\widetilde{E};\widetilde{\Omega})$$

holds for every Borel set $\widetilde{E} \subset \widetilde{\Omega}$.

Proof. Let $F \subset E = \varphi^{-1}(\widetilde{E})$ be a compact set. Because φ is a homeomorphism $\widetilde{F} = \varphi(F) \subset \widetilde{E}$ is also a compact set. Let $f \in C_0(\widetilde{\Omega}) \cap L^1_p(\widetilde{\Omega})$ be an arbitrary function such that $f \geq 1$ on $\widetilde{F}$. Then the composition $g = \varphi^*(f)$ belongs to $C_0(\Omega) \cap L^1_q(\Omega)$, $g \geq 1$ on F and

$$\|\varphi^*(f) \mid L^1_q(\Omega)\| \leq K_{p,q}(\varphi;\Omega)\|f \mid L^1_p(\Omega)\|.$$

Since the function $g = \varphi^*(f) \in C_0(\Omega) \cap L^1_q(\Omega)$ is an admissible function for the compact $F \subset E$, then

$$\operatorname{cap}_q^{1/q}(\varphi^{-1}(\widetilde{F});\Omega) \leq \|\varphi^*(f) \mid L^1_q(\Omega)\| \leq K_{p,q}(\varphi;\Omega)\|f \mid L^1_p(\Omega)\|.$$

Taking infimum over all functions $f \in C_0(\widetilde{\Omega}) \cap L^1_p(\widetilde{\Omega})$ such that $f \geq 1$ on $\widetilde{F}$ we have

$$\operatorname{cap}_q^{1/q}(\varphi^{-1}(\widetilde{F});\Omega) \leq K_{p,q}(\varphi;\Omega)\operatorname{cap}_p^{1/p}(\widetilde{F};\widetilde{\Omega})$$

for any compact set $\widetilde{F} \subset \widetilde{E} \subset \widetilde{\Omega}$.

Now for the Borel set $\widetilde{E} \subset \widetilde{\Omega}$ we have (by the definition of the p-capacity of Borel sets)

$$\operatorname{cap}_p^{1/p}(\widetilde{F};\widetilde{\Omega}) \leq \underline{\operatorname{cap}}_p^{1/p}(\widetilde{E};\widetilde{\Omega}) = \operatorname{cap}_p^{1/p}(\widetilde{E};\widetilde{\Omega}).$$

Hence

$$\operatorname{cap}_q^{1/q}(\varphi^{-1}(\widetilde{F});\Omega) \leq K_{p,q}(\varphi;\Omega)\operatorname{cap}_p^{1/p}(\widetilde{E};\widetilde{\Omega}).$$

Since $F = \varphi^{-1}(\widetilde{F})$ is an arbitrary compact set, $F \subset E$, E is a Borel set as a preimage of the Borel set $\widetilde{E}$ under the homeomorphism φ, then

$$\operatorname{cap}_q^{1/q}(\varphi^{-1}(\widetilde{E});\Omega) = \underline{\operatorname{cap}}_q^{1/q}(\varphi^{-1}(\widetilde{E});\Omega) = \sup_{F \subset E} \operatorname{cap}_q^{1/q}(F;\Omega) \leq K_{p,q}(\varphi;\Omega)\operatorname{cap}_p^{1/p}(\widetilde{E};\widetilde{\Omega}).$$

□

3. Composition operators and Ball's classes

In this section we consider applications of the geometric theory of composition operators on Sobolev spaces to nonlinear elasticity problems. These application build on a notion of an inner distortion which is used for a study of "inverse" composition operators. This notion gives a geometric interpretation of the integrability condition of $\operatorname{adj} D\varphi$ in the original definition of Ball's classes $A^+_{q,q'}(\Omega)$.

3.1. **Inverse composition operators.** Let Ω and $\widetilde{\Omega}$ be two bounded domains in $\mathbb{R}^n$ and $\varphi : \Omega \to \widetilde{\Omega}$ be a mapping of finite distortion of the class $W^1_{1,\mathrm{loc}}(\Omega;\widetilde{\Omega})$. We define the "normalized" inner s-distortion of φ at a point x as

$$K_s^I(x,\varphi) = \begin{cases} \frac{|J(x,\varphi)|^{\frac{1}{s}}}{l(D\varphi(x))}, & J(x,\varphi) \neq 0, \\ 0, & J(x,\varphi) = 0, \end{cases}$$

where $l(D\varphi(x))$ is defined as $\min\limits_{h=1} |D\varphi(x)\cdot h|$.

Its global integral version is called the inner (q,s)-distortion, $1 \leq s \leq q \leq \infty$:

$$K^I_{q,s}(\varphi;\Omega) = \|K^I_s(\varphi) \mid L_\kappa(\Omega)\|,\ 1/\kappa = 1/s - 1/q,\ (\kappa = \infty,\ \text{if } q = s).$$

Proposition 3.1. *Let a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ belongs to $L^1_q(\Omega;\widetilde{\Omega})$ and $J(x,\varphi) > 0$ for almost all $x \in \Omega$. Then*

$$\left(\int\limits_\Omega \left(\frac{|J(x,\varphi)|}{l(D\varphi(x))}\right)^{\frac{q}{q-1}}\ dx\right)^{\frac{q-1}{q}} = \left(\int\limits_\Omega |\operatorname{adj} D\varphi(x)|^{\frac{q}{q-1}}\ dx\right)^{\frac{q-1}{q}}.$$

Proof. Using the following equalities (see, for example, [10]):

$$(D\varphi(x))^{-1} = J^{-1}(x,\varphi)\operatorname{adj} D\varphi(x)$$

and

$$\min_{h=1} |D\varphi(x)\cdot h| = \left(\max_{h=1} |(D\varphi(x))^{-1}\cdot h|\right)^{-1}$$

we have

$$K^I_{q,1}(\varphi;\Omega) = \left(\int\limits_\Omega \left(\frac{|J(x,\varphi)|}{l(D\varphi(x))}\right)^{\frac{q}{q-1}}\ dx\right)^{\frac{q-1}{q}} = \left(\int\limits_\Omega |\operatorname{adj} D\varphi(x)|^{\frac{q}{q-1}}\ dx\right)^{\frac{q-1}{q}}.$$

□

The following theorem give the characterization of composition operators in Sobolev spaces in the terms of the inner (q,s)-distortion.

Theorem 3.2. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a Sobolev homeomorphism of finite distortion and belongs to $L^1_q(\Omega;\widetilde{\Omega})$. Then the inverse homeomorphism $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ generates a bounded composition operator*

$$\left(\varphi^{-1}\right)^* : L^1_q(\Omega) \to L^1_s(\widetilde{\Omega}),\ 1 \leq s < q < \infty, \tag{3.1}$$

if and only if $\varphi^{-1} \in L^1_s(\widetilde{\Omega};\Omega)$, possesses the Luzin N^{-1}-property (φ possesses the Luzin N-property) and

$$K^I_{q,s}(\varphi;\Omega) = \left(\int\limits_\Omega K^I_s(x,\varphi)^{\frac{qs}{q-s}}\ dx\right)^{\frac{q-s}{qs}} < \infty.$$

Proof. Necessity. Let the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ generates a bounded composition operator

$$\left(\varphi^{-1}\right)^{*} : L^1_q(\Omega) \to L^1_s(\widetilde{\Omega}),\ 1 \leq s < q < \infty.$$

Then by Theorem 2.1, the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ belongs to the Sobolev space $L^1_s(\widetilde{\Omega};\Omega)$, has finite distortion and

$$\left(\int\limits_{\widetilde{\Omega}} \left(\frac{|D\varphi^{-1}(y)|^q}{|J(y,\varphi^{-1})|}\right)^{\frac{s}{q-s}}\ dy\right)^{\frac{q-s}{qs}} < \infty,\ 1 \leq s < q < \infty.$$

Note, that in the case $q \geq n$, the homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ of the class $L^1_q(\Omega;\widetilde{\Omega})$ possesses the Luzin N-property [8] and in the case $1 \leq s < q < n$ the mapping φ^{-1} which generates a bounded composition operator

$$\left(\varphi^{-1}\right)^{*} : L^1_q(\Omega) \to L^1_s(\widetilde{\Omega}),\ 1 \leq s < q < n,$$

possesses the Luzin N^{-1}-property [34]. It means that $\varphi : \Omega \to \widetilde{\Omega}$ possesses the Luzin N-property for all $1 \leq s < q < \infty$.

Hence, under the conditions of the theorem, $\varphi^{-1} \in L^1_s(\widetilde{\Omega};\Omega)$, possesses the Luzin N^{-1}-property for all $1 \leq s < q < \infty$ and we can put $K^I_{q,s}(\varphi;\Omega) = 0$ on the set $Z = \{x \in \Omega : J(x,\varphi) = 0\}$. So

$$\begin{aligned}
K^I_{q,s}(\varphi;\Omega) &= \left(\int\limits_{\Omega\setminus Z} \left(\frac{|J(x,\varphi)|}{l(D\varphi(x))^s}\right)^{\frac{q}{q-s}}\ dx\right)^{\frac{q-s}{qs}} \\
&= \left(\int\limits_{\Omega\setminus Z} \left(\frac{|D\varphi^{-1}(\varphi(x))|^q}{|J(\varphi(x),\varphi^{-1})|}\right)^{\frac{s}{q-s}} |J(x,\varphi)|\ dx\right)^{\frac{q-s}{qs}} \\
&= \left(\int\limits_{\widetilde{\Omega}} \left(\frac{|D\varphi^{-1}(y)|^q}{|J(y,\varphi^{-1})|}\right)^{\frac{s}{q-s}}\ dy\right)^{\frac{q-s}{qs}} < \infty.
\end{aligned}$$

Sufficiency. Let $\varphi^{-1} \in L^1_q(\widetilde{\Omega};\Omega)$, possesses the Luzin N^{-1}-property and

$$K^I_{q,s}(\varphi;\Omega) = \left(\int\limits_{\Omega} \left(\frac{|J(x,\varphi)|}{l(D\varphi(x))^s}\right)^{\frac{q}{q-s}}\ dx\right)^{\frac{q-s}{qs}} < \infty.$$

Then

$$\left(\int\limits_{\widetilde{\Omega}} \left(\frac{|D\varphi^{-1}(y)|^q}{|J(y,\varphi^{-1})|}\right)^{\frac{s}{q-s}}\,dy\right)^{\frac{q-s}{qs}} =$$

$$\left(\int\limits_{\Omega\setminus Z} \left(\frac{|D\varphi^{-1}(\varphi(x))|^q}{|J(\varphi(x),\varphi^{-1})|}\right)^{\frac{s}{q-s}} |J(x,\varphi)|\,dx\right)^{\frac{q-s}{qs}}$$

$$= \left(\int\limits_{\Omega\setminus Z} \left(\frac{|J(x,\varphi)|}{l(D\varphi(x))^s}\right)^{\frac{q}{q-s}}\,dx\right)^{\frac{q-s}{qs}} = K^I_{q,s}(\varphi;\Omega) < \infty.$$

Then by Theorem 2.1 the mapping φ^{-1} generates a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_q(\Omega) \to L^1_s(\widetilde{\Omega}),\ \ 1 \leq s < q < \infty.$$

□

Now we give the description of Ball's classes $A^+_{q,q'}(\Omega;\widetilde{\Omega})$ in the terms of composition operators on Sobolev spaces.

Theorem 3.3. *The homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between bounded domains $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$ belongs to the Ball class $A^+_{q,q'}(\Omega;\widetilde{\Omega})$ for $q > n-1$, $1/q + 1/q' = 1$, if and only if $\varphi \in L^1_q(\Omega;\widetilde{\Omega})$, $J(x,\varphi) > 0$ for almost all $x \in \Omega$, possesses the Luzin N-property and the inverse mapping generates the bounded composition operator*

$$\left(\varphi^{-1}\right)^* : L^1_q(\Omega) \to L^1_1(\widetilde{\Omega}),$$

with $\| \left(\varphi^{-1}\right)^* \| \leq \| \operatorname{adj} D\varphi \mid L_{q'}(\Omega)\|$.

Proof. By Lemma 3.1 we have, that the inner integral distortion

$$K^I_{q,1}(\varphi;\Omega) = \left(\int\limits_{\Omega} \left(\frac{|J(x,\varphi)|}{l(D\varphi(x))}\right)^{\frac{q}{q-1}}\,dx\right)^{\frac{q-1}{q}}$$

$$= \left(\int\limits_{\Omega} |\operatorname{adj} D\varphi(x)|^{\frac{q}{q-1}}\,dx\right)^{\frac{q-1}{q}} < \infty.$$

Hence, by Theorem 3.2 the inverse mapping generates the bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_q(\Omega) \to L^1_1(\widetilde{\Omega}),$$

if and only if φ belongs to the Ball class $A^+_{q,q'}(\Omega;\widetilde{\Omega})$, $q > n-1$, $1/q + 1/q' = 1$. The estimate of the composition operator norm follows from [27, 34].

□

Now we consider two-sides estimates of composition operators. In [10] it was proved that the inequality

$$\|\varphi^*(f) \mid L^1_q(\Omega)\| \leq \|\varphi \mid L^1_q(\Omega)\| \cdot \|f \mid L^1_\infty(\widetilde{\Omega})\|, \ 1 \leq q < \infty,$$

holds for any $f \in L^1_\infty(\widetilde{\Omega})$ if and only if $\varphi \in L^1_q(\Omega)$.

Combining the previous theorem and this inequality we immediately obtain

Theorem 3.4. *Let a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between bounded domains $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$ belongs to the Ball class $A^+_{q,q'}(\Omega; \widetilde{\Omega})$ for $q > n-1$, $1/q + 1/q' = 1$. Then for any function $f \in L^1_\infty(\widetilde{\Omega})$ following inequalities*

$$\| \operatorname{adj} D\varphi \mid L_{q'}(\Omega)\|^{-1} \cdot \|f \mid L^1_1(\widetilde{\Omega})\| \leq \|\varphi^*(f) \mid L^1_q(\Omega)\| \leq \|\varphi \mid L^1_q(\Omega)\| \cdot \|f \mid L^1_\infty(\widetilde{\Omega})\|$$

hold.

This theorem demonstrates variations of the nonlinear elastic potential energy under weak quasiconformal deformations of elastic bodies.

3.2. **The weak regularity of inverse Sobolev mappings.** In the geometric theory of composition operators on Sobolev spaces the significant role plays the following composition duality property [27]. This property represents the weak inverse mapping theorem (in the part of regularity of inverse mappings) for Sobolev mappings.

Theorem 3.5. *Let a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$, $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$, induces a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \ \ n-1 < q \leq p < \infty.$$

Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ induces a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_{\tilde{q}}(\Omega) \to L^1_{\tilde{p}}(\widetilde{\Omega}), \ \ n-1 < \tilde{p} \leq \tilde{q} < \infty,$$

where $\tilde{p} = p/(p-n+1)$ and $\tilde{q} = q/(q-n+1)$.

In the present work we prove this property in the limit case $p = \infty$.

Theorem 3.6. *Let a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$, $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$, be a mappings of finite distortion, possesses the Luzin N-property and induces a bounded composition operator*

$$\varphi^* : L^1_\infty(\widetilde{\Omega}) \to L^1_q(\Omega), \ \ n-1 < q < \infty.$$

Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ induces a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_{\tilde{q}}(\Omega) \to L^1_1(\widetilde{\Omega}), \ \ \tilde{q} = q/(q-n+1).$$

Proof. Since $\varphi : \Omega \to \widetilde{\Omega}$ generates a bounded composition operator

$$\varphi^* : L^1_\infty(\widetilde{\Omega}) \to L^1_q(\Omega), \ \ n-1 < q < \infty,$$

then by [10] the mapping $\varphi \in L^1_q(\Omega; \widetilde{\Omega})$. Because φ possesses the Luzin N-property, then the inverse mapping belongs to $W^1_{1,\mathrm{loc}}(\widetilde{\Omega})$ and is a mapping of finite distortion

[10]. Denote by $Z = \{x \in \Omega \mid J(x,\varphi) = 0\}$ and S is the set from the change of variables formula (2.1), $|S| = 0$. Then [27]

$$|D\varphi^{-1}(y)| \leq \frac{|D\varphi(x)|^{n-1}}{|J(x,\varphi)|},$$

for almost all $x \in \Omega \setminus (S \cup Z)$, $y = \varphi(x) \in \widetilde{\Omega} \setminus \varphi(S \cup Z)$, and

$$|D\varphi^{-1}(y)| = 0 \text{ for almost all } y \in \varphi(S).$$

Then

$$\int\limits_{\widetilde{\Omega}} \left(\frac{|D\varphi^{-1}(y)|^{\tilde{q}}}{|J(y,\varphi^{-1})|}\right)^{\frac{1}{\tilde{q}-1}} dy = \int\limits_{\widetilde{\Omega}\setminus\varphi(S\cup Z)} \left(\frac{|D\varphi^{-1}(y)|^{\tilde{q}}}{|J(y,\varphi^{-1})|}\right)^{\frac{1}{\tilde{q}-1}} dy$$

$$\leq \int\limits_{\widetilde{\Omega}\setminus\varphi(S\cup Z)} \left(\left(\frac{|D\varphi(\varphi^{-1}(y))|^{n-1}}{|J(\varphi^{-1}(y),\varphi)|}\right)^{\tilde{q}} \cdot \frac{1}{|J(y,\varphi^{-1})|}\right)^{\frac{1}{\tilde{q}-1}} dy$$

$$= \int\limits_{\widetilde{\Omega}\setminus\varphi(S\cup Z)} \frac{|D\varphi(\varphi^{-1}(y))|^{q}}{|J(\varphi^{-1}(y),\varphi)|} dy = \int\limits_{\Omega\setminus(S\cup Z)} \frac{|D\varphi(x)|^q}{|J(x,\varphi)|}|J(x,\varphi)|\, dx$$

$$\leq \int\limits_{\Omega} |D\varphi(x)|^q \, dx < \infty.$$

Hence [27] $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ generates a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_{\tilde{q}}(\Omega) \to L^1_1(\widetilde{\Omega}),$$

where $\tilde{q} = q/(q-n+1)$. □

Corollary 3.7. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a Sobolev homeomorphism of bounded domains $\Omega, \widetilde{\Omega}$ such that $J(x,\varphi) > 0$ for almost all $x \in \Omega$. Then $\varphi \in L^1_n(\Omega;\widetilde{\Omega})$ if and only if $\varphi \in A^+_{n,n'}(\Omega;\widetilde{\Omega})$, $n' = n/(n-1)$.*

Proof. The inclusion

$$A^+_{n,n'}(\Omega;\widetilde{\Omega}) \subset L^1_n(\Omega;\widetilde{\Omega})$$

holds by the definition of Ball's class $A^+_{n,n'}(\Omega;\widetilde{\Omega})$. Now let $\varphi \in L^1_n(\Omega;\widetilde{\Omega})$. Then by Theorem 3.6 the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ induces a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_n(\Omega) \to L^1_1(\widetilde{\Omega}).$$

By Theorem 3.3 the mapping $\varphi \in A^+_{n,n'}(\Omega;\widetilde{\Omega})$, $n' = n/(n-1)$. □

The key point in proof of the regularity of mappings of Ball's classes plays the regularity of mappings which are inverse to Sobolev mappings. This topic arises in [37] and was studied by many authors, see, for example, [4, 10, 15, 16, 28]. In the present work we use the following theorem from [10]:

Theorem 3.8. *[10] Let a homeomorphism $\varphi:\Omega\to\widetilde{\Omega}$ between two domains $\Omega,\widetilde{\Omega}\subset\mathbb{R}^n$ belong to the Sobolev space $L^1_{n-1}(\Omega;\widetilde{\Omega})$, possess the Luzin N-property and have finite distortion. Then the inverse mapping φ^{-1} belongs to the Sobolev space $L^1_1(\widetilde{\Omega};\Omega)$.*

4. Measure and capacity distortion estimates

4.1. **Measure distortion estimates.** In this section we give the volume distortion property of mappings of Ball's classes. Let us recall the following theorem [34] in the convenient for us form, because in [34] this theorem was proved for general (not necessary homeomorphic mappings). Let $\varphi:\Omega\to\widetilde{\Omega}$ be a Sobolev mapping, then the inverse s-distortion function is defined [34] by

$$H_s(y)=\begin{cases}\left(\sum\limits_{x\in\varphi^{-1}(y)\setminus S, J(x,\varphi)\neq 0}\frac{|D\varphi(x)|^s}{|J(x,\varphi)|}\right)^{\frac{1}{s}}, \\ 0, \qquad\qquad\qquad\qquad \text{otherwise,}\end{cases}$$

where S is the set from the change of variables formula (2.1).

Theorem 4.1. *[34] Let a homeomorphism $\varphi:\Omega\to\widetilde{\Omega}$ generates a bounded composition operator*

$$\varphi^*: L^1_q(\widetilde{\Omega})\to L^1_s(\Omega),\ 1\leq s\leq q\leq n.$$

Then for any measurable set $\widetilde{A}\subset\widetilde{\Omega}$ the following inequality

$$|\varphi^{-1}(\widetilde{A})|^{\frac{1}{s}-\frac{1}{n}}\leq \|H_s\mid L_\kappa(\widetilde{A})\||\widetilde{A}|^{\frac{1}{q}-\frac{1}{n}},\ 1/\kappa=1/s-1/q, \tag{4.1}$$

holds.

Hence, we obtain

Theorem 4.2. *Let a homeomorphism φ belongs to the Ball's class $A^+_{q,q'}(\Omega;\widetilde{\Omega})$, $n-1<q\leq n$, $1/q+1/q'=1$, then the inequality*

$$|\varphi(A)|^{1-\frac{1}{n}}\leq\left(\frac{1}{|A|}\int\limits_A|\operatorname{adj} D\varphi(x)|^{\frac{q}{q-1}}\ dx\right)^{1-\frac{1}{q}}|A|^{1-\frac{1}{n}},\quad n-1<q\leq n,$$

holds for any measurable set $A\subset\Omega$.

Proof. By Theorem 3.3 the inverse mapping $\varphi^{-1}:\widetilde{\Omega}\to\Omega$ generates the bounded composition operator

$$\left(\varphi^{-1}\right)^*: L^1_q(\Omega)\to L^1_1(\widetilde{\Omega}).$$

Hence by [34] we have

$$|\varphi(A)|^{1-\frac{1}{n}}\leq\|H_1\mid L_\kappa(A)\||A|^{\frac{1}{q}-\frac{1}{n}},\ 1/\kappa=1-1/q,$$

Now we calculate the norm $\|H_1 \mid L_\kappa(A)\|$. Since $J(x,\varphi) > 0$ for almost all $x \in \Omega$ and φ possesses the Luzin N-property, then

$$\|H_1 \mid L_\kappa(A) = \left(\int\limits_A \left(\frac{|D\varphi^{-1}(\varphi(x))|}{|J(\varphi(x), \varphi^{-1})|} \right)^{\frac{q-1}{q}} dx \right)^{\frac{q}{q-1}}$$

$$= \left(\int\limits_A \left(\frac{|J(x,\varphi)|}{l(D\varphi(x))} \right)^{\frac{q}{q-1}} dx \right)^{\frac{q-1}{q}} = \left(\int\limits_A |\operatorname{adj} D\varphi(x)|^{\frac{q}{q-1}} dx \right)^{\frac{q-1}{q}}.$$

□

4.2. **Capacity distortion estimates.** In this section we prove that topological mappings (homeomorphism) of Ball's classes which possess the Luzin N-property are absolutely continuous with respect to the corresponding p-capacities, which considered as outer measures associated with Sobolev spaces. It refines corresponding results of [12, 26]. Recall that Borel sets are measurable for the p-capacity [19].

In the following theorem we give the capacitary distortion estimates of the mappings with integrable inner distortion.

Theorem 4.3. *Let a homeomorphism of finite distortion $\varphi : \Omega \to \widetilde{\Omega}$ belong to $L^1_q(\Omega)$, $1 < q < \infty$, possess the Luzin N-property and such that*

$$K^I_{q,s}(\varphi;\Omega) = \|K^I_s(\varphi) \mid L_\kappa(\Omega)\| < \infty, \; 1 \leq s < q < \infty,$$

where $1/\kappa = 1/s - 1/q$. Then for every Borel set $E \subset \Omega$ the inequality

$$\operatorname{cap}_s^{\frac{1}{s}}(\varphi(E);\widetilde{\Omega}) \leq K^I_{q,s}(\varphi;\Omega) \operatorname{cap}_q^{\frac{1}{q}}(E;\Omega)$$

holds.

Proof. By Theorem 3.2 the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ generates a bounded composition operator

$$(\varphi^{-1})^* : L^1_q(\Omega) \to L^1_s(\widetilde{\Omega}), 1 \leq s < q < \infty.$$

Hence, by Theorem 2.4 for any Borel set $E \subset \Omega$ the inequality

$$\operatorname{cap}_s^{\frac{1}{s}}(\varphi(E);\widetilde{\Omega}) \leq K^I_{q,s}(\varphi;\Omega) \operatorname{cap}_q^{\frac{1}{q}}(E;\Omega)$$

holds. □

Using this theorem we obtain that topological mappings of Ball's classes are absolutely continuous with respect to capacity, which is considered as an outer measure associated with the Sobolev spaces. This result refines results of [12, 26].

Theorem 4.4. *Let a homeomorphism $\varphi \in A^+_{q,q'}(\Omega;\widetilde{\Omega})$, $1/q + 1/q' = 1$, then the inequality*

$$\operatorname{cap}_1(\varphi(E);\widetilde{\Omega}) \leq \left(\int\limits_\Omega |\operatorname{adj} D\varphi(x)|^{\frac{q}{q-1}} dx \right)^{1-\frac{1}{q}} \operatorname{cap}_q^{\frac{1}{q}}(E;\Omega)$$

holds for any Borel set $E \subset \Omega$.

Proof. By Lemma 3.1 we have, that the inner integral distortion

$$K^I_{q,1}(\varphi;\Omega) = \left(\int\limits_{\Omega} \left(\frac{|J(x,\varphi)|}{l(D\varphi(x))}\right)^{\frac{q}{q-1}}\,dx\right)^{\frac{q-1}{q}} = \left(\int\limits_{\Omega} |\operatorname{adj} D\varphi(x)|^{\frac{q}{q-1}}\,dx\right)^{\frac{q-1}{q}} < \infty.$$

Hence by Theorem 4.3 we obtain the required inequality. □

Therefore we obtain that mappings of Ball's classes are absolutely continuous with respect to capacity.

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Manuscript received July 24 2022
revised October 15 2022

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