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QUALITATIVE QUESTIONS TO MIXED LOCAL-NONLOCAL ELLIPTIC OPERATORS

PRIYANK OZA AND JAGMOHAN TYAGI

ABSTRACT. We establish a Lyapunov-type inequality for a class of mixed local-nonlocal operator. We employ solution representation formula for the associated boundary value problem. Furthermore, as an application of the Lyapunov-type inequality, we show a positive lower bound for the generalized principal eigenvalue of the operator.

1. INTRODUCTION

We consider a class of mixed local-nonlocal equations:

$$\begin{cases} \alpha\Delta u - \beta(-\Delta)^s u + cu = 0 \text{ on } D, \\ \text{with } (\alpha,\beta)\text{-Dirichlet condition,} \end{cases} \tag{1.1}$$

where $\alpha, \beta \geq 0$ are constants, $D \subset \mathbb{R}^N$ is a bounded Lipschitz domain satisfying the uniform exterior sphere condition, $c \in C(D)$ is bounded and $0 < s < 1$. Here, (α,β)-Dirichlet condition is given as follows:

$$\begin{cases} u = 0 \text{ in } \partial D, & \text{if } \beta = 0 \\ u = 0 \text{ in } \mathbb{R}^N \setminus D, & \text{if } \beta \neq 0. \end{cases}$$

Very recently, the elliptic equations involving mixed local and non-local operators have gained much attention, see [4, 5] and the references therein. There also have been substantial developments on the regularity questions to mixed operators, see, for instance [11, 15, 22, 23] and the references therein. Recently, A. Biswas and M. Modasiya [6] obtained the Faber-Krahn inequality for mixed local-nonlocal operators. They established the solution representation formula for Dirichlet problem concerning mixed operators and then using it, they established the Faber-Krahn inequality.

Let us recall that Lyapunov inequality plays an important role in the analysis of differential equations. It has several applications such as in stability analysis, eigenvalue bounds and many others. It is very instrumental to get certain estimates

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on the first eigenvalue of the associated eigenvalue problem. A. M. Lyapunov [21] established a necessary condition for the problem:

$$\begin{cases} w'' + c(x)w = 0 \quad \text{in } (a,b), \\ w(a) = w(b) = 0, \end{cases}$$

to have a non-trivial solution, which is

$$\int_a^b |c| dx > \frac{4}{b-a},$$

see [8, 21]. Later, by replacing c with c^+, A. Wintner [28] proved that 4 is the optimal constant. There are significant improvements and generalizations to the classical Lyapunov inequality. This inequality has also been generalized to equations pertaining to quasilinear terms, see, for instance, [13, 16, 24]. We also mention the works [14, 17, 18, 26] and the references therein, which are devoted to the Lyapunov-type inequality for fractional equations. We point out that apart from the above works, there are other interesting works in this direction for partial differential equations (PDEs) as well, see, for instance, A. Cañada [10]. A Lyapunov-type inequality has also been established for p-Laplace equation [12], a class of singular elliptic PDEs [20] and for Pucci's extremal equation with gradient nonlinearity [27]. Lyapunov-type inequalities have a good number of applications, see, for instance, [1, 7, 9, 16, 25].

Let us recall the notion of generalized principal eigenvalue for $\alpha\Delta - \beta(-\Delta)^s$ in D in the sense of Berestycki et al. [3]:

$$\lambda^* = \sup\left\{\lambda : \exists u \in C(D), u > 0 \text{ and bounded in } \mathbb{R}^N, \alpha\Delta u - \beta(-\Delta)^s u + \lambda u \leq 0 \text{ in } D\right\}.$$

For the details, we refer to [6].

Motivated by the above results, we are interested to establish a Lyapunov-type inequality for mixed local-nonlocal equations. We leverage the solution representation formula established in [6] to discuss the inequality. Now, to formulate our result, let us consider the problem (1.1). Let us define a set

$$\mathcal{C} := \left\{c \in C(D) \setminus \{0\} \text{ s.t. (1.1) has nontrivial solutions}\right\}.$$

Theorem 1.1. Let $\alpha, \beta > 0$ in (1.1). Let $D \subset \mathbb{R}^N$ be a bounded Lipschitz domain satisfying a uniform exterior condition. Let $c \in \mathcal{C}$. Set

$$M = \inf_{\mathcal{C}} \sup_D c^+,$$

where c^+ denotes the positive part of the function c. Then $M = \lambda^*$, where λ^* is the generalized principal eigenvalue of $\alpha\Delta - \beta(-\Delta)^s$ in D.

Below, we state our main result on Lyapunov-type inequality for (1.1). This also establishes a positive lower bound for the generalized principal eigenvalue of the operator.

Theorem 1.2 (Lyapunov-type inequality)**.** Let $\alpha, \beta > 0$ in (1.1). Let D be a bounded Lipschitz domain in $\mathbb{R}^N$ satisfying a uniform exterior sphere condition. Let λ^* be the generalized principal eigenvalue of $\alpha\Delta - \beta(-\Delta)^s$ in D. Then

$$\lambda^* = M \geq \frac{1}{C(N,d)},$$

where $C(N,d)$ is a universal positive constant depending only on dimension N and the diameter d of D, which increases monotonically with d.

We mention that after the proof of this theorem, in Remark 3.1, we give a lower bound on $\sup_D |c|$ for any $c \in \mathcal{C}$, in terms of the first exit time of X_t from B, where X_t is the process associated with $\mathcal{L}$ (see, below Definition 2.3) and B is a ball containing D.

In the best of our knowledge, we are not aware of any results on Lyapunov type inequality for mixed operators.

Remark 1.3. When $\beta = 0$ in (1.1), Cañada et al. [10] established a Lyapunov-type inequality for (1.1). We refer to Remark 5 and Theorem 2.1 therein for the details.

The organization of this note is as follows. In Section 2, we list the basic definitions and several auxiliary results, which are pivotal tools in our paper. Section 3 is devoted to the proofs of our main results.

2. Preliminaries

First, we define the notion of viscosity solution for (1.1).

Definition 2.1 ([2])**.** Let $u : \mathbb{R}^N \longrightarrow \mathbb{R}$ be an upper semicontinuous (USC) function in $\overline{D}$. Then u is called a *viscosity subsolution* of (1.1) if for any $x \in D$ and C^2 function $\varphi : \overline{B}_\rho(x) \longrightarrow \mathbb{R}$, for some $B_\rho(x) \Subset D$ such that $\varphi(x) = u(x)$, $\varphi(y) > u(y)$ for $y \in \overline{B}_\rho(x) \setminus \{x\}$, we have

$$\alpha\Delta v(x) - \beta(-\Delta)^s v(x) + c(x)v(x) \geq f(x),$$

where

$$v := \begin{cases} \varphi & \text{in } B_\rho(x), \\ u & \text{in } \mathbb{R}^N \setminus B_\rho(x). \end{cases}$$

Moreover, we say that u satisfies $\alpha\Delta u - \beta(-\Delta)^s u + cu \geq f$ in D in the viscosity sense.

Definition 2.2 ([2])**.** Let $u : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a lower semicontinuous (LSC) function in $\overline{D}$. Then u is called a *viscosity supersolution* of (1.1) if for any $x \in D$ and C^2 function $\psi : \overline{B}_\rho(x) \longrightarrow \mathbb{R}$, for some $B_\rho(x) \Subset D$ such that $\psi(x) = u(x)$, $\psi(y) < u(y)$ for $y \in \overline{B}_\rho(x) \setminus \{x\}$, we have

$$\alpha\Delta w(x) - \beta(-\Delta)^s w(x) + c(x)w(x) \leq f(x),$$

where

$$w := \begin{cases} \psi & \text{in } B_\rho(x), \\ u & \text{in } \mathbb{R}^N \setminus B_\rho(x). \end{cases}$$

Moreover, we say that u satisfies $\alpha\Delta u - \beta(-\Delta)^s u + cu \leq f$ in D in the viscosity sense.

Definition 2.3. A continuous function u is said to be a *viscosity solution* to (1.1) if it is a subsolution as well as a supersolution of (1.1).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let X, B and Y be processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$ given as $X = B + Y$, where Y is a pure jump Lévy process and B is an N-dimensional Brownian motion, independent of Y, which runs twice as fast as standard Brownian motion. Here, X is a strong Markov process such that the generator of semigroup associated with X is given by

$$\mathcal{L} = \alpha\Delta - \beta(-\Delta)^s, \text{ for } 0 < s < 1, \text{ and } \alpha, \beta > 0.$$

In next theorem, let us mention the existence of a solution to mixed local-nonlocal equation and its representation formula, which is crucial to prove our results.

Theorem 2.4 (Theorem 1.1 [6])**.** Let $D \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Let $f \in C(\overline{D})$, and $g \in C(\mathbb{R}^N \setminus D)$ be bounded. Then there exists a unique bounded viscosity solution $u \in C(\mathbb{R}^N)$ to the problem:

$$\begin{cases} \Delta u - (-\Delta)^s u = -f & \text{in } D, \\ u = g & \text{in } \mathbb{R}^N \setminus D. \end{cases}$$

Moreover, u is given by

$$u(x) = E_x\left[\int_0^{\tau_D} f(X_t)dt\right] + E_x\left[g(X_{\tau_D})\right], \, x \in D.$$

Here τ_D is the first exit time of X from D, i.e.,

$$\tau_D = \inf\{t > 0 : X_t \notin D\},$$

and E denotes the expectation with respect to the measure $\mathbb{P}$.

Lemma 2.5 (Lemma 2.2 [6])**.** Let $D \subset \mathbb{R}^N$ be a bounded domain. Let $f \in L^p(D)$, for $p > \frac{N}{2}$. Let $u : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a bounded function such that

$$u(x) \leq E_x\big[u(X_{\tau_D})\big] + E_x\left[\int_0^{\tau_D} f(X_t)dt\right], \, x \in D.$$

Then there exists a positive constant $C = C(p, N, d)$ such that

$$\sup_D u \leq \sup_{\mathbb{R}^N \setminus D} u^+ + C\|f\|_{p,D},$$

where d denotes the diameter of D.

Next, we mention a maximum principle for narrow domains, which we use in the proof of Theorem 1.1.

Theorem 2.6 (Corollary 2.1 [6])**.** Let $c \in C(D)$. Let u be a bounded continuous viscosity subsolution to

$$\begin{cases} \Delta u - (-\Delta)^s u + cu = 0 & \text{in } D, \\ u \leq 0 & \text{in } \mathbb{R}^N \setminus D. \end{cases}$$

Then there exists a constant $\varepsilon = \varepsilon(d, \|c\|_\infty,\ \mathrm{diam}(D))$ such that if $|D| < \varepsilon$ then $u \leq 0$ in $\mathbb{R}^N$.

3. Proofs of our main results

Proof of Theorem 1.1. We first observe that $\lambda^* \in \mathcal{C}$. Let Ψ be the corresponding principal eigenfunction, i.e., $\Psi \in C(\mathbb{R}^N)$, bounded and positive in D, which satisfies

$$\begin{cases} \alpha\Delta\Psi - \beta(-\Delta)^s\Psi + \lambda^*\Psi = 0 & \text{in } D, \\ \Psi = 0 & \text{in } \mathbb{R}^N \setminus D. \end{cases}$$

Moreover, $\lambda^* > 0$, which can be seen as follows. Assume the contrary, i.e., $\lambda^* \leq 0$. It is clear using $\lambda^* \leq 0$ that Ψ is a viscosity subsolution to

$$\begin{cases} \alpha\Delta u - \beta(-\Delta)^s u = 0 & \text{in } D, \\ u = 0 & \text{in } \mathbb{R}^N \setminus D. \end{cases}$$

On the other hand, comparison principle (Theorem 5.2 [6]) infers that $\Psi \leq 0$ in D. This gives a contradiction. Hence $\lambda^* > 0$, which further yields by the definition that $M \leq \lambda^*$.

Our aim is to show that $M = \lambda^*$. Let if possible, $M < \lambda^*$. It implies that λ^* is not a lower bound of the set

$$\mathcal{S} := \left\{ \sup_D c^+ : c \in \mathcal{C} \right\}.$$

This along with the fact $\lambda^* \in \mathcal{C}$ yields that for some $\varepsilon > 0$, there exists $c \in \mathcal{C}$ such that

$$\alpha\Delta u - \beta(-\Delta)^s u + cu = 0 \text{ in } D,$$

and $c^+ \leq \lambda^* - \varepsilon$. Next, consider the set

$$D_u^+ = \{u > 0\} \cap D.$$

Now, let us define

$$t = \sup\{s \colon su - \Psi < 0 \text{ in } K\},$$

where K is some compact subset of D_u^+. Now, since $\Psi > 0$ in D, $\Psi = 0$ in $\mathbb{R}^N \setminus D$ and $u \leq 0$ in $\mathbb{R}^N \setminus D_u^+$, so this yields

$$tu - \Psi \leq 0 \text{ in } K \cup \left(\mathbb{R}^N \setminus D_u^+\right).$$

Also, observe that

$$\begin{aligned} &\big(\alpha\Delta - \beta(-\Delta)^s\big)(tu - \Psi) + \lambda^*(tu - \Psi) \\ &\qquad = \big(t\alpha\Delta - t\beta(-\Delta)^s\big)u - \big(\alpha\Delta - \beta(-\Delta)^s\big)\Psi + \lambda^*(tu - \Psi) \\ &\qquad \geq t\big(\alpha\Delta - \beta(-\Delta)^s + c^+\big)u - \big(\alpha\Delta - \beta(-\Delta)^s + \lambda^*\big)\Psi + \varepsilon tu \\ &\qquad \geq 0 \text{ in } D_u^+, \end{aligned}$$

since $\lambda^* \geq c^+ + \varepsilon$. Now, for any $\varepsilon' > 0$, we can take K large enough such that $|D_u^+ \setminus K| < \varepsilon'$, where $\varepsilon' = \varepsilon'(N, \mathrm{diam}(D_u^+))$. Further, by Theorem 2.6, it yields

$$tu \leq \Psi \text{ in } \mathbb{R}^N. \tag{3.1}$$

Now, we scale Ψ such that $\Phi := tu - \Psi$ attains its maximum value at some point x_0 of $\overline{D_u^+}$ with $\Phi(x_0) = 0$. By the continuity of u and that $\Psi > 0$ in D, we get that x_0 is an interior point of D_u^+. Note that we denote this scaled function by Ψ. Further, we observe that

$$0 = \alpha\Delta\Phi(x_0) - \beta(-\Delta)^s\Phi(x_0) + \lambda^*\Phi(x_0) \tag{3.2}$$
$$\leq -\beta \text{ P.V.} \int_{\mathbb{R}^N} \frac{\Phi(x_0) - \Phi(x)}{|x_0 - x|^{N+2s}} dx$$
$$= \beta \text{ P.V.} \int_{\mathbb{R}^N} \frac{\Phi(x)}{|x_0 - x|^{N+2s}} dx,$$

where in the second step, we used the fact that x_0 is a point of maxima inside $\overline{D_u^+ \setminus K}$ and $\Phi(x_0) = 0$. One may also see pp. 22–23 [19] for the details. Now, (3.1) together with (3.2) yields

$$\Phi \equiv 0 \text{ in } \mathbb{R}^N,$$

equivalently,

$$tu = \Psi \text{ in } \mathbb{R}^N.$$

This gives that

$$\alpha\Delta u - \beta(-\Delta)^s u + \lambda^* u = 0 \text{ in } D,$$

which is not possible. This yields that $D_u^+ = \emptyset$. Similarly, one can see that $D_u^- := \{u < 0\} \cap D = \emptyset$. Thus, u must be a trivial solution, which gives a contradiction. Hence, $M = \lambda^*$.

□

Proof of Theorem 1.2. By Theorem 2.4, we have that the viscosity solution of

$$\begin{cases} \alpha\Delta u - \beta(-\Delta)^s u = -f & \text{in } D \\ u = 0 & \text{in } \mathbb{R}^N \setminus D, \end{cases}$$

is given by

$$u(x) = E_x \left[\int_0^{\tau_D} f(X_t) dt \right], \, x \in D.$$

Further, by an application of Lemma 2.5, we get that u satisfies the following estimate:

$$u(x) \leq (\sup_D f) E_x[\tau_D]$$
$$\leq (\sup_D f) C(N, d), \, x \in D,$$

for some positive constant $C(N, d)$ depending only on N and d, where d denotes the diameter of D. Let Ψ be the principal eigenfunction of $\alpha\Delta - \beta(-\Delta)^s$, then we have

$$\alpha\Delta\Psi - \beta(-\Delta)^s\Psi + \lambda^*\Psi = 0 \text{ in } D.$$

This further gives

$$\alpha\Delta\Psi - \beta(-\Delta)^s\Psi = -\lambda^*\Psi \text{ in } D.$$

Then by Lemma 2.5, we have that Ψ satisfies

$$\begin{aligned}\sup_D \Psi &\le \sup_D(\lambda^*\Psi)\, C(N,d)\\ &\le C(N,d)\,\lambda^* \sup_D \Psi.\end{aligned}$$

Now, since $\Psi > 0$ in D, so we have

$$1 \le C(N,d)\lambda^*.$$

In other words,

$$\lambda^* \ge \frac{1}{C(N,d)}.$$

□

Remark 3.1. In particular, using the arguments of Lemma 3.1 [6], we see that

$$|u(x)| \le \sup_D |c| \sup_D |u| E_x[\tau_B], \text{ for } x \in D \text{ and } c \in \mathcal{C},$$

where B is a ball that contains D. Next, let $y \in D$ be a point, where $|u|$ attains its maximum in $\overline{D}$. This yields

$$|u(y)| \le \sup_D |c||u(y)|E_y[\tau_B],$$

equivalently,

$$\sup_D |c| \ge \frac{1}{E_y[\tau_B]},\ y \in D.$$

This immediately yields that

$$\lambda^* \ge \frac{1}{E_y[\tau_B]}, \text{ for any } y \in D.$$

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P. Oza
Indian Institute of Technology, Gandhinagar
E-mail address: priyank.k@iitgn.ac.in, priyank.oza3@gmail.com

J. Tyagi
Indian Institute of Technology, Gandhinagar
E-mail address: jtyagi@iitgn.ac.in, jtyagi1@gmail.com