A CHARACTERIZATION OF $\dot{W}^{1,p}(\mathbb{R}^d)$

RUPERT L. FRANK

Dedicated, in admiration, to V. Maz'ya on the occasion of his 85th birthday

ABSTRACT. For $1 < p < \infty$ we give a characterization of the Sobolev space $\dot{W}^{1,p}(\mathbb{R}^d)$ in terms of the oscillations of a function on balls of varying centers and radii. Our work is motivated both by the study of trace ideal properties of commutators with singular integral operators and by work of Nguyen and by Brezis, Van Schaftingen and Yung on derivative-free characterizations of Sobolev spaces.

1. Main result and discussion

By definition, the space $\dot{W}^{1,p}(\mathbb{R}^d)$ consists of all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ whose distributional gradient satisfies $\nabla f \in L^p(\mathbb{R}^d)$. (This space is denoted by $L^1_p(\mathbb{R}^d)$ in [27].) Our goal here is to discuss a necessary and sufficient criterion for the membership to $\dot{W}^{1,p}(\mathbb{R}^d)$ in the case $p > 1$ that does not involve derivatives. Throughout this paper, $d \geq 1$.

We denote $\mathbb{R}^{d+1}_+ := \mathbb{R}^d \times \mathbb{R}_+$ and, for $(a, r) \in \mathbb{R}^{d+1}_+$, we set $B_r(a) := \{x \in \mathbb{R}^d : |x - a| < r\}$. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, let

$$m_f(a, r) := \fint_{B_r(a)} \left| f(x) - \fint_{B_r(a)} f(y)\, dy \right| dx,$$

where $\fint_{B_r(a)} \ldots = |B_r(a)|^{-1} \int_{B_r(a)} \ldots$. Finally, let ν_p be the measure on $\mathbb{R}^{d+1}_+$ with

$$d\nu_p(a, r) = \frac{da\, dr}{r^{p+1}}.$$

Our main result is the following.

Theorem 1.1. *Let $1 < p < \infty$ and let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then $f \in \dot{W}^{1,p}(\mathbb{R}^d)$ if and only if $m_f \in L^p_{\mathrm{weak}}(\mathbb{R}^{d+1}_+, \nu_p)$, and*

$$(1.1) \qquad \|\nabla f\|^p_{L^p(\mathbb{R}^d)} \simeq \sup_{\kappa > 0} \kappa^p \nu_p(\{m_f > \kappa\}).$$

Moreover,

$$(1.2) \qquad \lim_{\kappa \to 0} \kappa^p \nu_p(\{m_f > \kappa\}) = c_{d,p} \|\nabla f\|^p_{L^p(\mathbb{R}^d)}$$

with

$$c_{d,p} = p^{-1} \left(\frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{d+2}{2})}{\Gamma(\frac{d+3}{2})} \right)^p .$$

Here and in what follows, we use the notations $\lesssim$, $\gtrsim$ and $\simeq$ to suppress constants that only depend on d and p.

Remarks. (a) One motivation of this work comes from the study of trace ideal properties of commutators with singular integral operators. This concerns the case $p = d > 1$ in Theorem 1.1. We will discuss the background in further detail in Subsection 1.1, but for now let us mention that in the context of commutator bounds, Rochberg and Semmes [32] introduced a discrete analogue of the condition $m_f \in L^d_{\mathrm{weak}}(\mathbb{R}^{d+1}_+, \nu_d)$ and raised the question of characterizing this condition more directly. Connes, Sullivan and Teleman, in the appendix of their paper [14] together with Semmes, announced that this discrete condition is equivalent to $f \in \dot{W}^{1,d}(\mathbb{R}^d)$ and sketched a proof. The recent paper by Lord, McDonald, Sukochev and Zanin [26], in conjunction with the results of Rochberg and Semmes [32], provides a complete proof under the additional assumption that $f \in L^\infty(\mathbb{R}^d)$. The latter proof, however, relies on rather deep results in operator theory and the theory of pseudo-differential operators. Our goal here is to provide a direct proof of (1.1), somewhat in the spirit of the sketch in [14]. In addition, we will prove (1.2), which is new and which, in turn, suggests a new result in the study of trace ideal properties of commutators; see Corollary 1.2. Finally, and importantly, we generalize the above results, which are restricted to $p = d > 1$ to general $p > 1$. We are most grateful to Jean Van Schaftingen for suggesting this after reading an earlier version of this manuscript that only concerned the case $p = d$.

(b) Another motivation comes from the papers [29, 8] by Nguyen and by Brezis, Van Schaftingen and Yung, which sparked an interest in finding characterizations of membership to Sobolev spaces that do not involve derivatives and which have led to a fast growing literature. We discuss this further in Subsection 1.2. Here we just mention that both (1.1) and (1.2) have their analogues in the corresponding formulas in [29, 8].

(c) The function m_f appears in the characterization of other function spaces. For instance, always assuming $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, one has

(1.3) $f \in BMO(\mathbb{R}^d)$ iff $\sup_{r>0} m_f(\cdot, r) \in L^\infty(\mathbb{R}^d)$,

(1.4) $f \in L^p(\mathbb{R}^d) + \mathbb{R}$ iff $\sup_{r>0} m_f(\cdot, r) \in L^p(\mathbb{R}^d)$,   provided $1 < p < \infty$,

(1.5) $f \in \dot{C}^s(\mathbb{R}^d)$ iff $\sup_{r>0} r^{-s} m_f(\cdot, r) \in L^\infty(\mathbb{R}^d)$,   provided $0 < s < 1$,

(1.6) $f \in \dot{W}^{1,p}(\mathbb{R}^d)$ iff $\sup_{r>0} r^{-1} m_f(\cdot, r) \in L^p(\mathbb{R}^d)$,   provided $1 < p \le \infty$.

Indeed, (1.3) is simply the definition; equivalence (1.4) can be deduced from [2, Proposition 8.10] (we are grateful to Mario Milman for showing us this argument, which improves that in [34, Section IV.2]); for (1.5) see [12] and [15, Theorem 6.3], and for (1.6) see [11] and [15, Theorem 6.2]. Note that all these classical results

involve a supremum with respect to $r > 0$. Closer to the criterion in Theorem 1.1 is the fact that

$$(1.7) \qquad f \in \dot{W}^{d/p,p}(\mathbb{R}^d) \qquad \text{iff} \qquad m_f \in L^p(\mathbb{R}_+^{d+1}, \nu_d), \qquad \text{provided } d < p < \infty.$$

This is of relevance in connection with the trace ideal properties mentioned in (b) and is at least implicitly contained in [32]. Seeger in [33] has identified, in great generality, function spaces defined in terms of m_f as special cases of Triebel–Lizorkin spaces. As far as we can see, however, the results there are restricted to (possibly mixed) Lebesgue norms of m_f and do not contain weak norms as in Theorem 1.1. The same applies to other derivative-free characterizations, for instance, the textbook characterization in [25, Theorem 11.75] as well as the more recent ones in [1, 36]. It seems somewhat surprising, to us at least, that the *strong* norms in (1.7) are replaced by a *weak* norm in the endpoint case $p = d$.

(d) We defined m_f in terms of an L^1-norm. Theorem 1.1 remains valid if we use an L^q norm with certain $1 \le q < \infty$, except, of course, that the implicit constant in (1.1) may depend on q and the value of $c_{d,p}$ in (1.2) changes; see Remark 2.2.

(e) It is worth singling out from Theorem 1.1 a sufficient condition for constancy of a function. Namely, if $1 < p < \infty$ and if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ satisfies $m_f \in L^p_{\text{weak}}(\mathbb{R}_+^{d+1}, \nu_p)$ and $\liminf_{\kappa \to 0} \kappa^p \nu_p(\{m_f > \kappa\}) = 0$, then f is constant. Related, but different conditions for constancy are discussed, for instance, in [5, 9].

(f) We have restricted ourselves in Theorem 1.1 to first order Sobolev spaces. It is natural to expect that similar results also hold in the higher order case where in the definition of m_f not only a constant, but a low degree polynomial needs to be subtracted. Many of the results mentioned in (c) extend to this case.

(g) It is noteworthy that the case $p = 1$ is excluded in Theorem 1.1. Our proof shows that (1.2) remains valid for sufficiently regular functions on $\mathbb{R}^d$ (Lemma 3.1) as well as that, if $m_f \in L^1_{\text{weak}}(\mathbb{R}_+^{d+1}, \nu_1)$, then $f \in \dot{BV}(\mathbb{R}^d)$. On the other hand, it is easy to see that there is an $f \in \dot{BV}(\mathbb{R}^d)$ (for instance, the characteristic function of a ball) for which $m_f \notin L^1_{\text{weak}}(\mathbb{R}_+^{d+1}, \nu_1)$. At present, no simple characterization of the condition $m_f \in L^1_{\text{weak}}(\mathbb{R}_+^{d+1}, \nu_1)$ seems to be available. In [31] Rochberg and Semmes show that for $d = 1$ this space strictly contains the Besov space $\dot{B}^1_{1,1}(\mathbb{R})$ and is strictly contained in a certain weak-type Besov space.

The remainder of this paper is organized as follows. In the following two subsections, we present our two motivations for this study, namely trace ideal properties in Subsection 1.1 and derivative-less characterizations of Sobolev spaces in Subsection 1.2. In Section 2, we prove the inequality $\ge$ in (1.1) and, in Section 3, we prove the inequality $\le$ in (1.1), as well as (1.2).

It is a pleasure to dedicate this paper, in great admiration, to V. Maz'ya, whose work on Sobolev spaces has inspired many, including the present author.

1.1. **Trace ideal properties of commutators.** Let us review the context in which the question answered by Theorem 1.1 arises. There is a substantial literature on boundedness and compactness properties of operators

$$[K, f] := Kf - fK,$$

where K is a Calderón–Zygmund singular integral operator and f is a function on $\mathbb{R}^d$. We identify f with the operator of multiplication by f. For simplicity, we assume that K is homogeneous and translation-invariant, and that its kernel is given by a function that is smooth away from the origin with mean value zero on spheres centered at the origin. Most of the results below hold under much weaker assumptions on K, but the present ones do include the important special case of the Hilbert transform if $d = 1$ and the Riesz transforms if $d \geq 2$. To avoid trivialities, we also assume $K \not\equiv 0$.

We will consider the operator $[K, f]$ on $L^2(\mathbb{R}^d)$. It is known that it is bounded if and only if $f \in BMO(\mathbb{R}^d)$, and it is compact if and only if $f \in CMO(\mathbb{R}^d)$, the closure in $BMO(\mathbb{R}^d)$ of compactly supported, smooth functions; see [28, 23] for $d = 1$ and [13, 35] for general d. (Our references throughout this subsection are far from complete, and in this specific case some rather concern the periodic than the whole space case.)

Having established criteria for compactness, the next questions concern quantitative versions of this property, expressed in the decay of singular values. We recall that the singular values of a compact operator T in a separable Hilbert space are the square roots of the eigenvalues, counting multiplicities, of the operator T^*T. The Schatten spaces $\mathcal{S}^p$ and $\mathcal{S}^p_{\mathrm{weak}}$ consist of those T for which the sequence of singular values belongs to ℓ^p and ℓ^p_{weak}, respectively. In dimension $d = 1$, it was shown by Peller [30] that, for $1 \leq p < \infty$, $[K, f] \in \mathcal{S}^p$ if and only if $f \in \dot{B}_{p,p}^{1/p}(\mathbb{R})$, the latter being a Besov space. (In fact, $\dot{B}_{p,p}^{1/p}(\mathbb{R}) = \dot{W}^{1/p,p}(\mathbb{R})$ if $p > 1$.) The higher dimensional case is somewhat different and it was shown by Janson and Wolff [24] that, for $d < p < \infty$, $[K, f] \in \mathcal{S}^p$ if and only $f \in \dot{B}_{p,p}^{d/p}(\mathbb{R}^d)$. The difference to the one-dimensional case is that, if $[K, f] \in \mathcal{S}^d$ for $d \geq 2$, then f is constant. The endpoint case $d = p$ was studied in more detail by Rochberg and Semmes [32] who showed that, again assuming $d \geq 2$, $[K, f] \in \mathcal{S}^d_{\mathrm{weak}}$ if and only if f belongs to a certain space $Osc^{d,\infty}(\mathbb{R}^d)$. They also improved on the Lorentz scale the Janson–Wolff condition for f to be constant.

One can show that a function f belongs to the space $Osc^{d,\infty}(\mathbb{R}^d)$ if and only if $m_f \in L^d_{\mathrm{weak}}(\mathbb{R}^{d+1}_+, \nu_d)$. We have not found this statement in the literature. Its proof is not difficult. The space $Osc^{d,\infty}(\mathbb{R}^d)$ is defined by the analogue of m_f with dyadic cubes instead of balls and considered not as a function of $(a, r) \in \mathbb{R}^{d+1}_+$, but as a sequence, indexed by dyadic cubes. By definition, $Osc^{d,\infty}(\mathbb{R}^d)$ is the space of functions f for which this sequence belongs to ℓ^d_{weak}.

In [31], Rochberg and Semmes address the question of whether the space $Osc^{d,\infty}(\mathbb{R}^d)$ coincides with some known function space. As mentioned in Remark (h) following Theorem 1.1, in dimension $d = 1$, they prove that this space strictly contains the Besov space $\dot{B}_{1,1}^1(\mathbb{R})$ and is strictly contained in a certain weak-type Besov space. In dimensions $d \geq 2$, they show that $Osc^{d,\infty}(\mathbb{R}^d) \supset \dot{W}^{1,d}(\mathbb{R}^d)$. Modulo the equivalence of the discrete condition defining $Osc^{d,\infty}(\mathbb{R}^d)$ and our continuous condition, this proves $\geq$ in (1.1). In fact, our proof of that inequality uses some ideas from their argument, but seems to us somewhat more direct.

As mentioned in Remark (a) after Theorem 1.1, the remaining inclusion $Osc^{d,\infty}(\mathbb{R}^d) \subset \dot{W}^{1,d}(\mathbb{R}^d)$ is stated as a theorem in the appendix of the paper [14] by Connes, Sullivan and Teleman, where they acknowledge a collaboration with Semmes. In particular, from our 'continuous' point of view the analogue of their 'discrete' equation (A3) is

$$(1.8) \qquad \limsup_{\kappa \to 0} \kappa^d \nu_d(\{m_f > \kappa\}) \gtrsim \|\nabla f\|_{L^d(\mathbb{R}^d)}^d \,.$$

In view of these results, our contribution in the present paper in the case $p = d$ is, on the one hand, to fill in the details in the somewhat sketchy presentation in the appendix of [14] and, on the other hand, to show that the asymptotic bound (1.8) can be replaced by the limit relation (1.2). (It is not clear to us whether one can expect a limit to exist in the discrete setting.) Our result (1.2) shows that (1.8) and thus [14, (A3)] hold with lim inf instead of lim sup.

Motivated by these results we obtain the following condition for constancy, which strengthends those due to Janson–Wolff [24] and Rochberg–Semmes [32]. We denote by $s_n([K,f])$ the sequence of singular values of $[K,f]$ in nonincreasing order and repeated according to multiplicities.

Corollary 1.2. *Let $d \geq 2$ and $f \in CMO(\mathbb{R}^d)$ with*

$$\liminf_{N \to \infty} N^{-1+1/d} \sum_{n=1}^{N} s_n([K,f]) = 0 \,.$$

Then f is constant.

Note that we do not assume a-priori that $f \in \dot{W}^{1,d}(\mathbb{R}^d)$ nor, equivalently, that $s.([K,f]) \in \ell_{\text{weak}}^d$. This is in contrast to the condition for constancy in Remark (e) following Theorem 1.1. The lim inf condition in the corollary is implied by the condition $\lim_{n \to \infty} n^{1/d} s_n([K,f]) = 0$. Since the latter condition is satisfied whenever $s.([K,f])$ belongs to a Lorentz space $\ell^{d,q}$ with $q < \infty$, the corollary is stronger than the results in [24, 32] (although, as we shall see, it can be proved using the methods in [24]).

Proof. We assume that $f \in CMO(\mathbb{R}^d)$ is not constant and aim at proving that the lim inf in the corollary is positive. In view of the inequality $\sum_{n=1}^{N} s_n([K,f]) \geq \sum_{n=1}^{N} |(\psi_n, [K,f]\varphi_n)|$ for all orthonormal $(\psi_n), (\varphi_n) \subset L^2(\mathbb{R}^d)$ (see, e.g, [21, Lemma 4.1]), it suffices to find such orthonormal systems with $|(\psi_n, [K,f]\varphi_n)| \gtrsim n^{-1/d}$. The functions φ_n are essentially constructed in [24, Section 3]. Indeed, the functions there are parametrized by the points ξ_j in the intersection of $\mathbb{Z}^d$ with a cone. Their nondecreasing rearrangement clearly behaves like $n^{-1/d}$. Moreover, it is shown there that $|\widehat{[K,f]\varphi_j}| \gtrsim n^{-1/d}$ on $B_\delta(\xi_j)$. We define ψ_n by $\widehat{\psi_n} := c_n \mathbb{1}_{B_\delta(\xi_j)} \operatorname{sgn} \widehat{[K,f]\varphi_n}$ with c_n chosen such that $\|\psi_n\|_{L^2} = 1$. Then $|(\psi_n, [K,f]\varphi_n) \gtrsim n^{-1/d}$, as claimed. $\square$

We finally mention the recent work of Lord, McDonald, Sukochev and Zanin [26] which concerns a particular operator K, namely the sign of the Dirac operator, that plays some role in noncommutative geometry. It is shown that, if $d \geq 2$ and $f \in L^\infty(\mathbb{R}^d)$, then $[K,f] \in \mathcal{S}_{\text{weak}}^d$ if and only if $f \in \dot{W}^{1,d}(\mathbb{R}^d)$. In fact, a simple approximation argument shows that the a-priori assumption $f \in L^\infty(\mathbb{R}^d)$ may be

replaced by $f \in BMO(\mathbb{R}^d)$, and then we deduce by the Rochberg–Semmes result [32] (in particular, the fact that the space $Osc^{d,\infty}(\mathbb{R}^d)$ is independent of K) that $Osc^{d,\infty}(\mathbb{R}^d) = \dot{W}^{1,d}(\mathbb{R}^d)$. This argument gives a complete proof (except for the approximation argument required to remove the boundedness assumption) of the first part of Theorem 1.1. It is, however, somewhat unsatisfactory that in order to deduce the real analysis statement in the theorem one needs to go through rather deep results in operator theory and the theory of pseudodifferential operators. This motivated us to look for a more direct proof, closer in spirit to the sketch in [14]. However, the intuition gained from the pseudodifferential perspective in [26] was also helpful in the present argument, in particular, in the proof of Lemma 3.1, which contains a local version of (1.2). We take this opportunity to thank Fedor Sukochev and Dmitriy Zanin for a fruitful discussion concerning [26]. Note added in proof: After the present paper was submitted for publication, the topic of trace ideal properties of commutators was further explored in [18, 19]

1.2. **Derivative-less characterizations of Sobolev spaces.** In the paper [10], Brezis, Seeger, Van Schaftingen and Yung, unifying and extending earlier work by Nguyen [29] (see also [4, 6, 7]) and by Brezis, Van Schaftingen and Yung [8], have shown the following fact, valid for all $1 < p < \infty$ and $\gamma \in \mathbb{R} \setminus \{0\}$. Denoting by $\tilde{\nu}_\gamma$ the measure on $\mathcal{X} := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x \neq y\}$ with $d\tilde{\nu}_\gamma(x,y) = |x-y|^{\gamma-d}\,dx\,dy$, a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to $\dot{W}^{1,p}(\mathbb{R}^d)$ if and only if $(f(x)-f(y))/|x-y|^{1+\gamma/p}$ belongs to $L^p_{\mathrm{weak}}(\mathcal{X}, \tilde{\nu}_\gamma)$, and

$$(1.9) \quad \|\nabla f\|^p_{L^p(\mathbb{R}^d)} \simeq \sup_{\kappa>0} \kappa^p \,\tilde{\nu}_\gamma(\{(x,y) \in \mathcal{X} : |f(x)-f(y)|/|x-y|^{1+\gamma/p} > \kappa\}).$$

Moreover, with an explicit constant $\tilde{c}_{d,p} \in \mathbb{R}_+$,

$$\lim \kappa^p \,\tilde{\nu}_\gamma(\{(x,y) \in \mathcal{X} : |f(x)-f(y)|/|x-y|^{1+\gamma/p} > \kappa\})$$
$$(1.10) \qquad = |\gamma|^{-1}\tilde{c}_{d,p}\,\|\nabla f\|^p_{L^p(\mathbb{R}^d)},$$

where one considers the limit $\kappa \to \infty$ for $\gamma > 0$ and $\kappa \to 0$ for $\gamma < 0$. We emphasize that the paper [10] contains many more results, including for instance a detailed analysis of the case $p = 1$. Clearly, (1.1) and (1.2) share some similarities with (1.9) and (1.10), respectively. In some vague sense one can think of $(x+y)/2$ and $|x-y|$ in (1.9) as our a and r, respectively. However, (1.9) and (1.10) are completely pointwise criteria, while the function m_f in (1.1) and (1.2) involves integrals. The similarity between (1.1)–(1.2) and (1.9)–(1.10) is also reflected in our proofs, namely, most clearly, in the one of (1.2), but also in the maximal function argument for $\geq$ in (1.1). We comment on this in more detail before the respective proofs.

Weak-type estimates were obtained, for instance, in [22]. The work [8] has led to many follow-up works and we refer to [10] for a partial bibliography. Let us mention, in particular, [16, Section 7], where it is shown that, if $1 < p < \infty$ and $f \in \dot{W}^{1,p}(\mathbb{R}^d)$, then

$$(1.11) \quad \|\nabla f\|^p_{L^p(\mathbb{R}^d)} \lesssim \liminf_{\kappa \to 0} \kappa^p \,\nu_0(\{(a,r) \in \mathbb{R}^{d+1}_+ : |(P_r f)(a)-f(a)|/r^{1+1/p} > \kappa\}),$$

where $P_r f = e^{-r\sqrt{-\Delta}}f$ denotes the Poisson extension of f. This is reminiscent of (1.2). Note, however, that the measure in $\mathbb{R}^{d+1}_+$ in (1.11) is the usual Lebesgue

measure ν_0 and not ν_p. Moreover, [16] only proves a one-sided inequality. For other related estimates with derivatives of harmonic and caloric extensions, see [17].

2. The lower bound on $\|\nabla f\|_{L^p}$

In this section we shall prove an upper bound on m_f for $f \in \dot{W}^{1,p}(\mathbb{R}^d)$ and deduce that, if $1 < p < \infty$, then $m_f \in L^p_{\text{weak}}(\mathbb{R}^{d+1}_+, \nu_p)$. We will use the (centered) maximal function, denoted by $\mathcal{M}$, somewhat in the spirit of [29] and [8, Remark 2.3] (see also [10, Proposition 2.1]).

Lemma 2.1. *If* $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^d)$, *then*

$$m_f(a,r) \lesssim r\mathcal{M}|\nabla f|(a) \qquad \text{for all } (a,r) \in \mathbb{R}^{d+1}_+ \,.$$

Proof. By the Poincaré inequality (see, e.g., [20, (7.45)]) we have

$$m_f(a,r) = \fint_{B_r(a)} \left| f(x) - \fint_{B_r(a)} f(y)\, dy \right| dx$$

$$\lesssim r \fint_{B_r(a)} |\nabla f(x)|\, dx \qquad \text{for all } (a,r) \in \mathbb{R}^{d+1}_+\,.$$

Since the right side is bounded from above by $r\mathcal{M}|\nabla f|(a)$, the lemma follows. $\square$

Proof of Theorem 1.1. First part. Let $1 < p < \infty$ and $f \in \dot{W}^{1,p}(\mathbb{R}^d)$. Then, by Lemma 2.1,

$$\nu_p(\{m_f > \kappa\}) \le \nu_p(\{(a,r) \in \mathbb{R}^{d+1}_+ : r\mathcal{M}|\nabla f|(a) > \kappa/C\}) \qquad \text{for all } \kappa > 0\,.$$

For fixed $a \in \mathbb{R}^d$, we compute

$$\int_0^\infty \mathbb{1}(r\mathcal{M}|\nabla f|(a) > \kappa/C)\, \frac{dr}{r^{p+1}} = p^{-1}\left(\frac{C\,\mathcal{M}|\nabla f|(a)}{\kappa} \right)^p$$

and, thus,

$$\nu_p(\{(a,r) \in \mathbb{R}^{d+1}_+ : r\mathcal{M}|\nabla f|(a) > \kappa/C\}) \le \frac{p^{-1}C^p}{\kappa^p} \int_{\mathbb{R}^d} (\mathcal{M}|\nabla f|(a))^p\, da\,.$$

The claimed bound now follows from the boundedness of the maximal function on $L^p(\mathbb{R}^d)$. (It is at this last step that the assumption $p > 1$ enters.) $\square$

We note that the above argument fits into the framework of [16, Appendix]. Indeed, we deduce from Lemma 2.1 and the boundedness of the maximal function that for the operator $T_t f(x) := t^{-1} m_f(x,t)$ the assumption [16, (9.1)] is satisfied. Therefore, [16, (9.2)] with $\gamma = -p$ gives the bound $\gtrsim$ in (1.1). We are grateful to Po-Lam Yung for this remark.

Remark 2.2. For $1 \le q < \infty$, $f \in L^q_{\text{loc}}(\mathbb{R}^d)$ and $(a,r) \in \mathbb{R}^{d+1}_+$, let

$$m_f^{(q)}(a,r) := \left(\fint_{B_r(a)} \left| f(x) - \fint_{B_r(a)} f(y)\, dy \right|^q dx \right)^{1/q}\,.$$

Then clearly $m_f^{(q)}(a, r)$ is nondecreasing in q. We claim that, if $1 < p < \infty$, if $f \in \dot{W}^{1,p}(\mathbb{R}^d)$ and if $1 \leq q < dp/(d-p)$ for $p < d$ and $1 \leq q < \infty$ for $p \geq d$, then $m_f^{(q)} \in L_{\text{weak}}^p(\mathbb{R}_+^{d+1}, \nu_p)$ and

$$(2.1) \qquad \sup_{\kappa > 0} \kappa^p \nu_p(\{m_f^{(q)} > \kappa\}) \lesssim \|\nabla f\|_{L^p(\mathbb{R}^d)}^p.$$

(In this and the following remark, implicit constants may also depend on q.) Indeed, given q as in the claim choose $1 \leq t < \min\{p, d\}$ such that $q < dt/(d-t)$ and follow the proof of [20, (7.45)]) to deduce that

$$m_f^{(q)}(a, r) \lesssim r \left(\fint_{B_r(a)} |\nabla f(x)|^t \, dx \right)^{1/t} \quad \text{for all } (a, r) \in \mathbb{R}_+^{d+1}.$$

Bounding the right side by $r(\mathcal{M}(|\nabla f|^t)(a)^{1/t}$ we can argue as in the proof of Theorem 1.1 above and obtain (2.1).

Remark 2.3. Another variation concerns the quantity, defined for $f \in L_{\text{loc}}^q(\mathbb{R}^d)$ and $(a, r) \in \mathbb{R}_+^{d+1}$,

$$\tilde{m}_f^{(q)}(a, r) := \left(\fint_{B_r(a)} \fint_{B_r(a)} |f(x) - f(y)|^q \, dy \, dx \right)^{1/q}.$$

Then clearly $\tilde{m}_f^{(q)}(a, r) \geq m_f^{(q)}(a, r)$. On the other hand, by adding and subtracting the mean of f on $B_r(a)$ and using the triangle inequality in L^q, we see that $\tilde{m}_f^{(q)}(a, r) \leq 2m_f^{(q)}(a, r)$. We deduce from Remark 2.2 that, if $1 < p < \infty$, $f \in \dot{W}^{1,p}(\mathbb{R}^d)$ and q as in that remark, then

$$(2.2) \qquad \sup_{\kappa > 0} \kappa^p \nu_p(\tilde{m}_f^{(q)} > \kappa) \lesssim \|\nabla f\|_{L^p(\mathbb{R}^d)}^p.$$

We are grateful to Jean Van Schaftingen for suggesting this argument, which simplifies significantly our original one.

3. THE UPPER BOUND ON $\|\nabla f\|_{L^p}$

In this section we prove that, if $f \in L_{\text{loc}}^1(\mathbb{R}^d)$ satisfies $m_f \in L_{\text{weak}}^p(\mathbb{R}_+^{d+1}, \nu_p)$ for some $1 < p < \infty$, then $f \in \dot{W}^{1,p}(\mathbb{R}^d)$ and the asymptotics (1.2) hold. Our proof uses some ideas from [29, 8], which, in turn, is inspired by [3].

We begin by computing the asymptotics of $\nu_p(\{m_f > \kappa\})$ as $\kappa \to 0$. The difference from the limit relation (1.2) in Theorem 1.1 is twofold. On the one hand, here we consider more regular functions, but on the other hand, we study a localized version of the asymptotics. The constant $c_{d,p}$ is defined in Theorem 1.1.

Lemma 3.1. *Let* $f \in L_{\text{loc}}^1(\mathbb{R}^d)$. *Let* $\Omega \subset \mathbb{R}^d$ *be a convex, open set and assume that* $f \in C^1(\Omega)$ *and* ∇f *is (globally) Lipschitz on* Ω. *Then for any bounded, open set* $\omega \subset \mathbb{R}^d$ *with* $\overline{\omega} \subset \Omega$,

$$\lim_{\kappa \to 0} \kappa^p \nu_p(\{m_f > \kappa\} \cap (\omega \times \mathbb{R}_+)) = c_{d,p} \int_\omega |\nabla f|^p \, dx.$$

If $\Omega = \mathbb{R}^d$ and ∇f is compactly supported, then the assertion remains valid for $\omega = \mathbb{R}^d$.

Proof. Step 1. Let us denote by A the Lipschitz constant of ∇f on Ω. We claim that

$$(3.1) \qquad \left| m_f(a, r) - c_d' \, r \, |\nabla f(a)| \right| \leq C A \, r^2 \qquad \text{for all } a \in \omega, \ r \leq \operatorname{dist}(\omega, \Omega^c)$$

with

$$c_d' := \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{d+2}{2})}{\Gamma(\frac{d+3}{2})}.$$

To prove this, we note that

$$|f(y) - f(x) - \nabla f(x) \cdot (y - x)| \leq A|x - y|^2 \qquad \text{for all } x, y \in \Omega.$$

(Here we used the convexity of Ω to write $f(y) - f(x) = \nabla f(\xi) \cdot (y - x)$ for some $\xi \in \Omega$ between x and y.) Now let a and r be as in (3.1). Then, for all $x \in \Omega$,

$$\left| f(x) - \fint_{B_r(a)} f(y) \, dy - \nabla f(x) \cdot (x - a) \right|$$

$$\leq \fint_{B_r(a)} |f(x) - f(y) - \nabla f(x) \cdot (x - y)| \, dy$$

$$\leq A \fint_{B_r(a)} |x - y|^2 \, dy$$

$$= A \left(|x - a|^2 + \tfrac{d}{d+2} r^2 \right),$$

so

$$\left| \left| f(x) - \fint_{B_r(a)} f(y) \, dy \right| - |\nabla f(a) \cdot (x - a)| \right|$$

$$\leq \left| f(x) - \fint_{B_r(a)} f(y) \, dy - \nabla f(x) \cdot (x - a) \right| + |(\nabla f(x) - \nabla f(a)) \cdot (x - a)|$$

$$\leq A \left(2|x - a|^2 + \tfrac{d}{d+2} r^2 \right)$$

and, integrating with respect to $x \in B_r(a)$,

$$\left| m_f(a, r) - \fint_{B_r(a)} |\nabla f(a) \cdot (x - a)| \, dx \right|$$

$$\leq \fint_{B_r(a)} \left| \left| f(x) - \fint_{B_r(a)} f(y) \, dy \right| - |\nabla f(a) \cdot (x - a)| \right| dx$$

$$\leq A \fint_{B_r(a)} \left(2|x - a|^2 + \tfrac{d}{d+2} r^2 \right) dx = \tfrac{3d}{d+2} A \, r^2.$$

It remains to note that

$$\fint_{B_r(a)} |\nabla f(a) \cdot (x - a)| \, dx = c_d' \, r \, |\nabla f(a)|,$$

since, for $v \in \mathbb{R}^d$,

$$\fint_{B_r(a)} |v \cdot (x - a)| \, dx = \frac{\int_0^r \rho^d \, d\rho}{\int_0^r \rho^{d-1} \, d\rho} \, \frac{\int_{\mathbb{S}^{d-1}} |v \cdot \omega| \, d\omega}{|\mathbb{S}^{d-1}|}$$

$$= \frac{d}{d+1} \, r \, |v| \, \frac{\int_0^\pi |\cos \theta| \sin^{d-2} \theta \, d\theta}{\int_0^\pi \sin^{d-2} \theta \, d\theta}$$

and, with B denoting the beta function,

$$\frac{\int_0^\pi |\cos \theta| \sin^{d-2} \theta \, d\theta}{\int_0^\pi \sin^{d-2} \theta \, d\theta} = \frac{\int_{-1}^1 |t| (1 - t^2)^{(d-3)/2} \, dt}{\int_{-1}^1 (1 - t^2)^{(d-3)/2} \, dt} = \frac{\int_0^1 (1 - u)^{(d-3)/2} \, du}{\int_0^1 u^{-1/2} (1 - u)^{(d-3)/2} \, du}$$

$$= \frac{B(1, \frac{d-1}{2})}{B(\frac{1}{2}, \frac{d-1}{2})} = \frac{1}{\sqrt{\pi}} \, \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})} \, .$$

This proves (3.1).

Step 2. It follows from (3.1) that, abbreviating $\delta := \mathrm{dist}(\omega, \Omega^c)$,

$$\{(a, r) \in \omega \times (0, \delta] : \ c_d' r |\nabla f(a)| - CAr^2 > \kappa\}$$

$$\subset \{(a, r) \in \omega \times (0, \delta] : \ m_f(a, r) > \kappa\}$$

(3.2) $$\subset \{(a, r) \in \omega \times (0, \delta] : \ c_d' r |\nabla f(a)| + CAr^2 > \kappa\} \, .$$

In this step we will show that

(3.3) $$\lim_{\kappa \to 0} \kappa^p \nu_p(\{(a, r) \in \omega \times \mathbb{R}_+ : \ c_d' r |\nabla f(a)| \pm CAr^2 > \kappa\}) = c_{d,p} \int_\omega |\nabla f(a)|^p \, da \, .$$

Since

$$\nu_p(\omega \times (\delta, \infty)) = |\omega| \int_\delta^\infty \frac{dr}{r^{p+1}} < \infty \, ,$$

this, together with (3.2), proves the assertion in the lemma.

The proof of (3.3) is elementary, but somewhat lengthy. The basic observation is that

$$\nu_p(\{(a, r) \in \omega \times \mathbb{R}_+ : \ c_d' r |\nabla f(a)| > \kappa\}) = \int_\omega \int_{\kappa/(c_d' |\nabla f(a)|)}^\infty \frac{dr}{r^{p+1}} \, da$$

$$= p^{-1} \int_\omega \left(\frac{c_d' |\nabla f(a)|}{\kappa} \right)^p \, da$$

$$= c_{d,p} \kappa^{-p} \int_\omega |\nabla f(a)|^p \, da \, .$$

To include the perturbation $\pm CAr^2$, we exploit the fact that, since r^{-p-1} is not integrable near $r = 0$, only the small r behavior of the bound in (3.1) is relevant and therefore the error term CAr^2 is negligible compared with the main term.

Let us give the details of the proof of (3.3). We begin with the $+$ case. We have $c_d' r |\nabla f(a)| + CAr^2 > \kappa$ if and only if $r > R$ with

$$R := \sqrt{\frac{\kappa}{CA} + \left(\frac{c_d' |\nabla f(a)|}{2CA} \right)^2} - \frac{c_d' |\nabla f(a)|}{2CA} \, .$$

Thus,

$$\int_0^\infty \mathbb{1}(c_d' r|\nabla f(a)| + CAr^2 > \kappa)\,\frac{dr}{r^{p+1}} = p^{-1}R^{-p}.$$

We rewrite

$$p^{-1}R^{-p} = p^{-1}\left(\frac{\sqrt{\frac{\kappa}{CA} + \left(\frac{c_d'|\nabla f(a)|}{2CA}\right)^2} + \frac{c_d'|\nabla f(a)|}{2CA}}{\frac{\kappa}{CA}}\right)^p$$

$$= \frac{c_{d,p}}{\kappa^p}\left(\sqrt{\frac{CA}{c_d'^2}\kappa + \left(\tfrac{1}{2}|\nabla f(a)|\right)^2} + \tfrac{1}{2}|\nabla f(a)|\right)^p.$$

Thus,

$$\nu_p(\{(a,r) \in \omega \times \mathbb{R}_+ :\ c_d' r|\nabla f(a)| + CAr^2 > \kappa\})$$

$$= \frac{c_{d,p}}{\kappa^p}\int_\omega \left(\sqrt{\frac{CA}{c_d'^2}\kappa + \left(\tfrac{1}{2}|\nabla f(a)|\right)^2} + \tfrac{1}{2}|\nabla f(a)|\right)^p da.$$

Dominated convergence (recalling that ω has finite measure) implies (3.3) with +.

We turn to the proof of (3.3) with -. Assuming $\kappa < (c_d'|\nabla f(a)|)^2/(4AC)$, we have $c_d' r|\nabla f(a)| - CAr^2 > \kappa$ if and only if $R_- < r < R_+$ with

$$R_\pm := \frac{c_d'|\nabla f(a)|}{2CA} \pm \sqrt{\left(\frac{c_d'|\nabla f(a)|}{2CA}\right)^2 - \frac{\kappa}{CA}}.$$

Thus, if $\kappa < (c_d'|\nabla f(a)|)^2/(4AC)$,

$$\int_0^\infty \mathbb{1}(c_d' r|\nabla f(a)| - CAr^2 > \kappa)\,\frac{dr}{r^{p+1}} = p^{-1}\left(R_-^{-p} - R_+^{-p}\right).$$

We rewrite, similarly as before,

$$p^{-1}R_-^{-p} = \frac{c_{d,p}}{\kappa^p}\left(\sqrt{\left(\tfrac{1}{2}|\nabla f(a)|\right)^2 - \frac{CA}{c_d'^2}\kappa} + \tfrac{1}{2}|\nabla f(a)|\right)^p.$$

Thus,

$$\kappa^p \nu_p(\{(a,r) \in \omega \times \mathbb{R}_+ :\ c_d' r|\nabla f(a)| - CAr^2 > \kappa\}) = I_1(\kappa) - I_2(\kappa),$$

where

$$I_1(\kappa) := c_{d,p}\int_{\omega \cap \{|\nabla f| > \sqrt{4CA\kappa}/c_d'\}} \left(\left(\left(\tfrac{1}{2}|\nabla f(a)|\right)^2 - \frac{CA}{c_d'^2}\kappa\right)^{\frac{1}{2}} + \tfrac{1}{2}|\nabla f(a)|\right)^p da,$$

$$I_2(\kappa) := \frac{\kappa^p}{p}\int_{\omega \cap \{|\nabla f| > \sqrt{4CA\kappa}/c_d'\}} \left(\frac{c_d'|\nabla f(a)|}{2CA} + \left(\left(\frac{c_d'|\nabla f(a)|}{2CA}\right)^2 - \frac{\kappa}{CA}\right)^{\frac{1}{2}}\right)^{-p} da.$$

By monotone convergence, $I_1(\kappa) \to c_{d,p}\|\nabla f\|_{L^p(\omega)}^p$. For $I_2(\kappa)$, we note that on $\{|\nabla f| > \sqrt{4CA\kappa}/c_d'\}$

$$\frac{c_d'|\nabla f(a)|}{2CA} + \left(\left(\frac{c_d'|\nabla f(a)|}{2CA}\right)^2 - \frac{\kappa}{CA}\right)^{1/2} \geq \sqrt{\frac{\kappa}{CA}},$$

so

$$\kappa^p p^{-1} \left(\frac{c_d'|\nabla f(a)|}{2CA} + \left(\left(\frac{c_d'|\nabla f(a)|}{2CA} \right)^2 - \frac{\kappa}{CA} \right)^{1/2} \right)^{-p} \leq \kappa^{p/2} p^{-1} (CA)^{p/2}.$$

Thus, using again the fact that ω has finite measure, $I_2(\kappa) \to 0$. This completes the proof of (3.3) with -.

Step 3. Finally, we assume that f is constant in $\{|x| \geq R_0\}$. Applying what we have proved so far with $\omega = \{|x| < 2R_0\}$, we obtain

$$\lim_{\kappa \to 0} \kappa^p \nu_p(\{m_f > \kappa\} \cap \{(a,r): \ |a| < 2R_0\})$$

$$= c_{d,p} \int_{|a| < 2R_0} |\nabla f|^p \, dx = c_{d,p} \int_{\mathbb{R}^d} |\nabla f|^p \, dx.$$

Thus, it suffices to prove that $\kappa^p \nu_p(\{m_f > \kappa\} \cap \{(a,r): \ |a| \geq 2R_0\}) \to 0$. The constancy assumption on f implies that $m_f(a,r) = 0$ for $|a| - r \geq R_0$. Thus, it suffices to consider the intersection of $\{m_f > \kappa\}$ with $\{R_0 + r > |a| \geq 2R_0\}$.

We write $f = g + c$ where g is supported in $\{|x| \leq R_0\}$ and c is a constant. Then

$$m_f(a,r) = \fint_{B_r(a)} \left| g(x) - \fint_{B_r(a)} g(y) \, dy \right| dx \leq 2|B_r(a)|^{-1} \int_{\mathbb{R}^d} |g(x)| \, dx =: \gamma r^{-d},$$

and we conclude that, if $m_f(a,r) > \kappa$, then $r < (\gamma/\kappa)^{1/d}$. Thus,

$$\int_{|a| \geq 2R_0} \int_{|a|-R_0}^{\infty} \mathbb{1}(m_f(r,a) > \kappa) \frac{dr}{r^{p+1}} \, da$$

$$\leq \int_{R_0 + (\gamma/\kappa)^{1/d} > |a| \geq 2R_0} \int_{|a|-R_0}^{\infty} \frac{dr}{r^{p+1}} \, da$$

$$= p^{-1} \int_{R_0 + (\gamma/\kappa)^{1/d} > |a| \geq 2R_0} \frac{da}{(|a| - R_0)^p}.$$

When $p > d$ this is uniformly bounded in κ, when $p = d$ it is bounded by a constant times $\ln_+(\gamma/(\kappa R_0^d))$ and when $p < d$ it is bounded by a constant times $(\gamma/\kappa)^{(d-p)/d}$. In any case, the bound, multiplied by κ^p, tends to zero as $\kappa \to 0$, as claimed. □

Lemma 3.2. *Let* $1 < p < \infty$. *There is a constant* $C_{d,p} < \infty$ *such that for all* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ *and all* $0 \leq \varphi \in L^1_c(\mathbb{R}^d)$ *with* $\int \varphi \, dx = 1$,

$$\sup_{\kappa > 0} \kappa^p \nu_p(\{m_{\varphi * f} > \kappa\}) \leq C_{d,p} \sup_{\kappa > 0} \kappa^p \nu_p(\{m_f > \kappa\}).$$

Proof. We bound, using Minkowski's inequality,

$$m_{\varphi * f}(a,r) = \fint_{B_r(a)} \left| \int_{\mathbb{R}^d} \varphi(z) \left(f(x-z) - \fint_{B_r(a)} f(y-z) \, dy \right) dz \right| dx$$

$$\leq \int_{\mathbb{R}^d} \varphi(z) \fint_{B_r(a)} \left| f(x-z) - \fint_{B_r(a)} f(y-z) \, dy \right| dx \, dz$$

$$= \int_{\mathbb{R}^d} \varphi(z) m_f(a-z, r) \, dz.$$

Passing to a norm in $L^p_{\text{weak}}(\mathbb{R}^{d+1}_+, \nu_p)$ that is equivalent to the quasi-norm in the statement of the lemma (this is possible since $p > 1$), we obtain the assertion from Minkowski's inequality. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 1.1. Second part. Throughout this proof, let $1 < p < \infty$ and let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ with $m_f \in L^p_{\text{weak}}(\mathbb{R}^{d+1}_+, \nu_p)$. We proceed in two steps.

Step 1. Let $0 \leq \varphi \in C^2_c(\mathbb{R}^d)$ with $\int \varphi \, dx = 1$ and set $\varphi_t(x) := t^{-d}\varphi(x/t)$. Note that $\varphi_t * f \in C^2(\mathbb{R}^d)$ with $D^2(\varphi_t * f) \in L^\infty_{\text{loc}}(\mathbb{R}^d)$. Thus, by Lemma 3.1, for any bounded, open set $\omega \subset \mathbb{R}^d$,

$$\lim_{\kappa \to 0} \kappa^p \nu_p(\{m_{\varphi_t * f} > \kappa\} \cap (\omega \times \mathbb{R}_+)) = c_{d,p} \int_\omega |\nabla(\varphi_t * f)|^p \, dx \,.$$

On the other hand, by Lemma 3.2,

$$\begin{aligned}
\lim_{\kappa \to 0} \kappa^p \nu_p(\{m_{\varphi_t * f} > \kappa\} \cap (\omega \times \mathbb{R}_+)) &\leq \liminf_{\kappa \to 0} \kappa^p \nu_p(\{m_{\varphi_t * f} > \kappa\}) \\
&\leq \sup_{\kappa > 0} \kappa^p \nu_p(\{m_{\varphi_t * f} > \kappa\}) \\
&\leq C_{d,p} \sup_{\kappa > 0} \kappa^p \nu_p(\{m_f > \kappa\}) \,.
\end{aligned}$$

Thus,

$$\int_\omega |\nabla(\varphi_t * f)|^p \, dx \leq c_{d,p}^{-1} C_{d,p} \sup_{\kappa > 0} \kappa^p \nu_p(\{m_f > \kappa\}) =: C' \,,$$

where the right side depends neither on t nor on ω. By monotone convergence, we conclude that $\nabla(\varphi_t * f) \in L^p(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} |\nabla(\varphi_t * f)|^p \, dx \leq C' \,.$$

By weak compactness (using again $p > 1$), we deduce that for a sequence $t_j \to 0$, $\nabla(\varphi_{t_j} * f) \rightharpoonup F$ in $L^p(\mathbb{R}^d)$. On the other hand, $\varphi_t * f \to f$ in $L^1_{\text{loc}}(\mathbb{R}^d)$ as $t \to 0$. Thus, for any $\Phi \in C^1_c(\mathbb{R}^d, \mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \Phi \cdot F \, dx \leftarrow \int_{\mathbb{R}^d} \Phi \cdot \nabla(\varphi_{t_j} * f) \, dx = -\int_{\mathbb{R}^d} (\nabla \cdot \Phi)\varphi_{t_j} * f \, dx \to -\int_{\mathbb{R}^d} (\nabla \cdot \Phi) f \, dx \,.$$

This proves that $f \in \dot{W}^{1,p}(\mathbb{R}^d)$ with $\nabla f = F$. Moreover, by weak convergence,

$$\begin{aligned}
\int_{\mathbb{R}^d} |\nabla f|^p \, dx &\leq \liminf_{j \to \infty} \int_{\mathbb{R}^d} |\nabla(\varphi_{t_j} * f)|^p \, dx \\
&\leq C' = c_{d,p}^{-1} C_{d,p} \sup_{\kappa > 0} \kappa^p \nu_p(\{m_f > \kappa\}) \,,
\end{aligned}$$

which proves the claimed upper bound on $\|\nabla f\|_{L^p(\mathbb{R}^d)}$ in (1.1).

Step 2. It remains to deduce the limit relation (1.2) for f. This follows by a density argument, using the fact that there is a sequence $(f_n) \subset C^2(\mathbb{R}^d)$ with ∇f_n compactly supported such that $\nabla f_n \to \nabla f$ in $L^p(\mathbb{R}^d)$; see, for instance, [25,

Theorem 11.43]. Note that $|m_{f_n} - m_f| \le m_{f_n - f}$. Thus, for any $\delta \in (0, 1)$,

$$\nu_p(\{m_f > \kappa\}) \le \nu_p(\{m_{f_n} + m_{f_n - f} > \kappa\})$$
$$\le \nu_p(\{m_{f_n} > (1-\delta)\kappa\}) + \nu_p(\{m_{f_n - f} > \delta\kappa\}),$$
$$\nu_p(\{m_{f_n} > \tfrac{\kappa}{1-\delta}\}) \le \nu_p(\{m_f + m_{f_n - f} > \tfrac{\kappa}{1-\delta}\})$$
$$\le \nu_p(\{m_f > \kappa\}) + \nu_p(\{m_{f_n - f} > \tfrac{\delta\kappa}{1-\delta}\}).$$

By the first part of Theorem 1.1, we deduce that

$$\limsup_{\kappa \to 0} \kappa^p \nu_p(\{m_f > \kappa\})$$
$$\le \limsup_{\kappa \to 0} \kappa^p \nu_p(\{m_{f_n} > (1-\delta)\kappa\}) + C\delta^{-p}\|\nabla(f_n - f)\|^p_{L^p(\mathbb{R}^d)},$$
$$\liminf_{\kappa \to 0} \kappa^p \nu_p(\{m_{f_n} > \tfrac{\kappa}{1-\delta}\})$$
$$\le \liminf_{\kappa \to 0} \kappa^p \nu_p(\{m_f > \kappa\}) + C\delta^{-p}(1-\delta)^p\|\nabla(f_n - f)\|^p_{L^p(\mathbb{R}^d)}.$$

On the other hand, by Lemma 3.1,

$$\lim_{\kappa \to 0} \kappa^p \nu_p(\{m_{f_n} > \alpha\kappa\}) = \alpha^{-p} c_{d,p}\|\nabla f_n\|^p_{L^p(\mathbb{R}^d)}.$$

Thus, we have shown that

$$\limsup_{\kappa \to 0} \kappa^p \nu_p(\{m_f > \kappa\})$$
$$\le (1-\delta)^{-p} c_{d,p}\|\nabla f_n\|^p_{L^p(\mathbb{R}^d)} + C\delta^{-p}\|\nabla(f_n - f)\|^p_{L^p(\mathbb{R}^d)},$$
$$(1-\delta)^p c_{d,p}\|\nabla f_n\|^d_{L^p(\mathbb{R}^d)}$$
$$\le \liminf_{\kappa \to 0} \kappa^p \nu_p(\{m_f > \kappa\}) + C\delta^{-p}(1-\delta)^p\|\nabla(f_n - f)\|^p_{L^p(\mathbb{R}^d)}.$$

Letting first $n \to \infty$ and then $\delta \to 0$, we obtain (1.2). This completes the proof of Theorem 1.1. $\qquad\square$

Similarly as in Section 2, let us view the above proof from within the framework of [16, Appendix]. Let again $T_t f(x) := t^{-1} m_f(x, t)$. Then (3.1) in the proof of Lemma 3.1 shows that assumption [16, (9.3)] with $g = c'_d|\nabla f|$ is satisfied for $f \in C^1(\mathbb{R}^d)$ with ∇f compactly supported. Therefore, [16, (9.4)] with $\gamma = -p$ gives (1.2) for such f. On the other hand, as far as we can see, [16] does not consider the question whether $m_f \in L^p_{\text{weak}}(\mathbb{R}^d, \nu_p)$ implies $f \in \dot{W}^{1,p}(\mathbb{R}^d)$. We are grateful to Po-Lam Yung for this remark.

Acknowledgements

References

[1] R. Alabern, J. Mateu and J. Verdera, *A new characterization of Sobolev spaces on $\mathbb{R}^n$*, Math. Ann. **354** (2012), 589–626.

[2] C. Bennett and R. Sharpley, *Interpolation of Operators*. Pure and Applied Mathematics, 129. Academic Press, Inc., Boston, MA, 1988.

[3] J. Bourgain, H. Brezis and P. Mironescu, *Another look at Sobolev spaces*, in: Optimal Control and Partial Differential Equations, IOS, Amsterdam, 2001, pp. 439–455.

[4] J. Bourgain and H.-M. Nguyen, *A new characterization of Sobolev spaces*, C. R. Math. Acad. Sci. Paris **343** (2006), 75–80.

[5] H. Brezis, *How to recognize constant functions. A connection with Sobolev spaces*, Uspekhi Mat. Nauk **57** (2002), (346), 59–74; English translation in Russian Math. Surveys **57** (2002), 693–708.

[6] H. Brezis and H.-M. Nguyen, *Non-local functionals related to the total variation and connections with image processing*, Ann. PDE **4** (2018): Paper No. 9, 77 pp.

[7] H. Brezis and H.-M. Nguyen, *Non-local, non-convex functionals converging to Sobolev norms*, Nonlinear Anal. **191** (2020): 111626, 9 pp.

[8] H. Brezis, J. Van Schaftingen and P.-L. Yung, *A surprising formula for Sobolev norms*. Proc. Natl. Acad. Sci. USA **118** (2021): Paper No. e2025254118, 6 pp.

[9] H. Brezis, J. Van Schaftingen and P.-L. Yung, *Going to Lorentz when fractional Sobolev, Gagliardo and Nirenberg estimates fail*, Calc. Var. Partial Differential Equations **60** (2021): Paper No. 129.

[10] H. Brezis, A. Seeger, J. Van Schaftingen and P.-L. Yung, *Families of functionals representing Sobolev norms*, Preprint (2021), arXiv:2109.02930.

[11] A. P. Calderón, *Estimates for singular integral operators in terms of maximal functions*, Studia Math. **44** (1972), 563–582.

[12] S. Campanato, *Proprietà di hölderianità di alcune classi di funzioni*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 175–188.

[13] R. R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) **103** (1976), 611–635.

[14] A. Connes, D. Sullivan and N. Teleman, *Quasiconformal mappings, operators on Hilbert space, and local formulae for characteristic classes*, Topology **33** (1994), 663–681.

[15] R. A. DeVore and R. C. Sharpley, *Maximal functions measuring smoothness*, Mem. Amer. Math. Soc. **47** (1984): no. 293.

[16] O. Dominguez and M. Milman, *New Brezis–Van Schaftingen–Yung–Sobolev type inequalities connected with maximal inequalities and one parameter families of operators*, Advances in Mathematics **411** Part A (2022): 108774.

[17] O. Dominguez, A. Seeger, B. Street, J. Van Schaftingen and P.-L. Yung *Spaces of Besov–Sobolev type and a problem on nonlinear approximation*, Preprint (2021), arXiv:2112.05539.

[18] R. L. Frank, F. Sukochov and D. Zanin, *Asymptotics of singular values for quantum derivatives*. Trans. Amer. Math. Soc. **376** (2023), 2047– 2088.

[19] R. L. Frank, F. Sukochov and D. Zanin, *Endpoint Schatten class properties of commutators*. Submitted.

[20] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Classics in mathematics, Springer, Berlin, 2001.

[21] I. C. Gohberg and M. G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Translations of Mathematical Monographs, vol. 18 Amer. Math. Soc., Providence, R.I. 1969.

[22] L. Greco and R. Schiattarella, *An embedding theorem for BV-functions*, Commun. Contemp. Math. **22** (2020): Article ID 1950032, 13 p.

[23] P. Hartman, *On completely continuous Hankel matrices*, Proc. Amer. Math. Soc. **9** (1958), 862–866.

[24] S. Janson and T. H. Wolff, *Schatten classes and commutators of singular integral operators*, Ark. Mat. **20** (1982), 301–310.

[25] G. Leoni, *A First Course in Sobolev Spaces*, Second edition. Graduate Studies in Mathematics, vol. 181, American Mathematical Society, Providence, RI, 2017.

[26] S. Lord, E. McDonald and F. Sukochev, D. Zanin, *Quantum differentiability of essentially bounded functions on Euclidean space*, J. Funct. Anal. **273** (2017), 2353–2387.

[27] V. Maz'ya, *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, Second, revised and augmented edition. Grundlehren der mathematischen Wissenschaften, vol. 342, Springer, Heidelberg, 2011.

[28] Z. Nehari, *On bounded bilinear forms*, Ann. of Math. (2) **65** (1957), 153–162.

[29] H.-M. Nguyen, *Some new characterizations of Sobolev spaces*, J. Funct. Anal. **237** (2006), 689–720.

[30] V. V. Peller, *Hankel operators of class $\mathfrak{S}_p$ and their applications (rational approximation, Gaussian processes, the problem of majorization of operators)*, Mat. Sb. (N.S.) **113** (155) (1980), no. 4 (12), 538–581, 637.

[31] R. Rochberg and S. Semmes, *End point results for estimates of singular values of singular integral operators*, in: Contributions to Operator Theory and its Applications (Mesa, AZ, 1987), Oper. Theory Adv. Appl., vol. 35, Birkhäuser, Basel, 1988, pp. 217–231.

[32] R. Rochberg, S. Semmes, *Nearly weakly orthonormal sequences, singular value estimates, and Calderon–Zygmund operators*, J. Funct. Anal. **86** (1989), 237–306.

[33] A. Seeger, *A note on Triebel–Lizorkin spaces*, Approximation and Function Spaces, Banach Center Publications **22** (1989), 391–400.

[34] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, With the assistance of Timothy S. Murphy. Princeton Mathematical Series, vol.43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[35] A. Uchiyama, *On the compactness of operators of Hankel type*, Tohoku Math. J. (2) **30** (1978), 163–171.

[36] D. Yang, W. Yuan and Y. Zhou, *A new characterization of Triebel–Lizorkin spaces on $\mathbb{R}^n$*, Publ. Mat. **57** (2013), 57–82.

RUPERT L. FRANK

Mathematisches Institut, Ludwig-Maximilans Universität München, Theresienstr. 39, 80333 München, Germany, and Munich Center for Quantum Science and Technology, Schellingstr. 4, 80799 München, Germany, and Mathematics 253-37, Caltech, Pasadena, CA 91125, USA

E-mail address: r.frank@lmu.de