Pure and Applied Functional Analysis
Volume 9, Number 1, 2024, 403–412

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

TURNPIKE PHENOMENON FOR PERTURBED DYNAMICAL SYSTEMS DETERMINED BY A SET-VALUED MAPPING

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Abstract. In this paper we study a weak version of the turnpike phenomenon for trajectories of perturbed discrete disperse dynamical systems generated by set-valued mappings.

1. Introduction

In [18, 19] A. M. Rubinov introduced a discrete disperse dynamical system generated by a set-valued mapping acting on a compact metric space, which were studied in [7, 18, 19, 24, 26–28]. This disperse dynamical system has prototype in the mathematical economics [13, 18, 19, 25]. In particular, it is an abstract extension of the classical von Neumann-Gale model [13, 18, 19, 25]. Our dynamical system is described by a compact metric space of states and a transition operator which is set-valued. Dynamical systems theory has been a rapidly growing area of research which has various applications to physics, engineering, biology and economics. In this theory one of the goals is to study the asymptotic behavior of the trajectories of a dynamical system. Usually in the dynamical systems theory a transition operator is single-valued. In [7, 16, 18, 19, 24, 26–28] and in the present paper we study dynamical systems with a set-valued transition operator. Such dynamical systems correspond to certain models of economic dynamics [13, 18, 19, 25].

Let (X, ρ) be a compact metric space and let $a : X \to 2^X \setminus \{\emptyset\}$ be a set-valued mapping whose graph

$$\mathrm{graph}(a) = \{(x, y) \in X \times X : \; y \in a(x)\}$$

is a closed subset of $X \times X$. For each nonempty subset $E \subset X$ set

$$a(E) = \cup\{a(x) : \; x \in E\} \text{ and } a^0(E) = E.$$

By induction we define $a^n(E)$ for any natural number n and any nonempty subset $E \subset X$ as follows:

$$a^n(E) = a(a^{n-1}(E)).$$

In this paper we study convergence and structure of trajectories of the perturbed dynamical system generated by the set-valued mapping a. Following [18, 19] this system is called a discrete dispersive dynamical system.

2010 *Mathematics Subject Classification.* 37B25, 49J53, 54E35, 93D30.
Key words and phrases. Compact metric space, global attractor, Lyapunov function, set-valued mapping, turnpike.

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called a trajectory of a (or just a trajectory if the mapping a is understood) if $x_{t+1} \in a(x_t)$ for all integers $t \geq 0$.

Let $T_2 > T_1$ be integers. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$ is called a trajectory of a (or just a trajectory if the mapping a is understood) if $x_{t+1} \in a(x_t)$ for all integers $t \in \{T_1, \dots, T_2 - 1\}$.

Define

$$\Omega(a) = \{z \in X : \text{ for each } \epsilon > 0 \text{ there is a trajectory } \{x_t\}_{t=0}^{\infty}$$

$$\text{such that } \liminf_{t\to\infty} \rho(z, x_t) \leq \epsilon\}. \tag{1.1}$$

Clearly, $\Omega(a)$ is a nonempty closed subset of (X, ρ). In the literature the set $\Omega(a)$ is called a global attractor of a. Note that in [18, 19] $\Omega(a)$ is called a turnpike set of a. This terminology is motivated by mathematical economics [13, 18, 19, 25].

For each $x \in X$ and each nonempty closed subset $E \subset X$ put

$$\rho(x, E) = \inf\{\rho(x, y) : y \in E\}.$$

It is clear that for each trajectory $\{x_t\}_{t=0}^{\infty}$ we have

$$\lim_{t\to\infty} \rho(x_t, \Omega(a)) = 0.$$

It is not difficult to see that if for a nonempty closed set $B \subset X$

$$\lim_{t\to\infty} \rho(x_t, B) = 0$$

for each trajectory $\{x_t\}_{t=0}^{\infty}$, then $\Omega(a) \subset B$.

Let $\phi : X \to R^1$ be a continuous function such that

$$\phi(z) \geq 0 \text{ for all } z \in X, \tag{1.2}$$

$$\phi(y) \leq \phi(x) \text{ for all } x \in X \text{ and all } y \in a(x). \tag{1.3}$$

It is clear that the function ϕ is a Lyapunov function for the dynamical system generated by the mapping a. It should be mentioned that in mathematical economics usually X is a subset of the finite-dimensional Euclidean space and ϕ is a linear functional on this space [13, 18, 19, 25]. Our goal in [28] was to study approximate solutions of the problem

$$\phi(x_T) \to \max,$$

$$\{x_t\}_{t=0}^{T} \text{ is a program satisfying } x_0 = x,$$

where $x \in X$ and a natural number T are given.

The following theorem was obtained in [28].

Theorem 1.1. *The following properties are equivalent:*

(1) *If a sequence $\{x_t\}_{t=-\infty}^{\infty} \subset X$ satisfies $x_{t+1} \in a(x_t)$ and $\phi(x_{t+1}) = \phi(x_t)$ for all integers t, then*

$$\{x_t\}_{t=-\infty}^{\infty} \subset \Omega(a).$$

(2) *For each $\epsilon > 0$ there exists a natural number $T(\epsilon)$ such that for each trajectory $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying $\phi(x_t) = \phi(x_{t+1})$ for all integers $t \geq 0$ the inequality $\rho(x_t, \Omega(a)) \leq \epsilon$ holds for all integers $t \geq T(\epsilon)$.*

For each bounded function $\psi : X \to R^1$ set

$$\|\psi\| = \sup\{|\psi(z)| : \ z \in X\}.$$

We denote by Card(A) the cardinality of a set A and suppose that the sum over empty set is zero.

For each $(x_1, x_1), \ (y_1, y_2) \in X \times X$ set

$$\rho_1((x_1, x_2), (y_1, y_2)) = \rho(x_1, y_1) + \rho(x_2, y_2).$$

For each $(x_1, x_2) \in X \times X$ and each nonempty closed subset $E \subset X \times X$ put

$$\rho_1((x_1, x_2), E) = \inf\{\rho_1((x_1, x_2), (y_1, y_2)) : \ (y_1, y_2) \in E\}.$$

In [28] we established the turnpike properties for approximate solutions of the problem

$$\phi(x_T) \to \max,$$

$$\{x_t\}_{t=0}^T \text{ is a program satisfying } x_0 = x,$$

where $x \in X$ and a natural number T are given. In [30] we established a weak version of the turnpike property which hold for all trajectories of our dynamical system which are of a sufficient length and which are not necessarily approximate solutions of the problem above. This result as well as the turnpike results of [28] usually hold for model of economic dynamics which are prototypes of our dynamical system [13,18,19,25]. In particular, it holds for von Neumann-Gale model generated by a monotone process of convex type which was studied in [17].

Namely, in [30] we prove the following result.

Theorem 1.2. *Assume that property (1) of Theorem 1.1 holds and that $\epsilon > 0$. Then there exists a natural number L such that for each integer $T > L$ and each trajectory $\{x_t\}_{t=0}^T$ the following inequality holds:*

$$Card(\{t \in \{0, \dots, T\} : \ \rho(x_t, \Omega(a)) > \epsilon\}) \leq L.$$

In [31] we showed that the turnpike property established in Theorem 1.2 is stable under small perturbations. More precisely, following result was obtained in [31].

Theorem 1.3. *Assume that property (1) of Theorem 1.1 holds and that $\epsilon > 0$. Then there exists a natural number Q and $\delta > 0$ such that for each integer $T > Q$, each function $\psi : X \to R^1$ satisfying*

$$|\psi(z) - \phi(z)| \leq \delta, \ z \in X$$

and each sequence $\{x_t\}_{t=0}^T$ such that for all integers $t = 0, \dots, T-1$,

$$\psi(x_{t+1}) \leq \psi(x_t)$$

and

$$\rho_1((x_t, x_{t+1}), graph(a)) \leq \delta$$

the following inequality holds:

$$Card(\{t \in \{0, \dots, T\} : \ \rho(x_t, \Omega(a)) > \epsilon\}) \leq Q.$$

It should be mentioned that turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [21]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path and a turnpike). This property was further investigated for optimal trajectories of models of economic dynamics. See, for example, [13, 19, 25] and the references mentioned there. Recently it was shown that the turnpike phenomenon holds for many important classes of problems arising in various areas of research [6,10–12,14,15,22,23,29]. For related infinite horizon problems see [1–5, 8, 9, 20, 25].

In this paper we obtain a strong version of Theorem 1.3 assuming that the following property holds which was introduced in [28].

(P1) If $x_1, x_2 \in \Omega(a)$ satisfies $\phi(x_1) = \phi(x_2)$, then $x_1 = x_2$.

Note that for models of economic growth which are prototype of our dynamical system property (P1) holds [13, 18, 19, 25].

We prove the following turnpike result.

Theorem 1.4. *Assume that property (P1) and property (1) of Theorem 1.1 hold and that $\epsilon > 0$. Then there exist natural numbers L, Q and $\delta > 0$ such that for every natural number $T > L$, every function $\psi : X \to R^1$ satisfying*

$$|\psi(z) - \phi(z)| \leq \delta, \ z \in X$$

and every sequence $\{x_t\}_{t=0}^T \subset X$ such that for all integers $t = 0, \ldots, T-1$,

$$\psi(x_{t+1}) \leq \psi(x_t)$$

and

$$\rho_1((x_t, x_{t+1}), graph(a)) \leq \delta$$

there exist nonnegative integers $a_i < b_i \leq T$, $i = 1, \ldots, q$, where $q \in \{1, \ldots, Q\}$ is an integer, and $z_i \in \Omega(a)$, $i = 1, \ldots, q$ such that

$$a_{i+1} > b_i, \ i \in \{1, \ldots, q\} \setminus \{q\},$$

for each $i \in \{1, \ldots, q\}$,

$$\rho(x_t, z_i) \leq \epsilon, \ t = a_i, \ldots, b_i$$

and that

$$Card(\{0, \ldots, T\} \setminus \cup_{i=1}^q \{a_i, \ldots, b_i\}) \leq L.$$

Theorem 1.4 is proved in Section 3 while Section 2 contains auxiliary results.

2. Auxiliary results

The following lemma was obtained in [31].

Lemma 2.1. *Assume that property (1) of Theorem 1.1 holds and that $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number L such that for each integer $T > 2L$ and each sequence $\{x_t\}_{t=0}^T$ which satisfies for all integers $t = 0, \ldots, T-1$,*

$$|\phi(x_{t+1}) - \phi(x_t)| \leq \delta$$

and

$$\rho_1((x_t, x_{t+1}), graph(a)) \leq \delta$$

the following inequality holds:

$$\rho(x_t, \Omega(a)) \le \epsilon,\ t = L, \dots, T - L.$$

Lemma 2.2. *Assume that property (1) of Theorem 1.1 and property (P1) hold and that $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number L such that for each integer $T > 2L$, each function $\psi : X \to R^1$ satisfying*

$$|\psi(z) - \phi(z)| \le \delta,\ z \in X$$

and each sequence $\{x_t\}_{t=0}^T \subset X$ such that for all integers $t = 0, \dots, T - 1$,

$$\psi(x_{t+1}) \le \psi(x_t),$$

$$\rho_1((x_t, x_{t+1}), graph(a)) \le \delta$$

and

$$\psi(x_0) \le \psi(x_T) + \delta$$

there exists $z \in \Omega(a)$ such that for each $t \in \{L, \dots, T - L\}$,

$$\rho(x_t, z) \le \epsilon.$$

Proof. Property (P1) implies that there exists

$$\epsilon_0 \in (0, \epsilon/4)$$

such that the following property holds:

(i) for each $z_1, z_2 \in \Omega(a)$ satisfying

$$|\phi(z_1) - \phi(z_2)| \le \epsilon_0$$

we have $\rho(z_1, z_2) \le \epsilon/4$.

By the uniform continuity of ϕ, there exists

$$\epsilon_1 \in (0, \epsilon_0/2)$$

such that the following property holds:

(ii) for each $y_1, y_2 \in X$ satisfying $\rho(y_1, y_2) \le \epsilon_1$ we have

$$|\phi(y_1) - \phi(y_2)| \le \epsilon_0/4.$$

Lemma 2.1 implies that there exist

$$\delta \in (0, \epsilon_1/16)$$

and a natural number L such that the following property holds:

(iii) for each integer $T > 2L$ and each sequence $\{x_t\}_{t=0}^T$ which satisfies for all integers $t = 0, \dots, T - 1$,

$$|\phi(x_{t+1}) - \phi(x_t)| \le 3\delta$$

and

$$\rho_1((x_t, x_{t+1}), \text{graph}(a)) \le \delta$$

we have

$$\rho(x_t, \Omega(a)) \le \epsilon_1,\ t = L, \dots, T - L.$$

Assume that an integer $T > 2L$, a function $\psi : X \to R^1$ satisfies

$$|\psi(z) - \phi(z)| \le \delta,\ z \in X \tag{2.1}$$

and that a sequence $\{x_t\}_{t=0}^T \subset X$ satisfies for all integers $t = 0, \dots, T-1$,

$$\psi(x_{t+1}) \le \psi(x_t), \tag{2.2}$$

$$\rho_1((x_t, x_{t+1}), \mathrm{graph}(a)) \le \delta \tag{2.3}$$

and

$$\psi(x_0) \le \psi(x_T) + \delta. \tag{2.4}$$

By (2.1), (2.2) and (2.4), for all $t = 0, \dots, T-1$,

$$|\psi(x_t) - \psi(x_0)| \le \delta,$$

$$|\psi(x_t) - \psi(x_{t+1})| \le \delta,$$

$$|\phi(x_t) - \phi(x_{t+1})| \le 3\delta. \tag{2.5}$$

Property (iii), (2.3) and (2.5) imply that

$$\rho(x_t, \Omega(a)) \le \epsilon_1, \ t = L, \dots, T-L. \tag{2.6}$$

Let

$$t_0, \ t \in \{L, \dots, T-L\}.$$

In view of (2.6), there exist

$$z_0, z_1 \in \Omega(a) \tag{2.7}$$

such that

$$\rho(x_0, z_0) \le \epsilon_1, \ \rho(x_t, z) \le \epsilon_1. \tag{2.8}$$

Property (ii) and (2.8) imply that

$$|\phi(x_{t_0}) - \phi(z_0)| \le \epsilon_0/4, \ |\phi(x_t) - \phi(z)| \le \epsilon_0/4. \tag{2.9}$$

By (2.2) and (2.4),

$$|\psi(x_t) - \psi(x_{t_0})| \le \delta. \tag{2.10}$$

By (2.1) and (2.10),

$$|\phi(x_t) - \phi(x_{t_0})| \le 3\delta. \tag{2.11}$$

It follows from (2.9) and (2.11),

$$\begin{aligned} |\phi(z_0) - \phi(z)| &\le |\phi(z_0) - \phi(x_{t_0})| + |\phi(x_{t_0}) - \phi(x_t)| + |\phi(x_t) - \phi(z_0)| \\ &\le \epsilon_0/4 + \epsilon_0/4 + 3\delta < \epsilon_0. \end{aligned} \tag{2.12}$$

Property (i), (2.7) and (2.12) imply that

$$\rho(z_0, z) \le \epsilon/4. \tag{2.13}$$

In view of (2.8) and (2.13),

$$\rho(z_0, x_t) \le \rho(z_0, z) + \rho(z, x_t) \le \epsilon/4 + \epsilon_1 < \epsilon.$$

Thus

$$z_0 \in \Omega(a)$$

and

$$\rho(z_0, x_t) < \epsilon, \ t \in \{L, \dots, T-L\}.$$

Lemma 2.2 is proved. □

3. Proof of Theorem 1.4

Lemma 2.2 implies that there exist

$$\delta \in (0, \min\{\epsilon, 1\})$$

and a natural number L_0 such that the following property holds:

(a) for each integer $T > 2L_0$, each function $\psi : X \to R^1$ satisfying

$$|\psi(z) - \phi(z)| \le \delta,\ z \in X \tag{3.1}$$

and each sequence $\{x_t\}_{t=0}^T \subset X$ such that for all integers $t = 0, \dots, T-1$,

$$\psi(x_{t+1}) \le \psi(x_t), \tag{3.2}$$

$$\rho_1((x_t, x_{t+1}), \mathrm{graph}(a)) \le \delta \tag{3.3}$$

and

$$\psi(x_0) \le \psi(x_T) + \delta \tag{3.4}$$

there exists $z \in \Omega(a)$ such that for each $t \in \{L_0, \dots, T - L_0\}$,

$$\rho(x_t, z) \le \epsilon.$$

Choose an integer

$$Q > 1 + 2\delta^{-1}(\|\phi\| + 1) \tag{3.5}$$

and an integer

$$L > 2L_0 + 2 + (4L_0 + 8)(1 + 2\delta^{-1}(\|\phi\| + 1)) \tag{3.6}$$

Assume that $T > L$ is an integer, $\psi : X \to R^1$ satisfies (3.1) and that a sequence $\{x_t\}_{t=0}^T \subset X$ satisfies (3.2) and (3.3).

By induction we define a strictly increasing finite sequence of integers $t_i \in [0, T]$, $i = 0, \dots, q$. Set

$$t_0 = 0.$$

If

$$\psi(x_T) \ge \psi(x_0) - \delta,$$

then set

$$t_1 = T$$

and the construction is completed.

Assume that

$$\psi(x_T) < \psi(x_0) - \delta.$$

Evidently, there exists an integer $t_1 \in (t_0, T]$ such that

$$\psi(x_{t_1}) < \psi(x_0) - \delta$$

and that if an integer S satisfies

$$t_0 < S < t_1,$$

then

$$\psi(x_S) \ge \psi(x_0) - \delta.$$

If $t_1 = T$, then the construction is completed.

Assume that k is a natural number and that we defined a strictly increasing sequence of nonnegative integers $t_0, \dots, t_k \in [0, T]$ such that

$$t_0 = 0,\ t_k \le T$$

and that for each $i \in \{0, \dots, k-1\}$,

$$\psi(x_{t_{i+1}}) < \psi(x_{t_i}) - \delta$$

and if an integer S satisfies $t_i < S < t_{i+1}$, then

$$\psi(x_S) \ge \psi(x_{t_i}) - \delta.$$

(It is not difficult to see that our assumption holds for $k = 1$.)

If $t_k = T$, then our construction is completed. Assume that $t_k < T$. If

$$\psi(x_T) \ge \psi(x_{t_k}) - \delta,$$

then we set $t_{k+1} = T$ and our construction is completed.

Assume that

$$\psi(x_T) < \psi(x_{t_k}) - \delta.$$

Clearly, there exists an integer

$$t_{k+1} \in (t_k, T]$$

such that

$$\psi(x_{t_{k+1}}) < \psi(x_{t_k}) - \delta$$

and that if an integer S satisfies

$$t_k < S < t_{k+1},$$

then

$$\psi(x_S) \ge \psi(x_{t_k}) - \delta.$$

It is clear that the assumption made for k also holds for $k+1$. Therefore by induction, we constructed the strictly increasing finite sequence of integers $t_i \in [0, T]$, $i = 0, \dots, q$, where q is a natural number such that

$$t_0 = 0,\ t_q = T \tag{3.7}$$

and that for each i satisfying $0 \le i < q - 1$,

$$\psi(x_{t_{i+1}}) < \psi(x_{t_i}) - \delta \tag{3.8}$$

and for each $i \in \{0, \dots, q-1\}$ and each integer S satisfies $t_i < S < t_{i+1}$, we have

$$\psi(x_S) \ge \psi(x_{t_i}) - \delta. \tag{3.9}$$

By (3.1), (3.5) and (3.8),

$$\begin{aligned} 2\|\phi\| + 2 \ge 2\|\psi\| &\ge \psi(x_{t_0}) - \psi(x_{t_{q-1}}) \\ &= \sum\{\psi(x_{t_i}) - \psi(x_{t_{i+1}}) :\ i \text{ is an integer},\ 0 \le i \le q-2\} \ge \delta(q-1) \end{aligned}$$

and

$$q \le 1 + 2\delta^{-1}(\|\phi\| + 1) < Q. \tag{3.10}$$

Set

$$E = \{i \in \{0, \dots, q-1\} :\ t_{i+1} - t_i \ge 2L_0 + 4\}. \tag{3.11}$$

Let

$$i \in E. \tag{3.12}$$

By (3.11) and (3.12),

$$t_{i+1} - 1 - t_i \geq 2L_0 + 3. \tag{3.13}$$

Equations (3.9) and (3.13) imply that

$$\psi(x_{t_{i+1}-1}) \geq \psi(x_{t_i}) - \delta. \tag{3.14}$$

Equations (3.1)-(3.3), (3.13), (3.14) and property (a) applied to the program $\{x_t\}_{t=t_i}^{t_{i+1}-1}$ imply that there exists

$$z_i \in \Omega(a)$$

such that

$$\rho(x_t, z_i) \leq \epsilon,\ t = t_i + L_0, \dots, t_{i+1} - 1 - L_0.$$

Set

$$a_i = t_i + L_0,\ b_i = t_{i+1} - L_0 - 1.$$

By (3.6), (3.7), (3.10), (3.11) and the equation above,

$$\begin{aligned}
&\mathrm{Card}(\{0, \dots, T\} \setminus \cup_{i \in E}\{a_i, \dots, b_i\}) \\
&\quad \leq \mathrm{Card}(\cup\{\{t_i, \dots, t_{i+1}\} :\ i \in \{0, \dots, q-1\} \setminus E\}) \\
&\qquad + \mathrm{Card}(\cup\{\{t_i, \dots, t_i + L_0 - 1\} \cup \{t_{i+1} - L_0, \dots, t_{i+1}\} :\ i \in E\}) \\
&\quad \leq (2L_0 + 4)(2 + 4\delta^{-1}(\|\phi\| + 1)) < L.
\end{aligned}$$

Theorem 1.4 is proved.

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Manuscript received November 21 2021
revised February 26 2022

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