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NONLINEAR POTENTIAL ESTIMATES FOR SUBLINEAR PROBLEMS WITH APPLICATIONS TO ELLIPTIC SEMILINEAR AND QUASILINEAR EQUATIONS

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Dedicated to Vladimir Maz'ya on the occasion of his 85th birthday

Abstract. We give a survey of nonlinear potential estimates and their applications obtained recently for positive solutions to sublinear problems of the type

$$u = \mathbf{G}(\sigma u^q) + f \quad \text{in } \Omega,$$

where $0 < q < 1$, $\sigma \geq 0$ is a Radon measure in Ω, $f \geq 0$ is a measurable function, and $\mathbf{G}$ is a linear integral operator with positive kernel G on $\Omega \times \Omega$. For quasi-metric (or quasi-metrically modifiable) kernels G, these bilateral pointwise estimates yield existence criteria and uniqueness of solutions $u \in L^q_{\rm loc}(\Omega, \sigma)$.

Applications are considered to semilinear elliptic equations involving the (fractional) Laplacian,

$$(-\Delta)^{\frac{\alpha}{2}} u = \sigma u^q + \mu \quad \text{in } \Omega, \qquad u = 0 \text{ in } \Omega^c.$$

Here $0 < q < 1$, μ, $\sigma \geq 0$ are Radon measures, and Ω is a bounded uniform domain in $\mathbb{R}^n$, if $0 < \alpha \leq 2$, or the entire space $\mathbb{R}^n$, a ball or half-space, if $0 < \alpha < n$.

Analogues of these results are presented for elliptic equations involving the p-Laplace operator on the entire space $\mathbb{R}^n$,

$$-\Delta_p u = \sigma u^q + \mu \quad \text{in } \mathbb{R}^n, \qquad \liminf_{x \to \infty} u(x) = 0,$$

where $0 < q < p - 1$, and $\mu, \sigma \geq 0$ are Radon measures. More general quasilinear equations with $\mathcal{A}$-Laplace operators $\mathrm{div}\mathcal{A}(x, \nabla u)$ in place of Δ_p are covered as well.

1. Introduction

1.1. **Sublinear problems.** We give a survey of recent developments in the theory of sublinear problems based on nonlinear potential methods ([8], [29], [30], [33], [34]). In particular, we focus on bilateral pointwise estimates of solutions, which yield existence criteria and uniqueness results, for sublinear integral equations of the type

$$u = \mathbf{G}(u^q d\sigma) + f, \quad 0 < u < \infty \quad d\sigma\text{-a.e.} \quad \text{in } \Omega, \tag{1.1}$$

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where $0 < q < 1$, $f \geq 0$ is a Borel measurable function, and $\sigma \in \mathcal{M}^+(\Omega)$, the cone of locally finite Radon measures in Ω. (The class of finite Radon measures, with $\|\mu\| := \mu(\Omega) < \infty$, is denoted by $\mathcal{M}^+_b(\Omega)$.)

Here Ω is a locally compact Hausdorff space with countable base, and $\mathbf{G}$ is a linear integral operator with nonnegative kernel G on Ω,

$$\mathbf{G}^\sigma f(x) = \mathbf{G}(f\, d\sigma)(x) := \int_\Omega G(x,y)\, f(y) d\sigma(y), \quad x \in \Omega,$$

where $f \in L^1_{\rm loc}(\Omega, \sigma)$. If $f \equiv 1$, we use the notation

$$\mathbf{G}\sigma(x) := \int_\Omega G(x,y)\, d\sigma(y), \quad x \in \Omega,$$

for the linear G-potential of $\sigma \in \mathcal{M}^+(\Omega)$.

Throughout this paper, we use the following conventions imposed on the kernels G.

Definition 1.1. A *kernel* G on Ω is understood to be a lower semicontinuous function $G\colon \Omega \times \Omega \to [0, +\infty]$. A kernel G is said to be positive if $G(x,y) > 0$ for all $x, y \in \Omega$.

We observe that, for equations (1.1), the sublinear case $0 < q < 1$ is quite different both from the linear case $q = 1$ and the superlinear case $q > 1$, treated, for instance, in [10] and [20], respectively. In particular, when $0 < q < 1$, no "smallness" assumptions on σ are needed in order for a nontrivial solution u to exist. Moreover, in contrast to the case $q \geq 1$, nontrivial solutions to the homogeneous equations ($f = 0$) are treated similarly to non-homogeneous equations ($f \neq 0$).

For quasi-metric kernels, or more general quasi-metrically modifiable kernels discussed below (see also [10], [17], [20]), we obtain matching bilateral pointwise estimates of solutions u to (1.1).

Definition 1.2. A positive kernel G on Ω is said to be *quasi-metric*, with quasi-metric constant $\kappa \geq \frac{1}{2}$, if G is symmetric, i.e., $G(x,y) = G(y,x)$ for all $x, y \in \Omega$, and $d(x,y) := \frac{1}{G(x,y)}$ satisfies the quasi-triangle inequality

$$d(x,y) \leq \kappa[d(x,z) + d(z,y)], \qquad \forall\, x, y, z \in \Omega. \tag{1.2}$$

Important examples of quasi-metric kernels include Riesz kernels of order α ($0 < \alpha < n$), $I_\alpha(x,y) = c(\alpha, n)|x-y|^{\alpha - n}$, i.e., Green's kernels of the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$, on the entire Euclidean space $\mathbb{R}^n$ ($n \geq 1$), as well as Green's kernels of the Laplace–Beltrami operator on complete, non-compact Riemannian manifolds M with nonnegative Ricci curvature (see, e.g., [15]).

The restriction that G is symmetric in Definition 1.2 can be relaxed.

Definition 1.3. A kernel G on Ω is said to be *quasi-symmetric* (QS), with quasi-symmetry constant $\mathfrak{a} \geq 1$, if

$$\mathfrak{a}^{-1} G(y,x) \leq G(x,y) \leq \mathfrak{a}\, G(y,x), \quad \forall x, y \in \Omega. \tag{1.3}$$

The quasi-symmetry condition (1.3) is often used below in combination with a weak version of the maximum principle.

Definition 1.4. A kernel G on Ω satisfies the *weak maximum principle* (WMP), with constant $\mathfrak{b} \geq 1$, if

$$\mathbf{G}\mu(x) \leq 1, \quad \forall x \in S_\mu \Longrightarrow \mathbf{G}\mu(x) \leq \mathfrak{b}, \quad \forall x \in \Omega, \tag{1.4}$$

for any $\mu \in \mathcal{M}^+(\Omega)$, where S_μ denotes the closed support of μ.

When $\mathfrak{b} = 1$, the kernel G is said to satisfy the (Frostman) maximum principle.

Remark 1.5. Quasi-metric kernels are known to satisfy the (WMP) with constant $\mathfrak{b} = 2\kappa$ (see [34, Lemma 2.1]).

Some of the results for sublinear problems (1.1) hold for quasi-metric kernels. In particular, bilateral pointwise estimates of solutions to (1.1) are given in terms of the linear potentials $\mathbf{G}\sigma$ and $\mathbf{G}(f\, d\sigma)$, as well as certain "intrinsic" nonlinear potentials $\mathbf{K}\sigma$ defined below, which depend on $q \in (0,1)$.

Nonlinear potential estimates lead to the existence criteria for *all* solutions (possibly unbounded) to (1.1). They complement earlier results on the existence of positive solutions $u \in L^q(\Omega, \sigma)$ (globally) in the homogeneous case $f = 0$, which were based on a sublinear version of Schur's lemma for (QS)&(WMP) kernels G obtained in [30].

Bilateral pointwise estimates also yield uniqueness of solutions to (1.1) in the sublinear case ($0 < q < 1$). We observe that the uniqueness property may fail when $q \geq 1$.

More generally, we consider quasi-metrically modifiable kernels G.

Definition 1.6. A positive kernel G is said to be *quasi-metrically modifiable*, with modifier $m \in C(\Omega)$, $m > 0$, if the modified kernel

$$\widetilde{G}(x,y) := \frac{G(x,y)}{m(x)\, m(y)}, \qquad x, y \in \Omega, \tag{1.5}$$

is quasi-metric, with quasi-metric constant $\tilde{\kappa}$.

A typical modifier for G is given by

$$g(x) := \min\{1, G(x, x_0)\}, \qquad x \in \Omega, \tag{1.6}$$

where $x_0 \in \Omega$ is a fixed pole, provided $g \in C(\Omega)$.

Quasi-metrically modifiable kernels have numerous applications to semi-linear elliptic PDE in domains Ω with a positive Green's function G, treated in the next section (see also [10], [11], [12], [16], [17], [30]).

We will discuss sharp lower estimates of (super) solutions, together with matching upper estimates of (sub) solutions, to equation (1.1), for quasi-metric, or quasi-metrically modifiable kernels, obtained in [34, Theorem 1.2]. As we will see below, lower estimates actually hold for (QS)&(WMP) kernels.

1.2. **Semilinear elliptic equations.** Main applications of the results obtained for (1.1) are concerned with sublinear elliptic equations of the type

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \sigma u^q + \mu, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{1.7}$$

where $0 < q < 1$, $0 < \alpha < n$, and $\mu, \sigma \in \mathcal{M}^+(\Omega)$, in a wide class of domains Ω in $\mathbb{R}^n$, or a Riemannian manifold, with Green's function G.

If $(-\Delta)^{\frac{\alpha}{2}}$ has a positive Green's function G in Ω, then applying Green's operator $\mathbf{G}$ to both sides, we obtain an equivalent problem where solutions u satisfy the integral equation (1.1) with $f = \mathbf{G}\mu$. If $\alpha = 2$, such solutions u to (1.7) in bounded C^2-domains Ω coincide with the so-called *very weak* solutions (see [25]).

Semilinear elliptic equations of this type have been extensively studied, especially in the classical case $\alpha = 2$, in bounded smooth domains Ω and on $\mathbb{R}^n$, for bounded solutions u, under substantial restrictions on the coefficients and data (see [4], [7], [23, Sec. 7.2.6], and the literature cited there).

On the entire space $\mathbb{R}^n$, sharp existence and uniqueness results were obtained by Brezis and Kamin [6] for *bounded* solutions $u > 0$ to the equation $-\Delta u = \sigma u^q$. The proof of the uniqueness property given in [6] under the assumption $\liminf_{x\to\infty} u(x) = 0$ is especially delicate. (Several simpler proofs are given in [6] under the more restrictive condition $\lim_{x\to\infty} u(x) = 0$.) Pointwise estimates of bounded entire solutions given in [6] have a gap between the lower and upper bounds.

Matching bilateral pointwise estimates were given recently in [33], [34] for all solutions $u \in L^q_{\mathrm{loc}}(\Omega, \sigma)$ to (1.7), with arbitrary $\mu, \sigma \in \mathcal{M}^+(\Omega)$. As a consequence, the uniqueness problem was solved, and sharp existence criteria were given for such solutions, in a certain class of bounded domains $\Omega \subset \mathbb{R}^n$ for $0 < \alpha \le 2$, and on the entire space $\mathbb{R}^n$ for $0 < \alpha < n$, as well as on complete, non-compact Riemannian manifolds M with nonnegative Ricci curvature (see [15]).

More precisely, for $(-\Delta)^{\frac{\alpha}{2}}$, with $0 < \alpha \le 2$ $(\alpha < n)$ in bounded uniform domains $\Omega \subset \mathbb{R}^n$, along with $\alpha = n = 2$ in finitely connected domains $\Omega \subset \mathbb{R}^2$, Green's kernels are known to be quasi-metrically modifiable (see [17], and the literature cited there). Hence, the general results for (1.1) are applicable to all solutions of (1.7) in these cases.

When $0 < \alpha < n$, we can treat equations (1.7) for "nice" domains Ω, such as the balls or half-spaces, where Green's kernel of $(-\Delta)^{\frac{\alpha}{2}}$ is known to be quasi-metrically modifiable (see [10]).

On the entire space $\Omega = \mathbb{R}^n$, the Green kernel, i.e., the Newtonian kernel if $\alpha = 2$, $n \ge 3$, and Riesz kernels of order α if $0 < \alpha < n$, are quasi-metric. Equations (1.7) in this case were treated earlier in [8] (existence and bilateral pointwise estimates for *minimal* solutions). More complete results, including bilateral pointwise estimates for *all* solutions, and consequently uniqueness of solutions, were obtained subsequently in [33] on $\mathbb{R}^n$, and in [34] for general quasi-metrically modifiable Green's kernels on Ω.

1.3. **Quasilinear elliptic equations.** We will also present analogous nonlinear potential estimates, as well as existence and uniqueness theorems, for quasilinear elliptic equations involving the p-Laplace operator, with lower order source terms,

$$(1.8) \qquad -\Delta_p u = \sigma u^q + \mu, \quad u > 0 \text{ in } \mathbb{R}^n, \qquad \liminf_{x\to\infty} u = 0,$$

in the sub-natural growth case $0 < q < p - 1$.

Bilateral pointwise estimates of all entire p-superharmonic solutions (or, equivalently, local renormalized solutions) were obtained in [33]. They involve Havin–Maz'ya–Wolff potentials $\mathbf{W}\sigma$, $\mathbf{W}\mu$, and intrinsic nonlinear potentials $\mathbf{K}\sigma$, discussed in Sec. 2 and Sec. 6 below.

These estimates were used very recently to establish existence and uniqueness of the so-called *reachable* p-superharmonic solutions, in joint work with Nguyen Cong Phuc [29].

More general quasilinear equations of the type (1.8), with $\mathcal{A}$-Laplace operators $\mathrm{div}\mathcal{A}(x, \nabla u)$ in place of Δ_p, under standard structural assumptions of order p on $\mathcal{A}(x, \xi)$ (see, e.g., [19]), and sub-natural growth terms, will be discussed as well.

1.4. **A brief outline of the paper.** Sec. 2 contains some preliminary notions and basic concepts used throughout the paper. We first consider certain weighted norm inequalities of $(1, q)$-type for linear integral operators $\mathbf{G}$ in Ω. We then define intrinsic nonlinear potentials $\mathbf{K}\sigma$, using localized versions of the $(1, q)$-type weighted norm inequalities. We also discuss the precise definitions of sub- and super-solutions, and introduce the notion of the Wiener capacity for general kernels G.

In Sec. 3, we state the main results on bilateral pointwise estimates, along with the existence and uniqueness results, for solutions to sublinear equations (1.1). We consider integral operators $\mathbf{G}$ with quasi-metric and quasi-metrically modifiable kernels G, first with data $f = \mathbf{G}\mu$, $\mu \in \mathcal{M}^+(\Omega)$, and then with arbitrary data $f \geq 0$.

In Sec. 4, we consider applications to semilinear elliptic problems of type (1.7) in uniform domains for $0 < \alpha \leq 2$, as well as the entire space, balls or half-spaces for $0 < \alpha < n$. We also treat similar problems involving linear uniformly elliptic operators with bounded measurable coefficients in non-tangentially accessible (NTA) domains, or more general uniform domains with Ahlfors regular boundary.

In Sec. 5, we focus on the major steps in the proofs of the main theorems stated in Sec. 3 and provide relevant comments. In particular, we discuss the key lemmas employed in the proofs of the lower estimates of super-solutions for (QS)&(WMP) kernels G, and upper estimates of sub-solutions for quasi-metric and quasi-metrically modifiable kernels.

Finally, in Sec. 6, we are concerned with analogous nonlinear potential estimates and their applications for quasilinear equations (1.8) involving the p-Laplacian, as well as more general $\mathcal{A}$-Laplace operators, in the case $0 < q < p - 1$. We first treat bilateral pointwise estimates for all solutions, and then discuss the notion of a reachable solution, and present the corresponding existence and uniqueness results.

2. Preliminaries

2.1. **Weighted norm inequalities of $(1, q)$-type.** In this subsection, we discuss certain weighted norm inequalities studied in [30], along with their localized versions used extensively below.

We recall that throughout this paper, we use the notation $\mathcal{M}^+(\Omega)$ for *locally finite* Radon measures in Ω, and $\mathcal{M}^+_b(\Omega)$ for *finite* Radon measures, with $\|\nu\| := \nu(\Omega) < \infty$ if $\nu \in \mathcal{M}^+_b(\Omega)$. All the kernels G are assumed to be nonnegative lower semicontinuous functions defined on $\Omega \times \Omega$.

For $\sigma \in \mathcal{M}^+(\Omega)$, $0 < q < 1$, and a kernel G on Ω, we consider weighted norm inequalities of $(1, q)$-type,

$$\|\mathbf{G}\nu\|_{L^q(\Omega,\sigma)} \le C\,\|\nu\|, \qquad \forall \nu \in \mathcal{M}^+_b(\Omega). \tag{2.1}$$

We denote by $\varkappa = \varkappa(G, q, \sigma)$ the *least constant* C in (2.1).

Clearly, (2.1) yields its L^1-version with $d\nu = f\,d\sigma$, $f \in L^1(\Omega, \sigma)$, that is,

$$\|\mathbf{G}^\sigma f\|_{L^q(\Omega,\sigma)} \le C\,\|f\|_{L^1(\Omega,\sigma)}, \qquad \forall f \in L^1(\Omega, \sigma). \tag{2.2}$$

Remark 2.1. It follows from [14, Lemma 3.I] and [30, Theorem 1.1] that, conversely, (2.2)$\Longrightarrow$(2.1), for (QS)&(WMP) kernels G, with a different constant C. However, it is easy to see that, without the (WMP) restriction, this implication may fail even for symmetric kernels G.

It is worth observing that inequality (2.2) is the end-point case $p = 1$ of the (p, q)-type weighted norm inequality

$$\|\mathbf{G}^\sigma f\|_{L^q(\Omega,\sigma)} \le C\,\|f\|_{L^p(\Omega,\sigma)}, \qquad \forall f \in L^p(\Omega, \sigma), \tag{2.3}$$

where $p \ge 1$ and $0 < q < p$.

For $p > 1$, $0 < q < p$, inequality (2.3) was characterized recently in [32], where it was shown that (2.3) holds, for kernels $G \ge 0$ that satisfy (QS)&(WMP) conditions, if and only if

$$\int_\Omega (\mathbf{G}\sigma)^{\frac{q}{p-q}}\,d\sigma < \infty. \tag{2.4}$$

In the more complicated case $p = 1$, condition (2.4) is only necessary, but not sufficient, for (2.3) to hold. In [30, Theorem 1.1], it was proved that, for (QS)&(WMP) kernels G, inequality (2.1), or equivalently (2.2), holds if and only if there exists a nontrivial super-solution $u \in L^q(\Omega, \sigma)$ of the homogeneous equation,

$$u \ge \mathbf{G}(u^q d\sigma)\ d\sigma\text{-a.e.} \quad \text{in } \Omega.$$

Moreover, the least constant $\varkappa$ in (2.1) satisfies the estimates

$$\|u\|^{1-q}_{L^q(\Omega,\sigma)} \le \varkappa \le C\,\|u\|^{1-q}_{L^q(\Omega,\sigma)}, \tag{2.5}$$

where $C = C(q, \mathfrak{a}, \mathfrak{b})$ is a positive constant, and $\mathfrak{a}$, $\mathfrak{b}$ are the constants in the conditions (QS), (WMP), respectively.

This can be viewed as a sublinear version of Schur's lemma (see [14]). The proof is based on the notion of the equilibrium measure associated with the Wiener capacity for kernels G discussed below (see [5], [13]).

In addition to estimates (2.5), we have ([30, Theorem 1.2]),

$$C_1\,\|\mathbf{G}\sigma\|_{L^{\frac{q}{1-q}}(\Omega,\sigma)} \le \varkappa \le C_2\,\|\mathbf{G}\sigma\|_{L^{\frac{q}{1-q},q}(\Omega,\sigma)}, \tag{2.6}$$

where $C_1 = C_1(q, \mathfrak{b})$ and $C_2 = C_2(q, \mathfrak{a}, \mathfrak{b})$ are positive constants. Here $L^{r,q}(\Omega, \sigma)$ $(0 < r < \infty, 0 < q < \infty)$ stands for the Lorentz space on Ω with respect to the measure $\sigma \in \mathcal{M}^+(\Omega)$.

2.2. **Intrinsic nonlinear potentials.** Let $0 < q < 1$ and $\sigma \in \mathcal{M}^+(\Omega)$. Suppose G is a kernel on Ω. In this section, we recall the definition of the intrinsic nonlinear potential $\mathbf{K}\sigma$ given in [34]. Together with the linear potential $\mathbf{G}\sigma$, it controls pointwise behavior of nontrivial solutions u to the homogeneous sublinear integral equation

$$u = \mathbf{G}(u^q d\sigma), \quad 0 < u < +\infty \quad d\sigma\text{-a.e., in } \Omega. \tag{2.7}$$

We define a "ball" $B = B(x,r)$ associated with G by

$$B(x,r) := \{y \in \Omega \colon \ G(x,y) > 1/r\}, \qquad x \in \Omega, \ r > 0. \tag{2.8}$$

Notice that if G is a quasi-metric kernel, then $B(x,r)$ is a *quasi-metric ball* with respect to the quasi-metric $d = 1/G$.

Remark 2.2. By Fubini's theorem, $\mathbf{G}\sigma$ can be represented in the form

$$\mathbf{G}\sigma(x) = \int_0^\infty \frac{\sigma(B(x,r))}{r^2}\, dr, \qquad x \in \Omega.$$

Let $d\sigma_B = \chi_B \, d\sigma$ be the restriction of σ to a ball B. We will need a localized version of inequality (2.1), namely,

$$\|\mathbf{G}\nu\|_{L^q(\Omega, \sigma_B)} \le C \, \|\nu\|, \qquad \forall \nu \in \mathcal{M}_b^+(\Omega), \tag{2.9}$$

By $\varkappa(B)$ we denote the least constant C in (2.9). We remark that by (2.6) with σ_B in place of σ, we deduce the following estimates of $\varkappa(B)$,

$$C_1 \, \|\mathbf{G}\sigma_B\|_{L^{\frac{q}{1-q}}(\Omega, \sigma_B)} \le \varkappa(B) \le C_2 \, \|\mathbf{G}\sigma_B\|_{L^{\frac{q}{1-q}, q}(\Omega, \sigma_B)},$$

where $C_1 = C_1(q, \mathfrak{b})$ and $C_2 = C_2(q, \mathfrak{a}, \mathfrak{b})$ are positive constants.

The constants $\varkappa(B)$ with $B = B(x,r)$ are used to construct the nonlinear potential $\mathbf{K}\sigma$, intrinsic to the sublinear problem (2.7),

$$\mathbf{K}\sigma(x) := \int_0^\infty \frac{[\varkappa(B(x,r))]^{\frac{q}{1-q}}}{r^2}\, dr, \qquad x \in \Omega. \tag{2.10}$$

We remark that nonlinear potentials of this type were introduced for the first time in [8] for Riesz kernels on $\Omega = \mathbb{R}^n$. In that case, $B = B(x,r)$ is a Euclidean ball of radius $r^{\frac{1}{n-\alpha}}$ centered at $x \in \mathbb{R}^n$.

Intrinsic nonlinear potentials $\mathbf{K}\sigma$ resemble nonlinear potentials introduced originally by Havin and Maz'ya in [27], but with $\sigma(B(x,r))$ used in place of $\varkappa(B(x,r))$.

More precisely, the Havin–Maz'ya–Wolff potential $\mathbf{W}_{\alpha,p}$ on $\mathbb{R}^n$ (often called Wolff potential) is defined, for $0 < \alpha < \frac{n}{p}$, $1 < p < \infty$, by

$$\mathbf{W}_{\alpha,p}\sigma(x) := \int_0^\infty \frac{[\sigma(B(x,\rho))]^{\frac{1}{p-1}}}{\rho^{\frac{n-\alpha p}{p-1}+1}}\, d\rho, \qquad x \in \mathbb{R}^n, \tag{2.11}$$

where $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$, and $B = B(x,\rho)$ is a Euclidean ball in $\mathbb{R}^n$ of radius ρ centered at x.

Notice that in the special case $p = 2$, the potential $\mathbf{W}_{\alpha,2}$ coincides, up to a constant multiple, with the linear Riesz potential $\mathbf{I}_{2\alpha}$ with kernel $I_{2\alpha}$ on $\mathbb{R}^n$.

Nonlinear potentials $\mathbf{W}_{\alpha,p}$ were used subsequently by Hedberg and Wolff [18] in relation to the spectral synthesis problem for Sobolev spaces (see [1], and the literature cited there). In the special case $\alpha = 1$, they are fundamental to the theory of quasilinear elliptic equations of p-Laplace type [22] (see also [19], [24], [26], and Sec. 6 below).

2.3. **Sub- and super-solutions.** Let $\mu, \sigma \in \mathcal{M}^+(\Omega)$ and $0 < q < 1$. A Borel measurable function $u\colon \Omega \to [0, +\infty]$ is called a nontrivial *super-solution* associated with the equation

$$u = \mathbf{G}(u^q d\sigma) + \mathbf{G}\mu \quad d\sigma\text{-a.e. in } \Omega, \tag{2.12}$$

if $u > 0$ $d\sigma$-a.e., and

$$\mathbf{G}(u^q d\sigma) + \mathbf{G}\mu \le u < +\infty \quad d\sigma\text{-a.e. in } \Omega. \tag{2.13}$$

A *sub-solution* is defined similarly as a Borel measurable function $u\colon \Omega \to [0, +\infty]$ such that

$$u \le \mathbf{G}(u^q d\sigma) + \mathbf{G}\mu < +\infty \quad d\sigma\text{-a.e. in } \Omega. \tag{2.14}$$

A nontrivial *solution* to (2.12) is both a sub-solution and a nontrivial super-solution.

If u is a (super) solution, it is easy to see that actually $u \in L^q_{\text{loc}}(\Omega, \sigma)$ ([30, Lemma 2.2]).

2.4. **The Wiener capacity.** Let G be a kernel on Ω. For $\mu \in \mathcal{M}^+(\Omega)$, we set

$$\mathbf{G}^*\mu(y) := \int_\Omega G(x, y)\, d\mu(x), \qquad y \in \Omega.$$

Notice that the operator $\mathbf{G}^*$ is a formal adjoint of $\mathbf{G}$.

Given a kernel G on Ω, a *symmetrized* kernel G^s is defined by

$$G^s(x, y) := G(x, y) + G(y, x), \qquad x, y \in \Omega.$$

Clearly, G^s is symmetric. If G is a (QS) kernel, then G^s is comparable to G:

$$\left(1 + \mathfrak{a}^{-1}\right) G(x, y) \le G^s(x, y) \le (1 + \mathfrak{a})\, G(x, y), \qquad \forall x, y \in \Omega.$$

The kernel G^s corresponds to the integral operator $\mathbf{G}^s := \mathbf{G} + \mathbf{G}^*$. For a (QS) kernel G, the least constants in the inequalities

$$\|\mathbf{G}\nu\|_{L^q(\Omega,\sigma)} \le \varkappa\, \|\nu\|, \qquad \forall \nu \in \mathcal{M}^+(\Omega),$$
$$\|\mathbf{G}^s\nu\|_{L^q(\Omega,\sigma)} \le \varkappa_s\, \|\nu\|, \qquad \forall \nu \in \mathcal{M}^+(\Omega),$$

are obviously equivalent: $\left(1 + \mathfrak{a}^{-1}\right) \varkappa \le \varkappa_s \le (1 + \mathfrak{a})\varkappa$.

If G is a (QS) kernel, then there is a nontrivial super-solution u, i.e., $\mathbf{G}(u^q d\sigma) + \mathbf{G}\mu \le u < \infty$ $d\sigma$ a.e. if and only if there is a nontrivial super-solution u_s to the symmetrized version, $\mathbf{G}^s(u_s^q d\sigma) + \mathbf{G}\mu \le u_s < \infty$ $d\sigma$ a.e. This is easy to see using a scaled version $u_s = c_s\, u$ with an appropriate constant $c_s > 0$. A similar conclusion is true for sub-solutions.

Moreover, if u_1 is a sub-solution, and u_2 is a nontrivial super-solution such that $u_1 \le u_2$ then there exists a solution u such that $u_1 \le u \le u_2$. This means that if G is a (QS) kernel, then by passing to G^s, without loss of generality we may assume that G is symmetric (see [34]).

Definition 2.3. The Wiener capacity $\text{cap}(K)$ of a compact set $K \subset \Omega$ is defined by

$$\text{cap}(K) := \sup \{\mu(K) \colon \mu \in \mathcal{M}^+(K),\ \mathbf{G}^*\mu(y) \le 1,\ \forall\, y \in S_\mu\}. \tag{2.15}$$

Remark 2.4. For positive symmetric kernels G, on any compact set $K \subset \Omega$ there exists an extremal measure μ (called equilibrium measure) such that $\mu(K) = \text{cap}(K) < \infty$ in (2.15) (see [5], [13]). Equilibrium measures play an important role in the proofs of the upper estimates of sub-solutions discussed below.

Definition 2.5. A measure $\sigma \in \mathcal{M}^+(\Omega)$ is said to be *absolutely continuous* with respect to the Wiener capacity if $\sigma(K) = 0$ whenever $\text{cap}(K) = 0$, for any compact set $K \subset \Omega$.

Remark 2.6. Let $0 < q < 1$ and $\sigma \in \mathcal{M}^+(\Omega)$. Let G be a kernel on Ω. If u is a notrivial super-solution for $\mathbf{G}^*$ in place of $\mathbf{G}$, i.e., $u > 0$ $d\sigma$-a.e. and $\mathbf{G}^*(u^q d\sigma) \le u < \infty$ $d\sigma$-a.e., then σ is absolutely continuous with respect to the Wiener capacity (see [30, Lemma 4.2]).

For (QS) kernels, the preceding remark is clearly true for nontrivial super-solutions $\mathbf{G}(u^q d\sigma) \le u < \infty$ $d\sigma$-a.e.

3. Main results for sublinear integral equations

3.1. **Quasi-metric kernels.** We state our main theorem for quasi-metric kernels G.

Theorem 3.1. *Let $\mu, \sigma \in \mathcal{M}^+(\Omega)$ ($\sigma \neq 0$) and $0 < q < 1$. Suppose G is a quasi-metric kernel on Ω with quasi-metric constant κ. Then the following statements hold.*

(i) Any nontrivial solution u to equation (2.12) *satisfies the bilateral pointwise estimates*

$$c\,[(\mathbf{G}\sigma(x))^{\frac{1}{1-q}} + \mathbf{K}\sigma(x)] + \mathbf{G}\mu(x) \le u(x), \tag{3.1}$$

$$u(x) \le\ C\,[(\mathbf{G}\sigma(x))^{\frac{1}{1-q}} + \mathbf{K}\sigma(x) + \mathbf{G}\mu(x)], \tag{3.2}$$

$d\sigma$-a.e. in Ω, where $c = c(q, \kappa)$, $C = C(q, \kappa)$ are positive constants. Moreover, such a solution u is unique.

(ii) Estimate (3.1) *holds for any nontrivial super-solution u at all $x \in \Omega$ such that*

$$\mathbf{G}(u^q d\sigma)(x) + \mathbf{G}\mu(x) \le u(x). \tag{3.3}$$

Similarly, (3.2) *holds for any sub-solution u at all $x \in \Omega$ such that*

$$u(x) \le \mathbf{G}(u^q d\sigma)(x) + \mathbf{G}\mu(x). \tag{3.4}$$

(iii) A nontrivial (super) solution u to (2.12) *exists if and only if the following three conditions hold:*

$$\int_a^\infty \frac{\sigma(B(x_0,r))}{r^2}\,dr < \infty, \tag{3.5}$$

$$\int_a^\infty \frac{[\varkappa(B(x_0,r))]^{\frac{q}{1-q}}}{r^2}\,dr < \infty, \tag{3.6}$$

$$\int_a^\infty \frac{\mu(B(x_0,r))}{r^2}\,dr < \infty, \tag{3.7}$$

for some (or, equivalently, all) $x_0 \in \Omega$ and $a > 0$. Any nontrivial solution u satisfies (3.1), (3.2) *at all $x \in \Omega$ such that*

$$u(x) = \mathbf{G}(u^q d\sigma)(x) + \mathbf{G}\mu(x). \tag{3.8}$$

In particular, (3.8) *holds $d\sigma$-a.e.*

We notice that, as in the linear case [12], given a solution u to (3.8) defined $d\sigma$-a.e., we can set

$$\tilde{u}(x) := \mathbf{G}(u^q d\sigma)(x) + \mathbf{G}\mu(x), \qquad \forall x \in \Omega. \tag{3.9}$$

Then clearly $\tilde{u} = u$ $d\sigma$-a.e., and

$$\tilde{u}(x) = \mathbf{G}(\tilde{u}^q d\sigma)(x) + \mathbf{G}\mu(x), \qquad \forall x \in \Omega.$$

Hence, the representative $\tilde{u}$ is a solution to (3.8) *everywhere* in Ω.

Remark 3.2. 1. Under the assumptions of Theorem 3.1, conditions (3.5)–(3.7), are equivalent to $\mathbf{G}\sigma < +\infty$, $\mathbf{K}\sigma < +\infty$, and $\mathbf{G}\mu < +\infty$ $d\sigma$-a.e. Other equivalent conditions are given in Lemma 5.9 and Corollary 5.16 below.

2. An analogue of Theorem 3.1 holds for equation (1.1) with arbitrary $f \geq 0$ (see Theorem 3.5 below). One only needs to replace $\mathbf{G}\mu$ with $\mathbf{G}(f^q d\sigma)+f$ in (3.1), (3.2), and the corresponding estimates for sub- and super-solutions. The term $\mathbf{G}(f^q d\sigma)$ is no longer needed in the special case $f = \mathbf{G}\mu$.

3.2. **Quasi-metrically modifiable kernels.** We next state our main theorem for quasi-metrically modifiable kernels G. In this case, Theorem 3.1 holds with $\widetilde{G}$ in place of G, which leads to matching lower and upper global estimates of solutions up to the boundary of Ω.

The modification procedure is applicable to Green's kernels G for $(-\Delta)^{\frac{\alpha}{2}}$ in some domains $\Omega \subset \mathbb{R}^n$, in particular, balls or half-spaces, if $0 < \alpha < n$, or uniform domains discussed below if $0 < \alpha \leq 2$. In all of these cases, we use modifiers g defined by (1.6). For bounded $C^{1,1}$-domains Ω if $0 < \alpha \leq 2$, as well as balls or half-spaces if $0 < \alpha < n$, it is known that $g(x) \approx [\mathrm{dist}(x, \Omega^c)]^{\frac{\alpha}{2}}$.

Suppose G is a quasi-metrically modifiable kernel, with modifier m, associated with the quasi-metric $\widetilde{d} = 1/\widetilde{G}$. We denote by $\widetilde{B}(x,r)$ a quasi-metric ball

$$\widetilde{B}(x,r) := \left\{ y \in \Omega\colon\ \widetilde{G}(x,y) > 1/r \right\}, \quad x \in \Omega,\ r > 0. \tag{3.10}$$

Let $d\tilde{\sigma} = m^{1+q} d\sigma$. For a Borel set $E \subseteq \Omega$, by $\tilde{\varkappa}(E) = \tilde{\varkappa}(E, \tilde{\sigma})$ we denote the least constant in the inequality

$$\|\widetilde{\mathbf{G}}\nu\|_{L^q(\Omega, \tilde{\sigma}_E)} \leq \tilde{\varkappa}(E)\, \|\nu\|, \qquad \forall\, \nu \in \mathcal{M}_b^+(\Omega). \tag{3.11}$$

Using the constants $\tilde{\varkappa}(\widetilde{B}(x,r))$, we construct the modified intrinsic potential $\widetilde{\mathbf{K}}\sigma$ defined by

$$\widetilde{\mathbf{K}}\sigma(x) := \int_0^\infty \frac{[\tilde{\varkappa}(\widetilde{B}(x,r))]^{\frac{q}{1-q}}}{r^2}\, dr, \qquad x \in \Omega. \tag{3.12}$$

Theorem 3.3. *Let $\mu, \sigma \in \mathcal{M}^+(\Omega)$ ($\sigma \neq 0$) and $0 < q < 1$. Suppose G is a quasi-metrically modifiable kernel with modifier m. Then any nontrivial solution u to equation* (2.12) *is unique and satisfies the bilateral pointwise estimates*

$$c\, m\, \left(\left[m^{-1}\mathbf{G}(m^q d\sigma)\right]^{\frac{1}{1-q}} + \widetilde{\mathbf{K}}\sigma \right) + \mathbf{G}\mu \le u, \tag{3.13}$$

$$u \le C\, m\, \left(\left[m^{-1}\mathbf{G}(m^q d\sigma)\right]^{\frac{1}{1-q}} + \widetilde{\mathbf{K}}\sigma \right) + C\, \mathbf{G}\mu, \tag{3.14}$$

$d\sigma$-a.e. in Ω, where c, C are positive constants which depend only on q and the quasi-metric constant $\tilde{\kappa}$ of the modified kernel $\widetilde{G}$.

The lower bound (3.13) *holds for any nontrivial super-solution u, whereas the upper bound* (3.14) *holds for any sub-solution u.*

Remark 3.4. 1. Under the assumptions of Theorem 3.3, a nontrivial (super) solution to (2.12) exists if and only if $\mathbf{G}(m^q d\sigma) < +\infty$, $\widetilde{\mathbf{K}}\sigma < +\infty$, and $\mathbf{G}\mu < +\infty$ $d\sigma$-a.e.

2. If G is quasi-metrically modifiable with modifier $m = g$ given by (1.6), a nontrivial (super) solution to (2.12) exists if and only if (Lemma 5.15 below),

$$\tilde{\varkappa}(\Omega) < \infty \quad \text{and} \quad \int_\Omega g\, d\mu < \infty. \tag{3.15}$$

3.3. **Equations with arbitrary data.** We now state the main theorem for equation (1.1) with arbitrary data $f \ge 0$ in place of $\mathbf{G}\mu$ (see [34, Theorem 6.1]).

Theorem 3.5. *Suppose $0 < q < 1$, G is a quasi-metric kernel on Ω, $\sigma \in \mathcal{M}^+(\Omega)$ ($\sigma \neq 0$) and $f \ge 0$ is a Borel measurable function in Ω. Then the following statements hold.*

(i) Any nontrivial solution u to equation (1.1) *is unique and satisfies the bilateral pointwise estimates*

$$c\, \left[(\mathbf{G}\sigma)^{\frac{1}{1-q}} + \mathbf{K}\sigma + \mathbf{G}(f^q d\sigma) \right] + f \le u, \tag{3.16}$$

$$u \le C\, \left[(\mathbf{G}\sigma)^{\frac{1}{1-q}} + \mathbf{K}\sigma + \mathbf{G}(f^q d\sigma) \right] + f, \tag{3.17}$$

$d\sigma$-a.e. in Ω, where $c = c(q, \kappa)$, $C = C(q, \kappa)$ are positive constants.

(ii) The lower estimate (3.16) *holds for any nontrivial super-solution, whereas the upper estimate in* (3.17) *holds for any sub-solution.*

(iii) A nontrivial solution u to (1.1) *exists if and only if*

$$\mathbf{G}\sigma < \infty, \ \mathbf{K}\sigma < \infty, \ f < \infty, \ \mathbf{G}(f^q d\sigma) < \infty, \quad d\sigma\text{-}a.e. \text{ in } \Omega. \tag{3.18}$$

Remark 3.6. 1. In the special case $f = \mathbf{G}\mu$ ($\mu \in \mathcal{M}^+(\Omega)$) of Theorem 3.5, the term $\mathbf{G}(f^q d\sigma)$ in (3.17) may be dropped if, at the same time, f is replaced with Cf. Moreover, the condition $\mathbf{G}(f^q d\sigma) < \infty$ $d\sigma$-a.e. in (3.18) is redundant (see Theorem 3.1 above).

2. An analogue of Theorem 3.5 holds for quasi-metrically modifiable kernels G. This gives an extension of Theorem 3.3 to solutions of equation (1.1) with arbitrary data $f \geq 0$. The corresponding estimates of solutions remain valid once we replace $\mathbf{G}\mu$ with $c\,\mathbf{G}(f^q d\sigma) + f$ in (3.13), and $C\,\mathbf{G}(f^q d\sigma) + f$ in (3.14), respectively. In the existence criteria discussed in Remark 3.4, it suffices to replace $\mathbf{G}\mu$ with $\mathbf{G}(f^q d\sigma)$.

4. Semilinear elliptic problems

4.1. **Equations with the fractional Laplace operator.** In the following definition of a *uniform domain* (or, equivalently, an interior NTA domain), we rely on the notions of the interior corkscrew condition and the Harnack chain condition. We refer to [17] for related definitions, in metric spaces, along with a discussion of quasi-metric properties, 3-G inequalities, and the uniform boundary Harnack principle (see also [2]).

Definition 4.1. A uniform domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain which satisfies the interior corkscrew condition and the Harnack chain condition.

Notice that uniform domains are not necessarily regular in the sense of Wiener. Bounded Lipschitz and non-tangentially accessible (NTA) domains are examples of regular uniform domains.

The next theorem ([34, Theorem 1.2]) is a direct consequence of Theorem 3.3 and the fact that Green's function G of $(-\Delta)^{\frac{\alpha}{2}}$ in a uniform domain Ω for $0 < \alpha \leq 2$ is quasi-metrically modifiable, with $m = g$ and quasi-metric constant $\tilde{\kappa}$ which does not depend on the choice of $x_0 \in \Omega$ (see [3], [17]).

Theorem 4.2. *Suppose* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *is a uniform domain. Suppose* G *is Green's kernel of* $(-\Delta)^{\frac{\alpha}{2}}$ *in* Ω, *where* $0 < \alpha \leq 2$, $\alpha < n$. *Define the modifier* $m = g$ *by* (1.6) *with pole* $x_0 \in \Omega$.

Let $0 < q < 1$, *and let* $\mu, \sigma \in \mathcal{M}^+(\Omega)$, *and* $d\tilde{\sigma} = g^{1+q} d\sigma$. *Then the following statements hold.*

(i) Any nontrivial solution u *to equation* (1.7) *is unique and satisfies estimates* (3.13), (3.14) $d\sigma$*-a.e., and at all* $x \in \Omega$ *where* (3.8) *holds.*

(ii) Any nontrivial super-solution u *satisfies the lower bound* (3.13), *and any sub-solution* u *satisfies the upper bound* (3.14).

(iii) A nontrivial (super) solution to (1.7) *exists if and only if* (3.15) *holds, for some (or, equivalently, all)* $x_0 \in \Omega$.

Remark 4.3. Uniqueness of solutions to sublinear problems of the type (1.7) was previously known only under heavy restrictions on solutions, coefficients, and data, for instance, for bounded solutions [6], [7], or finite energy solutions [31].

For specific domains $\Omega \subseteq \mathbb{R}^n$, Theorem 4.2 can be extended to the full range $0 < \alpha < n$.

Theorem 4.4. *Let* $0 < \alpha < n$. *Suppose* Ω *is the entire space* $\mathbb{R}^n$, *or a ball, or half-space in* $\mathbb{R}^n$. *Then an analogue of Theorem 4.2 holds.*

Remark 4.5. In all the cases listed in Theorem 4.4, Green's kernel G of $(-\Delta)^{\frac{\alpha}{2}}$, for $0 < \alpha < n$, is known to be either quasi-metric (Riesz kernel I_α of order α if

$\Omega = \mathbb{R}^n$), or quasi-metrically modifiable (Ω is a ball, or half-space), with modifier $m(x) = g(x) \approx [\text{dist}(x, \Omega^c)]^{\frac{\alpha}{2}}$ (see [10] and the literature cited there).

4.2. **Equations with uniformly elliptic operators.** A similar approach works also for Green's kernels G of uniformly elliptic, symmetric operators $\mathcal{L}$ in divergence form,

$$\mathcal{L}u = -\text{div}(A\nabla u), \quad A = (a_{ij}(x))_{i,j=1}^n, \quad a_{ij} = a_{ji}, \tag{4.1}$$

with real-valued coefficients $a_{ij} \in L^\infty(\Omega)$, in place of $(-\Delta)^{\frac{\alpha}{2}}$.

Then, as above, Green's kernel G of $\mathcal{L}$ is known to be quasi-metrically modifiable, with modifier $m = g$ defined by (1.6) with pole $x_0 \in \Omega$, under certain restrictions on Ω specified in the following theorem (see [12], [17], [34], and the literature cited there).

Theorem 4.6. *An analogue of Theorem 4.2, for operators $\mathcal{L}$ in place of $(-\Delta)^{\frac{\alpha}{2}}$, holds for linear uniformly elliptic operators $\mathcal{L}$ with bounded measurable coefficients given by* (4.1)*, and modifiers $m = g$, in NTA domains, as well as uniform domains with Ahlfors regular boundary.*

5. Outlines of the proofs of the main theorems

5.1. **Lower bounds for super-solutions: (QS)&(WMP) kernels.** Let G be a kernel on Ω. We first consider nontrivial super-solutions to the homogeneous equation (2.7), i.e., functions $u > 0$ $d\sigma$-a.e. such that (see Sec. 2.3)

$$\mathbf{G}(u^q d\sigma) \le u < +\infty \quad d\sigma\text{-a.e. in } \Omega. \tag{5.1}$$

We start with the following lower bound for super-solutions obtained in [16, Theorem 1.3] for (WMP) kernels.

Lemma 5.1. *Let $\sigma \in \mathcal{M}^+(\Omega)$ and $0 < q < 1$. Suppose G is a kernel on Ω which satisfies the* (WMP) *with constant $\mathfrak{b}$ in* (1.4)*. Then any nontrivial super-solution u satisfies the estimate*

$$u(x) \ge c\, [\mathbf{G}\sigma(x)]^{\frac{1}{1-q}}, \tag{5.2}$$

where $c = (1-q)^{\frac{1}{1-q}} \mathfrak{b}^{-\frac{q}{1-q}}$, for all $x \in \Omega$ such that $\mathbf{G}(u^q d\sigma)(x) \le u(x)$. In particular, (5.2) *holds $d\sigma$-a.e.*

There is another lower estimate for super-solutions ([34, Lemma 3.2]), which complements (5.2) in a crucial way. It holds for kernels G which satisfy both the (WMP) and (QS) conditions. Using a symmetrized kernel, we may assume without loss of generality that G is symmetric.

Lemma 5.2. *Let $\sigma \in \mathcal{M}^+(\Omega)$ and $0 < q < 1$. Suppose G is a symmetric kernel on Ω which satisfies the* (WMP) *with constant $\mathfrak{b}$. Then any nontrivial super-solution u satisfies the estimate*

$$u(x) \ge c\, \mathbf{K}\sigma(x), \tag{5.3}$$

where $c = (1-q)^{\frac{1}{1-q}} \mathfrak{b}^{-\frac{q}{1-q}}$, for all $x \in \Omega$ such that $u(x) \ge \mathbf{G}(u^q d\sigma)(x)$. In particular, (5.3) *holds $d\sigma$-a.e.*

For (QS)&(WMP) kernels, the constant c in (5.3) will depend on q, $\mathfrak{b}$, and additionally the constant $\mathfrak{a}$ in (1.3). Combining (5.2) with (5.3) and the trivial estimate $u \geq \mathbf{G}\mu$, we obtain the main lower estimate for any nontrivial super-solution u to (2.12).

Corollary 5.3. *Let $\mu, \sigma \in \mathcal{M}^+(\Omega)$ and $0 < q < 1$. Suppose G is a (QS)&(WMP) kernel on Ω with constants $\mathfrak{a}$, $\mathfrak{b}$ in (1.3), (1.4), respectively. Then any nontrivial super-solution u to (2.12) satisfies the estimate*

$$u(x) \geq c\,[(\mathbf{G}\sigma(x))^{\frac{1}{1-q}} + \mathbf{K}\sigma(x)] + \mathbf{G}\mu(x), \tag{5.4}$$

where $c = c(q, \mathfrak{a}, \mathfrak{b})$, for all $x \in \Omega$ such that

$$u(x) \geq \mathbf{G}(u^q d\sigma)(x) + \mathbf{G}\mu(x). \tag{5.5}$$

In particular, (5.4) holds $d\sigma$-a.e.

5.2. **Upper bounds for sub-solutions: quasi-metric kernels.** In this section, we discuss the main steps involved in the proof of the upper estimates (3.2) of sub-solutions associated with equation (2.12), for quasi-metric kernels G. They match the lower estimates of super-solutions obtained in Corollary 5.3.

We start with the following key estimate ([34, Lemma 5.4]).

Lemma 5.4. *Let G be a quasi-metric kernel on Ω with quasi-metric constant κ. Let $0 < q < 1$ and $\nu, \sigma \in \mathcal{M}^+(\Omega)$. Then, for all $x \in \Omega$,*

$$\mathbf{G}[(\mathbf{G}\nu)^q d\sigma](x) \leq C\,(\mathbf{G}\nu(x))^q \left[\mathbf{G}\sigma(x) + (\mathbf{K}\sigma(x))^{1-q}\right], \tag{5.6}$$

where $C = (2\kappa)^q$.

The following lemma, which is deduced from Lemma 5.4, yields the desired upper estimate for sub-solutions, but only $d\sigma$-a.e. The remaining difficulty is to prove the upper estimate for points x where possibly $u(x) = +\infty$; it is handled in Lemma 5.6 below.

Lemma 5.5. *Let G be a quasi-metric kernel on Ω with quasi-metric constant κ. Let $0 < q < 1$ and $\mu, \sigma \in \mathcal{M}^+(\Omega)$. Then any sub-solution $u \geq 0$ such that $u \leq \mathbf{G}(u^q d\sigma) + \mathbf{G}\mu < +\infty$ $d\sigma$-a.e., satisfies the estimate*

$$u(x) \leq C \left[(\mathbf{G}\sigma(x))^{\frac{1}{1-q}} + \mathbf{K}\sigma(x) + \mathbf{G}\mu(x)\right], \tag{5.7}$$

for all $x \in \Omega$ such that $u(x) \leq \mathbf{G}(u^q d\sigma)(x) + \mathbf{G}\mu(x) < +\infty$, where $C = (8\kappa)^{\frac{q}{1-q}}$. In particular, (5.7) holds $d\sigma$-a.e.

In what follows, we will repeatedly use the fact, mentioned in Remark 1.5 above, that a quasi-metric kernel with quasi-metric constant κ obeys the (WMP) with constant $\mathfrak{b} = 2\kappa$.

The next lemma is used to deduce estimate (5.7) for *all* $x \in \Omega$ such that $u(x) \leq \mathbf{G}(u^q d\sigma)(x) + \mathbf{G}\mu(x)$, including the case $u(x) = +\infty$.

Lemma 5.6. *Let G be a quasi-metric kernel on Ω with quasi-metric constant κ. Let $0 < q < 1$ and $\mu, \sigma \in \mathcal{M}^+(\Omega)$. Then the function*

$$h(x) := (\mathbf{G}\sigma(x))^{\frac{1}{1-q}} + \mathbf{K}\sigma(x) + \mathbf{G}\mu(x), \quad x \in \Omega, \tag{5.8}$$

satisfies the estimate

$$\mathbf{G}(h^q d\sigma)(x) \le C\, h(x), \qquad \forall x \in \Omega, \tag{5.9}$$

where C is a constant which depends only on q and κ.

Combining Lemma 5.5 and Lemma 5.6 yields the following corollary.

Corollary 5.7. *Let G be a quasi-metric kernel on Ω with quasi-metric constant κ. Let $0 < q < 1$ and $\mu, \sigma \in \mathcal{M}^+(\Omega)$. Then every sub-solution u for which $u \le \mathbf{G}(u^q d\sigma) + \mathbf{G}\mu < +\infty$ $d\sigma$-a.e. satisfies the estimate*

$$u(x) \le (8\kappa)^{\frac{q}{1-q}} \left[(\mathbf{G}\sigma(x))^{\frac{1}{1-q}} + \mathbf{K}\sigma(x) + \mathbf{G}\mu(x) \right], \tag{5.10}$$

for all $x \in \Omega$ such that $u(x) \le \mathbf{G}(u^q d\sigma)(x) + \mathbf{G}\mu(x)$. In particular, (5.10) *holds $d\sigma$-a.e.*

The following lemma provides bilateral pointwise estimates of solutions to (2.12) for quasi-metric kernels, together with the existence criteria.

Lemma 5.8. *Let $\mu, \sigma \in \mathcal{M}^+(\Omega)$ ($\sigma \ne 0$) and $0 < q < 1$. Suppose G is a quasi-metric kernel on Ω. Then a nontrivial solution u to* (2.12) *exists if and only if $\mathbf{G}\sigma < +\infty$, $\mathbf{K}\sigma < +\infty$, and $\mathbf{G}\mu < +\infty$ $d\sigma$-a.e., and satisfies the bilateral pointwise estimates*

$$c\,[(\mathbf{G}\sigma(x))^{\frac{1}{1-q}} + \mathbf{K}\sigma(x)] + \mathbf{G}\mu(x) \le u(x), \tag{5.11}$$

$$u(x) \le C\,[(\mathbf{G}\sigma(x))^{\frac{1}{1-q}} + \mathbf{K}\sigma(x) + \mathbf{G}\mu(x)], \tag{5.12}$$

$d\sigma$-a.e. in Ω, where c, C are positive constants which depend only on q and the quasi-metric constant κ of the kernel G.

Existence criteria can be stated in several equivalent forms using the following lemma.

Lemma 5.9. *Let $\mu, \sigma \in \mathcal{M}^+(\Omega)$ ($\sigma \ne 0$) and $0 < q < 1$. Suppose G is a quasi-metric kernel on Ω. Then the following conditions are equivalent:*
(i) $\mathbf{G}\sigma < +\infty$, $\mathbf{K}\sigma < +\infty$, and $\mathbf{G}\mu < +\infty$ $d\sigma$-a.e.
(ii) $\mathbf{G}\sigma \not\equiv +\infty$, $\mathbf{K}\sigma \not\equiv +\infty$, and $\mathbf{G}\mu \not\equiv +\infty$.
(iii) Conditions (3.5)–(3.7) *hold for some (or, equivalently, all) $x_0 \in \Omega$ and $a > 0$.*

Remark 5.10. An alternative criterion for the existence of (super) solutions in the case of quasi-metric kernels G is deduced in Corollary 5.16 below.

5.3. **Bilateral bounds: quasi-metrically modifiable kernels.** In this section, we give both lower estimates of super-solutions and upper estimates of sb-solutions to sublinear integral equations (2.12) with quasi-metrically modifiable kernels G and modifiers $m > 0$, as defined in the Introduction.

Examples of quasi-metrically modifiable kernels can be found in [3], [10], [12], [17]. In particular, for bounded domains $\Omega \subset \mathbb{R}^n$ satisfying the boundary Harnack principle, such as bounded Lipschitz, NTA or uniform domains, Green's kernels G for the Laplacian and fractional Laplacian (in the case $0 < \alpha \le 2$, $\alpha < n$) are quasi-metrically modifiable.

Let $0 < q < 1$ and $\mu, \sigma \in \mathcal{M}^+(\Omega)$. We discuss relations between solutions (as well as sub- and super-solutions) to the equations

$$u = \mathbf{G}(u^q d\sigma) + \mathbf{G}\mu, \quad 0 < u < \infty \quad d\sigma\text{-a.e. in } \Omega, \tag{5.13}$$

$$v = \widetilde{\mathbf{G}}(v^q d\tilde{\sigma}) + \widetilde{\mathbf{G}}\tilde{\mu}, \quad 0 < v < \infty \quad d\tilde{\sigma}\text{-a.e. in } \Omega, \tag{5.14}$$

where $\widetilde{G}$ is the modified kernel (1.5) with modifier m, and

$$v := \frac{u}{m}, \quad d\tilde{\sigma} := m^{1+q}\, d\sigma, \quad d\tilde{\mu} = m\, d\mu. \tag{5.15}$$

Clearly, equations (5.13) and (5.14) are equivalent.

Similarly, the following two weighted norm inequalities are equivalent,

$$\|\mathbf{G}\nu\|_{L^q(m\, d\sigma)} \le \tilde{\varkappa}(\Omega) \int_\Omega m\, d\nu, \qquad \forall\, \nu \in \mathcal{M}^+(\Omega), \tag{5.16}$$

$$\|\widetilde{\mathbf{G}}\nu\,\|_{L^q(\Omega, \tilde{\sigma})} \le \tilde{\varkappa}(\Omega)\, \|\nu\|, \qquad \forall\, \nu \in \mathcal{M}^+_b(\Omega). \tag{5.17}$$

In (5.16), without loss of generality we may assume $\int_\Omega m d\nu < \infty$.

Remark 5.11. The least constant $\tilde{\varkappa} = \tilde{\varkappa}(\Omega)$ is the same in (5.16) and (5.17), since the latter is an equivalent restatement of the former, in terms of $\widetilde{\mathbf{G}}$, $\tilde{\sigma}$ in place of $\mathbf{G}$, σ.

Let $\widetilde{B} = \widetilde{B}(x, r)$ be a quasi-metric ball in Ω associated with the quasi-metric $\tilde{d} = 1/\widetilde{G}$, i.e.,

$$\widetilde{B}(x, r) := \left\{ y \in \Omega\colon\ \widetilde{G}(x, y) > 1/r \right\}, \qquad x \in \Omega,\ r > 0. \tag{5.18}$$

We denote by $\tilde{\varkappa}(\widetilde{B})$ the least constant in the localized versions of inequalities (5.16), (5.17) with $\sigma_{\widetilde{B}}$ in place of σ, and $\tilde{\sigma}_{\widetilde{B}}$ in place of $\tilde{\sigma}$, respectively.

Then the modified intrinsic potential $\widetilde{\mathbf{K}}\sigma$ is defined by

$$\widetilde{\mathbf{K}}\sigma(x) := \int_0^\infty \frac{[\tilde{\varkappa}(\widetilde{B}(x, r))]^{\frac{q}{1-q}}}{r^2}\, dr, \qquad x \in \Omega. \tag{5.19}$$

The following lemma ([34, Lemma 5.1]) contains a lower bound for super-solutions to sublinear integral equations for quasi-metrically modifiable kernels G.

Lemma 5.12. *Let G be a* (QS) *kernel on Ω with quasi-symmetry constant $\mathfrak{a}$, such that the modified kernel $\widetilde{G}$, defined by* (1.5) *with modifier m, satisfies the* (WMP) *with constant $\mathfrak{b}$. Then any nontrivial super-solution u to* (5.13) *satisfies the estimate*

$$u \ge c\, m\, \left(\left[m^{-1} \mathbf{G}(m^q d\sigma) \right]^{\frac{1}{1-q}} + \widetilde{\mathbf{K}}\sigma \right) + \mathbf{G}\mu \qquad d\sigma\text{-a.e.}, \tag{5.20}$$

where $c = c(q, \mathfrak{a}, \mathfrak{b})$ is a positive constant.

The following lemma ([34, Lemma 5.2]) provides a matching upper bound for sub-solutions to sublinear integral equations, for quasi-metrically modifiable kernels G.

Lemma 5.13. *Let G be a quasi-metrically modifiable kernel on Ω with modifier m. Then any sub-solution u to* (5.13) *satisfies the estimate*

$$u \le C\, m\, \left(\left[m^{-1} \mathbf{G}(m^q d\sigma) \right]^{\frac{1}{1-q}} + \widetilde{\mathbf{K}}\sigma \right) + C\, \mathbf{G}\mu \quad d\sigma\text{-a.e.}, \tag{5.21}$$

where $C = C(q, \tilde{\kappa})$, *and* $\tilde{\kappa}$ *is the quasi-metric constant for the modified kernel* $\widetilde{G}$.

We next consider the modifiers $m = g$ given by

$$g(x) = g^{x_0}(x) := \min\{1, G(x, x_0)\}, \quad x \in \Omega, \tag{5.22}$$

where $x_0 \in \Omega$ is a fixed pole.

Let G be a quasi-metric kernel on Ω, so that $d := 1/G$ obeys the quasi-triangle inequality (1.2) with quasi-metric with constant κ. The proof of the following lemma ([34, Lemma 5.3]) is based on the so-called Ptolemy's inequality (see [10, Sec. 3]): for all $x_0, x, y, z \in \Omega$,

$$d(x, y)d(x_0, z) \leq 4\kappa^2 \Big[d(x, z)d(y, x_0) + d(x_0, x)d(z, y)\Big]. \tag{5.23}$$

Lemma 5.14. *Let* G *be a quasi-metric kernel on* Ω *with quasi-metric constant* κ. *Let* $x_0 \in \Omega$, *and let* $g(x) = \min\{1, G(x, x_0)\}$. *Then*

$$\widetilde{G}(x, y) = \frac{G(x, y)}{g(x)g(y)} \tag{5.24}$$

is a quasi-metric kernel on Ω *with quasi-metric constant* $4\kappa^2$. *In particular,* $\widetilde{G}$ *satisfies the* (WMP) *with constant* $\mathfrak{b} = 8\kappa^2$.

In the next lemma, we give a criterion for the existence of (super) solutions in the case of quasi-metrically modified kernels G.

Lemma 5.15. *Let* $\mu, \sigma \in \mathcal{M}^+(\Omega)$ *and* $0 < q < 1$. *Suppose* G *is a quasi-metrically modifiable kernel on* Ω *with modifier* $m = g \in C(\Omega)$ *defined by* (5.22). *Then there exists a nontrivial (super) solution to equation* (5.13) *if and only if conditions* (3.15) *hold, i.e.,*

$$\int_\Omega g \, d\mu < \infty \quad \text{and} \quad \tilde{\varkappa}(\Omega) < \infty, \tag{5.25}$$

where $\tilde{\varkappa}(\Omega)$ *is the least constant in the weighted norm inequality* (5.16) *with* $m = g$.

We now go back to quasi-metric kernels G. The following corollary is a direct consequence of Lemma 5.15, since in this case the modified kernels $\widetilde{G}$ defined by (5.24) are also quasi-metric by Lemma 5.14.

Corollary 5.16. *Let* $\mu, \sigma \in \mathcal{M}^+(\Omega)$ *and* $0 < q < 1$. *Suppose* G *is a quasi-metric kernel on* Ω *such that* $g \in C(\Omega)$, *where* g *is defined by* (5.22). *Then there exists a nontrivial (super) solution to equation* (5.13) *if and only if* (5.25) *holds.*

6. Quasilinear equations with sub-natural growth terms

6.1. **Nonlinear potential estimates.** In this section, we present bilateral pointwise estimates of solutions to quasilinear elliptic equations of the type

$$\begin{cases} -\Delta_p u = \sigma u^q + \mu, \quad 0 < u < \infty \quad d\sigma\text{-a.e.} \quad \text{in } \mathbb{R}^n, \\ \liminf\limits_{x \to \infty} u(x) = 0, \end{cases} \tag{6.1}$$

where μ, $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$, in the *sub-natural growth* case $0 < q < p - 1$.

Here all solutions u are understood to be p-superharmonic. This is a natural class of solutions to (6.1), since $\nu := \sigma u^q + \mu \geq 0$ in the sense of measures. We may assume here, without loss of generality, that $u \in L^q_{\rm loc}(\mathbb{R}^n, d\sigma)$, so that $\nu \in \mathcal{M}^+(\mathbb{R}^n)$ (see [33]).

The notion of a p-superharmonic function is discussed below in a more general setting of $\mathcal{A}$-superharmonic functions associated with quasilinear equations involving the $\mathcal{A}$-Laplace operator, namely,

$$-\mathrm{div}(\mathcal{A}(x, \mathrm{D}u)) = \nu \qquad \text{in} \quad \Omega, \tag{6.2}$$

where $\Omega \subseteq \mathbb{R}^n$, $\nu \in \mathcal{M}^+(\Omega)$, D is the generalized gradient defined below, and $\mathcal{A}$ obeys certain monotonicity and growth assumptions discussed below (see [19]).

Remark 6.1. For equations (6.2) with data $\omega \in \mathcal{M}^+(\Omega)$, the class of $\mathcal{A}$-superharmonic solutions coincides in a sense with the class of *local renormalized* solutions. We refer to [21] for the proof of this important fact, and the discussion of the literature on renormalized solutions.

We will present matching upper and lower estimates of solutions to (6.1) in terms of certain nonlinear potentials defined below. Our estimates hold for all p-superharmonic solutions u. In particular, they yield an existence criterion for solutions to (6.1).

These results, obtained recently in [33], are new even in the special case $\mu = 0$, i.e., for the equation

$$\begin{cases} -\Delta_p u = \sigma u^q, \quad u \geq 0 \quad \text{in} \ \mathbb{R}^n, \\ \liminf\limits_{x \to \infty} u(x) = 0. \end{cases} \tag{6.3}$$

Equation (6.3) was considered earlier in [8], but the upper pointwise estimate was obtained only for the *minimal* solution u. Moreover, we will discuss uniqueness results for *reachable* solutions u obtained very recently in [29].

We will use the notion of the p-capacity for compact sets $K \subset \mathbb{R}^n$.

Definition 6.2. Let $1 < p < \infty$ and $K \subset \mathbb{R}^n$ be a compact set. The p-capacity of K is defined by

$$\mathrm{cap}_p(K) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^p dx : \ u \geq 1 \text{ on } K, \quad u \in C_0^\infty(\mathbb{R}^n) \right\}. \tag{6.4}$$

Notice that the p-capacity on $\mathbb{R}^n$ is nontrivial only if $1 < p < n$.

We recall the following definition.

Definition 6.3. A measure $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$ is said to be absolutely continuous with respect to the p-capacity if $\sigma(K) = 0$ whenever $\mathrm{cap}_p(K) = 0$, for any compact set $K \subset \mathbb{R}^n$. In this case, we write $\sigma << \mathrm{cap}_p$.

We observe that the existence of a (super) solution to (6.1) yields $\sigma << \mathrm{cap}_p$. More precisely, it follows from [8, Lemma 3.6] that if u is a nontrivial super-solution to (6.3) in the case $0 < q \leq p-1$, then

$$\sigma(K) \leq \mathrm{cap}_p(K)^{\frac{q}{p-1}} \left(\int_K u^q d\sigma \right)^{\frac{p-1-q}{p-1}}, \tag{6.5}$$

for all compact sets $K \subset \mathbb{R}^n$.

Among our main tools are certain *nonlinear potentials* associated with (6.3). We recall that the Havin–Maz'ya–Wolff potential $\mathbf{W}_{\alpha,p}$ is defined, for $1 < p < \infty$ and $0 < \alpha < \frac{n}{p}$, by (2.11).

In the special case $\alpha = 1$ $(1 < p < n)$ used in Sec. 6, this nonlinear potential will be denoted by $\mathbf{W}_p$, i.e., for $\mu \in \mathcal{M}^+(\mathbb{R}^n)$, we set

$$\mathbf{W}_p\mu(x) := \int_0^\infty \left[\frac{\mu(B(x,\rho))}{\rho^{n-p}}\right]^{\frac{1}{p-1}} \frac{d\rho}{\rho}, \qquad x \in \mathbb{R}^n, \tag{6.6}$$

where $B = B(x, \rho)$ is a Euclidean ball in $\mathbb{R}^n$ of radius ρ centered at x.

For $\mu \in \mathcal{M}^+(\mathbb{R}^n)$, we consider the equation

$$\begin{cases} -\Delta_p u = \mu, \quad u \geq 0 \quad \text{in } \mathbb{R}^n, \\ \liminf\limits_{x\to\infty} u = 0, \end{cases} \tag{6.7}$$

The following important global estimate, together with its local counterpart, is due to T. Kilpeläinen and J. Malý [22]. Suppose $u \geq 0$ is a p-superharmonic solution to (6.7). Then

$$K^{-1}\mathbf{W}_p\mu(x) \leq u(x) \leq K\mathbf{W}_p\mu(x), \tag{6.8}$$

where $K = K(p, n)$ is a positive constant.

Moreover, it is known (see [28]) that a nontrivial solution u to (6.7) exists if and only if

$$\int_1^\infty \left[\frac{\mu(B(0,\rho))}{\rho^{n-p}}\right]^{\frac{1}{p-1}} \frac{d\rho}{\rho} < \infty. \tag{6.9}$$

This is equivalent to $\mathbf{W}_p\mu(x) < \infty$ for some $x \in \mathbb{R}^n$, or equivalently quasi-everywhere (q.e.) on $\mathbb{R}^n$ with respect to the p-capacity. In particular, (6.9) may hold only in the case $1 < p < n$, unless $\mu = 0$.

We next recall the definition the so-called *intrinsic* nonlinear potential $\mathbf{K}_{p,q}$ associated with (6.3), which was introduced in [8].

To define $\mathbf{K}_{p,q}\sigma$ for $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$, we first consider the weighted norm inequality

$$\left(\int_{\mathbb{R}^n} |\varphi|^q \, d\sigma\right)^{\frac{1}{q}} \leq C \,\|\Delta_p\varphi\|^{\frac{1}{p-1}}, \tag{6.10}$$

for all test functions φ which are p-superharmonic in $\mathbb{R}^n$, and such that $\liminf\limits_{x\to\infty} \varphi(x) = 0$. Here $-\Delta_p\varphi = \nu$ is the Riesz measure of φ, and without loss of generality we may assume $\|\Delta_p\varphi\| = \nu(\mathbb{R}^n) < \infty$.

By $\varkappa(\mathbb{R}^n)$ we denote the least constant in (6.10).

The nonlinear potential $\mathbf{K}_{p,q}\sigma$ is defined in terms of the localized version of (6.10), namely,

$$\left(\int_B |\varphi|^q \, d\sigma\right)^{\frac{1}{q}} \leq C \,\|\Delta_p\varphi\|^{\frac{1}{p-1}}, \tag{6.11}$$

where B is a Euclidean ball in $\mathbb{R}^n$, for the same class of p-superharmonic test functions φ in $\mathbb{R}^n$ as in (6.10).

By $\varkappa(B)$ we denote the least constant in (6.11) with σ_B in place of σ, where $\sigma_B = \sigma|_B$ is restricted to a ball $B = B(x, \rho)$. Then the *intrinsic* nonlinear potential $\mathbf{K}_{p,q}\sigma$ is defined by

$$\mathbf{K}_{p,q}\sigma(x) := \int_0^\infty \left[\frac{\varkappa(B(x,\rho))^{\frac{q(p-1)}{p-1-q}}}{\rho^{n-p}} \right]^{\frac{1}{p-1}} \frac{d\rho}{\rho}, \quad x \in \mathbb{R}^n. \tag{6.12}$$

As was noticed in [8], $\mathbf{K}_{p,q}\sigma \not\equiv +\infty$ if and only if

$$\int_1^\infty \left[\frac{\varkappa(B(0,\rho))^{\frac{q(p-1)}{p-1-q}}}{\rho^{n-p}} \right]^{\frac{1}{p-1}} \frac{d\rho}{\rho} < \infty. \tag{6.13}$$

If (6.13) holds, then actually $\mathbf{K}_{p,q}\sigma < +\infty$ $d\sigma$-a.e., and q.e. with respect to the p-capacity.

Remark 6.4. In the case $p = 2$ and $\Omega = \mathbb{R}^n$, the potential $\mathbf{K}_{2,q}$ is closely related to the nonlinear potential $\mathbf{K}$, defined by (2.10). Notice that in (2.10) in this special case, $B(x, r)$ stands for a quasi-metric ball with respect to the quasi-metric $d(x, y) = |x - y|^{n-2}$, $n \geq 3$, whereas in (6.12), $B(x, \rho)$ is a Euclidean ball. Hence, using the substitution $\rho = r^{\frac{1}{n-2}}$, we see that $\mathbf{K} = (n - 2)\mathbf{K}_{2,q}$.

We now state the main result of [33], which establishes global bilateral estimates and existence criteria for all solutions to (6.1). Besides (6.9), (6.13), we will use additionally the condition $\mathbf{W}_p\sigma \not\equiv +\infty$, i.e.,

$$\int_1^\infty \left[\frac{\sigma(B(0,\rho))}{\rho^{n-p}} \right]^{\frac{1}{p-1}} \frac{d\rho}{\rho} < \infty. \tag{6.14}$$

Theorem 6.5. *Let $1 < p < n$, $0 < q < p - 1$, and $\mu, \sigma \in M^+(\mathbb{R}^n)$. There exists a nontrivial p-superharmonic solution u to* (6.1) *if and only if conditions* (6.9), (6.13), *and* (6.14) *hold. Moreover, any such a solution u satisfies the estimates*

$$C_1 \left[(\mathbf{W}_p\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}_{p,q}\sigma(x) + \mathbf{W}_p\mu(x) \right] \leq u(x) \tag{6.15}$$

$$u(x) \leq C_2 \left[(\mathbf{W}_p\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}_{p,q}\sigma(x) + \mathbf{W}_p\mu(x) \right], \tag{6.16}$$

$d\sigma$-a.e. on $\mathbb{R}^n$, with positive constants $C_i = C_i(p, q, n)$ $(i = 1, 2)$.

If $n \leq p < \infty$, there are no nontrivial p-superharmonic solutions.

Remark 6.6. As in the case of sublinear problems discussed above, (6.15) holds for all nontrivial p-superharmonic super-solutions, whereas (6.16) holds for all p-superharmonic sub-solutions.

6.2. **Equations involving $\mathcal{A}$-Laplace operators.** Our next goal is to introduce the notion of a *reachable* solution to equation (6.7), and discuss criteria of existence and uniqueness for reachable solutions to equation (6.1) in the case $0 < q < p - 1$.

We actually consider more general quasilinear $\mathcal{A}$-Laplace operators in place of Δ_p.

Let $\mathcal{A}\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function such that the map $x \to \mathcal{A}(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^n$, and the map $\xi \to \mathcal{A}(x, \xi)$ is continuous for a.e. $x \in \mathbb{R}^n$.

We also assume that there are constants $0 < \alpha \le \beta < \infty$ and $1 < p < n$ such that for a.e. x in $\mathbb{R}^n$,

$$\begin{aligned} &\mathcal{A}(x,\xi)\cdot\xi \ge \alpha|\xi|^p, \quad |\mathcal{A}(x,\xi)| \le \beta|\xi|^{p-1}, \quad \forall\,\xi\in\mathbb{R}^n,\\ &[\mathcal{A}(x,\xi_1)-\mathcal{A}(x,\xi_2)]\cdot(\xi_1-\xi_2) > 0, \quad \forall\,\xi_1,\xi_2\in\mathbb{R}^n,\ \xi_1\ne\xi_2. \end{aligned} \tag{6.17}$$

For the uniqueness results, we will assume additionally

$$\mathcal{A}(x,\lambda\xi) = \lambda^{p-1}\mathcal{A}(x,\xi), \qquad \forall\,\xi\in\mathbb{R}^n,\ \lambda>0. \tag{6.18}$$

Condition (6.18) is often used in the literature ([19], [22]).

The special case $\mathcal{A}(x,\xi) = |\xi|^{p-2}\xi$ gives the p-Laplacian Δ_p.

For an open set $\Omega \subset \mathbb{R}^n$, it is well known that every weak solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ to the equation

$$-\mathrm{div}\mathcal{A}(x,\nabla u) = 0 \qquad \text{in } \Omega \tag{6.19}$$

has a continuous representative. Such continuous solutions are said to be $\mathcal{A}$-*harmonic* in Ω. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,dx \ge 0,$$

for all nonnegative $\varphi \in C_0^\infty(\Omega)$, i.e., $-\mathrm{div}\mathcal{A}(x,\nabla u) \ge 0$ in the distributional sense, then u is called a *super-solution* to (6.19) in Ω.

A function $u\colon \Omega \to (-\infty,\infty]$ is called $\mathcal{A}$-*superharmonic* if u is not identically infinite in each connected component of Ω, u is lower semicontinuous, and for all open sets D such that $\overline{D} \subset \Omega$, and all functions $h \in C(\overline{D})$, $\mathcal{A}$-harmonic in D, it follows that $h \le u$ on ∂D implies $h \le u$ in D.

It is well known that if u is an $\mathcal{A}$-superharmonic function, then for any $k > 0$, its truncation $u_k = \min\{u,k\}$ is $\mathcal{A}$-superharmonic as well. Moreover, $u_k \in W^{1,p}_{\mathrm{loc}}(\Omega)$ (see [19]). We will need the notion of the weak (generalized) gradient of u defined by

$$\mathrm{D}u := \lim_{k\to\infty} \nabla[\min\{u,k\}] \qquad \text{a.e. in } \Omega.$$

We observe that $\mathrm{D}u$ gives the usual distributional gradient ∇u if either $u \in L^\infty_{\mathrm{loc}}(\Omega)$ or $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Moreover, there exists a unique measure $\mu = \mu[u] \in \mathcal{M}^+(\Omega)$ called the *Riesz measure* of u such that

$$-\mathrm{div}\mathcal{A}(x,\mathrm{D}u) = \mu \qquad \text{in } \Omega. \tag{6.20}$$

Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$. We first treat the problems of existence and uniqueness of $\mathcal{A}$-superharmonic solutions to the equation

$$\begin{cases} -\mathrm{div}\,\mathcal{A}(x,\mathrm{D}u) = \mu, \quad u \ge 0 \quad \text{in } \mathbb{R}^n,\\ \liminf\limits_{x\to\infty} u = 0, \end{cases} \tag{6.21}$$

where $\mu \in \mathcal{M}^+(\mathbb{R}^n)$.

We notice that, under conditions (6.17), the Kilpeläinen–Malý estimates (6.8) hold for all $\mathcal{A}$-superharmonic solutions to (6.21), i.e.,

$$K^{-1}\mathbf{W}_p\mu(x) \le u(x) \le K\mathbf{W}_p\mu(x), \tag{6.22}$$

where $K = K(p,n,\alpha,\beta)$ is a positive constant (see [22], [24]).

Moreover, it is known [28] that a necessary and sufficient condition for the existence of an $\mathcal{A}$-superharmonic solution to (6.21) is given by (6.9), as in the case of equation (6.7) for the p-Laplacian.

We next observe that Theorem 6.5 has a complete analogue (see [33, Remark 4.3(2)]) for the equation

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \mathrm{D}u) = \sigma u^q + \mu, \quad 0 < u < \infty \ d\sigma\text{-a.e.} \quad \text{in } \mathbb{R}^n, \\ \liminf_{x \to \infty} u(x) = 0, \end{cases} \tag{6.23}$$

where μ, $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$.

A natural existence criterion for equation (6.23), together with bilateral pointwise estimates, is contained in the following theorem.

Remark 6.7. Some corrections in the proof of existence, provided in [29, Lemma 4.1], are needed in the comparison principle for $\mathcal{A}$-superharmonic functions in bounded domains Ω ([8, Lemma 5.2]) used in the constructions of solutions in [8] for $\mu = 0$, and [33] for $\mu \neq 0$.

Theorem 6.8. *Let $1 < p < n$, $0 < q < p - 1$, and $\mu, \sigma \in M^+(\mathbb{R}^n)$. Suppose that $\mathcal{A}$ satisfies conditions* (6.17)*. Then there exists a nontrivial $\mathcal{A}$-superharmonic solution u to* (6.23) *if and only if conditions* (6.9)*,* (6.13)*, and* (6.14) *hold.*

Moreover, any nontrivial solution u satisfies estimates (6.15) *with positive constants $C_i = C_i(p, q, n, \alpha, \beta)$ $(i = 1, 2)$.*

If $n \leq p < \infty$, then there exist no nontrivial $\mathcal{A}$-superharmonic solutions to (6.23).

An analogue of Remark 6.6 remains true for $\mathcal{A}$-superharmonic sub- and supersolutions of (6.23).

6.3. **Reachable solutions to basic quasilinear equations.** In this section, we will define *reachable* $\mathcal{A}$-superharmonic solutions to (6.21) for which existence is obtained under the sole condition (6.9), and uniqueness is ensured if additionally $\mu << \mathrm{cap}_p$. We recall that (6.9) is necessary for the existence of an $\mathcal{A}$-superharmonic solution.

We first notice the existence of a minimal solution to (6.21) if (6.9) holds and $\mu << \mathrm{cap}_p$.

Theorem 6.9. *Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$, where $\mu << \mathrm{cap}_p$. Suppose that* (6.9) *holds. Then there exists a minimal $\mathcal{A}$-superharmonic solution to equation* (6.21).

Remark 6.10. 1. It is not known if condition (6.9) alone is enough for the existence of the minimal solution in Theorem 6.9.

2. It is also not known whether, under condition (6.9) combined with $\mu << \mathrm{cap}_p$, an $\mathcal{A}$-superharmonic solution to (6.21) is unique, and hence coincides with the minimal solution. Some partial results in this direction will be discussed below.

We now consider the following notion of a reachable solution suitable for our purposes (see [9, Definition 2.3] in the case of bounded domains).

Definition 6.11. Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$. A function $u \colon \mathbb{R}^n \to [0, +\infty]$ is said to be an $\mathcal{A}$-superharmonic reachable solution to equation (6.21) if u is an $\mathcal{A}$-superharmonic

solution of (6.21), and there exist two sequences $\{u_i\}$ and $\{\mu_i\}$, $i = 1, 2, \dots$, such that
(i) Each $\mu_i \in \mathcal{M}^+(\mathbb{R}^n)$ is compactly supported in $\mathbb{R}^n$, and $\mu_i \le \sigma$;
(ii) Each u_i is an $\mathcal{A}$-superharmonic solution of (6.21) with μ_i in place of μ;
(iii) $u_i \to u$ a.e. in $\mathbb{R}^n$.

Remark 6.12. The requirement that $\mu_i \le \mu$ in Definition 6.11 is important in the proof of uniqueness in the case $\mu << \mathrm{cap}_p$.

The next theorem is the main result of [29] on reachable solutions to equation (6.21).

Theorem 6.13. *Suppose $\mu \in \mathcal{M}^+(\mathbb{R}^n)$, and* (6.9) *holds. Then there exists an $\mathcal{A}$-superharmonic reachable solution to* (6.21). *Moreover, if additionally $\mu << \mathrm{cap}_p$, then any $\mathcal{A}$-superharmonic reachable solution is unique and coincides with the minimal solution.*

Remark 6.14. For $\mu \in \mathcal{M}^+_b(\mathbb{R}^n)$, it is known [29, Theorem 3.12] that any $\mathcal{A}$-superharmonic solution u to (6.21) is unique, and coincides with the minimal $\mathcal{A}$-superharmonic solution.

The next theorem proved in [29] shows that all $\mathcal{A}$-superharmonic solutions to (6.21) are reachable, provided the condition $\liminf_{x\to\infty} u = 0$ is replaced with $\lim_{x\to\infty} u = 0$.

Theorem 6.15. *Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$, and $\mu << \mathrm{cap}_p$. Suppose that u is an $\mathcal{A}$-superharmonic solution to the equation*

$$\begin{cases} -\mathrm{div}\,\mathcal{A}(x, \mathrm{D}u) = \mu, \quad u \ge 0 \quad \text{in } \mathbb{R}^n, \\ \lim_{x\to\infty} u = 0, \end{cases} \tag{6.24}$$

Then u is the unique $\mathcal{A}$-superharmonic solution of (6.24), *which coincides with the minimal $\mathcal{A}$-superharmonic reachable solution of* (6.21).

6.4. **Existence of reachable solutions: sub-natural growth.** We now discuss criteria of existence for nontrivial *reachable* $\mathcal{A}$-superharmonic solutions u to equation (6.23). As above, we assume without loss of generality that $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n, \sigma)$, so that $\sigma u^q + \mu \in \mathcal{M}^+(\mathbb{R}^n)$, and $\sigma << \mathrm{cap}_p$.

We first consider homogeneous equations (6.23) with $\mu = 0$. By Theorem 6.8, there exists a nontrivial $\mathcal{A}$-superharmonic solution u if and only if $\mathbf{W}_p\sigma \not\equiv +\infty$, $\mathbf{K}_{p,q}\sigma \not\equiv +\infty$, i.e., conditions (6.9), (6.13) hold. This theorem is complemented by the following statement proved in [29, Theorem 4.2].

Theorem 6.16. *Let $0 < q < p - 1$, and let $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. Then the nontrivial minimal $\mathcal{A}$-superharmonic solution u of*

$$\begin{cases} -\mathrm{div}\,\mathcal{A}(x, \mathrm{D}u) = \sigma u^q, \quad u \ge 0 \quad \text{in } \mathbb{R}^n, \\ \liminf_{x\to\infty} u = 0, \end{cases} \tag{6.25}$$

constructed in the proof of [8, Theorem 1.1] *under conditions* (6.9), (6.13), *is in fact an $\mathcal{A}$-superharmonic reachable solution.*

The following statement ([29, Theorem 4.3]) provides an analogue of Theorem 6.16 for reachable solutions to inhomogeneous equations (6.23). In contrast to the construction of an $\mathcal{A}$-superharmonic solution (not necessarily reachable) in [33, Theorem 1.1], the proof for $\mu \neq 0$ is different, and relies on the extra assumption $\mu << \text{cap}_p$.

Theorem 6.17. *Let* $0 < q < p-1$*, and let* $\mu, \sigma \in \mathcal{M}^+(\mathbb{R}^n)$*, where* $\mu << \text{cap}_p$*. Then, under conditions* (6.9)*,* (6.13)*, and* (6.14)*, there exists a nontrivial minimal reachable* $\mathcal{A}$*-superharmonic solution of* (6.23)*.*

6.5. **Uniqueness of reachable solutions: sub-natural growth.** In conclusion, we discuss the uniqueness property for (reachable) solutions of (6.23). The following main theorem was obtained [29, Theorem 4.4].

Theorem 6.18. *Let* $0 < q < p-1$*, and let* $\mu, \sigma \in \mathcal{M}^+(\mathbb{R}^n)$*, where* $\mu << \text{cap}_p$*. Suppose* $\mathcal{A}$ *satisfies conditions* (6.17) *and* (6.18)*. Then nontrivial* $\mathcal{A}$*-superharmonic reachable solutions of* (6.23) *are unique.*

It is known that in some cases listed below the restriction to reachable solutions in this uniqueness property can be dropped.

Remark 6.19. 1. In the case $p = 2$ of Theorem 6.18, for linear uniformly elliptic operators $\mathcal{L}$ with bounded measurable coefficients given by (4.1), all $\mathcal{L}$-superharmonic solutions of (6.23) are unique, without the extra restriction $\mu << \text{cap}$.

2. All nontrivial $\mathcal{A}$-superharmonic solutions in Theorem 6.18 are unique if any one of the following conditions hold ([29, Corollary 4.5]):
(i) $\lim_{x\to\infty} u(x) = 0$;
(ii) $u \in L^q(\mathbb{R}^n, d\sigma)$ and $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$;
(iii) $|\nabla u| \in L^p(\mathbb{R}^n)$.

References

[1] D. R. Adams and L. I. Hedberg, *Function spaces and potential theory*, Grundlehren der math. Wissenschaften **314**, Berlin–Heidelberg–New York, Springer, 1996.
[2] H. Aikawa, *Boundary Harnack principle and Martin boundary for a uniform domain,* J. Math. Soc. Japan **53** (2001), 119–145.
[3] A. Ancona, *Some results and examples about the behaviour of harmonic functions and Green's functions with respect to second order elliptic operators,* Nagoya Math. J. **165** (2002), 123–158.
[4] L. Boccardo and L. Orsina, *Sublinear equations in* L^s, Houston J. Math. **20** (1994), 99–114.
[5] M. Brelot, *Lectures on potential theory*, Lectures on Math. **19**, Tata Institute, Bombay, 1960.
[6] H. Brezis and S. Kamin, *Sublinear elliptic equations on* $\mathbb{R}^n$, Manuscr. Math. **74** (1992), 87–106.
[7] H. Brezis and L. Oswald, *Remarks on sublinear elliptic equations*, J. Nonlin. Anal. TMA **10** (1986), 55–64.
[8] Dat T. Cao and I. E. Verbitsky, *Nonlinear elliptic equations and intrinsic potentials of Wolff type,* J. Funct. Anal. **272** (2017), 112–165.
[9] G. Dal Maso and A. Malusa, *Some properties of reachable solutions of nonlinear elliptic equations with measure data,* Ann. Scuola Norm. Super. Pisa, Ser. IV, **25** (1997), 375–396.
[10] M. Frazier, F. Nazarov, and I. Verbitsky, *Global estimates for kernels of Neumann series and Green's functions*, J. London Math. Soc. **90** (2014), 903–918.
[11] M. Frazier and I. Verbitsky, *Postive solutions to Schrödinger's equation and the exponential integrability of the balayage*, Ann. Inst. Fourier **67** (2017), 1393–1425.

[12] M. Frazier and I. Verbitsky, *Positive solutions and harmonic measure for Schrödinger operators in uniform domains*, Pure Appl. Funct. Anal. **7** (2022), 993–1023.

[13] B. Fuglede, *On the theory of potentials in locally compact spaces*, Acta Math. **103** (1960), 139–215.

[14] E. Gagliardo, *On integral transformations with positive kernel*, Proc. Amer. Math. Soc. **16** (1965), 429–434.

[15] A. Grigor'yan, Y. Sun, and I. Verbitsky, *Superlinear elliptic inequalities on manifolds,* J. Funct. Anal. **278** (2020), no. 9, Art. 108444.

[16] A. Grigor'yan and I. Verbitsky, *Pointwise estimates of solutions to nonlinear equations for nonlocal operators,* Ann. Scuola Norm. Super. Pisa, Ser. V, **20** (2020), 721–750.

[17] W. Hansen, *Uniform boundary Harnack principle and generalized triangle property*, J. Funct. Anal. **226** (2005), 452–484.

[18] L. I. Hedberg and T. H. Wolff, *Thin sets in nonlinear potential theory*, Ann. Inst. Fourier **33** (1983), 161–187.

[19] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Univ. Press, Oxford, 1993.

[20] N. J. Kalton and I. E. Verbitsky, *Nonlinear equations and weighted norm inequalities*, Trans. Amer. Math. Soc. **351** (1999), 3441–3497.

[21] T. Kilpeläinen, T. Kuusi, and A. Tuhola-Kujanpää, *Superharmonic functions are locally renormalized solutions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **28** (2011), 775–795.

[22] T. Kilpeläinen and J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172** (1994), 137–161.

[23] M. A. Krasnoselskii, *Positive solutions of operator equations*, P. Noordhoff, Groningen, 1964.

[24] T. Kuusi and G. Mingione, *Guide to nonlinear potential estimates*, Bull. Math. Sci. **4** (2014), 1–82.

[25] M. Marcus and L. Véron, *Nonlinear second order elliptic equations involving measures*, Walter de Gruyter, Berlin–Boston, 2014.

[26] V. Maz'ya, *Sobolev spaces, with applications to elliptic partial differential equations*, 2nd augm. ed., Grundlehren der math. Wissenschaften **342**, Springer, Berlin, 2011.

[27] V. G. Maz'ya and V. P. Havin, *Non-linear potential theory*, Uspekhi Mat. Nauk **27** (1972), 67–138; English transl. Russ. Math. Surveys **27** (1972), 71–148.

[28] N. C. Phuc and I. E. Verbitsky, *Quasilinear and Hessian equations of Lane–Emden type*, Ann. Math. **168** (2008), 859–914.

[29] N. C. Phuc and I. E. Verbitsky, *Uniqueness of entire solutions to quasilinear equations of p-Laplace type*, Math. in Engineering **5** (2023): Article 068.

[30] S. Quinn and I. E. Verbitsky, *A sublinear version of Schur's lemma and elliptic PDE*, Analysis & PDE **11** (2018), 439–466.

[31] A. Seesanea and I. E. Verbitsky, *Finite energy solutions to inhomogeneous nonlinear elliptic equations with sub-natural growth terms*, Adv. Calc. Var. **13** (2020), 53–74.

[32] I. E. Verbitsky, *Sublinear equations and Schur's test for integral operators*, In: 50 years with Hardy spaces, a tribute to Victor Havin, eds. A. Baranov et al., Birkhäuser, Operator Theory: Adv. Appl. **261** (2018), 465–482.

[33] I. E. Verbitsky, *Bilateral estimates of solutions to quasilinear elliptic equations with sub-natural growth terms*, Adv. Calc. Var. **16** (2023), 217–231

[34] I. E. Verbitsky, *Global pointwise estimates of positive solutions to sublinear equations*, St. Petersburg Math. J. reprint of Algebra i Analiz **34** (2023), 531–556

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