Pure and Applied Functional Analysis
Volume 9, Number 1, 2024, 69–91

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

EXTENDED POWER WEIGHTED RELLICH-TYPE INEQUALITIES WITH LOGARITHMIC REFINEMENTS

FRITZ GESZTESY, ISAAC MICHAEL, AND MICHAEL M. H. PANG

Dedicated, with great admiration, to Vladimir Maz'ya on the happy occasion of his 85th birthday

Abstract. The principal result in this paper is a proof of the logarithmically refined power weighted Rellich-type inequality in the form,

$$\begin{aligned}\int_{B_n(0;R)} d^n x\, |x|^{-\alpha} |(-\Delta f)(x)|^2 &\geqslant \beta_{n,\alpha} \int_{B_n(0;R)} d^n x\, |x|^{-\alpha-4} |f(x)|^2 \\ &+ \big\{\big[(n+\alpha)^2 + (n-\alpha-4)^2\big]/16\big\} \\ &\times \int_{B_n(0;R)} d^n x\, |x|^{-\alpha-4} \Bigg(\sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(\gamma/|x|)]^{-2}\Bigg) |f(x)|^2, \\ R \in (0,\infty),\ \alpha \in \mathbb{R},\ N \in \mathbb{N},\ &\gamma \in [e_N R, \infty),\ f \in C_0^\infty(B_n(0;R)\backslash\{0\}),\end{aligned}$$

where the constant

$$\beta_{n,\alpha} = \min_{j \in \mathbb{N}_0} \Big\{ \big| 4^{-1} \big[(n-2)^2 - (\alpha+2)^2\big] + j(j+n-2) \big|^2 \Big\}$$

is known to be optimal. Here $B_n(0;R)$ denotes the ball in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geqslant 2$, centered at the origin of radius $R \in (0,\infty)$, and the iterated logarithms $\ln_k(\,\cdot\,)$, $k \in \mathbb{N}$, and iterated exponentials e_j, $j \in \mathbb{N}_0$, are introduced below.

1. Introduction

To motivate our principal results on Rellich-type inequalities we start by briefly recalling the case of power weighted Hardy-type inequalities with repeated logarithmic refinements:

Theorem 1.1. *Let $R \in (0,\infty)$, $\alpha \in \mathbb{R}$, $N \in \mathbb{N}$, and $\gamma \in [e_N R, \infty)$. Then, for all $f \in C_0^\infty(B_n(0;R)\backslash\{0\})$, $n \in \mathbb{N}, n \geqslant 2$, one has*

$$\begin{aligned}\int_{B_n(0;R)} d^n x\, |x|^{-\alpha} |(\nabla f)(x)|^2 &\geqslant 4^{-1}(n-2-\alpha)^2 \int_{B_n(0;R)} d^n x\, |x|^{-\alpha-2} |f(x)|^2 \\ &+ 4^{-1} \int_{B_n(0;R)} d^n x\, |x|^{-\alpha-2} \Bigg(\sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(\gamma/|x|)]^{-2}\Bigg) |f(x)|^2,\end{aligned} \tag{1.1}$$

with optimal constants on the right-hand side of (1.1).

2020 *Mathematics Subject Classification.* Primary: 26D10, 34A40, 35A23; Secondary: 34L10.
Key words and phrases. Rellich inequalities, logarithmic refinements.

Here $B_n(0;R)$ denotes the ball in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geqslant 2$, centered at the origin of radius $R \in (0,\infty)$, the iterated logarithms $\ln_k(\,\cdot\,)$, $k \in \mathbb{N}$, are given by

$$\ln_1(\,\cdot\,) = \ln(\,\cdot\,), \quad \ln_{k+1}(\,\cdot\,) = \ln\big(\ln_k(\,\cdot\,)\big), \quad k \in \mathbb{N}, \tag{1.2}$$

and the iterated exponentials e_j, $j \in \mathbb{N}_0$, are introduced via

$$e_0 = 0, \quad e_{j+1} = e^{e_j}, \quad j \in \mathbb{N}_0. \tag{1.3}$$

Similarly, one has the following result.

Corollary 1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded, $n \in \mathbb{N}, n \geqslant 2$, and assume that $x_0 \in \Omega$, $\alpha \in (-\infty, n-2)$, $N \in \mathbb{N}$, and $\Gamma \in [e_N \operatorname{diam}(\Omega), \infty)$. Then, for all $f \in C_0^\infty(\Omega)$, one infers that*

$$\begin{aligned} \int_\Omega d^n x\, |x-x_0|^{-\alpha} |(\nabla f)(x)|^2 &\geqslant 4^{-1}(n-2-\alpha)^2 \int_\Omega d^n x\, |x-x_0|^{-\alpha-2} |f(x)|^2 \\ &+ 4^{-1} \int_\Omega d^n x\, |x-x_0|^{-\alpha-2} \Bigg(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\gamma/|x-x_0|)]^{-2}\Bigg) |f(x)|^2, \end{aligned} \tag{1.4}$$

with optimal constants on the right-hand side of (1.4).

Regarding (1.1), we note that the special unweighted case $\alpha = 0$ appeared in [32]. For additional references discussing logarithmic refinements, see, for instance, [2], [3], [4], [6], [9], [10], [18], [19], [23], [26], [27], [28], [31], [34], [36], [37], [38], [39, Chs. 2, 4], [40], [43], [45], [46], [50], [52], [53], [64], [65], [67], [68], [69], and the extensive literature cited therein. Some of these references treat the power weighted case $\alpha \neq 0$, and some handle the inequalities in the L^p-context.

Given the power-weighted Hardy-type case with (repeated) logarithmic refinements as a warmup, our principal new result on Rellich-type inequalities then reads as follows:

Theorem 1.3. *Let $R \in (0,\infty)$, $\alpha \in \mathbb{R}$, $N \in \mathbb{N}$ and $\gamma \in [e_N R, \infty)$. Then, for all $f \in C_0^\infty(B_n(0;R)\backslash\{0\})$, $n \in \mathbb{N}, n \geqslant 2$, one has*

$$\begin{aligned} \int_{B_n(0;R)} d^n x\, |x|^{-\alpha} |(-\Delta f)(x)|^2 &\geqslant \beta_{n,\alpha} \int_{B_n(0;R)} d^n x\, |x|^{-\alpha-4} |f(x)|^2 \\ &+ \big\{\big[(n+\alpha)^2 + (n-\alpha-4)^2\big]/16\big\} \\ &\times \int_{B_n(0;R)} d^n x\, |x|^{-\alpha-4} \Bigg(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\gamma/|x|)]^{-2}\Bigg) |f(x)|^2, \end{aligned} \tag{1.5}$$

where

$$\beta_{n,\alpha} = \min_{j \in \mathbb{N}_0} \Big\{ \big|4^{-1}\big[(n-2)^2 - (\alpha+2)^2\big] + j(j+n-2)\big|^2 \Big\}. \tag{1.6}$$

Theorem 1.3 implies the following result:

Corollary 1.4. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded, $n \in \mathbb{N}, n \geqslant 2$, and assume that $x_0 \in \Omega$, $\alpha \in (-\infty, n-4)$, $N \in \mathbb{N}$, and $\Gamma \in [e_N \operatorname{diam}(\Omega), \infty)$. Then, for all $f \in C_0^\infty(\Omega)$, one infers that*

$$\int_\Omega d^n x\, |x-x_0|^{-\alpha} |(-\Delta f)(x)|^2 \tag{1.7}$$

$$\geqslant \beta_{n,\alpha} \int_{\Omega} d^n x\, |x - x_0|^{-\alpha-4} |f(x)|^2 \\ + \big\{ \big[(n+\alpha)^2 + (n-\alpha-4)^2\big]/16 \big\} \\ \times \int_{\Omega} d^n x\, |x - x_0|^{-\alpha-4} \left(\sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(\Gamma/|x-x_0|)]^{-2} \right) |f(x)|^2.$$

Caldiroli and Musina [22, Theorem 3.1 and p. 151], considering the case without logarithmic refinements, proved that the constant $\beta_{n,\alpha}$ in (1.6) and (1.7) is optimal (see also [38], [39, Theorems 6.3.6, 6.5.2] in connection with inequality (1.7)).

For a variety of results in the context of Rellich-type inequalities with (repeated) logarithmic refinements, we refer, for instance, to [5], [7], [16], [17], [20], [38], [39, Chs. 6, 7], [53], [69], and the references cited therein. Once more, some of these references treat the power weighted case $\alpha \neq 0$, and some handle the inequalities in the L^p-context.

We note that the vast majority of these references, in both the Hardy and the Rellich context, do not discuss repeated logarithmic refinements and power weights simultaneously. The only notable exceptions known to us (in the multi-dimensional context) are Michael [50] in the Hardy context, and Barbatis [17], Ghoussoub and Moradifam [38], [39, Chs. 4, 6, 7] (see also Moradifam[53]), and Tertikas and Zographopoulos [69] (but we note that [17] and [69] employ a slightly different set of logarithmic terms), who all also discuss an infinite sequence of higher-order Hardy and Rellich-type inequalities. We will add some additional comments in the Rellich context in Remark 3.7.

Due to the enormity of the literature on (power weighted) Hardy and Rellich type inequalities without logarithmic refinements, we cannot possibly recall the pertinent literature here and just refer to a few selected references and some monographs in this context, such as, [8], [12], [13], [14], [15, Chs. 1, 3, 6], [22], [25], [29], [30], [33], [42], [44, Ch. 6], [47, Sects. 1.3, 2.7], [48], [49], [51], [56], [57], [58, Ch. 2, Sect. 21], [59], [60], [61, Sect. II.7], [62, Sects. 2.1.5, 2.3.1, 3.1, 3.3], [63], [66], [70], and the references cited therein.

Section 2 collects preliminaries on L^2-realizations of the Laplace and Laplace–Beltrami differential expressions and lists some elementary facts to be used in our principal Section 3 on power weighted Rellich-type inequalities with repeated logarithmic refinements. Appendix A introduces basic formulas regarding spherical coordinates and the Laplace–Beltrami differential expression. The proof of Lemma 2.1 is presented in Appendix B.

We conclude this section by pointing out some of the notation employed in this paper: We abbreviate $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, denote by $B_n(x_0, r_0)$ the open ball centered at $x_0 \in \mathbb{R}^n$, of radius $r_0 \in (0,\infty)$, and denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geqslant 2$. Additional notation is presented in Appendix A.

2. Preliminaries

Let $\mathbb{S}^{n-1}$ denote the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n, n \in \mathbb{N}, n \geqslant 2$, with $d^{n-1}\omega$ the usual volume measure on $\mathbb{S}^{n-1}$. Given the Laplace–Beltrami differential expression $-\Delta_{\mathbb{S}^{n-1}}$ in Appendix A, we abbreviate the associated nonnegative

Laplace–Beltrami operator[1] in $L^2(\mathbb{S}^{n-1}; d^{n-1}\omega)$, that is, the Friedrichs extension of $-\Delta_{\mathbb{S}^{n-1}}|_{C_0^\infty(\mathbb{S}^{n-1})}$, by the symbol $-\Delta_{\mathbb{S}^{n-1},F}$. We refer to Appendix A for more details and notational conventions in connection with spherical coordinates.

Let

$$\lambda_j = j(j+n-2), \quad j \in \mathbb{N}_0, \tag{2.1}$$

be the eigenvalues of $-\Delta_{\mathbb{S}^{n-1},F}$ of multiplicity

$$m(\lambda_j) = \frac{2j+n-2}{j+n-2}\binom{j+n-2}{n-2}, \quad j \in \mathbb{N}_0, \tag{2.2}$$

with corresponding eigenfunctions $\varphi_{j,\ell}$, $\ell \in \{1,\dots,m(\lambda_j)\}$, $j \in \mathbb{N}_0$. In particular, $\{\varphi_{j,\ell}\}_{\ell\in\{1,\dots,m(\lambda_j)\}, j\in\mathbb{N}_0}$ represents an orthonormal basis of $L^2(\mathbb{S}^{n-1}; d^{n-1}\omega)$. Let

$$\begin{split} F_{f,j,\ell}(r) = (\varphi_{j,\ell}, f(r,\cdot))_{L^2(\mathbb{S}^{n-1}, d^{n-1}\omega)} = \int_{\mathbb{S}^{n-1}} d^{n-1}\omega(\theta)\, \overline{\varphi_{j,\ell}(\theta)} f(r,\theta), \\ f \in C_0^\infty(\mathbb{R}^n\backslash\{0\}), \quad r>0, \quad \ell \in \{1,\dots,m(\lambda_j)\},\ j \in \mathbb{N}_0. \end{split} \tag{2.3}$$

We start with the following result:

Lemma 2.1. *Let $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, then the following items (i)–(iv) hold:*
(i) $F_{f,j,\ell} \in C_0^\infty((0,\infty))$ for all $\ell \in \{1,\dots,m(\lambda_j)\}$, $j \in \mathbb{N}_0$.
(ii) Let $T_{\mathbb{S}^{n-1},F} \geqslant 0$ be the Friedrichs extension in $L^2(\mathbb{R}^n; d^n x)$ of the symmetric operator defined on $C_0^\infty(\mathbb{R}^n\backslash\{0\})$ by

$$\begin{split} (T_{\mathbb{S}^{n-1},F}\, g)(x) = \big(-\Delta_{\mathbb{S}^{n-1}} g(r,\cdot)\big)(\theta), \quad g \in C_0^\infty(\mathbb{R}^n\backslash\{0\}), \\ x = r\omega \in \mathbb{R}^n\backslash\{0\},\ \omega \in \mathbb{S}^{n-1}. \end{split} \tag{2.4}$$

Then, for any $k \in \mathbb{N}_0$,

$$T^k_{\mathbb{S}^{n-1},F} f = \sum_{j\in\mathbb{N}_0} \sum_{\ell=1}^{m(\lambda_j)} \lambda_j^k F_{f,j,\ell}\varphi_{j,\ell} \ \text{ in } \ L^2(\mathbb{R}^n; d^n x). \tag{2.5}$$

(iii) Let $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Let $p_1,\dots,p_m \in C^\infty((0,\infty))$. Then, for all $\ell \in \{1,\dots,m(\lambda_j)\}$, $j \in \mathbb{N}_0$, one has

$$F_{p_1 \frac{\partial}{\partial r} p_2 \frac{\partial}{\partial r} \cdots p_{m-1} \frac{\partial}{\partial r} p_m T^k_{\mathbb{S}^{n-1},F} f, j, \ell} = \left(p_1 \frac{\partial}{\partial r} p_2 \frac{\partial}{\partial r} \cdots p_{m-1} \frac{\partial}{\partial r} p_m F_{f,j,\ell}(r)\right)\lambda_j^k. \tag{2.6}$$

(iv) One has, in the sense of $L^2(\mathbb{R}^n; d^n x)$,

$$\begin{split} (-\Delta f)(x) \\ = \sum_{j\in\mathbb{N}_0} \sum_{\ell=1}^{m(\lambda_j)} \bigg[-r^{1-n}\frac{\partial}{\partial r}\bigg(r^{n-1}\frac{\partial}{\partial r}\big(F_{f,j,\ell}(r)\big)\bigg) + \lambda_j r^{-2} F_{f,j,\ell}(r)\bigg]\varphi_{j,\ell}(\theta), \\ x = (r,\theta) \in \mathbb{R}^n\backslash\{0\}. \end{split} \tag{2.7}$$

[1]We will call $-\Delta$ the Laplacian to guarantee nonnegativity of the underlying $L^2(\mathbb{R}^n; d^n x)$-realization and analogously for the $L^2(\mathbb{S}^{n-1}; d^{n-1}\omega)$-realization of the Laplace–Beltrami operator $-\Delta_{\mathbb{S}^{n-1}}$.

The proof of Lemma 2.1 is deferred to Appendix B.

We continue with some elementary facts and for the remainder of this paper let $R \in (0, \infty)$ be fixed.

Lemma 2.2. *For all $\alpha \in \mathbb{R}$, $f \in C_0^\infty((0,R))$, f real-valued, one verifies that items (i)–(iii) hold:*

(i)

$$\begin{aligned} &\int_0^R dr\, r^{-\alpha+n-3} f''(r) f(r) \\ &\quad = \frac{1}{2}(-\alpha+n-3)(-\alpha+n-4) \int_0^R dr\, r^{-\alpha+n-5} f(r)^2 \\ &\qquad - \int_0^R dr\, r^{-\alpha+n-3} [f'(r)]^2. \end{aligned} \tag{2.8}$$

(ii)

$$\int_0^R dr\, r^{-\alpha+n-2} f''(r) f'(r) = -\frac{1}{2}(-\alpha+n-2) \int_0^R dr\, r^{-\alpha+n-3} [f'(r)]^2. \tag{2.9}$$

(iii)

$$\int_0^R dr\, r^{-\alpha+n-4} f'(r) f(r) = -\frac{1}{2}(-\alpha+n-4) \int_0^R dr\, r^{-\alpha+n-5} f(r)^2. \tag{2.10}$$

Proof. (*i*) One verifies that

$$\begin{aligned} &\int_0^R dr\, r^{-\alpha+n-3} f''(r) f(r) = -\int_0^R dr\, f'(r) \big(r^{-\alpha+n-3} f(r)\big)' \\ &\quad = -\int_0^R dr\, f'(r) \Big\{ (-\alpha+n-3) r^{-\alpha+n-4} f(r) + r^{-\alpha+n-3} f'(r) \Big\} \\ &\quad = -\int_0^R dr \left\{ \frac{1}{2}(-\alpha+n-3) r^{-\alpha+n-4} \big[f(r)^2\big]' + r^{-\alpha+n-3} [f'(r)]^2 \right\} \\ &\quad = \frac{1}{2}(-\alpha+n-3)(-\alpha+n-4) \int_0^R dr\, r^{-\alpha+n-5} f(r)^2 \\ &\qquad - \int_0^R dr\, r^{-\alpha+n-3} [f'(r)]^2. \end{aligned} \tag{2.11}$$

(*ii*) One computes,

$$\begin{aligned} &\int_0^R dr\, r^{-\alpha+n-2} f''(r) f'(r) = \frac{1}{2} \int_0^R dr\, r^{-\alpha+n-2} \frac{d}{dr} \Big[\big(f'(r)\big)^2 \Big] \\ &\quad = -\frac{1}{2}(-\alpha+n-2) \int_0^R dr\, r^{-\alpha+n-3} [f'(r)]^2. \end{aligned} \tag{2.12}$$

(*iii*) One obtains,

$$\begin{aligned}\int_0^R dr\, r^{-\alpha+n-4} f'(r)f(r) &= \frac{1}{2}\int_0^R dr\, r^{-\alpha+n-4}\frac{d}{dr}\big(f(r)^2\big) \\ &= -\frac{1}{2}(-\alpha+n-4)\int_0^R dr\, r^{-\alpha+n-5} f(r)^2. \end{aligned} \tag{2.13}$$

□

Lemma 2.3. *For all $\alpha \in \mathbb{R}$, $f \in C_0^\infty((0,R))$, f real-valued, and $\lambda \in [0,\infty)$, one infers that*

$$\begin{aligned}&\int_0^R r^{n-1} dr\, r^{-\alpha}\bigg[-r^{1-n}\frac{d}{dr}\bigg(r^{n-1}\frac{d}{dr}\big(f(r)\big)\bigg) + \lambda r^{-2} f(r)\bigg]^2 \\ &\quad = \int_0^R dr\, r^{-\alpha+n-1}[f''(r)]^2 \\ &\qquad + [2\lambda + (n-1)(\alpha+1)]\int_0^R dr\, r^{-\alpha+n-3}[f'(r)]^2 \\ &\qquad + \lambda[\lambda + (-\alpha+n-4)(\alpha+2)]\int_0^R dr\, r^{-\alpha+n-5} f(r)^2. \end{aligned} \tag{2.14}$$

Proof. One observes that

$$\begin{aligned}&\int_0^R r^{n-1} dr\, r^{-\alpha}\bigg[-r^{1-n}\frac{d}{dr}\bigg(r^{n-1}\frac{d}{dr}\big(f(r)\big)\bigg) + \lambda r^{-2} f(r)\bigg]^2 \\ &\quad = \int_0^R r^{n-1} dr\, r^{-\alpha}\big[-f''(r) - (n-1)r^{-1} f'(r) + \lambda r^{-2} f(r)\big]^2 \\ &\quad = \int_0^R dr\, r^{-\alpha+n-1}\big\{[f''(r)]^2 + 2(n-1)r^{-1} f''(r)f'(r) - 2\lambda r^{-2} f''(r)f(r) \\ &\qquad + (n-1)^2 r^{-2}[f'(r)]^2 - 2\lambda(n-1)r^{-3} f'(r)f(r) + \lambda^2 r^{-4} f(r)^2\big\} \\ &\quad = \int_0^R dr\, r^{-\alpha+n-1}[f''(r)]^2 + 2(n-1)\int_0^R dr\, r^{-\alpha+n-2} f''(r)f'(r) \\ &\qquad - 2\lambda\int_0^R dr\, r^{-\alpha+n-3} f''(r)f(r) + (n-1)^2\int_0^R dr\, r^{-\alpha+n-3}[f'(r)]^2 \\ &\qquad - 2\lambda(n-1)\int_0^R dr\, r^{-\alpha+n-4} f'(r)f(r) + \lambda^2\int_0^R dr\, r^{-\alpha+n-5} f(r)^2. \end{aligned} \tag{2.15}$$

Applying Lemma 2.2 (*i*)–(*iii*) to the integrals in (2.15), one obtains

$$\begin{aligned}&\int_0^R r^{n-1} dr\, r^{-\alpha}\bigg[-r^{1-n}\frac{d}{dr}\bigg(r^{n-1}\frac{d}{dr}\big(f(r)\big)\bigg) + \lambda r^{-2} f(r)\bigg]^2 \\ &\quad = \int_0^R dr\, r^{-\alpha+n-1}[f''(r)]^2 - (n-1)(-\alpha+n-2)\int_0^R dr\, r^{-\alpha+n-3}[f'(r)]^2 \\ &\qquad - \lambda(-\alpha+n-3)(-\alpha+n-4)\int_0^R dr\, r^{-\alpha+n-5} f(r)^2 \end{aligned} \tag{2.16}$$

$$
\begin{aligned}
&+ 2\lambda \int_0^R dr\, r^{-\alpha+n-3}[f'(r)]^2 + (n-1)^2 \int_0^R dr\, r^{-\alpha+n-3}[f'(r)]^2 \\
&+ \lambda(n-1)(-\alpha+n-4) \int_0^R dr\, r^{-\alpha+n-5} f(r)^2 \\
&+ \lambda^2 \int_0^R dr\, r^{-\alpha+n-5} f(r)^2 \\
= &\int_0^R dr\, r^{-\alpha+n-1}[f''(r)]^2 \\
&+ \big[2\lambda - (n-1)(-\alpha+n-2) + (n-1)^2\big] \int_0^R dr\, r^{-\alpha+n-3}[f'(r)]^2 \\
&+ \big[\lambda^2 + \lambda(n-1)(-\alpha+n-4) - \lambda(-\alpha+n-3)(-\alpha+n-4)\big] \\
&\qquad \times \int_0^R dr\, r^{-\alpha+n-5} f(r)^2 \\
= &\int_0^R dr\, r^{-\alpha+n-1}[f''(r)]^2 \\
&+ [2\lambda + (n-1)(\alpha+1)] \int_0^R dr\, r^{-\alpha+n-3}[f'(r)]^2 \\
&+ \big[\lambda^2 + \lambda(-\alpha+n-4)(\alpha+2)\big] \int_0^R dr\, r^{-\alpha+n-5} f(r)^2.
\end{aligned}
$$

□

3. The Principal Result

Definition 3.1. *For $k \in \mathbb{N}$ we introduce the iterated logarithms $\ln_k(\,\cdot\,)$ by*

$$\ln_1(\,\cdot\,) = \ln(\,\cdot\,), \quad \ln_{k+1}(\,\cdot\,) = \ln\big(\ln_k(\,\cdot\,)\big), \quad k \in \mathbb{N}. \tag{3.1}$$

Similarly, for $j \in \mathbb{N}_0$, we define the iterated exponentials e_j by

$$e_0 = 0, \quad e_{j+1} = e^{e_j}, \quad j \in \mathbb{N}_0. \tag{3.2}$$

Lemma 3.2 ([35], Theorem 3.1 *(iii)*)**.** *Let $\nu \in \mathbb{R}$, $N \in \mathbb{N}$. If $\gamma \in [e_N R, \infty)$, then, for all $f \in C_0^\infty((0,R))$, one obtains the following items (i) and (ii):*
(i)

$$
\begin{aligned}
\int_0^R dr\, r^{\nu} \big|f''(r)\big|^2 &\geqslant A(2,\nu) \int_0^R dr\, r^{\nu-4} |f(r)|^2 \\
&\quad + B(2,\nu) \int_0^R dr\, r^{\nu-4} \left(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\gamma/r)]^{-2} \right) |f(r)|^2,
\end{aligned} \tag{3.3}
$$

where

$$A(2,\nu) = (\nu-1)^2(\nu-3)^2/16 = (\nu^2-4\nu+3)^2/16, \tag{3.4}$$

$$B(2,\nu) = \big[(\nu-1)^2 + (\nu-3)^2\big]/16 = \big(\nu^2-4\nu+5\big)/8. \tag{3.5}$$

(*ii*)

$$\int_0^R dr\, r^{\nu}\big|f'(r)\big|^2 \geqslant \frac{1}{4}(\nu-1)^2\int_0^R dr\, r^{\nu-2}|f(r)|^2 + \frac{1}{4}\int_0^R dr\, r^{\nu-2}\Bigg(\sum_{k=1}^{N}\prod_{p=1}^{k}[\ln_p(\gamma/r)]^{-2}\Bigg)|f(r)|^2. \tag{3.6}$$

Corollary 3.3. *Let $\alpha\in\mathbb{R}$, $N\in\mathbb{N}$. If $\gamma\in[e_N R,\infty)$, then, for all $f\in C_0^\infty((0,R))$, one infers the following items (i) and (ii):*

(*i*)

$$\begin{aligned}&\int_0^R dr\, r^{-\alpha+n-1}\big|f''(r)\big|^2\\ &\quad\geqslant \big[(2+\alpha-n)^2(4+\alpha-n)^2/16\big]\int_0^R dr\, r^{-\alpha+n-5}|f(r)|^2\\ &\qquad+\big\{\big[(2+\alpha-n)^2+(4+\alpha-n)^2\big]/16\big\}\\ &\qquad\times\int_0^R dr\, r^{-\alpha+n-5}\Bigg(\sum_{k=1}^{N}\prod_{p=1}^{k}[\ln_p(\gamma/r)]^{-2}\Bigg)|f(r)|^2.\end{aligned} \tag{3.7}$$

(*ii*)

$$\begin{aligned}\int_0^R dr\, r^{-\alpha+n-3}\big|f'(r)\big|^2 dr &\geqslant \frac{1}{4}(4+\alpha-n)^2\int_0^R dr\, r^{-\alpha+n-5}|f(r)|^2\\ &\quad+\frac{1}{4}\int_0^R dr\, r^{-\alpha+n-5}\Bigg(\sum_{k=1}^{N}\prod_{p=1}^{k}[\ln_p(\gamma/r)]^{-2}\Bigg)|f(r)|^2.\end{aligned} \tag{3.8}$$

Proof. Corollary 3.3 follows from Lemma 3.2 by replacing ν by $-\alpha+n-1$ and by $-\alpha+n-3$. ☐

Lemma 3.4. *Let $\alpha\in\mathbb{R}$, $N\in\mathbb{N}$, and $\lambda\in[0,\infty)$. If $\gamma\in[e_N R,\infty)$, then, for all $f\in C_0^\infty((0,R))$, one concludes that*

$$\begin{aligned}&\int_0^R r^{n-1}dr\, r^{-\alpha}\bigg|-r^{1-n}\frac{d}{dr}\bigg(r^{n-1}\frac{d}{dr}\big(f(r)\big)\bigg)+\lambda r^{-2}f(r)\bigg|^2\\ &\quad\geqslant(\gamma_{n,\alpha}+\lambda)^2\int_0^R dr\, r^{-\alpha+n-5}|f(r)|^2\\ &\qquad+\Big\{\big\{\big[(n+\alpha)^2+(n-\alpha-4)^2\big]/16\big\}+(\lambda/2)\Big\}\\ &\qquad\times\int_0^R dr\, r^{-\alpha+n-5}\Bigg(\sum_{k=1}^{N}\prod_{p=1}^{k}[\ln_p(\gamma/r)]^{-2}\Bigg)|f(r)|^2,\end{aligned} \tag{3.9}$$

where

$$\gamma_{n,\alpha}=\big[(n-2)^2-(\alpha+2)^2\big]/4. \tag{3.10}$$

Proof. From Lemma 2.3 and Corollary 3.3 one infers that

$$\begin{aligned}
(3.11)\quad & \int_0^R r^{n-1} dr\, r^{-\alpha} \left| -r^{1-n}\frac{d}{dr}\Big(r^{n-1}\frac{d}{dr}\big(f(r)\big)\Big) + \lambda r^{-2} f(r)\right|^2 \\
& \geqslant \big\{\big[(2+\alpha-n)^2(4+\alpha-n)^2\big]/16\big\} \int_0^R dr\, r^{-\alpha+n-5}|f(r)|^2 \\
& + \big\{\big[(2+\alpha-n)^2+(4+\alpha-n)^2\big]/16\big\} \\
& \quad \times \int_0^R dr\, r^{-\alpha+n-5}\left(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\gamma/r)]^{-2}\right)|f(r)|^2 \\
& + [2\lambda+(n-1)(\alpha+1)]\bigg\{\frac{1}{4}(4+\alpha-n)^2 \int_0^R dr\, r^{-\alpha+n-5}|f(r)|^2 \\
& + \frac{1}{4}\int_0^R dr\, r^{-\alpha+n-5}\left(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\gamma/r)]^{-2}\right)|f(r)|^2\bigg\} \\
& + \big[\lambda^2+\lambda(-\alpha+n-4)(\alpha+2)\big]\int_0^R dr\, r^{-\alpha+n-5}|f(r)|^2 \\
& = \Big\{16^{-1}(2+\alpha-n)^2(4+\alpha-n)^2 \\
& \quad + 4^{-1}[2\lambda+(n-1)(\alpha+1)](4+\alpha-n)^2 \\
& \quad + \lambda^2+\lambda(-\alpha+n-4)(\alpha+2)\Big\}\int_0^R dr\, r^{-\alpha+n-5}|f(r)|^2 \\
& + \Big\{16^{-1}\big[(2+\alpha-n)^2+(4+\alpha-n)^2\big]+4^{-1}[2\lambda+(n-1)(\alpha+1)]\Big\} \\
& \quad \times \int_0^R dr\, r^{-\alpha+n-5}\left(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\gamma/r)]^{-2}\right)|f(r)|^2 \\
& = \Big\{16^{-1}\big[16\alpha^2+32n\alpha+16n^2+8\alpha^3+8n\alpha^2-8n^2\alpha-8n^3+\alpha^4 \\
& \quad -2n^2\alpha^2+n^4\big]+\lambda\big[2^{-1}(4+\alpha-n)^2+(-\alpha+n-4)(\alpha+2)\big]+\lambda^2\Big\} \\
& \quad \times \int_0^R dr\, r^{-\alpha+n-5}|f(r)|^2 + \big(1+\tfrac{1}{2}\alpha-\tfrac{1}{2}n+\tfrac{1}{8}\alpha^2+\tfrac{1}{8}n^2+\tfrac{1}{2}\lambda\big) \\
& \quad \times \int_0^R dr\, r^{-\alpha+n-5}\left(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\gamma/r)]^{-2}\right)|f(r)|^2 \\
& = \big[\gamma_{n,\alpha}^2+2\lambda\gamma_{n,\alpha}+\lambda^2\big]\int_0^R dr\, r^{-\alpha+n-5}|f(r)|^2 \\
& + \big\{16^{-1}\big[(n+\alpha)^2+(n-\alpha-4)^2\big]+(\lambda/2)\big\} \\
& \quad \times \int_0^R dr\, r^{-\alpha+n-5}\left(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\gamma/r)]^{-2}\right)|f(r)|^2.
\end{aligned}$$

□

Given these preparations, we are in a position to state our principal new result:

Theorem 3.5. *Let $R \in (0,\infty)$, $\alpha \in \mathbb{R}$, $N \in \mathbb{N}$ and $\gamma \in [e_N R, \infty)$. Then, for all $f \in C_0^\infty(B_n(0;R)\backslash\{0\})$, $n \in \mathbb{N}, n \geqslant 2$, one has*

$$\begin{aligned}
\int_{B_n(0;R)} d^n x\, |x|^{-\alpha} |(-\Delta f)(x)|^2 &\geqslant \beta_{n,\alpha} \int_{B_n(0;R)} d^n x\, |x|^{-\alpha-4} |f(x)|^2 \\
&\quad + \big\{\big[(n+\alpha)^2 + (n-\alpha-4)^2\big]/16\big\} \\
&\quad \times \int_{B_n(0;R)} d^n x\, |x|^{-\alpha-4} \Bigg(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\gamma/|x|)]^{-2}\Bigg) |f(x)|^2,
\end{aligned} \tag{3.12}$$

where

$$\beta_{n,\alpha} = \min_{j\in\mathbb{N}_0} \Big\{ \big| 4^{-1}\big[(n-2)^2 - (\alpha+2)^2\big] + j(j+n-2)\big|^2 \Big\}, \tag{3.13}$$

Proof. Without loss of generality, we may assume that f is real-valued. Since $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, Lemma 2.1 (iv) implies that

$$\begin{aligned}
|x|^{-\alpha/2}(-\Delta f)(x) = \sum_{j\in\mathbb{N}_0} \sum_{\ell=1}^{m(\lambda_j)} |x|^{-\alpha/2} \bigg[&- r^{1-n} \frac{\partial}{\partial r}\bigg(r^{n-1} \frac{\partial}{\partial r}\big(F_{f,j,\ell}(r)\big)\bigg) \\
&+ \lambda_j r^{-2} F_{f,j,\ell}(r) \bigg] \varphi_{j,\ell}(\theta)
\end{aligned} \tag{3.14}$$

in $L^2(\mathbb{R}^n; d^n x)$. Thus, applying Lemma 3.4, one infers that

$$\begin{aligned}
\int_{B_n(0;R)} d^n x\, |x|^{-\alpha} |(-\Delta f)(x)|^2 &= \int_0^R r^{n-1} dr \int_{\mathbb{S}^{n-1}} d^{n-1}\omega(\theta) \sum_{j\in\mathbb{N}_0} \sum_{\ell=1}^{m(\lambda_j)} r^{-\alpha} \\
&\quad \times \bigg| - r^{1-n} \frac{\partial}{\partial r}\bigg(r^{n-1} \frac{\partial}{\partial r}(F_{f,j,\ell}(r))\bigg) + \lambda_j r^{-2} F_{f,j,\ell}(r)\bigg|^2 |\varphi_{j,\ell}(\theta)|^2 \\
&= \sum_{j\in\mathbb{N}_0} \sum_{\ell=1}^{m(\lambda_j)} \int_0^R r^{n-1} dr\, r^{-\alpha} \bigg| - r^{1-n} \frac{\partial}{\partial r}\bigg(r^{n-1} \frac{\partial}{\partial r}(F_{f,j,\ell}(r))\bigg) \\
&\qquad + \lambda_j r^{-2} F_{f,j,\ell}(r)\bigg|^2 \\
&\geqslant \sum_{j\in\mathbb{N}_0} \sum_{\ell=1}^{m(\lambda_j)} (\gamma_{n,\alpha} + \lambda_j)^2 \int_0^R r^{n-1} dr\, r^{-\alpha-4} |F_{f,j,\ell}(r)|^2 \\
&\quad + \sum_{j\in\mathbb{N}_0} \sum_{\ell=1}^{m(\lambda_j)} \Big\{ \big\{\big[(n+\alpha)^2 + (n-\alpha-4)^2\big]/16\big\} + (\lambda_j/2) \Big\} \\
&\qquad \times \int_0^R r^{n-1} dr\, r^{-\alpha-4} \Bigg(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\gamma/r)]^{-2} \Bigg) |F_{f,j,\ell}(r)|^2
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \min\big\{(\gamma_{n,\alpha}+\lambda_j)^2 \,\big|\, j\in\mathbb{N}_0\big\} \sum_{j\in\mathbb{N}_0}\sum_{\ell=1}^{m(\lambda_j)} \int_0^R r^{n-1}dr\, r^{-\alpha-4}|F_{f,j,\ell}(r)|^2 \\
&\quad + \min\Big\{\big\{\big[(n+\alpha)^2+(n-\alpha-4)^2\big]/16\big\}+(\lambda_j/2)\,\Big|\, j\in\mathbb{N}_0\Big\} \\
&\quad \times \sum_{j\in\mathbb{N}_0}\sum_{\ell=1}^{m(\lambda_j)} \int_0^R r^{n-1}dr\, r^{-\alpha-4}\Bigg(\sum_{k=1}^N\prod_{p=1}^k[\ln_p(\gamma/r)]^{-2}\Bigg)|F_{f,j,\ell}(r)|^2 \\
&= \beta_{n,\alpha}\int_0^R r^{n-1}dr\int_{\mathbb{S}^{n-1}} d^{n-1}\omega(\theta)\sum_{j\in\mathbb{N}_0}\sum_{\ell=1}^{m(\lambda_j)} r^{-\alpha-4}|F_{f,j,\ell}(r)|^2|\varphi_{j,\ell}(\theta)|^2 \\
&\quad + \big\{\big[(n+\alpha)^2+(n-\alpha-4)^2\big]/16\big\}\int_0^R r^{n-1}dr\int_{\mathbb{S}^{n-1}} d^{n-1}\omega(\theta) \\
&\quad \times \sum_{j\in\mathbb{N}_0}\sum_{\ell=1}^{m(\lambda_j)} r^{-\alpha-4}\Bigg(\sum_{k=1}^N\prod_{p=1}^k[\ln_p(\gamma/r)]^{-2}\Bigg)|F_{f,j,\ell}(r)|^2|\varphi_{j,\ell}(\theta)|^2 \\
&= \beta_{n,\alpha}\int_{B_n(0;R)} d^nx\,|x|^{-\alpha-4}|f(x)|^2 + \big\{\big[(n+\alpha)^2+(n-\alpha-4)^2\big]/16\big\} \\
&\quad \times \int_{B_n(0;R)} d^nx\,|x|^{-\alpha-4}\Bigg(\sum_{k=1}^N\prod_{p=1}^k[\ln_p(\gamma/|x|)]^{-2}\Bigg)|f(x)|^2. \qquad (3.15)
\end{aligned}
$$

Here we used the abbreviation $\gamma_{n,\alpha}$ introduced in (3.10). □

Remark 3.6. It was shown by Caldiroli and Musina [22, Theorem 3.1 and p. 151] (considering the case without logarithmic refinements) that the constant $\beta_{n,\alpha}$ in (3.12) is optimal. ◇

Remark 3.7. For $\alpha\in(-\infty,n-4)$, (3.12) was proved in [38, Theorem 3.15] (see also [39, Theorem 6.3.6]). We note that in [38, Theorem 3.15], (3.12) was proved for f belonging to the larger space of functions, namely for $f\in C_0^\infty(B_n(0;R))$. For this larger function space, their range of $\alpha\in(-\infty,n-4)$ is optimal.

In fact, if $\alpha\in[n-4,\infty)$, then even the basic power weighted Rellich inequality,

$$\int_{B_n(0;R)} d^nx\,|x|^{-\alpha}|(-\Delta f)(x)|^2 \geqslant C\int_{B_n(0;R)} d^nx\,|x|^{-\alpha-4}|f(x)|^2 \qquad (3.16)$$

does not hold for all $f\in C_0^\infty(B_n(0;R))$ with any $C>0$. To see this let $\alpha\in[n-4,\infty)$, $R=1$, and $f_0\in C_0^\infty(B_n(0;1))$ be a radially symmetric function satisfying

$$f_0(x)\equiv f_0(|x|)=\begin{cases}1, & 0\leqslant|x|\leqslant 1/4,\\ 0, & 3/4\leqslant|x|<1.\end{cases} \qquad (3.17)$$

Then $-\Delta f_0\in C_0^\infty(B_n(0;1)\backslash\{0\})$, hence the left-hand side of (3.16) is finite. But on the right-hand side of (3.16) one obtains

$$\int_{B_n(0;1)} d^nx\,|x|^{-\alpha-4}|f_0(x)|^2 \geqslant \omega(\mathbb{S}^{n-1})\int_0^{1/4} r^{-\alpha+n-5}dr=\infty. \qquad (3.18)$$

Moreover, since [38, Theorem 3.15] (see also [39, Theorem 6.3.6] and [53, Theorem 3.9]) does not imply (3.12) when $f \in C_0^\infty(B_n(0;R)\backslash\{0\})$ and $\alpha \in [n-4,\infty)$, Theorem 3.5 is new for $\alpha \in [n-4,\infty)$.

On the other hand, Theorem 3.5 implies [38, Theorem 3.15] (see also [39, Theorem 6.3.6] and [53, Theorem 3.9]) as a corollary, that is, if (3.12) holds for all $f \in C_0^\infty(B_n(0;R)\backslash\{0\})$, then it holds for all $f \in C_0^\infty(B_n(0;R))$ provided that $\alpha \in (-\infty, n-4)$. To see this, let $\alpha \in (-\infty, n-4)$, $R=1$ (without loss of generality), and $f \in C_0^\infty(B_n(0;1))$. Let $\varphi \in C^\infty(\mathbb{R})$ be a non-decreasing function satisfying

$$\varphi(r) = \begin{cases} 0, & r \leqslant 1, \\ 1, & r \geqslant 2. \end{cases} \tag{3.19}$$

For $0 < \varepsilon < 1/4$ let

$$\varphi_\varepsilon(r) = \varphi(r/\varepsilon) = \begin{cases} 0, & r \leqslant \varepsilon, \\ 1, & r \geqslant 2\varepsilon, \end{cases} \tag{3.20}$$

and put

$$f_\varepsilon(x) = f(x)\varphi_\varepsilon(|x|), \quad x \in B_n(0;1), \tag{3.21}$$

so that $f_\varepsilon \in C_0^\infty(B_n(0;1)\backslash\{0\})$. Without loss of generality, we will assume that f is real-valued for the remainder of Remark 3.7. Writing

$$D_k = \frac{\partial}{\partial x_k}, \quad k = 1,\dots,n. \tag{3.22}$$

Then, for $\varepsilon \in (0,1/4)$ and $k \in \{1,\dots,n\}$,

$$\begin{aligned} |D_k\varphi_\varepsilon(|x|)| &= \big|\varepsilon^{-1}x_k|x|^{-1}\varphi'(|x|/\varepsilon)\big| \leqslant C_1\varepsilon^{-1}, \\ \big|D_k^2\varphi_\varepsilon(|x|)\big| &\leqslant \big|\varepsilon^{-1}(x_1^2+\cdots+x_{i-1}^2+x_{i+1}^2+\cdots+x_n^2)|x|^{-3/2}\varphi'(|x|/\varepsilon)\big| \\ &\quad + \big|\varepsilon^{-2}x_k^2|x|^2\varphi''(|x|/\varepsilon)\big| \\ &\leqslant C_2\varepsilon^{-3/2} + C_3\varepsilon^{-2} \leqslant C_4\varepsilon^{-2}, \end{aligned} \tag{3.23}$$

for some $C_1, C_2, C_3, C_4 \in (0,\infty)$, hence

$$\begin{aligned} &\big|D_k^2 f_\varepsilon(x)\big|^2 \\ &\quad = \Big[(D_k^2 f(x))\varphi_\varepsilon(|x|) + 2(D_k f(x))(D_k\varphi_\varepsilon(|x|)) + f(x)(D_k^2\varphi_\varepsilon(|x|))\Big]^2 \\ &\quad = \big(D_k^2 f(x)\big)^2\varphi_\varepsilon(|x|)^2 + 4(D_k f(x))(D_k^2 f(x))\varphi_\varepsilon(|x|)(D_k\varphi_\varepsilon(|x|)) \\ &\qquad + 2f(x)(D_k^2 f(x))\varphi_\varepsilon(|x|)(D_k^2\varphi_\varepsilon(|x|)) + 4\big(D_k f(x)\big)^2\big(D_k\varphi_\varepsilon(|x|)\big)^2 \\ &\qquad + 4f(x)(D_k f(x))(D_k\varphi_\varepsilon(|x|))(D_k^2\varphi_\varepsilon(|x|)) + f(x)^2\big(D_k^2\varphi_\varepsilon(|x|)\big)^2. \end{aligned} \tag{3.24}$$

Thus,

$$\begin{aligned} &\bigg|\int_{B_n(0;1)} d^n x\, |x|^{-\alpha}|(-\Delta f)(x)|^2 - \int_{B_n(0;1)} d^n x\, |x|^{-\alpha}|(-\Delta f_\varepsilon)(x)|^2\bigg| \\ &\quad \leqslant \bigg|\int_{B_n(0;1)} d^n x\, |x|^{-\alpha}|(-\Delta f)(x)|^2\big(1-\varphi_\varepsilon(|x|)^2\big)\bigg| \end{aligned}$$

$$
\begin{aligned}
&+ 4C_1\varepsilon^{-1}\int_{B_n(0;2\varepsilon)} d^n x\, |x|^{-\alpha}\bigg|\sum_{k=1}^{n}(D_k f(x))\big(D_k^2 f(x)\big)\bigg| \\
&+ 2C_4\varepsilon^{-2}\int_{B_n(0;2\varepsilon)} d^n x\, |x|^{-\alpha}\bigg|\sum_{k=1}^{n} f(x)\big(D_k^2 f(x)\big)\bigg| \\
&+ 4C_1^2\varepsilon^{-2}\int_{B_n(0;2\varepsilon)} d^n x\, |x|^{-\alpha}\bigg|\sum_{k=1}^{n}(D_k f(x))^2\bigg| \\
&+ 4C_1C_4\varepsilon^{-3}\int_{B_n(0;2\varepsilon)} d^n x\, |x|^{-\alpha}\bigg|\sum_{k=1}^{n} f(x)(D_k f(x))\bigg| \\
&+ nC_4^2\varepsilon^{-4}\int_{B_n(0;2\varepsilon)} d^n x\, |x|^{-\alpha}|f(x)|^2 \\
&= I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} + I_{4,\varepsilon} + I_{5,\varepsilon} + I_{6,\varepsilon}. \qquad (3.25)
\end{aligned}
$$

By dominated convergence, one infers that

$$
\lim_{\varepsilon\downarrow 0} I_{1,\varepsilon} = 0. \qquad (3.26)
$$

In addition, there exists a constant $D = D(f, C_1, C_2, C_3, C_4, n) > 0$ such that, for $k = 2, \dots, 6$,

$$
\begin{aligned}
0 \leqslant I_{k,\varepsilon} &\leqslant D\varepsilon^{-4}\int_{B_n(0;2\varepsilon)} d^n x\, |x|^{-\alpha} \\
&= D\omega(\mathbb{S}^{n-1})\varepsilon^{-4}\int_0^{2\varepsilon} r^{-\alpha+n-1}dr \\
&= 2^{-\alpha+n}(n-\alpha)^{-1}D\omega(\mathbb{S}^{n-1})\varepsilon^{-\alpha+n-4} \\
&\underset{\varepsilon\downarrow 0}{\longrightarrow} 0. \qquad (3.27)
\end{aligned}
$$

Thus, by (3.25), (3.26), (3.27), one obtains that

$$
\lim_{\varepsilon\downarrow 0}\int_{B_n(0;1)} d^n x\, |x|^{-\alpha}|(-\Delta f_\varepsilon)(x)|^2 = \int_{B_n(0;1)} d^n x\, |x|^{-\alpha}|(-\Delta f)(x)|^2. \qquad (3.28)
$$

Again, applying the dominated convergence theorem, one concludes that

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\int_{B_n(0;1)} d^n x\, |x|^{-\alpha-4}|f_\varepsilon(x)|^2 = \int_{B_n(0;1)} d^n x\, |x|^{-\alpha-4}|f(x)|^2, \\
&\lim_{\varepsilon\downarrow 0}\int_{B_n(0;1)} d^n x\, |x|^{-\alpha-4}\Bigg(\sum_{k=1}^{N}\prod_{p=1}^{k}[\ln_p(\gamma/|x|)]^{-2}\Bigg)|f_\varepsilon(x)|^2 \\
&\quad = \int_{B_n(0;1)} d^n x\, |x|^{-\alpha-4}\Bigg(\sum_{k=1}^{N}\prod_{p=1}^{k}[\ln_p(\gamma/|x|)]^{-2}\Bigg)|f(x)|^2. \qquad (3.29)
\end{aligned}
$$

Since $f_\varepsilon \in C_0^\infty(B_n(0;1)\backslash\{0\})$, (3.12) holds for f_ε by Theorem 3.5. Thus (3.28) and (3.29) imply that (3.12) also holds for $f \in C_0^\infty(B_n(0;1))$. $\diamond$

As a consequence of Remark 3.7, one has the following result:

Corollary 3.8. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded, $n \in \mathbb{N}, n \geqslant 2$, and assume that $x_0 \in \Omega$, $\alpha \in (-\infty, n-4)$, $N \in \mathbb{N}$, and $\Gamma \in [e_N \operatorname{diam}(\Omega), \infty)$. Then, for all $f \in C_0^\infty(\Omega)$, one infers that*

$$\begin{aligned}&\int_\Omega d^n x\, |x-x_0|^{-\alpha} |(-\Delta f)(x)|^2 \\ &\quad \geqslant \beta_{n,\alpha} \int_\Omega d^n x\, |x-x_0|^{-\alpha-4} |f(x)|^2 \\ &\qquad + \big\{\big[(n+\alpha)^2 + (n-\alpha-4)^2\big]/16\big\} \\ &\qquad \times \int_\Omega d^n x\, |x-x_0|^{-\alpha-4} \Bigg(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\Gamma/|x-x_0|)]^{-2}\Bigg) |f(x)|^2, \end{aligned} \tag{3.30}$$

where $\beta_{n,\alpha}$, given by (3.13)*, is optimal.*

Proof. By Theorem 3.5, (3.30) holds for all $f \in C_0^\infty(B_n(x_0; \operatorname{diam}(\Omega))\backslash\{x_0\})$. Since $\alpha \in (-\infty, n-4)$, the discussion in Remark 3.7 implies that (3.30) holds also for all $f \in C_0^\infty(B_n(x_0; \operatorname{diam}(\Omega)))$. The obvious fact $C_0^\infty(\Omega) \subseteq C_0^\infty(B_n(x_0; \operatorname{diam}(\Omega)))$ then implies (3.30). Once again, optimality of $\beta_{n,\alpha}$ was already proven in [22, Theorem 3.1 and p. 151] for the smaller spaces of functions $f \in C_0^\infty(B_n(x_0; R)\backslash\{x_0\})$, $R > 0$, but in the context of balls $B_n(0; R)$ this was also proven in [38], [39, Theorems 6.3.6, 6.5.2]. □

We conclude with elementary applications of the inequalities (3.12) and (3.30) to lower semiboundedness of differential operators naturally connected to these inequalities. Introducing the preminimal operator $\dot{T}_{\alpha,\gamma,R,min}$ in $L^2(B_n(0; R); d^n x)$ by

$$\begin{aligned}\big(\dot{T}_{\alpha,\gamma,R,min} f\big)(x) &= \big(-\Delta\, |x|^{-\alpha}(-\Delta) f\big)(x) + |x|^{-\alpha-4} \Bigg[-\beta_{n,\alpha} \\ &\quad - \big\{\big[(n+\alpha)^2 + (n-\alpha-4)^2\big]/16\big\} \Bigg(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\gamma/|x|)]^{-2}\Bigg]\Bigg) f(x), \\ &\quad R \in (0,\infty),\ \alpha \in \mathbb{R},\ N \in \mathbb{N},\ \gamma \in [e_N R, \infty),\ x \in B_n(0; R)\backslash\{0\}, \\ f \in \operatorname{dom}\big(\dot{T}_{\alpha,\gamma,R,min}\big) &= C_0^\infty(B_n(0; R)\backslash\{0\}),\end{aligned} \tag{3.31}$$

with $\beta_{n,\alpha}$ given by (3.13). Integrating by parts in the left-hand side of (3.12) then yields that $\dot{T}_{\alpha,\gamma,R,min}$ is bounded from below by zero, that is,

$$\dot{T}_{\alpha,\gamma,R,min} \geqslant 0. \tag{3.32}$$

Thus, defining the minimal operator associated with $\dot{T}_{\alpha,\gamma,R,min}$ as the (operator) closure of $\dot{T}_{\alpha,\gamma,R,min}$ in $L^2(B_n(0; R); d^n x)$,

$$T_{\alpha,\gamma,R,min} = \overline{\dot{T}_{\alpha,\gamma,R,min}}, \tag{3.33}$$

and denoting by $T_{\alpha,\gamma,R,F}$ the Friedrichs extension of $T_{\alpha,\gamma,R,min}$, one infers by general principles that

$$T_{\alpha,\gamma,R,min} \geqslant 0, \quad T_{\alpha,\gamma,R,F} \geqslant 0. \tag{3.34}$$

The analogous result can be proved in connection with integrations by parts in (3.30) upon introducing the following operators in $L^2(\Omega; d^n x)$, $\Omega \subset \mathbb{R}^n$ open and bounded, $n \in \mathbb{N}, n \geqslant 2$,

$$
\begin{aligned}
(3.35)\quad & \big(\dot{T}_{\alpha,\Gamma,\Omega,min} f\big)(x) = \big(-\Delta\, |x-x_0|^{-\alpha} (-\Delta) f\big)(x) + |x-x_0|^{-\alpha-4} \Bigg[-\beta_{n,\alpha} \\
&\quad - \big\{ \big[(n+\alpha)^2 + (n-\alpha-4)^2\big]/16 \big\} \Bigg(\sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(\Gamma/|x-x_0|)]^{-2} \Bigg]\Bigg) f(x), \\
&\quad R \in (0,\infty),\ \alpha \in (-\infty, n-4),\ N \in \mathbb{N},\ \Gamma \in [e_N \operatorname{diam}(\Omega), \infty), \\
&\quad x_0 \in \Omega,\ x \in \Omega\backslash\{x_0\}, \\
& f \in \operatorname{dom}\big(\dot{T}_{\alpha,\Gamma,\Omega,min}\big) = C_0^\infty(\Omega), \\
(3.36)\quad & T_{\alpha,\Gamma,\Omega,min} = \overline{\dot{T}_{\alpha,\Gamma,\Omega,min}},
\end{aligned}
$$

and the Friedrichs extension $T_{\alpha,\Gamma,\Omega,F}$ of $T_{\alpha,\Gamma,\Omega,min}$, such that

$$
T_{\alpha,\Gamma,\Omega,min} \geqslant 0, \quad T_{\alpha,\Gamma,\Omega,F} \geqslant 0. \tag{3.37}
$$

Appendix A. Spherical Coordinates and the Laplace–Beltrami Differential Expression

In this appendix we summarize some conventions regarding spherical coordinates and the Laplace–Beltrami operator on the unit sphere $\mathbb{S}^{n-1}$ in dimensions $n \in \mathbb{N}$, $n \geqslant 2$, following [11, Chs. 2, 3], [24, Ch. 1], and [41, Ch. 2].

Assuming $n \in \mathbb{N}$, $n \geqslant 2$, cartesian and polar coordinates (cf. e.g., [21]) on $\mathbb{R}^n\backslash\{0\}$ are given by

$$
\begin{aligned}
(A.1)\quad & x = (x_1, \dots, x_n) \in \mathbb{R}^n\backslash\{0\}, \\
& x = r\omega,\ \omega = \omega(\theta) = \omega(\theta_1, \theta_2, \dots, \theta_{n-1}) = x/|x| \in \mathbb{S}^{n-1}, \\
& x_k \in \mathbb{R},\ 1 \leqslant k \leqslant n,\ r = |x| \in (0,\infty),\ \theta_1 \in [0, 2\pi),\ \theta_j \in [0,\pi],\ 2 \leqslant j \leqslant n-1,
\end{aligned}
$$

where (cf., e.g., [21], [24, Sect. 1.5])

$$
\begin{cases}
x_1 = r\sin(\theta_1) \prod_{j=2}^{n-1} \sin(\theta_j), \\
x_2 = r\cos(\theta_1) \prod_{j=2}^{n-1} \sin(\theta_j), \\
\vdots \\
x_{n-1} = r\cos(\theta_{n-2}) \sin(\theta_{n-1}), \\
x_n = r\cos(\theta_{n-1}).
\end{cases} \tag{A.2}
$$

The surface measure $d^{n-1}\omega$ on $\mathbb{S}^{n-1}$ and the volume element in $\mathbb{R}^n$ then read

$$
d^{n-1}\omega(\theta) = d\theta_1 \prod_{j=2}^{n-1} [\sin(\theta_j)]^{j-1} d\theta_j, \quad d^n x = r^{n-1} dr\, d^{n-1}\omega(\theta), \tag{A.3}
$$

in particular, the area ω_n of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ is given by (cf. [54, p. 2])

$$\omega\big(\mathbb{S}^{n-1}\big) = \int_{\mathbb{S}^{n-1}} d^{n-1}\omega(\theta) = 2\pi^{n/2}/\Gamma(n/2), \tag{A.4}$$

with $\Gamma(\,\cdot\,)$ the Gamma function (see, e.g., [1, Ch. 6]).

Next, we recall the expression of the Laplace–Beltrami differential expression on $\mathbb{S}^{n-1}$ in spherical coordinates. From [11, p. 94], [24, Lemma 1.4.2], one obtains the recursion

$$\begin{aligned} -\Delta_{\mathbb{S}^1} &= -\frac{\partial^2}{\partial\theta_1^2}, \\ -\Delta_{\mathbb{S}^2} &= -\frac{1}{\sin(\theta_2)}\frac{\partial}{\partial\theta_2}\bigg(\sin(\theta_2)\frac{\partial}{\partial\theta_2}\bigg) - \frac{1}{\sin^2(\theta_2)}\frac{\partial^2}{\partial\theta_1^2}, \\ -\Delta_{\mathbb{S}^{n-1}} &= -\frac{\partial^2}{\partial\theta_{n-1}^2} - (n-2)\cot(\theta_{n-1})\frac{\partial}{\partial\theta_{n-1}} - [\sin(\theta_{n-1})]^{-2}\Delta_{\mathbb{S}^{n-2}}, \\ &\hspace{9cm} n \geqslant 3. \end{aligned} \tag{A.5}$$

Explicitly (cf. [24, p. 19]),

$$\begin{aligned} -\Delta_{\mathbb{S}^{n-1}} &= -[\sin(\theta_{n-1})]^{2-n}\frac{\partial}{\partial\theta_{n-1}}\bigg[[\sin(\theta_{n-1})]^{n-2}\frac{\partial}{\partial\theta_{n-1}}\bigg] \\ &\quad - \sum_{j=1}^{n-2}\bigg(\prod_{k=j+1}^{n-1}[\sin(\theta_k)]^{-2}\bigg)[\sin(\theta_j)]^{1-j}\frac{\partial}{\partial\theta_j}\bigg[[\sin(\theta_j)]^{j-1}\frac{\partial}{\partial\theta_j}\bigg] \\ &= -\sum_{j=1}^{n-1}\bigg(\prod_{k=1}^{j-1}[\sin(\theta_{n-k})]^{-2}\bigg)[\sin(\theta_{n-j})]^{1-j-n} \\ &\qquad \times \frac{\partial}{\partial\theta_{n-j}}\bigg[[\sin(\theta_{n-j})]^{n-j-1}\frac{\partial}{\partial\theta_{n-j}}\bigg]. \end{aligned} \tag{A.6}$$

Finally, we recall the connection of $-\Delta_{\mathbb{S}^{n-1}}$ with the Laplace differential expression[2] in $\mathbb{R}^n$

$$\begin{aligned} -\Delta_n &= -\frac{\partial^2}{\partial r^2} - (n-1)r^{-1}\frac{\partial}{\partial r} - r^{-2}\Delta_{\mathbb{S}^{n-1}} \\ &= -r^{1-n}\frac{\partial}{\partial r}\bigg(r^{n-1}\frac{\partial}{\partial r}\bigg) - r^{-2}\Delta_{\mathbb{S}^{n-1}}. \end{aligned} \tag{A.7}$$

Appendix B. Proof of Lemma 2.1

We will break up the proof of Lemma 2.1 in a few steps.

Lemma B.1. *Let $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, then*

$$f = \sum_{j\in\mathbb{N}_0}\sum_{\ell=1}^{m(\lambda_j)} F_{f,j,\ell}\,\varphi_{j,\ell} \ \text{ in } L^2(\mathbb{R}^n; d^nx). \tag{B.1}$$

[2] For clarity we indicate the space dimension $n \in \mathbb{N}$ as a subscript in the Laplacian $-\Delta_n$ in this appendix.

Proof. Since by hypothesis $\{\varphi_{j,\ell}\}_{\ell\in\{1,\dots,m(\lambda_j)\},j\in\mathbb{N}_0}$ represents an orthonormal basis of $L^2(\mathbb{S}^{n-1};d^{n-1}\omega)$, one obtains for all $r\in(0,\infty)$, $J\in\mathbb{N}_0$,

$$\left\|f(r,\,\cdot\,)-\sum_{j=0}^{J}\sum_{\ell=1}^{m(\lambda_j)}F_{f,j,\ell}\,\varphi_{j,\ell}\right\|^2_{L^2(\mathbb{S}^{n-1};d^{n-1}\omega)}\leqslant\|f(r,\,\cdot\,)\|^2_{L^2(\mathbb{S}^{n-1};d^{n-1}\omega)}\tag{B.2}$$

and

$$\lim_{J\to\infty}\left\|f(r,\,\cdot\,)-\sum_{j=0}^{J}\sum_{\ell=1}^{m(\lambda_j)}F_{f,j,\ell}\,\varphi_{j,\ell}\right\|^2_{L^2(\mathbb{S}^{n-1};d^{n-1}\omega)}=0.\tag{B.3}$$

Next, suppose that $\operatorname{supp}(f)\subseteq\{x\in\mathbb{R}^n\backslash\{0\}\,|\,|x|\in[a,b]\}$ for some $a,b\in(0,\infty)$, $a<b$. Then an application of the dominated convergence theorem yields

$$\begin{aligned}&\lim_{J\to\infty}\left\|f-\sum_{j=0}^{J}\sum_{\ell=1}^{m(\lambda_j)}F_{f,j,\ell}\,\varphi_{j,\ell}\right\|^2_{L^2(\mathbb{R}^n;d^nx)}\\&\quad=\lim_{J\to\infty}\int_a^b r^{n-1}dr\int_{\mathbb{S}^{n-1}}d^{n-1}\omega(\theta)\left|f(r,\theta)-\sum_{j=0}^{J}\sum_{\ell=1}^{m(\lambda_j)}F_{f,j,\ell}(r)\,\varphi_{j,\ell}(\theta)\right|^2\\&\quad=0.\end{aligned}\tag{B.4}$$

□

Lemma B.2. *Let $f\in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ and $p\in C^\infty((0,\infty))$. Then for all $r\in(0,\infty)$, $\ell\in\{1,\dots,m(\lambda_j)\}$, $j\in\mathbb{N}_0$, one obtains*

$$F_{pf,j,\ell}(r)=p(r)F_{f,j,\ell}(r),\tag{B.5}$$

$$F_{T_{\mathbb{S}^{n-1},F}f,j,\ell}(r)=\lambda_jF_{f,j,\ell}(r),\tag{B.6}$$

$$F_{\frac{\partial}{\partial r}f,j,\ell}(r)=F'_{f,j,\ell}(r).\tag{B.7}$$

Proof. (i) One computes

$$\begin{aligned}F_{pf,j,\ell}(r)&=\int_{\mathbb{S}^{n-1}}d^{n-1}\omega(\theta)\,\overline{\varphi_{j,\ell}(\theta)}p(r)f(r,\theta)\\&=p(r)\int_{\mathbb{S}^{n-1}}d^{n-1}\omega(\theta)\overline{\varphi_{j,\ell}(\theta)}f(r,\theta)=p(r)F_{f,j,\ell}(r).\end{aligned}\tag{B.8}$$

(ii) One confirms that

$$\begin{aligned}F_{T_{\mathbb{S}^{n-1},F}f,j,\ell}(r)&=\int_{\mathbb{S}^{n-1}}d^{n-1}\omega(\theta)\,\overline{\varphi_{j,\ell}(\theta)}(T_{\mathbb{S}^{n-1},F}f)(r,\theta)\\&=\int_{\mathbb{S}^{n-1}}d^{n-1}\omega(\theta)\,\overline{\varphi_{j,\ell}(\theta)}(-\Delta_{\mathbb{S}^{n-1}}f)(r,\theta)\\&=\int_{\mathbb{S}^{n-1}}d^{n-1}\omega(\theta)\,\overline{(-\Delta_{\mathbb{S}^{n-1}}\varphi_{j,\ell})(\theta)}f(r,\theta)\\&=\lambda_j\int_{\mathbb{S}^{n-1}}d^{n-1}\omega(\theta)\,\overline{\varphi_{j,\ell}(\theta)}f(r,\theta)=\lambda_jF_{f,j,\ell}(r).\end{aligned}\tag{B.9}$$

(iii) Without loss of generality, we assume $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ to be real-valued. Since $(\partial f/\partial r) \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, an application of the dominated convergence theorem implies for $r \in (0,\infty)$,

$$\begin{aligned}
&\lim_{h\to 0} h^{-1}[F_{f,j,\ell}(r+h) - F_{f,j,\ell}(r)] \\
&\quad = \lim_{h\to 0}\int_{\mathbb{S}^{n-1}} d^{n-1}\omega(\theta)\,\overline{\varphi_{j,\ell}(\theta)} h^{-1}[f(r+h,\theta) - f(r,\theta)] \\
&\quad = \lim_{h\to 0}\int_{\mathbb{S}^{n-1}} d^{n-1}\omega(\theta)\,\overline{\varphi_{j,\ell}(\theta)} \frac{\partial f}{\partial r}(r+\eta(\theta)h,\theta) \quad (\text{for } 0 \leqslant \eta(\theta) \leqslant 1) \\
&\quad = \int_{\mathbb{S}^{n-1}} d^{n-1}\omega(\theta)\,\overline{\varphi_{j,\ell}(\theta)} \frac{\partial f}{\partial r}(r,\theta) = F_{\frac{\partial}{\partial r}f,j,\ell}(r).
\end{aligned} \tag{B.10}$$

□

Lemma B.3. *Let $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, then the following items (i)–(iii) hold:*
(i) $F_{f,j,\ell} \in C_0^\infty((0,\infty))$ for all $\ell \in \{1,\dots,m(\lambda_j)\}$, $j \in \mathbb{N}_0$.
(ii) For any $k \in \mathbb{N}_0$,

$$T^k_{\mathbb{S}^{n-1},F} f = \sum_{j\in\mathbb{N}_0}\sum_{\ell=1}^{m(\lambda_j)} \lambda_j^k F_{f,j,\ell}\,\varphi_{j,\ell} \ \text{ in } L^2(\mathbb{R}^n; d^n x). \tag{B.11}$$

(iii) Let $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $p_1,\dots,p_m \in C^\infty((0,\infty))$. Then, for all $\ell \in \{1,\dots,m(\lambda_j)\}$, $j \in \mathbb{N}_0$, one has

$$F_{p_1\frac{\partial}{\partial r}p_2\frac{\partial}{\partial r}\cdots p_{m-1}\frac{\partial}{\partial r}p_m T^k_{\mathbb{S}^{n-1},F}f,j,\ell} = \bigg(p_1\frac{\partial}{\partial r}p_2\frac{\partial}{\partial r}\cdots p_{m-1}\frac{\partial}{\partial r}p_m F_{f,j,\ell}(r)\bigg)\lambda_j^k. \tag{B.12}$$

Proof. *(i)* This follows from Lemma B.2 *(iii)*.
(ii) The case $k=0$ follows from Lemma B.1. Since $T^r_{\mathbb{S}^{n-1},F} f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, $r \in \mathbb{N}_0$, Lemma B.1 and (B.6) imply

$$\begin{aligned}
T^k_{\mathbb{S}^{n-1},F} f &= \sum_{j\in\mathbb{N}_0}\sum_{\ell=1}^{m(\lambda_j)} F_{T^k_{\mathbb{S}^{n-1},F}f,j,\ell}\,\varphi_{j,\ell} = \sum_{j\in\mathbb{N}_0}\sum_{\ell=1}^{m(\lambda_j)} \lambda_j F_{T^{k-1}_{\mathbb{S}^{n-1},F}f,j,\ell}\,\varphi_{j,\ell} \\
&= \cdots = \sum_{j\in\mathbb{N}_0}\sum_{\ell=1}^{m(\lambda_j)} \lambda_j^k F_{f,j,\ell}\,\varphi_{j,\ell}, \quad k \in \mathbb{N}.
\end{aligned} \tag{B.13}$$

(iii) Repeatedly using (B.6), one obtains

$$F_{T^k_{\mathbb{S}^{n-1},F}f,j,\ell} = \lambda_j^k F_{f,j,\ell}, \quad k \in \mathbb{N}_0, \tag{B.14}$$

and repeatedly using (B.5) and (B.7) in Lemma B.2 one gets

$$\begin{aligned}
&F_{p_m T^k_{\mathbb{S}^{n-1},F}f,j,\ell} = p_m F_{T^k_{\mathbb{S}^{n-1},F}f,j,\ell} = p_m F_{f,j,\ell}\lambda_j^k, \\
&F_{\frac{\partial}{\partial r}p_m T^k_{\mathbb{S}^{n-1},F}f,j,\ell} = \frac{\partial}{\partial r}F_{p_m T^k_{\mathbb{S}^{n-1},F}f,j,\ell} = \frac{\partial}{\partial r}\big(p_m F_{f,j,\ell}\big)\lambda_j^k, \\
&F_{p_{m-1}\frac{\partial}{\partial r}p_m T^k_{\mathbb{S}^{n-1},F}f,j,\ell} = p_{m-1} F_{\frac{\partial}{\partial r}p_m T^k_{\mathbb{S}^{n-1},F}f,j,\ell}
\end{aligned}$$

$$= \left(p_{m-1}\frac{\partial}{\partial r}p_m F_{f,j,\ell}\right)\lambda_j^k,$$

$$\vdots$$

$$F_{p_1\frac{\partial}{\partial r}p_2\frac{\partial}{\partial r}\cdots p_{m-1}\frac{\partial}{\partial r}p_m T^k_{\mathbb{S}^{n-1},F}f,j,\ell} = \left(p_1\frac{\partial}{\partial r}p_2\frac{\partial}{\partial r}\cdots p_{m-1}\frac{\partial}{\partial r}p_m F_{f,j,\ell}\right)\lambda_j^k. \tag{B.15}$$

□

Lemma B.4. *Let* $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$.
(i) *Let* $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ *and* $p_1, \dots, p_m \in C^\infty((0,\infty))$. *Then*

$$\begin{aligned} &p_1\frac{\partial}{\partial r}p_2\frac{\partial}{\partial r}\cdots p_{m-1}\frac{\partial}{\partial r}p_m T^k_{\mathbb{S}^{n-1},F}f \\ &\quad = \sum_{j\in\mathbb{N}_0}\sum_{\ell=1}^{m(\lambda_j)}\left(p_1\frac{\partial}{\partial r}p_2\frac{\partial}{\partial r}\cdots p_{m-1}\frac{\partial}{\partial r}p_m F_{f,j,\ell}\right)\lambda_j^k\,\varphi_{j,\ell} \ \text{ in } L^2(\mathbb{R}^n; d^n x). \end{aligned} \tag{B.16}$$

(ii) *Let* $r \in \mathbb{N}$, *then, in the sense of* $L^2(\mathbb{R}^n; d^n x)$,

$$(-\Delta)^r f = \sum_{j\in\mathbb{N}_0}\sum_{\ell=1}^{m(\lambda_j)}\left[-r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) + r^{-2}T_{\mathbb{S}^{n-1},F}\right]^r (F_{f,j,\ell}\,\varphi_{j,\ell}). \tag{B.17}$$

Proof. (i) Given $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, one notes that

$$p_1\frac{\partial}{\partial r}p_2\frac{\partial}{\partial r}\cdots p_{m-1}\frac{\partial}{\partial r}p_m T^k_{\mathbb{S}^{n-1},F}f \in C_0^\infty(\mathbb{R}^n\backslash\{0\}), \tag{B.18}$$

and hence applying Lemma B.1 and Lemma B.3 (iii) one obtains

$$\begin{aligned} &p_1\frac{\partial}{\partial r}p_2\frac{\partial}{\partial r}\cdots p_{m-1}\frac{\partial}{\partial r}p_m T^k_{\mathbb{S}^{n-1},F}f \\ &\quad = \sum_{j\in\mathbb{N}_0}\sum_{\ell=1}^{m(\lambda_j)} F_{p_1\frac{\partial}{\partial r}p_2\frac{\partial}{\partial r}\cdots p_{m-1}\frac{\partial}{\partial r}p_m T^k_{\mathbb{S}^{n-1},F}f,j,\ell}\,\varphi_{j,\ell} \\ &\quad = \sum_{j\in\mathbb{N}_0}\sum_{\ell=1}^{m(\lambda_j)}\left(p_1\frac{\partial}{\partial r}p_2\frac{\partial}{\partial r}\cdots p_{m-1}\frac{\partial}{\partial r}p_m F_{f,j,\ell}\right)\lambda_j^k\,\varphi_{j,\ell} \ \text{ in } L^2(\mathbb{R}^n; d^n x). \end{aligned} \tag{B.19}$$

(ii) One expands $(-\Delta)^r f$, $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, into a finite sum of terms of the form

$$p_1\frac{\partial}{\partial r}p_2\frac{\partial}{\partial r}\cdots p_{m-1}\frac{\partial}{\partial r}p_m T^k_{\mathbb{S}^{n-1},F}f, \tag{B.20}$$

where $k \in \mathbb{N}_0$ and $p_1, \dots, p_m \in C^\infty((0,\infty))$. Denoting the finite sum by $\sum_\nu$, one has

$$\begin{aligned} (-\Delta)^r f &= \sum_\nu p_{1,\nu}\frac{\partial}{\partial r}p_{2,\nu}\frac{\partial}{\partial r}\cdots p_{m(\nu)-1,\nu}\frac{\partial}{\partial r}p_{m(\nu),\nu}T^{k(\nu)}_{\mathbb{S}^{n-1},F}f \\ &= \sum_\nu\sum_{j\in\mathbb{N}_0}\sum_{\ell=1}^{m(\lambda_j)}\left(p_{1,\nu}\frac{\partial}{\partial r}p_{2,\nu}\frac{\partial}{\partial r}\cdots p_{m-1(\nu),\nu}\frac{\partial}{\partial r}p_{m(\nu),\nu}F_{f,j,\ell}\right)\lambda_j^{k(\nu)}\,\varphi_{j,\ell} \\ &\qquad\qquad \text{(by item } (i)) \end{aligned} \tag{B.21}$$

$$= \sum_{j\in\mathbb{N}_0} \sum_{\ell=1}^{m(\lambda_j)} \sum_{\nu} p_{1,\nu} \frac{\partial}{\partial r} p_{2,\nu} \frac{\partial}{\partial r} \cdots p_{m(\nu)-1,\nu} \frac{\partial}{\partial r} p_{m(\nu),\nu} T_{\mathbb{S}^{n-1},F}^{k(\nu)} (F_{f,j,\ell}\varphi_{j,\ell})$$

$$= \sum_{j\in\mathbb{N}_0} \sum_{\ell=1}^{m(\lambda_j)} (-\Delta)^r (F_{f,j,\ell}\varphi_{j,\ell})$$

$$= \sum_{j\in\mathbb{N}_0} \sum_{\ell=1}^{m(\lambda_j)} \bigg[- r^{1-n} \frac{\partial}{\partial r} \bigg(r^{n-1} \frac{\partial}{\partial r} \bigg) + r^{-2} T_{\mathbb{S}^{n-1},F} \bigg]^r (F_{f,j,\ell}\, \varphi_{j,\ell}).$$

□

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972.

[2] Adimurthi, N. Chaudhuri, and M. Ramaswami, *An improved Hardy-Sobolev inequality and its application*, Proc. Amer. Math. Soc. **130** (2001), 489–505.

[3] Adimurthi and M. J. Esteban, *An improved Hardy-Sobolev inequality in $W^{1,p}$ and its application to Schrödinger operators*, Nonlin. Diff. Eq. Appl. **12** (2005), 243–263.

[4] Adimurthi, S. Filippas, and A. Tertikas, *On the best constant of Hardy–Sobolev inequalities*, Nonlin. Anal. **70** (2009), 2826–2833.

[5] Adimurthi, M. Grossi, and S. Santra, *Optimal Hardy–Rellich inequalities, maximum principle and related eigenvalue problem*, J. Funct. Anal. **240** (2006), 36–83.

[6] Adimurthi and K. Sandeep, *Existence and non-existence of the first eigenvalue of the perturbed Hardy-Sobolev operator*, Proc. Roy. Soc. Edinburgh **A 132** (2002), 1021–1043.

[7] Adimurthi and S. Santra, *Generalized Hardy–Rellich inequalities in critical dimension and its applications*, Comm. Contemp. Math. **11** (2009), 367—394.

[8] W. Allegretto, *Nonoscillation theory of elliptic equations of order* $2n$, Pac. J. Math. **64** (1976), 1–16.

[9] A. Alvino, R. Volpicelli, and B. Volzone, *On Hardy inequalities with a remainder term*, Ric. Mat. **59** (2010), 265–280.

[10] H. Ando and T. Horiuchi, *Missing terms in the weighted Hardy–Sobolev inequalities and its application*, Kyoto J. of Math. **52** (2012), 759—796.

[11] K. Atkinson and W. Han, *Spherical Harmonics and Approximations on the Unit Sphere: An Introduction*, Lecture Notes in Math., Vol. 2044, Springer, 2012.

[12] F. Avkhadiev, *On Rellich's inequalities in Euclidean spaces*, Russ. Math. **62** (2018), 71–75.

[13] F. Avkhadiev, *Selected results and open problems on Hardy–Rellich and Poincaré–Friedrichs inequalities*, Anal. Math. Phys. **11** (2021): No. 134, 20pp.

[14] F. G. Avkhadiev, R. G. Nasibullin, and I. K. Shafigullin, *Hardy-type inequalities with power and logarithmic weights in domains of the Euclidean space*, Russian Math. **55** (2011), 76–79.

[15] A. A. Balinsky, W. D. Evans, and R. T. Lewis, *The Analysis and Geometry of Hardy's Inequality*, Universitext, Springer, Cham, 2015.

[16] G. Barbatis, *Improved Rellich inequalities for the polyharmonic operator*, Indiana Univ. Math. J. **55** (2006), 1401–1422.

[17] G. Barbatis, *Best constants for higher-order Rellich inequalities in $L^p(\Omega)$*, Math Z. **255** (2007), 877–896.

[18] G. Barbatis, S. Fillipas, and A. Tertikas, *Series expansion for L^p Hardy inequalities*, Indiana Univ. Math. J. **52** (2003), 171–190.

[19] G. Barbatis, S. Fillipas, and A. Tertikas, *Refined geometric L^p Hardy inequalities*, Commun. Contemp. Math. **5** (2003), 869–881.

[20] G. Barbatis and A. Tertikas, *On a class of Rellich inequalities*, J. Comp. Appl. Math. **194** (2006), 156–172.

[21] L. E. Blumenson, *A derivation of n-dimensional spherical coordinates*, Amer. Math. Monthly **67** (1960), 63–66.

[22] P. Caldiroli and R. Musina, *Rellich inequalities with weights*, Calc. Var. **45** (2012), 147–164.

[23] C. Cowan, *Optimal Hardy inequalities for general elliptic operators with improvements*, Commun. Pure Appl. Anal. **9** (2010), 109–140.

[24] F. Dai and Y. Xu, *Approximation Theory and Harmonic Analysis on Spheres and Balls*, Springer, New York, 2013.

[25] E. B. Davies and A. M. Hinz, *Explicit constants for Rellich inequalities in $L_p(\Omega)$*, Math. Z. **227** (1998), 511–523.

[26] A. Detalla, T. Horiuchi, and H. Ando, *Sharp remainder terms of Hardy–Sobolev inequalities*, Math. J. Ibaraki Univ. **37** (2005), 39–52.

[27] B. Devyver, *A spectral result for Hardy inequalities*, J. Math. Pures Appl. **102** (2014), 813–853.

[28] N. T. Duy, N. Lam, and N. A. Triet, *Improved Hardy and Hardy–Rellich type inequalities with Bessel pairs via factorizations*, J. Spectral Th. **10** (2020), 1277–1302.

[29] D. E. Edmunds and W. D. Evans, *The Rellich inequality*, Rev. Mat. Complut. **29** (2016), 511–530.

[30] D. E. Edmunds, W. D. Evans, and R. T. Lewis, *Inequalities of Rellich type*, Funct. Anal. Appl. **55** (2021), 130–139.

[31] S. Fillipas, and A. Tertikas, *Optimizing improved Hardy inequalities*, J. Funct. Anal. **192** (2002), 186–233; Corrigendum: J. Funct. Anal. **255** (2008): 2095; see also [55].

[32] F. Gesztesy, *On non-degenerate ground states for Schrödinger operators*, Rep. Math. Phys. **20** (1984), 93–109.

[33] F. Gesztesy and L. L. Littlejohn, *Factorizations and Hardy–Rellich-type inequalities*, in: Non-Linear Partial Differential Equations, Mathematical Physics, and Stochastic Analysis. The Helge Holden Anniversary Volume, F. Gesztesy, H. Hanche-Olsen, E. R. Jakobsen, Yu. Lyubarskii, N. H. Risebro, and K. Seip (eds.), EMS Congress Reports, EMS, ETH–Zürich, 2018, pp. 207–226.

[34] F. Gesztesy, L. L. Littlejohn, I. Michael, and M. M. H. Pang, *Radial and logarithmic refinements of Hardy's inequality*, St. Petersburg Math. J. **30** (2019), 429–436.

[35] F. Gesztesy, L. L. Littlejohn, I. Michael, and M. M. H. Pang, *A sequence of weighted Birman–Hardy–Rellich inequalities with logarithmic refinements*, arXiv:2003.12894, Integral Eq. Operator Th. **94** (2022): 13.

[36] K. T. Gkikas and G. Psaradakis, *Optimal non-homogeneous improvements for the series expansion of Hardy's inequality*, arXiv1805.10935, Comm. Contemp. Math., to appear.

[37] N. Ghoussoub and A. Moradifam, *On the best possible remaining term in the Hardy inequality*, Proc. Nat. Acad. Sci. **105** (2008), 13746–13751.

[38] N. Ghoussoub and A. Moradifam, *Bessel pairs and optimal Hardy and Hardy–Rellich inequalities*, Math. Ann. **349** (2011), 1–57.

[39] N. Ghoussoub and A. Moradifam, *Functional Inequalities: New Perspectives and New Applications*, Math. Surveys Monographs, Vol. 187, Amer. Math. Soc., Providence, RI, 2013.

[40] N. Ghoussoub and F. Robert, *Sobolev inequalities for the Hardy–Schrödinger operator: extremals and critical dimensions*, Bull. Math. Sci. **6** (2016), 89–144.

[41] L. Hermi, *On the Spectrum of the Dirichlet Laplacian and Other Elliptic Operators*, Ph.D. Thesis, University of Missouri, Columbia, 1999.

[42] A. M. Hinz, *Topics from spectral theory of differential operators*, in: Spectral Theory of Schrödinger Operators, R. del Rio and C. Villegas-Blas (eds.), Contemp. Math. **340**, Amer. Math. Soc., Providence, R.I., 2004, pp. 1–50.

[43] N. Ioku and M. Ishiwata, *A scale invariant form of a critical Hardy inequality*, Int. Math. Res. Notes **2015** (2015), 8830–8846.

[44] J. Kinnunen, J. Lehrbäck, and A. Vähäkangas, *Maximal Function Methods for Sobolev Spaces*, Math. Surveys Monographs, Vol. 257, Amer. Math. Soc., Providence, RI, 2021.

[45] S. Machihara, T. Ozawa, and H. Wadade, *Hardy type inequalities on balls*, Tohoku Math. J. **65** (2013), 321–330.

[46] S. Machihara, T. Ozawa, and H. Wadade, *Scaling invariant Hardy inequalities of multiple logarithmic type on the whole space*, J. Inequal. Appls. **2015** (2015): 281.
[47] V. Maz'ya, *Sobolev Spaces. With Applications to Elliptic Partial Differential Equations*, Grundlehren math. Wissensch., Vol. 342, 2nd rev. and augmented ed., Springer, Berlin, 2011.
[48] G. Metafune, M. Sobajima, and C. Spina, *Weighted Calderón–Zygmund and Rellich inequalities in L^p*, Math. Ann. **361** (2015), 313–366.
[49] G. Metafune, L. Negro, M. Sobajima, and C. Spina, *Rellich inequalities in bounded domains*, Math. Ann. **379** (2021), 765–824.
[50] I. B. Michael, *On Birman–Hardy–Rellich-type Inequalities*, Ph.D. Thesis, Baylor University, 2019.
[51] E. Mitidieri, *A simple approach to Hardy inequalities*, Math. Notes **67** (2000), 479–486.
[52] Y. Mizuta and T. Shimomura, *Hardy–Sobolev inequality for Sobolev functions in central Herz–Morrey spaces*, Math. Nachr. **294** (2021), 1148–1159.
[53] A. Moradifam, *Optimal weighted Hardy–Rellich inequalities on $H^2 \cap H^1_0$*, J. London Math. Soc. **85** (2012), 22–40.
[54] C. Müller, *Spherical Harmonics*, Lecture Notes in Math., vol. 17, Springer, Berlin, 1966.
[55] R. Musina, *A note on the paper "Optimizing improved Hardy inequalities" by S. Filippas and A. Tertikas*, J. Funct. Anal. **256** (2009), 2741–2745.
[56] R. Musina, *Weighted Sobolev spaces of radially symmetric functions*, Ann. Mat. **193** (2014), 1629–1659.
[57] R. Musina, *Optimal Rellich–Sobolev constants and their extremals*, Diff. Integral Eq. **27** (2014), 570–600.
[58] B. Opic and A. Kufner, *Hardy-Type Inequalities*, Pitman Research Notes in Mathematics Series, vol. 219. Longman Scientific & Technical, Harlow, 1990.
[59] M. P. Owen, *The Hardy–Rellich inequality for polyharmonic operators*, Proc. Roy. Soc. Edinburgh Sect. **A 129** (1999), 825–839.
[60] F. Rellich, *Halbbeschränkte Differentialoperatoren höherer Ordnung*, in: Proceedings of the International Congress of Mathematicians 1954, Vol. III, J. C. H. Gerretsen and J. de Groot (eds.), E. P. Nordhoff, Groningen, North-Holland Publ. Co., Amsterdam, 1956, pp. 243–250.
[61] F. Rellich, *Perturbation Theory of Eigenvalue Problems*, Gordon and Breach, 1969.
[62] M. Ruzhansky and D. Suragan, *Hardy Inequalities on Homogeneous Groups. 100 Years of Hardy Inequalities*, Progr. Math., Vol. 327, Birkhäuser, Springer, Cham, Switzerland, 2019.
[63] M. Ruzhansky and N. Yessirkegenov, *Factorizations and Hardy–Rellich inequalities on stratified groups*, J. Spectral Th. **10** (2020), 1361–1411.
[64] M. Sano, *Extremal functions of generalized critical Hardy inequalities*, J. Diff. Eq. **267** (2019), 2594–2615.
[65] M. Sano and F. Takahashi, *Sublinear eigenvalue problems with singular weights related to the critical Hardy inequality*, Electronic J. Diff. Eq. **2016** (2016): No. 212,
[66] B. Simon, *Hardy and Rellich inequalities in non-integral dimension*, J. Operator Th. **9** (1983), 143–146.
[67] M. Solomyak, *A remark on the Hardy inequalities*, Integral Eq. Operator Th. **19** (1994), 120–124.
[68] F. Takahashi, *A simple proof of Hardy's inequality in a limiting case*, Arch. Math. **104** (2015), 77–82.
[69] A. Tertikas and N. B. Zographopoulos, *Best constants in the Hardy–Rellich inequalities and related improvements*, Adv. Math. **209** (2007), 407–459.
[70] D. Yafaev, *Sharp constants in the Hardy–Rellich inequalities*, J. Funct. Anal. **168** (1999), 121–144.

Manuscript received September 13 2022
revised March 6 2023

F. Gesztesy
Department of Mathematics, Baylor University, Sid Richardson Bldg., 1410 S. 4th Street, Waco, TX 76706, USA
E-mail address: Fritz_Gesztesy@baylor.edu

I. Michael
Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, USA
E-mail address: imichael@lsu.edu

M. M. H. Pang
Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: pangm@missouri.edu