Pure and Applied Functional Analysis
Volume 9, Number 1, 2024, 337–355

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

GAUSS-GREEN FORMULAS ON DOMAINS WITH NON-RECTIFIABLE BOUNDARIES

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ABSTRACT. We discuss versions of the Gauss-Green theorem valid for a bounded domain Ω that is not of finite perimeter. Thus in the statement of the divergence theorem for a vector field F, the boundary term pairs F with a distribution μ more singular than a measure. We investigate geometrical conditions on Ω that lead to information on μ, including regularity and localization properties. We see how some of these results refine pioneering work of Harrison and Norton.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and set $\chi_\Omega(x) = 1$ for $x \in \Omega$, 0 for $x \in \mathbb{R}^n \setminus \Omega$. We have the $\mathbb{R}^n$-valued distribution,

$$\nabla \chi_\Omega = \mu \in \mathcal{E}'(\mathbb{R}^n), \tag{1.1}$$

supported on $\partial\Omega$, and basic distribution theory gives

$$\langle \operatorname{div} F, \chi_\Omega \rangle = -\langle F, \mu \rangle, \tag{1.2}$$

for each vector field $F \in C^\infty(\mathbb{R}^n)$. This is a very general version of the Gauss-Green formula.

Several important, related questions arise. For one, it is of extreme interest to extend (1.2) to a much broader class of vector fields F. A related matter is to place the distribution μ in a smaller class of distributions, such as Sobolev spaces. For example, we clearly have

$$\mu \in H^{-1,\infty}(\mathbb{R}^n), \tag{1.3}$$

a result essentially equivalent to the assertion that (1.2) extends to all $F \in H^{1,1}(\mathbb{R}^n)$, but we want to do better. A third important question is to investigate what sharper information on μ and on extensions of (1.2) one has under various geometric hypotheses on $\partial\Omega$.

Fundamental work of deGiorgi and Federer addressed these issues in the setting of finite-perimeter domains. These are domains for which μ in (1.1) is a finite $\mathbb{R}^n$-valued measure. It was shown that this holds if and only if the measure-theoretic boundary $\partial_*\Omega$ (a subset of $\partial\Omega$) has finite $(n-1)$-dimensional Hausdorff measure ($\mathcal{H}^{n-1}(\partial_*\Omega) < \infty$). In such a case, the Radon-Nikodym theorem gives

$$\mu = \nu\sigma, \tag{1.4}$$

2020 *Mathematics Subject Classification.* 46F10, 46N20.
Key words and phrases. Gauss-Green formula, distributional derivative, box dimension.

where σ is a positive Borel measure on $\partial\Omega$, ν is $\mathbb{R}^n$-valued, and $|\nu(x)| = 1$ for σ-a.e. x. Then (1.2) can be written

$$\int_{\Omega} \operatorname{div} F \, dx = \int_{\partial\Omega} \nu \cdot F \, d\sigma, \tag{1.5}$$

first for $F \in C^\infty(\mathbb{R}^n)$. This result extends to F satisfying

$$F \in C(\mathbb{R}^n), \quad \operatorname{div} F \in L^1(\mathbb{R}^n). \tag{1.6}$$

In fact, using a mollifier we get $F_k = \varphi_k * F \in C^\infty(\mathbb{R}^n)$,

$$\begin{aligned} &F_k \longrightarrow F \ \text{ locally uniformly,} \\ &\operatorname{div} F_k = \varphi_k * \operatorname{div} F \longrightarrow \operatorname{div} F \ \text{ in } \ L^1(\mathbb{R}^n). \end{aligned} \tag{1.7}$$

Applying (1.5) to F_k gives

$$\int_{\Omega} \operatorname{div} F_k \, dx = \int_{\partial\Omega} \nu \cdot F_k \, d\sigma, \tag{1.8}$$

and taking $k \to \infty$ and using (1.7) gives (1.5) for all F satisfying (1.6). Expositions of the theory of finite-perimeter domains are given in [6], [4], and [18], including proofs that

$$\sigma = \mathcal{H}^{n-1} \lfloor \partial_* \Omega, \tag{1.9}$$

and that $\partial_*\Omega$ is countably rectifiable.

There are results extending (1.5) to much less regular F under additional hypotheses on Ω, such as Ahlfors regularity, of use in the analysis of layer potentials. See for example [12] and [13]. In this note we are pursuing the opposite direction, examining domains that are rougher than finite-perimeter domains.

Let us return for now to general bounded open Ω, and consider the following extension of (1.2), beyond $F \in H^{1,1}(\mathbb{R}^n)$. Namely, assume

$$F \in L^1(\mathbb{R}^n), \quad \operatorname{div} F \in L^1(\mathbb{R}^n). \tag{1.10}$$

Using a mollifier to obtain $F_k = \varphi_k * F$, as above, we have

$$\int_{\Omega} \operatorname{div} F_k \, dx = \langle F_k, \mu \rangle, \tag{1.11}$$

and $\operatorname{div} F_k = \varphi_k * \operatorname{div} F \to \operatorname{div} F$ in L^1-norm as $k \to \infty$, hence

$$\int_{\Omega} \operatorname{div} F_k \, dx \longrightarrow \int_{\Omega} \operatorname{div} F \, dx, \tag{1.12}$$

as $k \to \infty$. By (1.11), $\langle F_k, \mu \rangle$ also converges to the right side of (1.12) as $k \to \infty$, so $\mu \in H^{-1,1}(\mathbb{R}^n)$ extends to a bounded linear functional on the Banach space $V_1(\mathbb{R}^n)$ of vector fields satisfying (1.10), and in that sense we have an extension of (1.2) to this Banach space $V_1(\mathbb{R}^n)$:

$$\mu \in V_1(\mathbb{R}^n)' \ \text{ and } \ \int_{\Omega} \operatorname{div} F \, dx = \langle F, \mu \rangle, \quad \forall \, F \in V_1(\mathbb{R}^n). \tag{1.13}$$

To put this another way, the space $\mathcal{V}(\mathbb{R}^n)$ of smooth vector fields on $\mathbb{R}^n$ with compact support is dense in $V_1(\mathbb{R}^n)$, and μ has a unique extension from a continuous linear functional on $\mathcal{V}(\mathbb{R}^n)$ to a bounded linear functional on $V_1(\mathbb{R}^n)$, given by

$$\langle F, \mu \rangle = \int_\Omega \operatorname{div} F \, dx.$$

Further extensions, involving

$$F \in L^p(\mathbb{R}^n), \quad \operatorname{div} F \in \mathcal{M}(\mathbb{R}^n), \tag{1.14}$$

the space of finite signed Borel measures on $\mathbb{R}^n$, are given in [1], for general open Ω, following work on finite-perimeter domains in [2], [3], and other works cited there.

Now (1.13) might seem to be a strictly stronger result than (1.5), applied to F satisfying (1.6). After all, (1.13) applies to a larger class of domains Ω and to a larger class of vector fields F. However, (1.5) has the advantage that the right side clearly applies strictly to the *restriction* of F to $\partial\Omega$. Generally, if $\alpha \in \mathcal{E}'(\mathbb{R}^n)$ and supp $\alpha \subset K$, compact, one might have $F \in C^\infty(\mathbb{R}^n)$, satisfying $F|_K = 0$ but $\langle F, \alpha \rangle \neq 0$. (Example: $K = \{p\}$, $\alpha = (\partial/\partial x_1)\delta_p$.) It is important to investigate when such a phenomenon can be shown not to arise for $\alpha = \mu$, given by (1.1), and when F is somewhat less regular than C^∞.

Here is one basic case, yielding localization of μ on $\partial\Omega$.

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and define μ by* (1.1)*. Then, if F has compact support,*

$$\begin{aligned} F \in \operatorname{Lip}(\mathbb{R}^n),\ F\big|_{\partial\Omega} = 0 &\Longrightarrow \int_\Omega \operatorname{div} F \, dx = 0 \\ &\Longrightarrow \langle F, \mu \rangle = 0. \end{aligned} \tag{1.15}$$

Proof. For $k \in \mathbb{N}$, define $\rho_k : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} \rho_k = \quad & 0, && \text{for } |\lambda| \le 2^{-k}, \\ & \lambda - 2^{-k}, && \text{for } \lambda \ge 2^{-k}, \\ & \lambda + 2^{-k}, && \text{for } \lambda \le -2^{-k}, \end{aligned} \tag{1.16}$$

and set

$$F_k = \rho_k \circ F, \tag{1.17}$$

where ρ_k is applied componentwise to $F(x)$. Then each $F_k \in \operatorname{Lip}(\mathbb{R}^n)$, and, as $k \to \infty$,

$$F_k \longrightarrow F \ \text{ locally uniformly}, \quad \nabla F_k \longrightarrow \nabla F, \ \text{ boundedly and a.e.} \tag{1.18}$$

Also, each F_k vanishes on a neighborhood of $\partial\Omega$, so it is elementary that

$$\int_\Omega \operatorname{div} F_k \, dx = 0, \quad \forall\, k \in \mathbb{N}. \tag{1.19}$$

Letting $k \to \infty$, we have $\int_\Omega \operatorname{div} F \, dx = 0$, i.e., the first implication in (1.15), and this leads to the second implication, via (1.13). ☐

In turn, this leads to the following.

Corollary 1.2. *In the setting of Proposition 1.1, there is a uniquely defined*

$$\mu^{\#} \in \mathrm{Lip}(\partial\Omega)', \tag{1.20}$$

satisfying, for each $\mathbb{R}^n$*-valued* $f \in \mathrm{Lip}(\partial\Omega)$*,*

$$\langle f, \mu^{\#}\rangle = \langle F, \mu\rangle, \quad \forall\, F \in \mathrm{Lip}(\mathbb{R}^n) \text{ such that } F\big|_{\partial\Omega} = f. \tag{1.21}$$

Proof. First, given a compact $K \subset \mathbb{R}^n$, each $f \in \mathrm{Lip}(K)$ has an extension to $F \in \mathrm{Lip}(\mathbb{R}^n)$, given, e.g., by the Whitney extension theorem. The fact that $\mu^{\#}$ is well defined then follows by applying Proposition 1.1 to $F_1 - F_2$, given two extensions $F_j \in \mathrm{Lip}(\mathbb{R}^n)$ of f. □

Combining Corollary 1.2 with (1.13), we have

$$\int_\Omega \operatorname{div} F \, dx = \langle f, \mu^{\#}\rangle, \tag{1.22}$$

for each $f \in \mathrm{Lip}(\partial\Omega)$, and each extension $F \in \mathrm{Lip}(\mathbb{R}^n)$.

In a pioneering work, [11] took this further, defining

$$\mu^{\#} \in \mathrm{Lip}^r(\partial\Omega)', \tag{1.23}$$

with $r \in (0,1)$, for a class of bounded open $\Omega \subset \mathbb{R}^n$ satisfying further geometric conditions essentially related to the "box dimension" of $\partial\Omega$. Here, given $r \in (0,1]$ and a bounded function f in a set $S \subset \mathbb{R}^n$ (maybe valued in $\mathbb{R}^k$), we say

$$f \in \mathrm{Lip}^r(S) \Longleftrightarrow |f(x) - f(y)| \le C|x-y|^r, \tag{1.24}$$

for all $x, y \in S$. Thus $\mathrm{Lip}^1(S) = \mathrm{Lip}(S)$. We set

$$\|f\|_{\mathrm{Lip}^r(S)} = \|f\|_{\mathrm{lip}^r(S)} + \sup_S |f|, \tag{1.25}$$

with

$$\|f\|_{\mathrm{lip}^r(S)} = \sup_{x \ne y \in S} \frac{|f(x) - f(y)|}{|x-y|^r}. \tag{1.26}$$

The purpose of this note is to present some more results along these lines. Our hypotheses differ from those of [11] in several respects. For one, [11] works under the hypothesis that $\partial\Omega$ is a topological manifold (of topological dimension $n-1$). We do not make that hypothesis. Our basic geometric hypothesis on Ω is

$$\int_\Omega \delta(x)^{r-1} \, dx < \infty, \tag{1.27}$$

where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. This is related to but weaker than the hypothesis in [11] that $\partial\Omega$ be "d-summable," with $d = n - 1 + r$. The relationship is discussed in §3. On the other hand, [11] treats vector fields F (or rather, in their setting, $(n-1)$-forms) that are "d-flat," a class that contains Lip^r.

Given a bounded open set $\Omega \subset \mathbb{R}^n$, the functional $\mu^{\#} \in \mathrm{Lip}^r(\partial\Omega)'$ is constructed in §2 by a process similar to that used in [10] (there in the setting of $n = 2$ and $\partial\Omega$ a Jordan curve). A Whitney extension operator $\mathcal{W}$ is shown to have the property

$$\mathcal{W} : \mathrm{Lip}^r(\partial\Omega) \longrightarrow C(\overline{\Omega}) \cap H^{1,1}(\Omega), \tag{1.28}$$

provided (1.27) holds. In fact, for $f \in \mathrm{Lip}^r(\partial\Omega)$,

$$\int_\Omega |\nabla \mathcal{W} f(x)|\, dx \le C\Bigl(\int_\Omega \delta(x)^{r-1}\, dx\Bigr) \|f\|_{\mathrm{lip}^r(\partial\Omega)}. \tag{1.29}$$

Then $\mu^\#$ is defined by

$$\langle f, \mu^\# \rangle = \int_\Omega \operatorname{div} \mathcal{W} f(x)\, dx. \tag{1.30}$$

This is shown to be independent of choices inherent in the construction of $\mathcal{W}$, in Proposition 2.2.

To tie in $\mu^\#$ in (1.23) with $\mu^\#$ in (1.20), we need to face the fact that $\mathrm{Lip}(\partial\Omega)$ is not dense in $\mathrm{Lip}^r(\partial\Omega)$, in the norm topology, when $r < 1$. This issue is dealt with in Propositions 2.7–2.8. It is shown that, for each $f \in \mathrm{Lip}^r(\partial\Omega)$, there exist $f_k \in \mathrm{Lip}(\partial\Omega)$, satisfying

$$\|f_k\|_{\mathrm{Lip}^r(\partial\Omega)} \le A < \infty, \quad \|f_k - f\|_{C^0(\partial\Omega)} \to 0, \tag{1.31}$$

and, whenever this holds,

$$\langle f, \mu^\# \rangle = \lim_{k\to\infty} \langle f_k, \mu^\# \rangle. \tag{1.32}$$

A key to this is a refinement of the estimate (1.29), to

$$\int_\Omega |\nabla \mathcal{W} f(x)|\, dx \le C\omega_{r,\Omega}(\varepsilon) \|f\|_{\mathrm{lip}^r(\partial\Omega)} + \frac{C}{\varepsilon} m(\Omega)\, \|f\|_{c^0(\partial\Omega)}, \tag{1.33}$$

valid for all $\varepsilon \in (0, 1]$. Here,

$$\omega_{r,\Omega}(\varepsilon) = \int_{\{x\in\Omega:\delta(x)<\varepsilon\}} \delta(x)^{r-1}\, dx, \tag{1.34}$$

having the property that $\omega_{r,\Omega}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Also, we use the notation

$$\|f\|_{C^0(S)} = \sup_S |f|, \quad \|f\|_{c^0(S)} = \inf_{a\in\mathbb{R}^k} \|f - a\|_{C^0(S)}, \tag{1.35}$$

for bounded $f : S \to \mathbb{R}^k$. As we will see in §4, it is useful to know that the constants C on the right side of (1.33) are independent of Ω (given n).

In §3 we discuss the geometrical significance of the hypothesis (1.27), and relate it to the box dimension and box counting function of $\partial\Omega$. We show that the hypothesis of [11] that $\partial\Omega$ is d-summable, with $d = n - 1 + r$, is equivalent to the validity of (1.27) plus the following:

$$m(\partial\Omega) = 0, \quad \text{and} \quad \int_{\Omega^-} \delta(x)^{r-1}\, dx < \infty, \tag{1.36}$$

where $\Omega^- = B_R \setminus \overline{\Omega}$, given an open ball $B_R \supset \overline{\Omega}$. We discuss examples of bounded open sets $\Omega \subset \mathbb{R}^n$ that satisfy (1.27) but not (1.36).

In §4 we seek conditions on a sequence of domains $\Omega_j \subset \mathbb{R}^n$ such that

$$\langle F, \mu_j \rangle \longrightarrow \langle F, \mu \rangle \tag{1.37}$$

(with $\mu_j = \nabla\chi_{\Omega_j}$), with particular attention to which spaces of vector fields F this holds for. One simple result is that if

$$F \in \mathrm{Lip}^r(\mathbb{R}^n), \quad \operatorname{div} F \in L^1(\mathbb{R}^n), \tag{1.38}$$

and Ω, Ω_j all satisfy (1.27), then

$$\langle F, \mu - \mu_j \rangle = \int\limits_{\Omega\triangle\Omega_j} \operatorname{div} F \, dx, \tag{1.39}$$

which tends to 0 as $j \to \infty$ provided

$$m(\Omega\triangle\Omega_j) \longrightarrow 0. \tag{1.40}$$

However, it is of greater interest to know when (1.37) holds for all $F \in \mathrm{Lip}^r(\mathbb{R}^n)$. Proposition 4.2 states that if all Ω_j lie in some ball B_R, $R < \infty$, and if (1.27) holds uniformly, in the sense that there exist $\omega(\varepsilon)$ so that, for all $j \in \mathbb{N}$, $\varepsilon \in (0, 1]$,

$$\omega_{r,\Omega_j}(\varepsilon) \le \omega(\varepsilon), \quad \omega(\varepsilon) \to 0, \tag{1.41}$$

and if (1.40) holds, then (1.37) holds for all $F \in \mathrm{Lip}^r(\mathbb{R}^n)$. The validity of the estimate (1.33), with C independent of Ω, plays a key role in the proof.

2. Gauss-Green with Lip^r boundary values

Here we extend $\mu^\#$ from a continuous linear functional on $\mathrm{Lip}(\partial\Omega)$ to one on $\mathrm{Lip}^r(\partial\Omega)$, under a metric condition on Ω, which we derive below. One tool we use is the Whitney extension map, which we now recall (cf. [17], or [15], Appendix C).

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set. Whitney's construction says there exist $C, M \in (0, \infty)$ and a partition of unity $\{\Phi_j : j \ge 1\}$ on Ω such that each $\Phi_j \in C_0^\infty(\Omega)$, and furthermore the following hold.

(a) Each $x \in \Omega$ is in the support of at most M of the Φ_j.

(b) For each $\delta > 0$, if $x \in \operatorname{supp} \Phi_j$ and $\operatorname{dist}(x, \partial\Omega) = \delta$, then

$$\operatorname{diam} \operatorname{supp} \Phi_j \le \frac{\delta}{2}, \tag{2.1}$$

and

$$|\nabla\Phi_j(x)| \le \frac{C}{\delta}. \tag{2.2}$$

Having this, and given $r \in (0, 1]$, we construct

$$\mathcal{W} : \mathrm{Lip}^r(\partial\Omega) \longrightarrow C(\overline{\Omega}) \cap C^\infty(\Omega) \tag{2.3}$$

as follows. For each $j \in \mathbb{N}$, let y_j be a point in $\partial\Omega$ of minimal distance from $\operatorname{supp} \Phi_j$. Then, for $f \in \mathrm{Lip}^r(\partial\Omega)$, set

$$\mathcal{W}f(x) = \sum_j f(y_j)\Phi_j(x), \quad x \in \Omega. \tag{2.4}$$

Since this sum is locally finite, we clearly have $\mathcal{W} : \mathrm{Lip}^r(\partial\Omega) \to C^\infty(\Omega)$. Now suppose $x \in \Omega,\ z \in \partial\Omega$, and $|x - z| = \delta$. Then

$$\begin{aligned} x \in \operatorname{supp}\Phi_j &\Longrightarrow |x - y_j| \le C\delta \\ &\Longrightarrow |z - y_j| \le C\delta \\ &\Longrightarrow |f(y_j) - f(z)| \le C\delta^r, \end{aligned} \tag{2.5}$$

so

$$\begin{aligned} \mathcal{W}f(x) &= f(z) + \sum_j \bigl\{f(y_j) - f(z)\bigr\}\Phi_j(x) \\ &= f(z) + O(\delta^r). \end{aligned} \tag{2.6}$$

This implies

$$\mathcal{W} : \mathrm{Lip}^r(\partial\Omega) \to C(\overline{\Omega}), \quad \mathcal{W}f\big|_{\partial\Omega} = f. \tag{2.7}$$

We next estimate $\nabla v(x)$, for $v = \mathcal{W}f,\ x \in \Omega$. Noting that

$$\sum_j \nabla\Phi_j(x) \equiv 0 \quad \text{on} \quad \Omega, \tag{2.8}$$

we have

$$\nabla v(x) = \sum_j \bigl\{f(y_j) - f(z)\bigr\}\nabla\Phi_j(x), \quad \forall\, z \in \partial\Omega. \tag{2.9}$$

For each $x \in \Omega$, there are at most M terms in this sum, for which $x \in \operatorname{supp}\Phi_j$. Say $x \in \operatorname{supp}\Phi_\ell$, and pick $z = y_\ell$. It follows from (2.1)–(2.2) that

$$\begin{aligned} |\nabla v(x)| &\le \sum_j |f(y_j) - f(y_\ell)| \cdot |\nabla\Phi_j(x)| \\ &\le C\delta(x)^{r-1}\|f\|_{\mathrm{lip}^r(\partial\Omega)}, \end{aligned} \tag{2.10}$$

where the lip^r seminorm is defined in (1.26), and

$$\delta(x) = \operatorname{dist}(x, \partial\Omega). \tag{2.11}$$

This establishes the following result.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and take $r \in (0,1]$. Then*

$$\mathcal{W} : \mathrm{Lip}^r(\partial\Omega) \longrightarrow C(\overline{\Omega}) \cap H^{1,1}(\Omega), \tag{2.12}$$

provided

$$\int_\Omega \delta(x)^{r-1}\, dx < \infty. \tag{2.13}$$

Remark. A condition equivalent to (2.13) is

$$\int_0^1 m\Bigl(\bigl\{x \in \Omega : \delta(x) < t\bigr\}\Bigr)t^{r-1}\,\frac{dt}{t} < \infty, \tag{2.14}$$

i.e.,

$$m\Bigl(\bigl\{x \in \Omega : \delta(x) < t\bigr\}\Bigr) \le \beta(t)t^{1-r}, \quad \int_0^1 \frac{\beta(t)}{t}\, dt < \infty, \tag{2.15}$$

where m denotes Lebesgue measure on $\mathbb{R}^n$. See §3 for a further discussion of this condition.

We are now ready for a definition.

Definition. Given that the bounded open set $\Omega \subset \mathbb{R}^n$ satisfies (2.13), we define $\mu^\# \in \mathrm{Lip}^r(\partial\Omega)'$ by

$$\langle f, \mu^\# \rangle = \int_\Omega \operatorname{div} \mathcal{W} f(x)\, dx, \tag{2.16}$$

for $\mathbb{R}^n$-valued $f \in \mathrm{Lip}^r(\partial\Omega)$.

Constructing the partition of unity $\{\Phi_j\}$ and the extension map $\mathcal{W}$ involves choices. The following important result implies, among other things, that $\mu^\#$ is independent of such choices.

Proposition 2.2. *Assume the bounded open set $\Omega \subset \mathbb{R}^n$ satisfies* (2.13), *and take $f \in \mathrm{Lip}^r(\partial\Omega)$. Then, for $\mathbb{R}^n$-valued G,*

$$G \in C(\overline{\Omega}) \cap H^{1,1}(\Omega),\ G\big|_{\partial\Omega} = f \Longrightarrow \int_\Omega \operatorname{div} G\, dx = \langle f, \mu^\# \rangle. \tag{2.17}$$

Proof. Considering $H = G - \mathcal{W}f$, it suffices to prove the following. □

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set. Given $\mathbb{R}^n$-valued $H \in C(\overline{\Omega}) \cap H^{1,1}(\Omega)$, we have*

$$H\big|_{\partial\Omega} = 0 \Longrightarrow \int_\Omega \operatorname{div} H\, dx = 0. \tag{2.18}$$

Proof. Define $H_k = \rho_k \circ H$, as in (1.16)–(1.17). Then (cf. [7], Lemmas 7.6–7.7)

$$\begin{aligned} &H_k \longrightarrow H \ \text{ uniformly on } \overline{\Omega}, \quad H_k \in H_0^{1,1}(\Omega), \\ &\nabla H_k(x) \longrightarrow \nabla H(x) \ \text{ a.e., and } \ |\nabla H_k(x)| \le |\nabla H(x)|, \end{aligned} \tag{2.19}$$

so

$$H_k \longrightarrow H \ \text{ in } \ H^{1,1}(\Omega). \tag{2.20}$$

Hence

$$\int_\Omega \operatorname{div} H\, dx = \lim_{k\to\infty} \int_\Omega \operatorname{div} H_k\, dx = 0. \tag{2.21}$$

□

We next record the following useful property of $\mathcal{W}$.

Proposition 2.4. *If $\Omega \subset \mathbb{R}^n$ is a bounded open set, and $0 < r \le 1$,*

$$\mathcal{W} : \mathrm{Lip}^r(\partial\Omega) \longrightarrow \mathrm{Lip}^r(\overline{\Omega}). \tag{2.22}$$

Proof. Take $f \in \mathrm{Lip}(\partial\Omega)$, $v = \mathcal{W}f$. We already have $v \in C(\overline{\Omega})$. Also (2.6) gives

$$|v(x) - f(z)| \le C|x - z|^r, \quad \text{for } \ x \in \Omega,\ z \in \partial\Omega, \tag{2.23}$$

and (2.10) gives

$$|\nabla v(x)| \le C\delta(x)^{r-1}, \quad x \in \Omega, \tag{2.24}$$

with $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. With these results in hand, take

$$x, y \in \Omega, \quad h = |x - y|. \tag{2.25}$$

We consider two cases:

$$\begin{aligned} &(a) \qquad && \delta(x) \ge 2h, \\ &(b) \qquad && \delta(x) < 2h. \end{aligned}$$

In case (a), the line segment $\ell(t) = ty + (1-t)x$ from x to y has the property that $\delta(\ell(t)) \ge h$ for each $t \in [0,1]$, so

$$|v(x) - v(y)| \le Ch \cdot h^{r-1} = Ch^r, \quad \text{in case (a)}. \tag{2.26}$$

In case (b), one also has $\delta(y) < 3h$. Pick

$$x_0, y_0 \in \partial\Omega, \quad |x - x_0| = \delta(x), \ |y - y_0| = \delta(y). \tag{2.27}$$

Then

$$\begin{aligned} |v(x) - v(y)| &\le |v(x) - f(x_0)| + |f(x_0) - f(y_0)| + |f(y_0) - v(y)| \\ &\le Ch^r, \quad \text{in case (b)}. \end{aligned} \tag{2.28}$$

This yields (2.22). □

There is the following related result. Let $K \subset \mathbb{R}^n$ be compact. Say $K \subset B_R(0)$, and consider $\Omega = B_R(0) \setminus K$. The analysis behind Proposition 2.4, plus a cut-off near $\partial B_R(0)$ yields a continuous map

$$\widetilde{\mathcal{W}} : \mathrm{Lip}^r(K) \longrightarrow \mathrm{Lip}^r(\mathbb{R}^n), \quad \widetilde{\mathcal{W}} f\big|_K = f. \tag{2.29}$$

Consequently, in the setting of Proposition 2.4, we have

$$\widetilde{\mathcal{W}} : \mathrm{Lip}^r(\partial\Omega) \longrightarrow \mathrm{Lip}^r(\mathbb{R}^n), \quad \widetilde{\mathcal{W}} f\big|_{\overline{\Omega}} = \mathcal{W} f. \tag{2.30}$$

Note that the case $r = 1$ of (2.29) was invoked in the proof of Corollary 1.2, which we can rephrase as

$$\langle f, \mu^\# \rangle = \langle \widetilde{\mathcal{W}} f, \mu \rangle, \quad \mu = \nabla \chi_\Omega, \quad \forall\, f \in \mathrm{Lip}(\partial\Omega). \tag{2.31}$$

Here is another useful consequence of (2.29).

Proposition 2.5. *Given $K \subset \mathbb{R}^n$ compact, $s \in (0,1)$, and $f \in \mathrm{Lip}^s(K)$, there exist f_k satisfying*

$$f_k \in \mathrm{Lip}(K), \ \{f_k\} \ \text{bounded in } \mathrm{Lip}^s(K), \quad f_k \to f \ \text{in } \mathrm{Lip}^r\text{-norm}, \ \forall\, r < s. \tag{2.32}$$

Proof. Apply a standard mollifier argument to $v = \widetilde{\mathcal{W}} f$, obtaining $v_k \in \mathrm{Lip}(\mathbb{R}^n)$ having properties analogous to those stated in (2.32), and set $f_k = v_k|_K$. □

The following result ties in $\mu^\#$ as defined in (2.16) with its debut in Corollary 1.2.

Proposition 2.6. *Take* $r \in (0,1)$ *and assume* $\Omega \subset \mathbb{R}^n$ *is a bounded open set satisfying the condition* (2.13). *Take*

$$f \in \mathrm{Lip}^s(\partial\Omega), \quad s > r. \tag{2.33}$$

Then there exist $f_k \in \mathrm{Lip}(\partial\Omega)$ *satisfying* (2.32), *with* $K = \partial\Omega$. *For any such sequence,*

$$\langle f, \mu^\# \rangle = \lim_{k\to\infty} \langle f_k, \mu^\# \rangle. \tag{2.34}$$

While Proposition 2.6 is a conveniently established consequence of Propositions 2.1–2.5, it is useful to sharpen it. We start with a sharpening of the estimate

$$\|\mathcal{W}f\|_{H^{1,1}(\Omega)} \le C\|f\|_{\mathrm{Lip}^r(\partial\Omega)} \tag{2.35}$$

implicit in (2.12). To get it, we complement (2.10) with the observations that $v = \mathcal{W}f$ satisfies

$$|v(x)| \le C\|f\|_{C^0(\partial\Omega)}, \quad |\nabla v(x)| \le C\delta(x)^{-1}\|f\|_{c^0(\partial\Omega)}, \tag{2.36}$$

with the C^0-norm and c^0-seminorm given by (1.34). Hence, for all $\varepsilon \in (0,1]$,

$$\begin{aligned} \|\nabla v\|_{L^1(\Omega)} &= \int\limits_{\{\delta(x)<\varepsilon\}} |\nabla v(x)|\,dx + \int\limits_{\{\delta(x)\ge\varepsilon\}} |\nabla v(x)|\,dx \\ &\le C\omega(\varepsilon)\|f\|_{\mathrm{lip}^r(\partial\Omega)} + \frac{C}{\varepsilon}\|f\|_{c^0(\partial\Omega)}, \end{aligned} \tag{2.37}$$

where, for Ω satisfying (2.13),

$$\omega(\varepsilon) = \int\limits_{\{x\in\Omega:\delta(x)<\varepsilon\}} \delta(x)^{r-1}\,dx. \tag{2.38}$$

Note that

$$\omega(\varepsilon) \longrightarrow 0, \quad \text{as } \varepsilon \to 0. \tag{2.39}$$

These estimates yield the following useful complement to Proposition 2.1.

Proposition 2.7. *Assume* Ω *satisfies* (2.13). *Take*

$$f,\ f_k \in \mathrm{Lip}^r(\partial\Omega), \tag{2.40}$$

satisfying

$$\|f_k\|_{\mathrm{Lip}^r(\partial\Omega)} \le A < \infty, \quad \|f_k - f\|_{C^0(\partial\Omega)} \to 0. \tag{2.41}$$

Then

$$\mathcal{W}f_k \longrightarrow \mathcal{W}f, \quad \text{in } H^{1,1}(\Omega)\text{-norm}. \tag{2.42}$$

This leads to the following sharpening of Proposition 2.6.

Proposition 2.8. *Take* $r \in (0,1)$ *and assume* $\Omega \subset \mathbb{R}^n$ *is a bounded open set satisfying* (2.13). *Take* $f \in \mathrm{Lip}^r(\partial\Omega)$. *Then there exist* $f_k \in \mathrm{Lip}(\partial\Omega)$ *satisfying* (2.41). *For any such sequence,*

$$\langle f, \mu^\# \rangle = \lim_{k\to\infty} \langle f_k, \mu^\# \rangle. \tag{2.43}$$

Here is a variant of Proposition 2.2, which can also be compared with (1.13).

Proposition 2.9. *Assume the bounded open set $\Omega \subset \mathbb{R}^n$ satisfies* (2.13). *Then*

$$
\begin{aligned}
&F \in \mathrm{Lip}^r(\mathbb{R}^n),\ \operatorname{div} F \in L^1(\mathbb{R}^n),\ f = F\big|_{\partial\Omega} \\
&\Longrightarrow \int_\Omega \operatorname{div} F\, dx = \langle f, \mu^\# \rangle.
\end{aligned}
\tag{2.44}
$$

Proof. A mollifier argument involving $F_k = \varphi_k * F$ as in (1.7) yields $F_k \in C^\infty(\mathbb{R}^n)$,

$$
\begin{gathered}
F_k \to F \text{ in } C^0(\mathbb{R}^n), \quad F_k \text{ bounded in } \mathrm{Lip}^r(\mathbb{R}^n), \\
\operatorname{div} F_k = \varphi_k * \operatorname{div} F \to \operatorname{div} F \text{ in } L^1(\mathbb{R}^n).
\end{gathered}
\tag{2.45}
$$

We have

$$
f_k = F_k|_{\partial\Omega} \longrightarrow f \text{ in } C^0(\partial\Omega), \quad f_k \text{ bounded in } \mathrm{Lip}^r(\partial\Omega),
\tag{2.46}
$$

hence

$$
\int_\Omega \operatorname{div} F_k\, dx = \langle f_k, \mu^\# \rangle \to \langle f, \mu^\# \rangle,
\tag{2.47}
$$

the first identity by Corollary 1.2. Meanwhile,

$$
\int_\Omega \operatorname{div} F_k\, dx \longrightarrow \int_\Omega \operatorname{div} F\, dx,
\tag{2.48}
$$

and we have (2.44). □

Remark. The only role of the Lip^r hypothesis on F in (2.44) is to guarantee (2.46). Thus we could weaken this hypothesis to

$$
F \in \mathrm{Lip}^r(\mathcal{O}), \quad \text{for some open } \ \mathcal{O} \supset \partial\Omega,
\tag{2.49}
$$

and still obtain the conclusion in (2.44). Even more generally, we could simply hypothesize (2.46).

We complement the construction of $\mu^\#$ with one of μ^b, as follows. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and take $\Omega_- = \mathbb{R}^n \setminus \overline{\Omega}$. Assume $\overline{\Omega} \subset B_R$, an open ball of radius $R < \infty$. Apply the Whitney construction described above, with Ω replaced by Ω_-, to obtain a continuous extension map $\mathrm{Lip}^r(\partial\Omega) \to \mathrm{Lip}^r(\overline{\Omega}_-) \cap C^\infty(\Omega_-)$, and follow this with multiplication by a function $K \in C_0^\infty(\mathbb{R}^n)$, satisfying $K = 1$ on a neighborhood of $\overline{\Omega}$, $K = 0$ outside B_R, to get

$$
\mathcal{W} : \mathrm{Lip}^r(\partial\Omega) \longrightarrow \mathrm{Lip}^r(\overline{\Omega}_-) \cap C^\infty(\Omega_-), \quad \forall\, r \in (0,1].
\tag{2.50}
$$

Now assume

$$
\int_{B_R \setminus \overline{\Omega}} \delta(x)^{s-1}\, dx < \infty.
\tag{2.51}
$$

As shown in §3, there are cases where (2.13) and (2.51) hold for different ranges of r and s. For the sake of argument, assume

$$
r \le s.
\tag{2.52}
$$

Parallel to Proposition 2.1, if (2.51) holds, then

$$\mathcal{W} : \mathrm{Lip}^s(\partial\Omega) \longrightarrow \mathrm{Lip}^s(\overline{\Omega}_-) \cap H^{1,1}(\Omega_-). \tag{2.53}$$

This leads to the following

Definition. Given that $\Omega \subset B_R$ and that $B_R \setminus \overline{\Omega}$ satisfies (2.51), we define $\mu^b \in \mathrm{Lip}^s(\partial\Omega)'$ by

$$\langle f, \mu^b \rangle = -\int\limits_{\Omega_-} \operatorname{div} \mathcal{W} f \, dx, \tag{2.54}$$

for $\mathbb{R}^n$-valued f on $\partial\Omega$.

Parallel to Proposition 2.2, we have

$$\int\limits_{\Omega_-} \operatorname{div} \mathcal{W} f(x) \, dx = \int\limits_{\Omega_-} \operatorname{div} F \, dx, \tag{2.55}$$

for such f as in (2.54), whenever $F \in C(\overline{\Omega}_-) \cap H^{1,1}(\Omega_-)$ has compact support and $F|_{\partial\Omega_-} = f$. Note that $\Omega \cup \partial\Omega \cup \Omega_- = \mathbb{R}^n$ and this is a disjoint union. Hence $\chi_\Omega + \chi_{\Omega_-} = 1$ a.e. on $\mathbb{R}^n$ provided $m(\partial\Omega) = 0$, so

$$m(\partial\Omega) = 0 \Longrightarrow \langle f, \mu^\# \rangle = \langle f, \mu^b \rangle, \tag{2.56}$$

for all $f \in \mathrm{Lip}(\partial\Omega)$.

There is also an analogue of (2.37) for $\|\nabla \mathcal{W} f\|_{L^1(B_R \setminus \overline{\Omega})}$. Furthermore, we have an analogue of Proposition 2.8, yielding, for $f \in \mathrm{Lip}^s(\partial\Omega)$,

$$\langle f, \mu^b \rangle = \lim_{k \to \infty} \langle f_k, \mu^b \rangle, \tag{2.57}$$

whenever $f_k \in \mathrm{Lip}(\partial\Omega)$, $\|f_k\|_{\mathrm{Lip}^s(\partial\Omega)} \le A < \infty$, and $\|f_k - f\|_{C^0(\partial\Omega)} \to 0$. This leads to the validity of (2.56) whenever $f \in \mathrm{Lip}^s(\partial\Omega)$, given (2.13), (2.51), and (2.52). Also, By Proposition 2.8, in the setting of (2.50)–(2.56), $\mu^\#$ is the unique linear extension of μ^b from $\mathrm{Lip}^s(\partial\Omega)$ to $\mathrm{Lip}^r(\partial\Omega)$ satisfying

$$|\langle f, \mu^\# \rangle| \le C\omega(\varepsilon) \|f\|_{\mathrm{lip}^r(\partial\Omega)} + \frac{C}{\varepsilon} \|f\|_{c^0(\partial\Omega)}, \tag{2.58}$$

for all $\varepsilon \in (0, 1]$, $f \in \mathrm{Lip}^r(\partial\Omega)$, where $\omega(\varepsilon)$ is given by (2.38).

We record a Gauss-Green formula involving Ω_-, though it does not use the results of (2.50)–(2.58).

Proposition 2.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume Ω satisfies* (2.13) *and*

$$m(\partial\Omega) = 0. \tag{2.59}$$

Set $\Omega_- = \mathbb{R}^n \setminus \overline{\Omega}$, and assume

$$F \in \mathrm{Lip}^r(\mathbb{R}^n), \quad \operatorname{div} F \in L^1(\mathbb{R}^n), \quad \operatorname{supp} F \ \textit{compact}, \tag{2.60}$$

and set $f = F|_{\partial\Omega}$. *Then*

$$\int_{\Omega_-} \operatorname{div} F\,dx = -\langle f, \mu^{\#}\rangle. \tag{2.61}$$

Proof. We have

$$\begin{aligned} 0 = \int_{\mathbb{R}^n} \operatorname{div} F\,dx &= \int_{\Omega} \operatorname{div} F\,dx + \int_{\Omega_-} \operatorname{div} F\,dx \\ &= \langle f, \mu^{\#}\rangle + \int_{\Omega_-} \operatorname{div} F\,dx. \end{aligned} \tag{2.62}$$

□

3. The geometric condition on Ω

As derived in (2.13), the geometric hypothesis on the bounded open set $\Omega \subset \mathbb{R}^n$ used for the results of §2, related to applying $\mu^{\#}$ to $\operatorname{Lip}^r(\partial\Omega)$, is

$$\int_{\Omega} \delta(x)^{r-1}\,dx < \infty, \tag{3.1}$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$, or equivalently

$$M_\Omega(t) \le \beta(t) t^{1-r}, \quad \int_0^1 \frac{\beta(t)}{t}\,dt < \infty, \tag{3.2}$$

where

$$M_\Omega(t) = m\Big(\big\{x \in \Omega : \delta(x) < t\big\}\Big) = m(\mathcal{O}_t). \tag{3.3}$$

Note that $M_\Omega(t) \le \widetilde{\mathcal{M}}_{\partial\Omega}(t)$, defined by

$$\widetilde{\mathcal{M}}_{\partial\Omega}(t) = m\Big(\big\{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial\Omega) < t\big\}\Big) = m(\widetilde{\mathcal{O}}_t). \tag{3.4}$$

For each $t > 0$, the set $\widetilde{\mathcal{O}}_t$ contains $\partial\Omega$ and also points in $\mathbb{R}^n \setminus \overline{\Omega}$, while $\mathcal{O}_t$ is disjoint from these sets. The infimum of all $d > 0$ such that

$$m(\widetilde{\mathcal{O}}_t) \le C_d t^{n-d}, \quad \forall\, t \in (0, 1], \tag{3.5}$$

is called the box dimension of $\partial\Omega$ (B-dim($\partial\Omega$)). We see that

$$\begin{aligned} \text{B-dim}(\partial\Omega) < n - 1 + r \Longrightarrow &\int_0^1 \widetilde{\mathcal{M}}_{\partial\Omega}(t) t^{r-1}\,\frac{dt}{t} < \infty \\ &\Longrightarrow (3.2). \end{aligned} \tag{3.6}$$

The terminology "box dimension" arises as follows. Given $t > 0$, tile $\mathbb{R}^n$ by n-dimensional cubes (boxes) $Q_{\alpha t}$, of edge t, the edges being parallel to the coordinate axes. We define the box-counting function of the compact set $\partial\Omega$ as

$$N_{\partial\Omega}(t) = \text{ number of boxes } Q_{\alpha t} \text{ that intersect } \partial\Omega. \tag{3.7}$$

There exists $C = C_n < \infty$ such that

$$m(\widetilde{\mathcal{O}}_t) \le C N_{\partial\Omega}(t) t^n, \quad N_{\partial\Omega}(t) t^n \le m(\widetilde{\mathcal{O}}_{Ct}). \tag{3.8}$$

Hence B-dim$(\partial\Omega)$ is the infimum of all $d > 0$ such that

$$N_{\partial\Omega}(t) \le C_d t^{-d}, \quad \forall\, t \in (0,1]. \tag{3.9}$$

Basic material on the box dimension can be found in [5]. We mention that B-dim$(\partial\Omega)$ is greater than or equal to the Hausdorff dimension of $\partial\Omega$, which in turn is $\ge n-1$ when $\Omega \subset \mathbb{R}^n$ is a (nonempty) bounded open set.

The estimates in (3.8) also give

$$\int_0^1 \widetilde{\mathcal{M}}_{\partial\Omega}(t) t^{r-1}\,\frac{dt}{t} \approx \int_0^1 N_{\partial\Omega}(t) t^{n-1+r}\,\frac{dt}{t}. \tag{3.10}$$

The hypothesis that this be finite, i.e., that

$$\int_0^1 N_{\partial\Omega}(t) t^{n-1+r}\,\frac{dt}{t} < \infty, \tag{3.11}$$

constitutes the hypothesis in [11] that $\partial\Omega$ be d-summable, with $d = n-1+r$. The analysis above shows that (3.11) holds if and only if we have (3.1) plus two other conditions, namely

$$m(\partial\Omega) = 0, \tag{3.12}$$

and

$$\int\limits_{\Omega_-} \delta(x)^{r-1}\,dx < \infty, \tag{3.13}$$

where $\Omega_- = B_R \setminus \overline{\Omega}$, B_R being some open ball that contains $\overline{\Omega}$. This quantifies the extent to which the condition (3.11) is stronger than (3.1).

Here is an example of a bounded open set $\Omega \subset \mathbb{R}^2$ for which (3.1) applies but (3.11) does not, produced as a modification of the planar domain illustrated in Figure 5.1 in Chapter 5 of [16]. The shaded region Ω winds like a tail infinitely often about an oval Σ, which is its inner boundary. (The goal there was to discuss whether a point $z_0 \in \Sigma$ is a regular point for the Dirichlet problem on Ω.) As the tail of Ω winds about Σ, it gets progressively thinner. One can construct this set Ω so that the tail thins exponentially fast, so that, for $t \le 1/2$,

$$M_\Omega(t) \le Ct\,\log\frac{1}{t}, \tag{3.14}$$

hence (3.1) and (3.2) hold for all $r > 0$. Now modify this construction, simply by taking Σ to be a Koch snowflake (of Hausdorff dimension and box dimension $d_K = (\log 4)/(\log 3)$, cf. [5], §9.2.) One can still arrange that (3.14) hold. But since $\partial\Omega \supset \Sigma$, (3.11) fails, for $r \le d_K - 1$.

In this example, $\mathbb{R}^2 \setminus \partial\Omega$ has three connected components, Ω, Ω_0, and Ω_1, where Ω_1 is the unbounded component and Ω_0 is the bounded region for which $\partial\Omega_0 = \Sigma$. We have

$$\nabla\chi_\Omega = \mu,\ \nabla\chi_{\Omega_j} = \mu_j, \quad \mu + \mu_0 + \mu_1 = 0. \tag{3.15}$$

The distribution μ_0 is more singular than μ, as far as its action on the Lip^r scale is concerned.

One can readily produce related examples, replacing the Koch snowflake by fatter fractals, for example, or moving up in dimension.

For a variant, one can start with the graph of

$$y = \sin \frac{1}{x}, \quad 0 < x \le \pi, \tag{3.16}$$

which, as $x \searrow 0$, snakes toward the vertical line segment $\{(0, y) : -1 \le y \le 1\}$. Now alter this to a curve that similarly snakes toward an arc of the Koch snowflake, or some other fractal, such as $\{(u(y), y) : -1 \le y \le 1\}$, where $u : [-1, 1] \to \mathbb{R}$ is continuous but quite rough. Then thicken up the curve, to a tail of rapidly decreasing thickness, to obtain Ω. One can arrange that such Ω satisfy (3.1) for all $r > 0$, while $B_R \setminus \overline{\Omega}$ (with R sufficiently large that $\overline{\Omega} \subset B_R$) satisfies (3.1) only for r in some interval bounded away from 0. In this example, $\mathbb{R}^2 \setminus \partial\Omega$ has only two connected components.

For a third example, let $B_1 = B_1(0) \subset \mathbb{R}^n$ be the open unit ball, and let $\{p_j : j \in \mathbb{N}\}$ be a dense subset of B_1. Take a sequence r_j satisfying

$$r_j \searrow 0, \quad \sum_{j \ge 1} r_j^{n-1} < \infty, \quad \sum_{j \ge 1} r_j^n < 1. \tag{3.17}$$

Inductively, pick balls $B_{\rho_j}(p_j)$ as follows:

$$0 < \rho_j \le r_j, \quad B_{\rho_j}(p_j) \subset B_1 \setminus \bigcup_{k<j} B_{\rho_k}(p_k). \tag{3.18}$$

If $p_j \in \cup_{k<j} B_{\rho_k}(p_k)$, skip it. Now form the open set

$$\Omega = \bigcup_j B_{\rho_j}(p_j). \tag{3.19}$$

By construction,

$$\begin{aligned} &\Omega \subset B_1, \quad \overline{\Omega} = \overline{B}_1, \quad \text{and} \\ &m(\Omega) < m(B), \quad \text{hence} \quad m(\partial\Omega) > 0. \end{aligned} \tag{3.20}$$

In this case, we have

$$M_\Omega(t) \le Ct, \quad \forall\, t \in (0, 1], \tag{3.21}$$

with $C = A_{n-1} \sum_j \rho_j^{n-1}$, A_{n-1} denoting the area of S^{n-1}. By contrast,

$$\widetilde{M}_{\partial\Omega}(t) \ge m(\partial\Omega) > 0, \quad \forall\, t \in (0, 1], \tag{3.22}$$

Hence (3.1) holds for all $r > 0$, but (3.11) fails for all $r \in (0, 1)$, since (3.12) fails. On the other hand, here

$$\Omega_- = B_R \setminus \overline{B}_1 \tag{3.23}$$

also satisfies an estimate like (3.21), and (3.13) holds for all $r > 0$. Actually, in this case both Ω and Ω_- are finite-perimeter domains. We have

$$\partial_*\Omega = \bigcup_{j \ge 1} \partial B_{\rho_j}(p_j), \quad \partial_*\Omega_- = \partial\Omega_- = \partial B_1 \cup \partial B_R. \tag{3.24}$$

4. Variation of $\mu = \nabla\chi_\Omega$ with Ω

Here we study the dependence of the distribution $\mu = \nabla\chi_\Omega$ on Ω, with particular attention to when, and in what topology, we might have

$$\mu_j \longrightarrow \mu, \quad \text{for} \quad \mu_j = \nabla\chi_{\Omega_j}. \tag{4.1}$$

For this it is useful to keep track of how estimates on μ depend on Ω. We begin with the following observations.

First, in the estimates on a Whitney partition of unity on Ω described in (a)–(b) of §2, the constants M and C may depend on the dimension n, but they are otherwise independent of the open set $\Omega \subset \mathbb{R}^n$. (See [17], [15].) Consequently, if $v = \mathcal{W}f$, the estimate (2.10) on $|\nabla v(x)|$ involves a constant that is independent of Ω. The same goes for the estimates in (2.36). Hence we can reformulate the estimate (2.37) as

$$\int_\Omega |\nabla v(x)|\,dx \le C\omega_{r,\Omega}(\varepsilon)\|f\|_{\mathrm{lip}^r(\partial\Omega)} + \frac{C}{\varepsilon}m(\Omega)\|f\|_{c^0(\partial\Omega)}, \quad \forall\,\varepsilon \in (0,1], \tag{4.2}$$

where

$$\omega_{r,\Omega}(\varepsilon) = \int_{\{x\in\Omega:\delta(x)<\varepsilon\}} \delta(x)^{r-1}\,dx, \tag{4.3}$$

and C in (4.2) is independent of Ω. As a corollary, one has

$$|\langle f, \mu^{\#}\rangle| \le C\omega_{r,\Omega}(\varepsilon)\|f\|_{\mathrm{lip}^r(\partial\Omega)} + \frac{C}{\varepsilon}m(\Omega)\|f\|_{c^0(\partial\Omega)}, \tag{4.4}$$

for $f \in \mathrm{Lip}^r(\partial\Omega)$, given that (2.13) holds. Hence, by Proposition 2.9 plus (1.13),

$$|\langle F, \mu\rangle| \le C\omega_{r,\Omega}(\varepsilon)\|F\|_{\mathrm{lip}^r(\mathbb{R}^n)} + \frac{C}{\varepsilon}m(\Omega)\|F\|_{c^0(\mathbb{R}^n)}, \tag{4.5}$$

given

$$F \in \mathrm{Lip}^r(\mathbb{R}^n), \quad \operatorname{div} F \in L^1(\mathbb{R}^n). \tag{4.6}$$

Here is one simple comparison of μ with μ_j. Given F satisfying (4.6),

$$\begin{aligned}\langle F, \mu\rangle - \langle F, \mu_j\rangle &= \int_\Omega \operatorname{div} F\,dx - \int_{\Omega_j} \operatorname{div} F\,dx \\ &= \int_{\Omega\setminus\Omega_j} \operatorname{div} F\,dx - \int_{\Omega_j\setminus\Omega} \operatorname{div} F\,dx.\end{aligned} \tag{4.7}$$

Hence

$$|\langle F, \mu\rangle - \langle F, \mu_j\rangle| \le \int_{\Omega\triangle\Omega_j} |\operatorname{div} F|\,dx, \tag{4.8}$$

where

$$\Omega\triangle\Omega_j = (\Omega \setminus \Omega_j) \cup (\Omega_j \setminus \Omega). \tag{4.9}$$

This leads to the following convergence result.

Proposition 4.1. *Let Ω and Ω_j be bounded open sets in $\mathbb{R}^n$. Assume F satisfies (4.6). If*

$$m(\Omega \triangle \Omega_j) \longrightarrow 0, \tag{4.10}$$

as $j \to \infty$, then $\langle F, \mu_j \rangle \to \langle F, \mu \rangle$.

Proof. If (4.10) holds, each subsequence of (j) has a further subsequence on which $\chi_{\Omega \triangle \Omega_j} \to 0, m$-a.e. Then the dominated convergence theorem applies to the right side of (4.8). □

If the hypothesis on $\operatorname{div} F$ in (4.6) is strengthened to

$$\operatorname{div} F \in L^p(\mathbb{R}^n), \quad 1 < p \le \infty, \tag{4.11}$$

we get a rate of convergence:

$$|\langle F, \mu \rangle - \langle F, \mu_j \rangle| \le \| \operatorname{div} F\|_{L^p(\Omega \triangle \Omega_j)} \, m(\Omega \triangle \Omega_j)^{1/p'}. \tag{4.12}$$

We now aim for a convergence result valid for all $F \in \mathrm{Lip}^r(\mathbb{R}^n)$, without an extra hypothesis on $\operatorname{div} F$, such as given in (4.6). Istead, the domains Ω and Ω_j will satisfy an appropriate geometric hypothesis. As a first step in formulating the result, we extend μ from a continuous linear functional on the space of F satisfying (4.6) to a linear functional on $\mathrm{Lip}^r(\mathbb{R}^n)$, by

$$\langle F, \mu \rangle = \langle F|_{\partial\Omega}, \mu^{\#} \rangle, \tag{4.13}$$

under the hypothesis that Ω satisfies (2.13). We also assume Ω_j satisfy (2.13), and similarly bring in $\mu_j^{\#}$ and extend μ_j. Our geometrical hypothesis on these domains is that

$$\omega_{r,\Omega}(\varepsilon), \, \omega_{r,\Omega_j}(\varepsilon) \le \omega(\varepsilon), \quad \forall \, j, \tag{4.14}$$

where $\omega(\varepsilon)$ satisfies

$$\omega(\varepsilon) \longrightarrow 0, \quad \text{as} \;\; \varepsilon \to 0. \tag{4.15}$$

We also assume $\Omega, \Omega_j \subset B_R(0)$, for all j, so $m(\Omega), m(\Omega_j) \le A_n R^n$. In such a case, we have from (4.4) and its analogue for Ω_j that

$$|\langle F, \mu - \mu_j \rangle| \le 2C\omega(\varepsilon) \|F\|_{\mathrm{lip}^r(\mathbb{R}^n)} + 2\frac{C}{\varepsilon} A_n R^n \|F\|_{c^0(\mathbb{R}^n)}, \tag{4.16}$$

for all $F \in \mathrm{Lip}^r(\mathbb{R}^n)$. Using these estimates, we can establish the following convergence result.

Proposition 4.2. *Let Ω, Ω_j be open sets in $\mathbb{R}^n$, all contained in $B_R(0)$. Take $r \in (0, 1)$. Assume* (2.13) *holds, uniformly, and more precisely that* (4.14) *holds, with $\omega(\varepsilon)$ satisfying* (4.15). *Furthermore, assume the estimate* (4.10) *on $\Omega \triangle \Omega_j$ holds. Then, as $j \to \infty$,*

$$\langle F, \mu_j \rangle \longrightarrow \langle F, \mu \rangle, \quad \forall \, F \in \mathrm{Lip}^r(\mathbb{R}^n). \tag{4.17}$$

Proof. We can assume F is supported in $B_{2R}(0)$. Apply the standard mollifier argument to F, obtaining $F_k = \varphi_k * F \in C^\infty(\mathbb{R}^n)$, satisfying

$$\|F_k\|_{\mathrm{Lip}^r} \le \|F\|_{\mathrm{Lip}^r}, \quad \|F_k\|_{C^0} \le \|F\|_{C^0}, \quad \|F_k - F\|_{C^0} = \delta_k \to 0. \tag{4.18}$$

By Proposition 4.1, if (4.10) holds, then

$$\lim_{j\to\infty} \langle F_k, \mu - \mu_j \rangle = 0, \quad \forall\, k. \tag{4.19}$$

Meanwhile, by (4.16), applied to $F - F_k$ (plus (4.18)),

$$\begin{aligned} |\langle F - F_k, \mu - \mu_j \rangle| &\le 2C\omega(\varepsilon)\|F - F_k\|_{\mathrm{Lip}^r} + 2\frac{C}{\varepsilon} A_n R^n \|F - F_k\|_{C^0} \\ &\le 4C\omega(\varepsilon)\|F\|_{\mathrm{Lip}^r} + 2\frac{C}{\varepsilon} A_n R^n \delta_k, \end{aligned} \tag{4.20}$$

for all j. Thus,

$$\limsup_{j\to\infty} |\langle F, \mu - \mu_j \rangle| \le 4\omega(\varepsilon)\|F\|_{\mathrm{Lip}^r} + 2\frac{C}{\varepsilon} A_n R^n \delta_k, \quad \forall\, k, \tag{4.21}$$

and for all $\varepsilon \in (0, 1]$. Taking $k \to \infty$, we have

$$\limsup_{j\to\infty} |\langle F, \mu - \mu_j \rangle| \le 4\omega(\varepsilon)\|F\|_{\mathrm{Lip}^r}, \quad \forall\, \varepsilon \in (0, 1], \tag{4.22}$$

and then taking $\varepsilon \to 0$ yields (4.17). □

References

[1] G.-Q. Chen, G. Comi, and M. Torres, *Cauchy fluxes and Gauss-Green formula for divergence-measure fields over general open sets*, Arch. Rat. Mech. Anal. 233 (2019), 87–166.

[2] G.-Q. Chen, M. Torres, and W. Ziemer, *Gauss-Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws*, Commun. Pure Appl. Math. 62 (2009), 242–304.

[3] G. Comi and K. Payne, *On locally essentially bounded divergence measure fields and sets of locally finite perimeter*, Adv. Calc. Var. 2 (2018), 179–217.

[4] L. Evans and R. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, 1992.

[5] K. Falconer, *Fractal Geometry*, J. Wiley, 1990.

[6] H. Federer, *Geometric Measure Theory*, Springer, 1969.

[7] D. Gilbarg and N. Trudinger, *Elliptic Boundary Problems of Second Order* (2nd ed.), Springer, 1983.

[8] J. Harrison, *Continuity of the integral as a function of the domain*, J. Geometric Anal. 8 (1998), 769–795.

[9] J. Harrison, *Isomorphisms of differential forms and cochains*, J. Geometric Anal. 8 (1998), 797–807.

[10] J. Harrison and A. Norton, *Geometric integration on fractal curves in the plane*, Indiana Univ. Math. J. 40 (1991), 567–594.

[11] J. Harrison and A. Norton, *The Gauss-Green theorem for fractal boundaries*, Duke Math. J. 67 (1992), 575–588.

[12] S. Hofmann, M. Mitrea, and M. Taylor, *Singular integrals and elliptic boundary problems on regular SKT domains*, International Math. Res. Notices (2010), 2567–2865.

[13] D. Mitrea, I. Mitrea, and M. Mitrea, *Geometric Harmonic Analysis*, Vols. 1–5, Springer, 2023.

[14] M. Silhavy, *The divergence theorem for divergence measure vector fields on sets with fractal boundaries*, Math. Mech. Solids 14 (2009), 445–455.

[15] M. Taylor, *Measure Theory and Integration*, Amer. Math Soc., Providence RI, 2006.

[16] M. Taylor, *Partial Differential Equations*, Vol. 1, Springer, New York, 1996 (2nd ed., 2011).

[17] H. Whitney, *Geometric Integration Theory*, Princeton Univ. Press, 1957.

[18] W. Ziemer, *Weakly Differentiable Functions*, Springer, 1989.

Manuscript received September 15 2022
revised March 5 2022

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