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$L_*^{1\le q<\infty}$-LIFT OF L^1-SPACE VIA THE FRACTIONAL HEAT EQUATION

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Abstract. This paper presents a geometric capacity analysis of the weak solution $u = R_\alpha f$ to the fractional heat equation

$$\begin{cases}(\partial_t + (-\Delta_x)^\alpha)u(t,x) = 0 & \forall \ \ (t,x) \in \mathbb{R}^{1+n}_+;\\ u(0,x) = f(x) & \forall \ \ x \in \mathbb{R}^n,\end{cases}$$

subject to $f \in L^1(\mathbb{R}^n)$ & $R_\alpha f \in L_*^{1\le q<\infty}(\mathbb{R}^{1+n}_+, \mu)$ — the weak q-Lebesgue space on the upper-half space $\mathbb{R}^{1+n}_+$ with respect to a given nonnegative Radon measure μ. After stating and validating Theorem 1.1 & Corollary 1.2, we also address additional aspects of these two results.

1. Statement of Theorem 1.1 & Corollary 1.2

In taking the limit of p to 1 in Theorem 3.1 [9], we will prove the following theorem.

Theorem 1.1. *Let $0 < \alpha < 1 \le q < \infty$ and $\mu \in \mathcal{M}_+(\mathbb{R}^{1+n}_+)$. Then*

$$R_\alpha : L^1(\mathbb{R}^n) \to L^q_*(\mathbb{R}^{1+n}_+, \mu) \text{ is continuous}$$
$$\iff \big(\mu(K)\big)^{\frac{1}{q}} \lesssim C_1^{(R_\alpha)}(K) \ \ \forall \ \ \text{compact} \ \ K \subset \mathbb{R}^{1+n}_+.$$

Consequently,

$$\sup_{\text{compact } K\subset\mathbb{R}^{1+n}_+} \frac{\big(\mu(K)\big)^{\frac{1}{q}}}{C_1^{(R_\alpha)}(K)} < \infty \tag{1.1}$$
$$\Longrightarrow \|R_\alpha 1_\Omega\|_{L^q_*(\mathbb{R}^{1+n}_+,\mu)} \lesssim L^{(n)}(\Omega) \ \ \forall \ \ \text{measurable} \ \ \Omega \subset \mathbb{R}^n.$$

In this paper, we use the following conventions:

- $U \lesssim V$ stands for $U \le cV$ with a constant $c > 0$; moreover $U \approx V$ means $U \lesssim V \lesssim U$.

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- $\mathcal{M}_+(\mathbb{R}^{1+n}_+)$ is the class of all nonnegative Radon measures μ with the total variation $\|\mu\| < \infty$ on the upper-half-space $\mathbb{R}^{1+n}_+ = (0,\infty)\times\mathbb{R}^n$ equipped with the $(1+n)$-dimensional Lebesgue measure $L^{(1+n)}$.
- $(-\Delta_x)^\alpha$ denotes the fractional power of the spatial Laplacian which is defined by

$$(-\Delta_x)^\alpha u(\cdot,x) = \mathcal{F}^{-1}(|\xi|^{2\alpha}\mathcal{F}u(\cdot,\xi))(x) \ \ \forall\ x\in\mathbb{R}^n,$$

  where $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ is its inverse.
- If $\mathbb{R}^n \ni y \mapsto dy$ is the differential of the n-dimensional Lebesgue measure $L^{(n)}$, then

$$u(t,x) = R_\alpha f(t,x) = e^{-t(-\Delta_x)^\alpha} f(x) = \int_{\mathbb{R}^n} K_t^{(\alpha)}(x-y)f(y)dy,$$

  along with

$$\begin{cases} R_\alpha 1_\Omega(t,x) = \int_\Omega K_t^{(\alpha)}(x-y)dy \ \forall \text{ indicator } 1_\Omega \text{ of Borel set } \Omega\subset\mathbb{R}^n;\\ \int_\Omega R_\alpha 1_\Omega(t,x)\,dx = \text{the fractional heat content of } \Omega \text{ at time } t\in(0,\infty) \ \ (\text{cf.}[7, 26]);\end{cases}$$

  solves the heat equation of fractional order:

$$\begin{cases} \big(\partial_t + (-\Delta_x)^\alpha\big)u(t,x) = 0 & \forall\ \ (t,x)\in\mathbb{R}^{1+n}_+;\\ u(0,x) = f(x) & \forall\ \ x\in\mathbb{R}^n.\end{cases}$$

- With the help of the standard gamma function

$$(0,\infty)\ni z\mapsto \Gamma(z) = \int_0^\infty x^z e^{-x}dx,$$

  the fractional heat kernel

$$K_t^{(\alpha)}(x) := (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\cdot y - t|y|^{2\alpha}}dy \ \ \ \forall\ (t,x)\in\mathbb{R}^{1+n}_+$$

  is the fundamental solution to the heat equation of fractional order - namely -

$$\begin{cases} \big(\partial_t + (-\Delta_x)^\alpha\big)K_t^{(\alpha)}(x) = 0 & \forall\ \ (t,x)\in\mathbb{R}^{1+n}_+;\\ K_0^{(\alpha)}(x) = \delta_0(x) = \text{the Diract mass at the origin} & \forall\ \ x\in\mathbb{R}^n.\end{cases}$$

  Respectively, the middle-point $\alpha = 2^{-1}$ and the endpoint $\alpha = 1$ of $K_t^{(\alpha)}(x)$ lead to the standard Poisson kernel

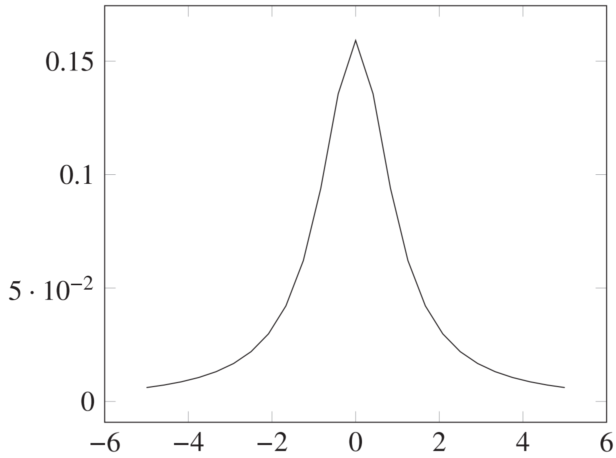
$$K_t^{(\frac{1}{2})}(x) = \pi^{-\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)t(t^2+|x|^2)^{-\frac{n+1}{2}}$$

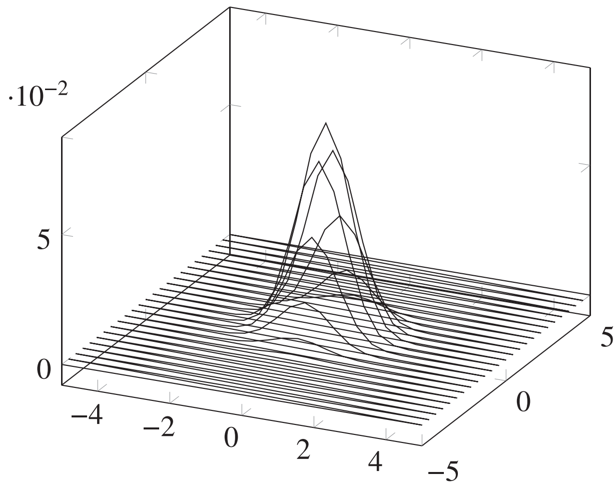
  whose situation when $n=1$ & $t=1$ is graphically as shown below and the heat kernel

$$K_t^{(1)}(x) = (4\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}}$$

  whose situation when $n=2$ & $t=2^{-2}$ is graphically shown below

According to [12, 14, 20, 21, 22, 23, 27, 28, 25, 33], there is no explicit formula valid for $K_t^{(\alpha)}(x) > 0$ except $\alpha = 2^{-1}$, but there is (see e.g. [2, 3, 8, 13, 17, 31, 32])





$$\begin{cases} K_t^{(\alpha)}(x) \approx t(t^{\frac{1}{2\alpha}} + |x|)^{-2\alpha-n} & \forall\ (t,x) \in \mathbb{R}_+^{1+n}; \\ \int_{\mathbb{R}^n} K_t^{(\alpha)}(x)\,dx = 1 & \forall\ t \in (0,\infty); \\ \lim_{t\to 0} K_t^{(\alpha)}(x) = \delta_0(x) & \forall\ x \in \mathbb{R}^n. \end{cases}$$

- $L_*^q(\mathbb{R}_+^{1+n}, \mu)$ stands for the weak L^q-space on $\mathbb{R}_+^{1+n}$ with respect to μ - i.e. - it consists of all functions g on $\mathbb{R}_+^{1+n}$ with

$$\|g\|_{L_*^q(\mathbb{R}_+^{1+n},\mu)} := \sup_{\lambda>0} \lambda\Big(\mu(\{(t,x) \in \mathbb{R}_+^{1+n} :\ |g(t,x)| > \lambda\}\Big)^{\frac{1}{q}} < \infty. \tag{1.2}$$

Naturally, if $\mu = L^{(1+n)}$, then $L_*^q(\mathbb{R}_+^{1+n}, \mu)$ is simply written as $L_*^q(\mathbb{R}_+^{1+n})$.

- Given a compact set $K \subset \mathbb{R}_+^{1+n}$ and its characteristic function 1_K, let

$$C_1^{(R_\alpha)}(K) := \inf\Big\{\|f\|_{L^1(\mathbb{R}^n)} :\ C_0^\infty(\mathbb{R}^n) \ni f \ge 0\ \&\ R_\alpha f \ge 1_K\Big\}, \tag{1.3}$$

where $C_0^\infty(\mathbb{R}^n)$ comprises all infinitely smooth functions f on $\mathbb{R}^n$ with compact support. Naturally, if O is an open subset of $\mathbb{R}^{1+n}_+$, then

$$C_1^{(R_\alpha)}(O) := \sup_{\text{compact } K\subset O} C_1^{(R_\alpha)}(K),$$

and hence for an arbitrary set $A \subset \mathbb{R}^{1+n}_+$, one has

$$C_1^{(R_\alpha)}(A) := \inf_{\text{open } O\supset A} C_1^{(R_\alpha)}(O).$$

Importantly, Theorem 1.1 deduces the (n,α)-order isocapacitary inequality as described below.

Corollary 1.2. *If* $0 < \alpha < 1$, $0 < \tau_1 < \tau_2 < \infty$, $L^{(n+1)}$ *is the Lebesgue measure on* $\mathbb{R}^{1+n}_+$, K *is a compact subset of* $[\tau_1, \tau_2]\times\mathbb{R}^n$, *and* $H^{(n,\alpha)}(K)$ *denotes the* (n,α)*-order Hausdorff capacity of* K *defined by*

$$\begin{cases} H_\infty^{(n,\alpha)}(K) = \inf\Big\{\sum_{j=1}^\infty r_j^n : \ K \subset \cup_{j=1}^\infty B_{r_j}^{(\alpha)}(t_j,x_j)\Big\}; \\ B_{r_j}^{(\alpha)}(t_j,x_j) = \Big\{(s,y): \ r_j^{2\alpha} < s-t_j < 2r_j^{2\alpha} \ \& \ |y-x_j| < 2^{-1}r_j\Big\}, \end{cases}$$

then

$$L^{(1+n)}(K) \lesssim \big(C_1^{(R_\alpha)}(K)\big)^{\frac{2\alpha+n}{n}} \lesssim \big(H^{(n,\alpha)}(K)\big)^{\frac{2\alpha+n}{n}}. \tag{1.4}$$

2. Validation of Theorem 1.1 & Corollary 1.2

This section is devoted to proving Theorem 1.1 & Corollary 1.2.

Proof of Theorem 1.1. It suffices to check the equivalence which readily derives (1.1).

On the one hand, suppose that

$$R_\alpha : L^1(\mathbb{R}^n) \to L^q_*(\mathbb{R}^{1+n}_+,\mu) \quad \text{is continuous.}$$

Then

$$\|R_\alpha f\|_{L^q_*(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|f\|_{L^1(\mathbb{R}^n)} \ \ \forall \ \ f \in C_0^\infty(\mathbb{R}^n). \tag{2.1}$$

If

$$\begin{cases} K \text{ is a compact subset of } \mathbb{R}^{1+n}_+; \\ 0 \le f \in C_0^\infty(\mathbb{R}^n); \\ R_\alpha f \ge 1_K, \end{cases}$$

then (2.1) & (1.2) derive

$$\big(\mu(K)\big)^{\frac{1}{q}} \le \|R_\alpha f\|_{L^q_*(\mathbb{R}^{1+n}_+,\mu)} \lesssim \|f\|_{L^1(\mathbb{R}^n)},$$

and hence the first formula of (1.3) implies

$$\big(\mu(K)\big)^{\frac{1}{q}} \lesssim C_1^{(R_\alpha)}(K). \tag{2.2}$$

On the other hand, assume that (2.2) holds for any compact set $K\subset\mathbb{R}^{1+n}_+$. If

$$\begin{cases} f\in C_0^\infty(\mathbb{R}^n);\\ \lambda>0;\\ \mathrm{L}(R_\alpha f,\lambda):=\{(t,x)\in\mathbb{R}^{1+n}_+ : |R_\alpha f(t,x)|>\lambda\},\end{cases}$$

then (2.2), coupled with the first formula of (1.3), implies

$$\mu\big(\mathrm{L}(R_\alpha f,\lambda)\big)\lesssim\Big(C_1^{(R_\alpha)}\big(\mathrm{L}(R_\alpha f,\lambda)\big)\Big)^q\lesssim\|f/\lambda\|^q_{L^1(\mathbb{R}^n)},$$

and hence (2.1) follows from (1.2). □

Proof of Corollary 1.2. The left-hand inequality of (1.4) follows from both the necessary part "$\Longrightarrow$" of Theorem 1.1 with

$$\mu=L^{(1+n)}\big|_{[\tau_1,\tau_2]\times\mathbb{R}^n} - \text{the restriction of } L^{(1+n)} \text{ to } [\tau_1,\tau_2]\times\mathbb{R}^n$$

and the continuity of

$$R_\alpha: L^1(\mathbb{R}^n)\to L_*^{\frac{2\alpha+n}{n}}([t_1,t_2]\times\mathbb{R}^n).$$

Needless to say, we are required to verify the last continuity. Given

$$\begin{cases}(\lambda,t,r,x)\in(0,\infty)\times[\tau_1,\tau_2]\times(0,\infty)\times\mathbb{R}^n;\\ \epsilon=\frac{2\alpha n}{2\alpha+n}<n;\\ q=\frac{n}{n-\epsilon}=\frac{n+2\alpha}{n},\end{cases}$$

we utilize the approximate estimate of $K_t^{(\alpha)}(y-x)$ to estimate

$$\begin{aligned}\lambda&<|R_\alpha f(t,x)|\\ &\le\int_{\mathbb{R}^n}|K_t^{(\alpha)}(y-x)||f(y)|dy\\ &\lesssim\int_{\mathbb{R}^n}t(t^{\frac{1}{2\alpha}}+|y-x|)^{-2\alpha-n}|f(y)|dy\\ &\lesssim\int_{\mathbb{R}^n}t(t^{\frac{1}{2\alpha}}+|y-x|)^{-2\alpha-\epsilon}(t^{\frac{1}{2\alpha}}+|y-x|)^{\epsilon-n}|f(y)|dy\\ &\lesssim t^{-\frac{\epsilon}{2\alpha}}\int_{\mathbb{R}^n}|y-x|^{\epsilon-n}|f(y)|dy.\end{aligned}$$

Since the ϵ-Riesz potential operator

$$\mathbb{R}^n\ni x\mapsto I_\epsilon f(x):=\int_{\mathbb{R}^n}|y-x|^{\epsilon-n}f(y)dy$$

continuously maps $L^1(\mathbb{R}^n)$ to $L_*^{\frac{n}{n-\epsilon}}(\mathbb{R}^n)$ (cf. [1, Theorem 5.1] or [29, Lemma 2.1])), we have

$$L^{(n)}\Big(\big\{x\in\mathbb{R}^n:|I_\epsilon f(x)|>\lambda t^{\frac{\epsilon}{2\alpha}}\big\}\Big)\lesssim\left(\frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda t^{\frac{\epsilon}{2\alpha}}}\right)^q,$$

thereby finding

$$\begin{aligned}
&L^{(1+n)}\Big(\big\{(t,x)\in[\tau_1,\tau_2]\times\mathbb{R}^n:\lambda<|R_\alpha f(t,x)|\big\}\Big)\\
&\lesssim\int_{\tau_1}^{\tau_2}L^{(n)}\Big(\big\{x\in\mathbb{R}^n:|I_\epsilon f(x)|>\lambda t^{\frac{\epsilon}{2\alpha}}\big\}\Big)dt\\
&\lesssim\int_{\tau_1}^{\tau_2}\left(\frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda t^{\frac{\epsilon}{2\alpha}}}\right)^q dt\\
&\lesssim\Big(\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}\Big)^q\int_{\tau_1}^{\tau_2}t^{-\frac{\epsilon q}{2\alpha}}dt.
\end{aligned}$$

So, we get the desired result:

$$\|R_\alpha f\|_{L_*^{\frac{2\alpha+n}{n}}([\tau_1,\tau_2]\times\mathbb{R}^n)}\lesssim\|f\|_{L^1(\mathbb{R}^n)}. \tag{2.3}$$

The right-hand inequality of (1.4) follows from [9, Corollary 1(i)]:

$$H_\infty^{(n,\alpha)}(K)\gtrsim C_1^{(R_\alpha)}(K)\ \ \forall\ \text{ compact set }\ K\subset\mathbb{R}_+^{1+n}.$$

□

3. Beyond Theorem 1.1 & Corollary 1.2

This section presents six perspectives on geometric capacity analysis of Theorem 1.1 & Corollary 1.2.

Remark 3.1. If $\Omega\subset\mathbb{R}^{n\ge2}$ is a uniformly $C^{1,1}$-regular bounded domain - namely - there exist two positive constants c_1,c_2 such that for any $x\in\partial\Omega$ (the boundary of Ω) the set

$$\partial\Omega\cap\{y\in\mathbb{R}^n:|y-x|<c_1\}$$

is a graph of a $C^{1,1}$-function ϕ with

$$\|\nabla\phi\|_{L^\infty(\mathbb{R}^{n-1})}\le c_2,$$

and if $H^{(n-1)}$ is the $(n-1)$-dimensional Hausdorff measure, then [26, Theorem 1.2] & [15] give the following three-fold assertion.

- If $0<\alpha<2^{-1}$, then

$$\begin{cases}\lim_{t\to0}t^{-1}\int_{\mathbb{R}^n\setminus\Omega}R_\alpha 1_\Omega(t,x)\,dx=\frac{\sin(\pi\alpha)\Gamma(\alpha+\frac{n}{2})}{\alpha^{-1}2^{-2\alpha}\pi^{1+\frac{n}{2}}}\int_\Omega\int_{\mathbb{R}^n\setminus\Omega}\frac{dxdy}{|y-x|^{2\alpha+n}};\\ L^{(n)}(\Omega)\le\kappa_{n,\alpha}\left(\int_\Omega\int_{\mathbb{R}^n\setminus\Omega}\frac{dxdy}{|y-x|^{2\alpha+n}}\right)^{\frac{n}{n-2\alpha}}\ \text{ for a sharp constant }\kappa_{n,\alpha}.\end{cases} \tag{3.1}$$

  Actually, the second inequality of (3.1) is the fractional $0<2\alpha<1$ isoperimetric inequality.
- If $\alpha=2^{-1}$, then

$$\lim_{t\to0}(-t\ln t)^{-1}\int_{\mathbb{R}^n\setminus\Omega}R_{\frac12}1_\Omega(t,x)\,dx=\pi^{-1}H^{(n-1)}(\partial\Omega). \tag{3.2}$$

  Thus, (3.2) exists as the limiting case $\alpha\to2^{-1}$ of the first formula in (3.1).

- If $2^{-1}<\alpha<1$, then

$$\begin{cases}\lim_{t\to 0} t^{-\frac{1}{2\alpha}}\int_{\mathbb{R}^n\setminus\Omega} R_\alpha 1_\Omega(t,x)\,dx = \pi^{-1}\Gamma\big(\frac{2\alpha-1}{2\alpha}\big)H^{(n-1)}(\partial\Omega);\\ \sup_{t\in(0,\infty)} t^{-\frac{1}{2\alpha}}\int_{\mathbb{R}^n\setminus\Omega} R_\alpha 1_\Omega(t,x)\,dx \le \pi^{-1}\Gamma\big(\frac{2\alpha-1}{2\alpha}\big)H^{(n-1)}(\partial\Omega).\end{cases}\tag{3.3}$$

So, the second inequality of (3.3) may be treated as a nice addition to Corollary 1.2.

Remark 3.2. A measurable set $\Omega\subset\mathbb{R}^{n\ge 2}$ is said to be of finite perimeter provided that the distributional gradient $D1_\Omega$ of 1_Ω is a vector-valued Radon measure on $\mathbb{R}^n$ with finite total variation (cf. (1.2))

$$|D1_\Omega|(\mathbb{R}^n) := \sup\left\{\int_E \mathrm{div}\vec{\psi}(x)\,dx:\ \vec{\psi}\in C_0^1(\mathbb{R}^n,\mathbb{R}^n)\ \&\ \|\vec{\psi}\|_{L^\infty(\mathbb{R}^n)}\le 1\right\}<\infty.$$

Of course, if Ω has a smooth boundary $\partial\Omega$, then $|D1_\Omega|(\mathbb{R}^n)$ coincides with $H^{(n-1)}(\partial\Omega)$; see also [6] for a concept of the perimeter on a measurable space with respect to a σ-additive function and its variational mean curvature.

- In accordance with [5, Theorem] or [16, Theorem 2.1], if $\Omega\subset\mathbb{R}^n$ is a measurable set of finite perimeter, then there exists a so-called variational mean curvature $h\in L^1(\mathbb{R}^n)$ such that the variational inequality

$$|D1_\Omega|(\mathbb{R}^n)-\int_\Omega h(x)\,dx \le |D1_\Lambda|(\mathbb{R}^n)-\int_\Lambda h(x)\,dx$$

holds for any Lebesgue measurable set

$$\Lambda\subset\mathbb{R}^n\quad\text{with}\quad |D1_\Lambda|(\mathbb{R}^n)<\infty.$$

Moreover, [4] derives another variational mean curvature $h_\Omega\in L^1(\mathbb{R}^n)$ such that

$$\|1_\Omega h_\Omega\|_{L^1(\mathbb{R}^n)}\le\|h_\Omega\|_{L^1(\mathbb{R}^n)}=|D1_\Omega|(\mathbb{R}^n)=\|1_{\mathbb{R}^n\setminus\Omega}h_\Omega\|_{L^1(\mathbb{R}^n)}\le\|h\|_{L^1(\mathbb{R}^n)}.$$

However, (2.3) indicates that $\|1_\Omega h_\Omega\|_{L^1(\mathbb{R}^n)}$ has a nice lower bound as given below

$$\|1_\Omega h_\Omega\|_{L^1(\mathbb{R}^n)}\gtrsim\|R_\alpha 1_\Omega h_\Omega\|_{L_*^{\frac{2\alpha+n}{n}}([\tau_1,\tau_2]\times\mathbb{R}^n)}.$$

- Interestingly, [16, Remark 2.3] derives

$$\|1_\Omega h\|_{L^1(\mathbb{R}^n)}\ge|D1_\Omega|(\mathbb{R}^n).$$

This last inequality corresponds to (2.3)-induced inequality

$$\|1_\Omega h\|_{L^1(\mathbb{R}^n)}\gtrsim\|R_\alpha 1_\Omega h\|_{L_*^{\frac{2\alpha+n}{n}}([\tau_1,\tau_2]\times\mathbb{R}^n)}$$

whose right-hand-side may be more or less treated as a fractional variant of the perimeter of Ω.

- Even more interestingly, if h is continuous at $x\in\partial\Omega$ which is smooth near x, then the classical mean curvature (the arithmetic average of $n-1$ principle curvatures) $H_{\partial\Omega}(x)$ of $\partial\Omega$ at x is given by $(n-1)^{-1}h(x)$, and hence the above-achieved function $(n-1)^{-1}h$ may be regarded as a variational mean curvature of $\partial\Omega$. Consequently, if

$$
\begin{aligned}
f(x) &= (-\Delta_x)^{-\alpha}h(x)\\
&= \left(\frac{\Gamma(\frac{n-2\alpha}{2})}{\pi^{\frac{4\alpha+n}{2}}2^{4\alpha}\Gamma(\alpha)}\right)\int_{\mathbb{R}^n} h(y)|x-y|^{2\alpha-n}dy \qquad (3.4)\\
&= \left(\frac{\Gamma(\frac{n-2\alpha}{2})}{\pi^{\frac{4\alpha+n}{2}}2^{4\alpha}\Gamma(\alpha)}\right)I_{2\alpha}h(x),
\end{aligned}
$$

or equivalently,

$$h(x) = (-\Delta_x)^{\alpha}f(x),$$

then the well-known fractional weak Sobolev inequality holds

$$\|(-\Delta_x)^{-\alpha}h\|_{L_*^{\frac{n}{n-2\alpha}}(\mathbb{R}^n)} = \|f\|_{L_*^{\frac{n}{n-2\alpha}}(\mathbb{R}^n)} \lesssim \|h\|_{L^1(\mathbb{R}^n)}, \qquad (3.5)$$

and hence $(n-1)^{-1}(-\Delta_x)^{\alpha}$ exists in the sense of the variational mean curvature operator (cf. [30]). Evidently, (3.5) may be viewed as an analytic version of the geometric inequality described in the second inequality in (3.1).

Remark 3.3. As a kind of the limiting case $1 < p \le q \to 1$ of Adams' [1, Theorem 4.1], we find that if $0 < \beta \le \alpha < 1$ and K is the same as in Corollary 1.2, then there holds the capacity inequality

$$C_1^{(R_\beta)}(K) \lesssim C_1^{(R_\alpha)}(K). \qquad (3.6)$$

As a matter of fact, for

$$
\begin{cases}
C_0^\infty(\mathbb{R}^n) \ni f \ge 0;\\
R_\alpha f \ge 1_K;\\
0 < \tau_1 \le t \le \tau_2 < \infty,
\end{cases}
$$

we estimate

$$
\begin{aligned}
R_\beta f(x) &\approx \int_{\mathbb{R}^n} t(t^{\frac{1}{2\beta}} + |y-x|)^{-2\beta-n}f(y)dy\\
&= \int_{\mathbb{R}^n} t\big(t^{\frac{1}{2\alpha}}t^{\frac{1}{2\beta}-\frac{1}{2\alpha}} + |y-x|\big)^{-2\alpha-n}\big(t^{\frac{1}{2\beta}} + |y-x|\big)^{2(\alpha-\beta)}f(y)dy\\
&\ge \int_{\mathbb{R}^n} t\big(t^{\frac{1}{2\alpha}}\tau_2^{\frac{1}{2\beta}-\frac{1}{2\alpha}} + |y-x|\big)^{-2\alpha-n}\tau_1^{\frac{\alpha-\beta}{\beta}}f(y)dy\\
&\gtrsim R_\alpha f(x)\\
&\ge 1_K(x),
\end{aligned}
$$

thereby arriving at (3.6) through the definitions of

$$C_1^{(R_\alpha)}(K) \ \ \& \ \ C_1^{(R_\beta)}(K).$$

Remark 3.4. The S_α-case of [24, Theorem 1.1] can be successfully driven to the limit $p\to 1$. More precisely, for $g\in L^{1+n}(\mathbb{R}^{1+n}_+)$ suppose

$$\begin{aligned} S_\alpha g(t,x) &:= \int_0^t e^{-(t-s)(-\Delta_x)^\alpha} g(s,x)ds \\ &= \int_{\mathbb{R}^n}\left(\int_0^t K^{(\alpha)}_{t-s}(x-y)g(s,y)ds\right)dy \ \ \forall\ \ (t,x)\in\mathbb{R}^{1+n}_+, \end{aligned}$$

which is the weak solution $u(t,x)$ of the inhomogeneous heat equation of fractional order:

$$\big(\partial_t+(-\Delta_x)^\alpha\big)u(t,x)=g(t,x)\ \ \forall\ \ (t,x)\in\mathbb{R}^{1+n}_+.$$

- Firstly, an argument similar to that for Theorem 1.1 derives that

$$\begin{aligned} &S_\alpha: L^1(\mathbb{R}^{1+n}_+)\to L^q_*(\mathbb{R}^{1+n}_+,\mu) \text{ is continuous} \\ &\Longleftrightarrow \big(\mu(K)\big)^{\frac1q}\lesssim C_1^{(S_\alpha)}(K):=\inf\left\{\|g\|_{L^1(\mathbb{R}^{1+n}_+)}:\ C_0^\infty(\mathbb{R}^{1+n}_+)\ni g\ge 0\ \&\ S_\alpha g\ge 1_K\right\} \\ &\quad\forall\ \text{ compact }\ K\subset\mathbb{R}^{1+n}_+, \end{aligned}$$

and, consequently,

$$\sup_{\text{compact }K\subset\mathbb{R}^{1+n}_+}\frac{\big(\mu(K)\big)^{\frac1q}}{C_1^{(S_\alpha)}(K)}<\infty \tag{3.7}$$
$$\Longrightarrow \|S_\alpha 1_\Omega\|_{L^q_*(\mathbb{R}^{1+n}_+,\mu)}\lesssim L^{(1+n)}(\Omega)\ \ \forall\ \text{ measurable }\ \Omega\subset\mathbb{R}^{1+n}_+.$$

Obviously, (3.7) is the S_α-analogue of (1.1).

- Secondly, in order to get an $S^{(\alpha)}$-analogue of Corollary 1.2, we choose

$$\mu=L^{(1+n)}\big|_{[\tau_1,\tau_2]\times\mathbb{R}^n}\ \ \&\ \ 0<\tau_1<\tau_2<\infty,$$

thereby applying

$$\begin{cases}\tau_2-\tau_1\le|t-s|\le\tau_1+\tau_2 & \forall\ \ \tau_1\le t\le\tau_2\ \&\ 0\le s\le t;\\ g(-s,y):=g(s,y) & \forall\ \ (s,y)\in\mathbb{R}^{1+n}_+,\end{cases}$$

to obtain

$$\begin{aligned} |S_\alpha g(t,x)| &\lesssim \int_{\mathbb{R}^n}\int_0^t (t-s)\big((t-s)^{\frac{1}{2\alpha}}+|y-x|\big)^{-2\alpha-n}|g(s,y)|dsdy \\ &\lesssim \int_{\mathbb{R}^{1+n}}\big(|t-s|+|y-x|\big)^{2(1-\alpha)-(1+n)}|g(s,y)|dsdy \\ &=: I^{(1+n)}_{2(1-\alpha)}|g|(t,x)\ \ \big(\text{the } 2(1-\alpha)\text{-Reisz potential of } |g|\big). \end{aligned}$$

This in turn derives not only the weak-type embedding inequality (cf. (2.3))

$$\|S_\alpha g\|_{L_*^{\frac{1+n}{1+n-2(1-\alpha)}}([\tau_1,\tau_2]\times\mathbb{R}^n)}\lesssim\|g\|_{L^1(\mathbb{R}^{1+n}_+)},$$

but also its geometric version - the isocapacitary inequality

$$L^{(1+n)}(K)\lesssim\big(C_1^{(S_\alpha)}(K)\big)^{\frac{1+n}{1+n-2(1-\alpha)}}\ \ \forall\ \text{ compact set }\ K\subset[\tau_1,\tau_2]\times\mathbb{R}^n.$$

- Thirdly, in a manner similar to establishing (3.6) & [9, Corollary 1(i)], we can achieve that if $0<\beta\le\alpha<1$ then

$$C_1^{(S_\beta)}(K)\lesssim C_1^{(S_\alpha)}(K)\lesssim H_\infty^{(n,\alpha)}(K)\ \ \forall\ \text{ compact set }\ K\subset[\tau_1,\tau_2]\times\mathbb{R}^n.$$

Remark 3.5. In a similar manner, we can push [18, Theorem 3.1(i)] to the limit $p\to 1$. More precisely, if

$$P_\alpha f(t,x)=\frac{\Gamma(\frac{n+2\alpha}{2})}{\pi^{\frac{n}{2}}\Gamma(\alpha)}\int_{\mathbb{R}^n}\frac{t^{2\alpha}f(y)}{(t^2+|x-y|^2)^{\frac{n+2\alpha}{2}}}dy\ \ \forall\ \ (t,x)\in\mathbb{R}^{1+n}_+$$

is the Caffarelli-Silvestre solution $u(t,x)$ to the elliptic partial differential equation (cf. [13])

$$\begin{cases}\operatorname{div}_{(t,x)}\big(t^{1-2\alpha}\nabla_{(t,x)}u(t,x)\big)=0 & \forall\ \ (t,x)\in\mathbb{R}^{1+n}_+;\\ u(0,x)=f(x) & \forall\ \ x\in\mathbb{R}^n,\end{cases}$$

then

$$(-\Delta)^\alpha f(x)=\frac{\Gamma(\alpha)}{2\pi^{2\alpha}\Gamma(1-\alpha)}\lim_{t\to0}\frac{f(x)-P_\alpha f(t,x)}{(2\alpha)^{-1}t^{2\alpha}}\ \ \forall\ \ x\in\mathbb{R}^n,$$

gives an important explanation of the fractional Laplacian $(-\Delta)^\alpha$, and, consequently, if

$$C_1^{(P_\alpha)}(K):=\inf\Big\{\|f\|_{L^1(\mathbb{R}^n)}:\ C_0^\infty(\mathbb{R}^n)\ni f\ge0\ \&\ P_\alpha f\ge 1_K\Big\}$$

is the P_α-type capacity of a given compact set $K\subset\mathbb{R}^{1+n}_+$, then Theorem 1.1 & Corollary 1.2 as well as (3.6) hold for P_α. □

Remark 3.5. Since (3.4) ensures that under $f\in C_0^\infty(\mathbb{R}^n)$ there holds

$$\begin{aligned}\|(-\Delta)^\alpha_x f\|_{L^q(\mathbb{R}^n)}&\approx\big\|I_{1-2\alpha}\nabla_x f\big\|_{L^q(\mathbb{R}^n)}\\ &\approx\Big\|\int_{\mathbb{R}^n}\frac{\nabla_x f(x)}{|x-y|^{n+2\alpha-1}}\,dx\Big\|_{L^q(\mathbb{R}^n)}\ \ \forall\ \ (\alpha,q)\in(0,2^{-1})\times(1,\infty),\end{aligned}$$

according to Maz'ya's [19, Theorem 1.7], if $\Phi:\ \mathbb{R}^n\to[0,\infty)$ is continuous, $(\alpha,q)\in(0,2^{-1})\times(1,\infty)$, and μ is a nonnegative Radon measure on $\mathbb{R}^n$, then the imbedding inequality

$$\big\|I_{1-2\alpha}\nabla_x f\big\|_{L^q(\mathbb{R}^n,\mu)}\lesssim\|\Phi\nabla_x f\|_{L^1(\mathbb{R}^n)}\ \ \forall\ \ f\in C_0^\infty(\mathbb{R}^n)\tag{3.8}$$

amounts to the isoperimetric inequality

$$\Big\|\int_{\partial\Omega}\frac{\mathbf{n}_y\,dH^{(n-1)}(y)}{|x-y|^{n+2\alpha-1}}\Big\|_{L^q(\mathbb{R}^n,\mu)}\lesssim\int_{\partial\Omega}\Phi\,dH^{n-1}\quad\forall\text{ bounded open }\Omega\subset\mathbb{R}^n\text{ with smooth }\partial\Omega,\tag{3.9}$$

where $\mathbf{n}_y$ is the unit outer normal vector at $y\in\partial\Omega$. Consequently, we have the forthcoming weak-type analogue of the equivalence (3.8)$\Longleftrightarrow$(3.9).

- If

$$\big\|I_{1-2\alpha}\nabla_x f\big\|_{L^q_*(\mathbb{R}^n,\mu)} \lesssim \|\Phi\nabla_x f\|_{L^1(\mathbb{R}^n)} \quad \forall \ f\in C_0^\infty(\mathbb{R}^n), \tag{3.10}$$

then choosing f in (3.10) as a mollification of 1_Ω derives that the weak-type isoperimetric inequality

$$\left\|\int_{\partial\Omega}\frac{\mathbf{n}_y\,dH^{n-1}(y)}{|x-y|^{n+2\alpha-1}}\right\|_{L^q_*(\mathbb{R}^n,\mu)} \lesssim \int_{\partial\Omega}\Phi\,dH^{(n-1)} \tag{3.11}$$

$$\forall \ \text{bounded open } \Omega\subset\mathbb{R}^n \text{ with smooth } \partial\Omega.$$

- Conversely, suppose that (3.11) (which is weaker than (3.9)) is valid for any bounded open $\Omega\subset\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Note that (cf. [11, p. 202, Theorem 5.18])

$$\|I_{1-2\alpha}\nabla_x f\|_{L^q_*(\mathbb{R}^n,\mu)} \approx \sup_{0<\mu(E)<\infty}\big(\mu(E)\big)^{\frac{1-q}{q}}\int_E |I_{1-2\alpha}\nabla_x f|\,d\mu(x). \tag{3.12}$$

So, the co-area formula, (3.12), the L^1-Minkowski inequality, and (3.11) deduce

$$\begin{aligned}
&\big\|I_{1-2\alpha}\nabla_x f\big\|_{L^q_*(\mathbb{R}^n,\mu)}\\
&\approx \left\|\int_{\mathbb{R}}\left(\int_{\{x\in\mathbb{R}^n:f(x)=t\}}\frac{\mathbf{n}_y\,dH^{(n-1)}(y)}{|x-y|^{n+2\alpha-1}}\right)dt\right\|_{L^q_*(\mathbb{R}^n,\mu)}\\
&\approx \sup_{0<\mu(E)<\infty}\big(\mu(E)\big)^{\frac{1-q}{q}}\int_E\left|\int_{\mathbb{R}}\left(\int_{\{y\in\mathbb{R}^n:f(y)=t\}}\frac{\mathbf{n}_y\,dH^{(n-1)}(y)}{|x-y|^{n+2\alpha-1}}\right)dt\right|d\mu(x)\\
&\lesssim \int_{\mathbb{R}}\left(\sup_{0<\mu(E)<\infty}\big(\mu(E)\big)^{\frac{1-q}{q}}\int_E\left|\int_{\{y\in\mathbb{R}^n:f(y)=t\}}\frac{\mathbf{n}_y\,dH^{(n-1)}(y)}{|x-y|^{n+2\alpha-1}}\right|d\mu(x)\right)dt\\
&\approx \int_{\mathbb{R}}\left\|\int_{\{y\in\mathbb{R}^n:f(y)=t\}}\frac{\mathbf{n}_y\,dH^{(n-1)}(y)}{|x-y|^{n+2\alpha-1}}\right\|_{L^q_*(\mathbb{R}^n,\mu)}dt\\
&\lesssim \int_{\mathbb{R}}\left(\int_{\{x\in\mathbb{R}^n:f(x)=t\}}\Phi\,dH^{(n-1)}\right)dt\\
&\approx \int_{\mathbb{R}^n}\Phi(x)|\nabla_x f(x)|\,dx\\
&= \|\Phi\nabla_x f\|_{L^1(\mathbb{R}^n)} \quad \forall \ f\in C_0^\infty(\mathbb{R}^n),
\end{aligned}$$

whence yielding (3.10) (which is weaker than (3.8)).

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