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THE EXTENSION PROPERTY FOR DOMAINS WITH ONE SINGULAR POINT

PEKKA KOSKELA AND ZHENG ZHU

Abstract. An arbitrary outward cuspidal domain is shown to be bi-Lipschitz equivalent to a Lipschitz outward cuspidal domain via a global transformation. This allows us to extend earlier Sobolev extension results on Lipschitz outward cuspidal domains from the work of Maz'ya and Poborchi to general outward cuspidal domains. We also establish a limit case of the extension results on outward cuspidal domains.

1. Introduction

A bounded domain $\Omega \subset \mathbb{R}^n$ is said to be a Sobolev (p, q)-extension domain for $1 \le q \le p \le \infty$, if there exists a bounded extension operator

$$E : W^{1,p}(\Omega) \to W^{1,q}(\mathbb{R}^n),$$

i.e., for every $u \in W^{1,p}(\Omega)$, we have $E(u) \in W^{1,q}(\mathbb{R}^n)$ with

$$\|E(u)\|_{W^{1,q}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)}$$

for a constant C independent of u. The smallest constant in the inequality above is denoted by $\|E\|$. Studies on these extensions can be found in [1, 7, 9, 11, 15, 16, 19, 20, 21, 22, 26, 30, 31, 32]. In some of the above references, one considers the Sobolev spaces with the homogeneous seminorms instead of the full Sobolev norms. This gives the same class of extension domains when $q = p$ by results in [10] for our bounded setting. The situation is more complicated when $q < p$ but the extension property with seminorms still implies the above extension property. The converse also holds when $q \ge n$ or $p < qn/(n - q)$ but not in general [17].

In [4, 29], Calderón and Stein proved that if $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, then there exists a bounded linear extension operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$, for all $k \ge 1$ and $1 \le p \le \infty$. Here $W^{k,p}(\Omega)$ is the Banach space of those L^p-integrable functions whose weak derivatives up to order k belong to $L^p(\Omega)$. In [11], Jones introduced the notion of (ε, δ)-domains which forms a generalization of Lipschitz domains in the sense of the flexible cone condition [3]. Jones proved that, for every (ε, δ)-domain, there exists a bounded linear extension operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$, for all $k \ge 1$ and $1 \le p \le \infty$. This has motivated the search

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for geometric characterizations for Sobolev extension domains. The necessary and sufficient geometric conditions for simply connected planar Sobolev $(2,2)$-extension domains were obtained in [31]. By a more recent sequence of results in [12, 13, 14, 26], more complicated geometric characterizations in terms of the sub-hyperbolic metric are given for simply connected planar Sobolev (p,p)-extension domains, for all $1 \leq p \leq \infty$. Furthermore, by [27], geometric characterizations in terms of the sub-hyperbolic metric of planar simply connected extension domains are also known in the case of the Sobolev spaces $L^{k,p}(\Omega)$, $2 < p < \infty$, of those locally integrable functions whose k-th order distributional partial derivatives belong to $L^p(\Omega)$, endowed with the homogeneous seminorm. However, no characterizations are available yet for Sobolev (p,q)-extension domains, for most $1 \leq q \leq p \leq \infty$.

In this paper, we consider Sobolev extension properties for a class of bounded quasiconvex Euclidean domains with a single singular boundary point. A domain $\Omega \subset \mathbb{R}^n$ is called quasiconvex, if for all $x, y \in \Omega$ there exists a curve $\gamma_{x,y} \subset \Omega$ connecting x, y with

$$\mathrm{length}(\gamma_{x,y}) \leq C|x-y|$$

for a constant C independent of x and y. We study the outward cuspidal domains defined by setting

(1.1)

$$\Omega^n_\psi := \left\{z=(t,x)\in(0,1]\times\mathbb{R}^{n-1}; |x|<\psi(t)\right\}\cup\{z=(t,x)\in[1,2)\times\mathbb{R}^{n-1}; |x|<\psi(1)\},$$

where $\psi\colon (0,1]\to(0,\infty)$ is a left-continuous and increasing function. (Left-continuity is required just to ensure Ω^n_ψ to be open. The term "increasing" is used in the non-strict sense.) If the left-continuous and increasing function ψ is Lipschitz, that is, there exists a large enough constant $C > 1$ such that for every $s, t \in (0,1]$,

$$|\psi(s)-\psi(t)| \leq C|s-t|,$$

Ω^n_ψ is called a Lipschitz outward cuspidal domain. This kind of model domains have been studied by Maz'ya and Poborchi's [19, 20, 21, 22, 23, 24, 25]. Our terminology here is not standard: when e.g. $\psi(t) = t^2$, one could also call our cusp a Hölder cusp since the boundary can be represented as the graph of a function with Hölder exponent $1/2$ close to the tip. Examples of Lipschitz cuspidal domains for which the Sobolev extension operators can be realized via a composition operators were given by Gol'dshtein and Sitnikov in [7], also see [16]. Returning to the general case, from now on, every left-continuous and increasing function $\psi : (0,1] \to (0,\infty)$ will be called a cuspidal function. Our class of domains Ω^n_ψ was introduced in [5]. It was shown in [5], that, for an arbitrary cuspidal function ψ, the Sobolev space $W^{1,p}(\Omega^n_\psi)$ coincides with the Hajłasz-Sobolev space $M^{1,p}(\Omega^n_\psi)$ for all $1 < p \leq \infty$. See [8] for the definition of the Hajłasz-Sobolev space $M^{1,p}(\Omega)$. One may ask a natural question:

For which cuspidal functions ψ, the outward cuspidal domain Ω^n_ψ is a Sobolev (p,q)-extension domain for given $1 \leq q \leq p \leq \infty$?

The first observation is that Ω^n_ψ is a Sobolev (∞,∞)-extension domain, for every cuspidal function ψ. This follows from a result in [9] since Ω^n_ψ is always quasiconvex for an arbitrary cuspidal function ψ.

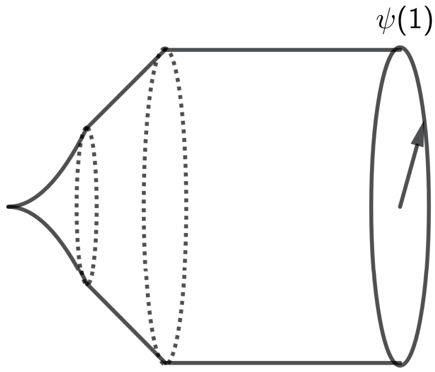

FIGURE 1. An outward cuspidal domain Ω^n_ψ

In [20, 21, 19], for certain Lipschitz cuspidal functions ψ, Maz'ya and Poborchi used integrability conditions on ψ to characterize the Sobolev extension property for Ω^n_ψ. By transferring an outward cuspidal domain onto a Lipschitz outward cuspidal domain via a global bi-Lipschitz transformation, we obtain a more general version of their result.

Theorem 1.2. *Let $\psi : (0,1] \to (0,\infty)$ be a cuspidal function such that $\psi\big|_{(0,1]} > 0$ and the function $\psi(t)/t$ is nondecreasing on $(0,1]$ with $\lim_{t\to 0}\psi(t)/t = 0$. Then the following conclusions hold.*
(1) : *If*

$$\int_0^1 \left(\frac{t^s}{\psi(t)}\right)^{\frac{n}{s-1}} \frac{dt}{t} < \infty, \tag{1.3}$$

then there exists a bounded linear extension operator E_1 from $W^{1,p}(\Omega^n_\psi)$ to $W^{1,q}(\mathbb{R}^n)$ whenever $\frac{1+(n-1)s}{n} \le p < \infty$ and $1 \le q \le \frac{np}{1+(n-1)s}$, and a bounded linear extension operator E_2 from $W^{1,p}(\Omega^n_\psi)$ to $W^{1,q}(\mathbb{R}^n)$ whenever $\frac{1+(n-1)s}{2+(n-2)s} \le p < \infty$ and $1 \le q \le \frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p}$.
(2) : *If*

$$\int_0^1 \left(\frac{t^s}{\psi(t)}\right)^{\frac{n}{s-1}} \left|\log\left(\frac{\psi(t)}{t}\right)\right|^{-\alpha} \frac{dt}{t} < \infty \tag{1.4}$$

with $\alpha = \frac{(n-2)p}{p+1-n}$, then there exists a bounded linear extension operator E_3 from $W^{1,p}(\Omega^n_\psi)$ to $W^{1,q}(\mathbb{R}^n)$ whenever $\frac{(n-1)^2 s+(n-1)}{n} \le p < \infty$ and $1 \le q \le n-1$.

Furthermore, under the doubling condition

$$\psi(2t) \le C\psi(t), \text{ for } t \in \left(0, \frac{1}{2}\right), \tag{1.5}$$

on ψ, the statements converse to (1) and (2) also hold.

If in addition we have

$$t^s/C \leq \psi(t) \leq Ct^s$$

with $1 < s < \infty$ and a positive constant C independent of t for the cuspidal function ψ in the preceding theorem, we will see in Remark 3.17 below that the outward cuspidal domain Ω^n_ψ can be mapped onto the "polynomial" outward cuspidal domain $\Omega^n_{t^s} := \Omega^n_{\hat{\psi}}$ with $\hat{\psi}(t) = t^s$ via a global bi-Lipschitz transformation. It follows from [7, 16] that in this case the extension operators E_1 and E_2 from above can be replaced by ones generated by reflections.

Let $n \geq 3$. Theorem 1.2 yields that $\Omega^n_{t^s}$ is a Sobolev (p,q)-extension domain, whenever $1 \leq q < n-1$ and $(n-1)q/(n-1-q) \leq p \leq \infty$. Our bi-Lipschitz transformation method allows us to extend this result from the case of t^s to arbitrary cuspidal functions. This result can be regarded as a limit case of Theorem 1.2 and also of certain theorems by Maz'ya and Poborchi in [19]. It also provides an intuitive reason as to why the boundary of a (p,q)-extension domain can have positive volume when q is small, see [15].

Theorem 1.6. *Let $3 \leq n < \infty$ and $\psi : (0,1] \to (0,\infty)$ be a cuspidal function. Then the corresponding outward cuspidal domain Ω^n_ψ is a Sobolev (p,q)-extension domain, whenever $1 \leq q < n-1$ and $(n-1)q/(n-1-q) \leq p \leq \infty$.*

The sharpness part of Theorem 1.2 also yields the sharpness of Theorem 1.6.

Proposition 1.7. *For arbitrary $n-1 \leq p < \infty$, there exists $1 < s_1 < \infty$ such that $\Omega^n_{t^s}$ is not a Sobolev $(p, n-1)$-extension domain when $s > s_1$. For arbitrary $1 \leq q < n-1$ and $q \leq p < (n-1)q/(n-1-q)$, there exists $1 < s_2 < \infty$ such that $\Omega^n_{t^s}$ is not a Sobolev (p,q)-extension domain when $s > s_2$.*

The paper is organized as follows. Section 2 contains definitions and preliminary results. Section 3 contains proofs of all results presented above. Section 4 contains some further discussion and a conjecture.

2. Definitions and Preliminaries

In this note, $\Omega \subset \mathbb{R}^n$ is always a bounded domain. C will refer to constants that depend on various parameters and may differ even in a chain of inequalities. The Euclidean distance between points $x, y \in \mathbb{R}^n$ is denoted by $|x-y|$. The open n-dimensional ball of radius r centered at the point x is denoted by $B^n(x,r)$.

Let us give the definition of the Sobolev space $W^{1,p}(\Omega)$.

Definition 2.1. We define the first order Sobolev space $W^{1,p}(\Omega)$, $1 \leq p \leq \infty$, as the set

$$\{\, u \in L^p(\Omega); \nabla u \in L^p(\Omega; \mathbb{R}^n) \,\} \,.$$

Here $\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$ is the weak (or distributional) gradient of the integrable function u.

The Sobolev space $W^{1,p}(\Omega)$ is equipped with the norm:

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \||\nabla u|\|_{L^p(\Omega)}$$

for $1 \leq p \leq \infty$, where $\|f\|_{L^p(\Omega)}$ denotes the usual L^p-norm for $p \in [1,\infty]$. Let us give the definition of Sobolev extension domains.

Definition 2.2. Let $1 \le q \le p \le \infty$. A bounded domain $\Omega \subset \mathbb{R}^n$ is said to be a Sobolev (p, q)-extension domain, if there is a bounded operator E from $W^{1,p}(\Omega)$ to $W^{1,q}(\mathbb{R}^n)$ such that, for every $u \in W^{1,p}(\Omega)$, $E(u) \in W^{1,q}(\mathbb{R}^n)$ with $E(u)\big|_{\Omega} \equiv u$ and

$$\|E(u)\|_{W^{1,q}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)}$$

for a positive constant C independent of u.

An outward cuspidal domain Ω^n_ψ has a singular point on the boundary. However, it still has some nice geometric properties. For example, it satisfies the following *segment condition*.

Definition 2.3. We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies the segment condition if every $x \in \partial\Omega$ has a neighborhood U_x and a nonzero vector y_x such that if $z \in \overline{\Omega} \cap U_x$, then $z + ty_x \in \Omega$ for $0 < t < 1$.

The following lemma tells us that Sobolev functions on a domain with the segment condition can be approximated by globally smooth functions. See [2, Theorem 3.22].

Lemma 2.4. *If a domain $\Omega \subset \mathbb{R}^n$ satisfies the segment condition, then the set of restrictions to Ω of functions in $C^\infty_0(\mathbb{R}^n)$ is dense in $W^{1,p}(\Omega)$ for $1 \le p < \infty$. In short, $C^\infty_0(\mathbb{R}^n) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for $1 \le p < \infty$.*

3. Proofs of Theorem 1.2, Theorem 1.6 and Proposition 1.7

As mentioned in the introduction, every outward cuspidal domain Ω^n_ψ is a Sobolev (∞, ∞)-extension domain. We show that every outward cuspidal domain is globally bi-Lipschitz equivalent to a Lipschitz outward cuspidal domain. For the proof of this result, we first introduce a Lipschitz cuspidal function $\hat{\psi}$ generated by a given cuspidal function ψ.

Lemma 3.1. *Let $\psi : (0, 1] \to (0, \infty)$ be an arbitrary cuspidal function. Then the following claims hold.*
(1) : *For every $0 < \hat{t} < 1$, there exists a unique pair $(t_{\hat{t}}, r_{\hat{t}})$ with $0 < t_{\hat{t}} < 1$, $\psi(t_{\hat{t}}) \le r_{\hat{t}} \le \lim_{s \to t^+_{\hat{t}}} \psi(s)$ and $t_{\hat{t}} + r_{\hat{t}} = (1 + \psi(1))\hat{t}$.*
(2) : *The function $\hat{\psi} : (0, 1] \to (0, \infty)$ defined by setting*

$$\hat{\psi}(\hat{t}) = r_{\hat{t}} \text{ for every } \hat{t} \in (0, 1]$$

is a Lipschitz cuspidal function.

Proof. Let $(t, r) \in (0, 1] \times (0, \psi(1))$ be a pair of positive numbers such that for every $x \in \mathbb{R}^{n-1}$ with $|x| = r$, we have $(t, x) \in \partial\Omega^n_\psi$. Define a function T on $\partial\Omega^n_\psi$ by setting

$$T(t, x) := t + |x|$$

for every $(t, x) \in \partial\Omega^n_\psi$. Since ψ is left-continuous and increasing on $(0, 1]$, we always have $T(t_1, x_1) < T(t_2, x_2)$ for every $(t_1, x_1), (t_2, x_2) \in \partial\Omega^n_\psi$ with $0 < t_1 < t_2 < 1$. By the same reason, we have $T(t, x_1) < T(t, x_2)$ for every $(t, x_1), (t, x_2) \in \partial\Omega^n_\psi$ with $0 < t < 1$ and

$$\psi(t) \le |x_1| < |x_2| \le \lim_{s \to t^+} \psi(s).$$

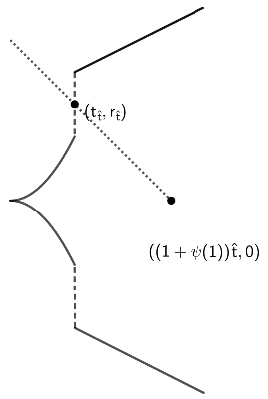

FIGURE 2. Bi-Lipschitz transformations

Hence, for every $0 < \hat{t} < 1$, there exists a unique pair $(t_{\hat{t}}, r_{\hat{t}})$ with $0 < t_{\hat{t}} < 1$, $\psi(t_{\hat{t}}) \leq r_{\hat{t}} \leq \lim_{s\to t_{\hat{t}}^+} \psi(s)$ and $t_{\hat{t}} + r_{\hat{t}} = (1+\psi(1))\hat{t}$. Now, let us show $\hat{\psi}$ is a Lipschitz cuspidal function. By the definition, it is easy to see $\hat{\psi}$ is increasing. Hence, it suffices to show that it is Lipschitz. Define a measurable subset $A \subset (0,1]$ by setting

$$A := \{\hat{t} \in (0,1] : r_{\hat{t}} \geq t_{\hat{t}}\}.$$

Let $\hat{t}_1, \hat{t}_2 \in (0,1]$ be arbitrary. According to their locations, we divide the following argument into three cases. First, let us assume $\hat{t}_1, \hat{t}_2 \in A$. Since $\hat{t} \sim r_{\hat{t}}$ for every $\hat{t} \in A$, by the definition, we have

$$|\hat{\psi}(\hat{t}_1) - \hat{\psi}(\hat{t}_2)| = |r_{\hat{t}_1} - r_{\hat{t}_2}| \leq C|\hat{t}_1 - \hat{t}_2|. \tag{3.2}$$

Next, we assume $\hat{t}_1, \hat{t}_2 \in (0,1] \setminus A$. For every $\hat{t} \in (0,1] \setminus A$, we have $\hat{t} \sim t_{\hat{t}}$. Hence, the triangle inequality implies

$$|\hat{\psi}(\hat{t}_1) - \hat{\psi}(\hat{t}_2)| = |r_{\hat{t}_1} - r_{\hat{t}_2}| \leq C|\hat{t}_1 - \hat{t}_2| + C|t_{\hat{t}_1} - t_{\hat{t}_2}| \leq C|\hat{t}_1 - \hat{t}_2|. \tag{3.3}$$

Finally, we assume $\hat{t}_1 \in A$ and $\hat{t}_2 \in (0,1] \setminus A$. Since both $t_{\hat{t}}$ and $r_{\hat{t}}$ are continuous with respect to $\hat{t}$, there exists $\hat{s} \in (0,1]$ between $\hat{t}_1$ and $\hat{t}_2$ with $t_{\hat{s}} = r_{\hat{s}}$. By (3.2) and (3.3), the triangle inequality implies

$$|\hat{\psi}(\hat{t}_1) - \hat{\psi}(\hat{t}_2)| \leq |\hat{\psi}(\hat{t}_1) - \hat{\psi}(\hat{s})| + |\hat{\psi}(\hat{s}) - \hat{\psi}(\hat{t}_2)| \leq C|\hat{t}_1 - \hat{s}| + C|\hat{s} - \hat{t}_2| \leq C|\hat{t}_1 - \hat{t}_2|. \tag{3.4}$$

By combining (3.2), (3.3) and (3.4), we conclude that $\hat{\psi}$ is Lipschitz. □

We are ready to prove the following proposition which asserts that an arbitrary outward cuspidal domain is globally bi-Lipschitz equivalent to a Lipschitz outward cuspidal domain.

Proposition 3.5. *For a given cuspidal function ψ, let $\hat{\psi}$ be the Lipschitz cuspidal function defined in Lemma 3.1. Then there exists a global bi-Lipschitz transformation $\mathcal{O} : \mathbb{R}^n \to \mathbb{R}^n$ with $\mathcal{O}(\Omega^n_\psi) = \Omega^n_{\hat{\psi}}$.*

Proof. Let $\psi : (0,1] \to (0,\infty)$ be a cuspidal function. Let $\hat{\psi} : (0,1] \to (0,\infty)$ be the corresponding Lipschitz cuspidal function defined in Lemma 3.1. We extend the definition of ψ to the entire real line $\mathbb{R}$ by setting $\psi(t) = 0$ for every $t \le 0$. Without loss of generality, we may assume $2\psi(1) < 1$. Otherwise, we consider the cuspidal function $\tilde{\psi}$ defined by setting

$$\tilde{\psi}(t) := \frac{\psi(t)}{2\psi(1)} \text{ for } t \in (-\infty, 1).$$

Obviously, Ω^n_ψ and $\Omega^n_{\tilde{\psi}}$ are globally bi-Lipschitz equivalent.

We define

$$U_1 := \left\{(t,x) \in (-\infty, 1+\psi(1)] \times \mathbb{R}^{n-1} : |x| \le 1+\psi(1)-t\right\}, U_2 := \Omega^n_\psi \setminus U_1,$$
$$U_3 := \left\{(t,x) \in (-\infty,\infty) \times \mathbb{R}^{n-1} : |x| \ge \max\{\psi(1), 1+\psi(1)-t\}\right\},$$
$$U_4 := \left\{(t,x) \in [2,\infty) \times \mathbb{R}^{n-1} : |x| \le \psi(1)\right\}.$$

Obviously, we have $\mathbb{R}^n = \bigcup_{i=1}^4 U_i$. Then we define a global transformation $\mathcal{O} : \mathbb{R}^n \to \mathbb{R}^n$ by setting

$$\mathcal{O}(t,x) := \begin{cases} \left(\left(\frac{1}{1+\psi(1)}\right)(t+|x|), x\right), & (t,x) \in U_1, \\ \left(\left(\frac{t}{1+|x|-\psi(1)} + \frac{2(|x|-\psi(1))}{1+|x|-\psi(1)}\right), x\right), & (t,x) \in U_2, \\ (t+|x|-\psi(1), x), & (t,x) \in U_3, \\ (t,x), & (t,x) \in U_4. \end{cases} \tag{3.6}$$

The global homeomorphism $\mathcal{O}$ is differentiable almost everywhere and there exists a positive constant $C > 1$ such that for almost every $z \in \mathbb{R}^n$, we have

$$\frac{1}{C} \le |D\mathcal{O}(z)| \le C.$$

Hence, $\mathcal{O} : \mathbb{R}^n \to \mathbb{R}^n$ is a global bi-Lipschitz transformation. By a simple computation, we obtain

$$\mathcal{O}(U_1 \cap \Omega^n_\psi) = \{(s,y) \in (0,1] \times \mathbb{R}^{n-1} : |y| < \hat{\psi}(s)\}$$

and

$$\mathcal{O}(U_2) = \{(s,y) \in (1,2) \times \mathbb{R}^{n-1} : |y| < \hat{\psi}(1)\}.$$

Hence $\mathcal{O}(\Omega^n_\psi) = \Omega^n_{\hat{\psi}}$. □

Let us prove Theorem 1.2.

Proof of Theorem 1.2. Let $\psi : (0,1] \to (0,\infty)$ be a cuspidal function such that $\psi(t)/t$ is nondecreasing in $(0,1]$ with $\lim_{t\to 0} \psi(t)/t = 0$. Let $\hat{\psi}$ be the corresponding Lipschitz cuspidal function, defined as in Lemma 3.1. Following the notation from Lemma 3.1, for every $\hat{t} \in (0,1]$, we have

$$t_{\hat{t}} + r_{\hat{t}} = (1+\psi(1))\hat{t} \text{ and } \hat{\psi}(\hat{t}) = r_{\hat{t}}.$$

Since ψ is left-continuous and increasing, for every $\hat{t} \in \left(0, \frac{1}{1+\psi(1)}\right)$, we have

$$\psi(t_{\hat{t}}) \le r_{\hat{t}} < \psi\left((1+\psi(1))\hat{t}\right).$$

Hence, we have

$$\lim_{\hat{t}\to 0}\frac{\hat{\psi}(\hat{t})}{\hat{t}} \leq C\lim_{\hat{t}\to 0}\frac{\psi(\hat{t})}{\hat{t}} = 0$$

and

$$\frac{\hat{\psi}(\hat{t})}{\hat{t}} = \frac{r_{\hat{t}}}{\hat{t}} = \frac{(1+\psi(1))\hat{t} - t_{\hat{t}}}{\hat{t}} = (1+\psi(1))\left(1 - \frac{t_{\hat{t}}}{t_{\hat{t}} + r_{\hat{t}}}\right). \tag{3.7}$$

By (3.7), to show that $\hat{\psi}(\hat{t})/\hat{t}$ is increasing, it suffices to show that

$$\frac{r_{\hat{t}_1}}{t_{\hat{t}_1}} \leq \frac{r_{\hat{t}_2}}{t_{\hat{t}_2}} \ \text{ for } \ 0 < \hat{t}_1 \leq \hat{t}_2 < 1. \tag{3.8}$$

Since ψ is non-decreasing and $\hat{t}_1 \leq \hat{t}_2$, we always have $r_{\hat{t}_1} \leq r_{\hat{t}_2}$. Hence, if $t_{\hat{t}_1} = t_{\hat{t}_2}$, we immediately obtain inequality (3.8). If $t_{\hat{t}_1} < t_{\hat{t}_2}$, the fact that ψ is non-decreasing implies $r_{\hat{t}_1} \leq \psi(t_{\hat{t}_2}) \leq r_{\hat{t}_2}$. Since ψ has at most countably many points of discontinuity, for every $\epsilon > 0$, we can find a point $t_\epsilon \in (t_{\hat{t}_1}, t_{\hat{t}_1} + \epsilon)$ of continuity of ψ with $r_{\hat{t}_1} \leq \psi(t_\epsilon)$. The fact that $\psi(t)/t$ is non-decreasing implies

$$\frac{r_{\hat{t}_1}}{t_{\hat{t}_1}} \leq \lim_{\epsilon\to 0}\frac{\psi(t_\epsilon)}{t_{\hat{t}_1}+\epsilon} \leq \lim_{\epsilon\to 0}\frac{\psi(t_\epsilon)}{t_\epsilon} \leq \frac{\psi(t_{\hat{t}_2})}{t_{\hat{t}_2}} \leq \frac{r_{\hat{t}_2}}{t_{\hat{t}_2}}.$$

(1)**:** We show that the assumption that original cuspidal function ψ satisfies the integrability condition (1.3) implies that the corresponding Lipschitz cuspidal function $\hat{\psi}$ also satisfies the integrability condition (1.3). Define a measurable subset $B \subset (0,1]$ by setting

$$B := \{\hat{t} \in (0,1] : t_{\hat{t}} \geq r_{\hat{t}}\}.$$

Then, for every $\hat{t} \in B$, we have $\hat{t} \sim t_{\hat{t}}$ and $\hat{\psi}(\hat{t}) = r_{\hat{t}} \geq \psi(t_{\hat{t}})$. Hence, we have

$$\int_B \left(\frac{\hat{t}^s}{\hat{\psi}(\hat{t})}\right)^{\frac{n}{s-1}} \frac{d\hat{t}}{\hat{t}} \leq C\int_0^1 \left(\frac{t_{\hat{t}}^s}{\psi(t_{\hat{t}})}\right)^{\frac{n}{s-1}} \frac{dt_{\hat{t}}}{t_{\hat{t}}} < \infty. \tag{3.9}$$

If $\hat{t} \in (0,1] \setminus B$, we have $\hat{\psi}(\hat{t}) = r_{\hat{t}} \sim \hat{t}$. Then we have

$$\int_{(0,1]\setminus B} \left(\frac{\hat{t}^s}{\hat{\psi}(\hat{t})}\right)^{\frac{n}{s-1}} \frac{d\hat{t}}{\hat{t}} \leq C\int_0^1 \hat{t}^{n-1} d\hat{t} < \infty. \tag{3.10}$$

Hence, by combining the last two inequalities, we obtain that $\hat{\psi}$ satisfies the integrability condition that

$$\int_0^1 \left(\frac{\hat{t}^s}{\hat{\psi}(\hat{t})}\right)^{\frac{n}{s-1}} \frac{d\hat{t}}{\hat{t}} < \infty.$$

In conclusion, we have shown that $\hat{\psi}$ satisfies all assumptions of the theorems due to Maz'ya and Poborchi in [19, page 304 and 312]. Hence there exists a bounded linear extension operator $\widetilde{E}_1 : W^{1,p}(\Omega^n_{\hat{\psi}}) \to W^{1,q}(\mathbb{R}^n)$ whenever $\frac{1+(n-1)s}{n} \leq p < \infty$ and $1 \leq q \leq \frac{np}{1+(n-1)s}$. Then, for $\frac{1+(n-1)s}{n} \leq p < \infty$, we define an extension operator E_1 on $W^{1,p}(\Omega^n_\psi)$ by setting

$$E_1(u)(x) := \widetilde{E}_1\left(u \circ \mathcal{O}^{-1}\right)(\mathcal{O}(x))$$

for every function $u \in W^{1,p}(\Omega^n_\psi)$ and every $x \in \mathbb{R}^n$. By the facts that a bi-Lipschitz transformation preserves first order Sobolev spaces and that $\widetilde{E}_1 : W^{1,p}(\Omega^n_{\hat{\psi}}) \to W^{1,q}(\mathbb{R}^n)$ is a bounded linear extension operator whenever $\frac{1+(n-1)s}{n} \leq p < \infty$ and $1 \leq q \leq \frac{np}{1+(n-1)s}$, we obtain that $E_1 : W^{1,p}(\Omega^n_\psi) \to W^{1,q}(\mathbb{R}^n)$ is also a bounded linear extension operator whenever $\frac{1+(n-1)s}{n} \leq p < \infty$ and $1 \leq q \leq \frac{np}{1+(n-1)s}$. The theorem from [19, page 312] tells us that there exists a bounded linear extension operator $\widetilde{E}_2 : W^{1,p}(\Omega^n_{\hat{\psi}}) \to W^{1,q}(\mathbb{R}^n)$ whenever $\frac{1+(n-1)s}{2+(n-2)s} \leq p < \infty$ and $1 \leq q \leq \frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p}$. Then, for every $\frac{1+(n-1)s}{2+(n-2)s} \leq p < \infty$, we define an extension operator E_2 on $W^{1,p}(\Omega^n_\psi)$ by setting

$$E_2(u)(x) := \widetilde{E}_2\left(u \circ \mathcal{O}^{-1}\right)(\mathcal{O}(x))$$

for every function $u \in W^{1,p}(\Omega^n_\psi)$ and $x \in \mathbb{R}^n$. By the same reason as above, $E_2 : W^{1,p}(\Omega^n_\psi) \to W^{1,q}(\mathbb{R}^n)$ is a bounded linear extension operator whenever $\frac{1+(n-1)s}{2+(n-2)s} \leq p < \infty$ and $1 \leq q \leq \frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p}$.

(2) : By an argument similar with the first case (1), we conclude that if the original cuspidal function ψ satisfies the integrability condition (1.4) then the corresponding Lipschitz cuspidal function $\hat{\psi}$ also satisfies the integrability condition (1.4). Hence, $\hat{\psi}$ satisfies all the assumptions of the theorem from [19, page 308]. This theorem tells us that there exists a bounded linear extension operator $\widetilde{E}_3 : W^{1,p}(\Omega^n_{\hat{\psi}}) \to W^{1,q}(\mathbb{R}^n)$ whenever $\frac{(n-1)^2s+(n-1)}{n} \leq p < \infty$ and $1 \leq q \leq n-1$. For every $\frac{(n-1)^2s+(n-1)}{n} \leq p < \infty$, we define an extension operator E_3 on $W^{1,p}(\Omega^n_\psi)$ by setting

$$E_3(u)(x) := \widetilde{E}_3\left(u \circ \mathcal{O}^{-1}\right)(\mathcal{O}(x))$$

for every function $u \in W^{1,p}(\Omega^n_\psi)$ and $x \in \mathbb{R}^n$. By the same reason as above, $E_3 : W^{1,p}(\Omega^n_\psi) \to W^{1,q}(\mathbb{R}^n)$ is a bounded linear extension operator for every $\frac{(n-1)^2s+(n-1)}{n} \leq p < \infty$ and $1 \leq q \leq n-1$.

Next, let us show the necessity of integrability conditions (1.3) and (1.4). First, let us show that if the doubling condition (1.5) holds for a cuspidal function ψ, it also holds for its corresponding Lipschitz cuspidal function $\hat{\psi}$. Since $\hat{\psi}$ is Lipschitz and increasing, it suffices to show that there exists a positive constant $C > 1$ such that for every $\hat{t} \in \left(0, \frac{1}{2(1+\psi(1))}\right]$, we have

$$\hat{\psi}(2\hat{t}) \leq C\hat{\psi}(\hat{t}). \tag{3.11}$$

Let $\hat{t} \in \left(0, \frac{1}{2(1+\psi(1))}\right]$ be arbitrary. There exists a unique pair $(t_{\hat{t}}, r_{\hat{t}})$ with

$$\psi(t_{\hat{t}}) \leq r_{\hat{t}} \leq \lim_{s \to t_{\hat{t}}^+} \psi(s) \tag{3.12}$$

and

$$t_{\hat{t}} + r_{\hat{t}} = (1+\psi(1))\hat{t}. \tag{3.13}$$

Moreover, there exists a unique pair $(t_{2\hat{t}}, r_{2\hat{t}})$ with

$$\psi(t_{2\hat{t}}) \le r_{2\hat{t}} \le \lim_{s\to t_{2\hat{t}}^+} \psi(s) \tag{3.14}$$

and

$$t_{2\hat{t}} + r_{2\hat{t}} = (1+\psi(1))2\hat{t}. \tag{3.15}$$

By the definition in Lemma 3.1, we have

$$\hat{\psi}(2\hat{t}) = r_{2\hat{t}} \text{ and } \hat{\psi}(\hat{t}) = r_{\hat{t}}.$$

If $r_{2\hat{t}} \le 2r_{\hat{t}}$, then (3.11) holds with $C = 2$. Hence, we assume $r_{2\hat{t}} > 2r_{\hat{t}}$. By (3.13) and (3.15), we have $t_{2\hat{t}} < 2t_{\hat{t}}$. Since ψ is increasing and satisfies inequality (1.5), by (3.12) and (3.14), we have

$$r_{2\hat{t}} \le \psi(2t_{\hat{t}}) \le C\psi(t_{\hat{t}}) \le Cr_{\hat{t}}.$$

We have showed that inequality (3.11) holds for every $\hat{t} \in \left(0, \frac{1}{2(1+\psi(1))}\right]$. The fact that Ω^n_ψ is globally bi-Lipschitz equivalent to $\Omega^n_{\hat{\psi}}$ implies that Ω^n_ψ and $\Omega^n_{\hat{\psi}}$ have the same Sobolev extension properties. By the results due to Maz'ya and Poborch in [19, pages 304 and 312], if $\Omega^n_{\hat{\psi}}$ is a Sobolev (p,q)-extension domain with

$$\left(\frac{1+(n-1)s}{n} \le p < \infty, 1 \le q \le \frac{np}{1+(n-1)s}\right)$$

or

$$\left(\frac{1+(n-1)s}{2+(n-2)s} \le p < \infty, 1 \le q \le \frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p}\right),$$

then $\hat{\psi}$ satisfies the integrability condition

$$\int_0^1 \left(\frac{t^s}{\hat{\psi}(t)}\right)^{\frac{n}{s-1}} \frac{dt}{t} < \infty.$$

By the theorem from [19, page 308], if $\Omega^n_{\hat{\psi}}$ is a Sobolev (p,q)-extension domain with $\frac{(n-1)^2 s+(n-1)}{n} \le p < \infty$ and $1 \le q \le n-1$, then $\hat{\psi}$ satisfies the integrability condition

$$\int_0^1 \left(\frac{t^s}{\hat{\psi}(t)}\right)^{\frac{n}{s-1}} \left|\log\left(\frac{\hat{\psi}(t)}{t}\right)\right|^{-\alpha} \frac{dt}{t} < \infty$$

for $\alpha = \frac{(n-2)p}{p+1-n}$. By the definition of $\hat{\psi}$, for every $t \in (0,1]$, we always have

$$\psi(t) \ge \hat{\psi}\left(\frac{1}{1+\psi(1)} t\right). \tag{3.16}$$

Hence, if the corresponding Lipschitz cuspidal function $\hat{\psi}$ satisfies the integrability conditions (1.3) and (1.4), the original cuspidal function ψ also satisfies them. □

Remark 3.17. For the cuspidal function ψ, assume additionally that there exists $1 < s < \infty$ and a constant $C > 1$ such that for every $t \in (0,1)$, we have

$$t^s/C \le \psi(t) \le Ct^s.$$

Then for every $\hat{t} \in (0,1)$, we have $r_{\hat{t}} \leq C t^s_{\hat{t}}$. Hence, by the definition of $\hat{\psi}$ in Lemma 3.1 and (3.16), there exists a constant $C_1 > 1$ such that for every $\hat{t} \in (0,1)$ we have

$$\hat{t}^s / C_1 \leq \hat{\psi}(\hat{t}) \leq C_1 \hat{t}^s.$$

Since $\hat{\psi}$ is Lipschitz, there exists a global bi-Lipschitz transformation $\tilde{\mathcal{O}} : \mathbb{R}^n \to \mathbb{R}^n$ with $\tilde{\mathcal{O}}(\Omega^n_{\hat{\psi}}) = \Omega^n_{t^s}$. Hence, $\tilde{\mathcal{O}} \circ \mathcal{O} : \mathbb{R}^n \to \mathbb{R}^n$ is a bi-Lipschitz transformation with $\tilde{\mathcal{O}} \circ \mathcal{O}(\Omega^n_{\psi}) = \Omega^n_{t^s}$.

Let us prove Theorem 1.6.

Proof of Theorem 1.6. As we mentioned, every outward cuspidal domain is a Sobolev (∞,∞)-extension domain. Hence, it suffices to deal with the case $1 \leq q \leq p < \infty$. We first prove the result for Lipschitz outward cuspidal domains and then extend the result to arbitrary outward cuspidal domains via the global bi-Lipschitz equivalence method established in Proposition 3.5.

Let $1 \leq q < n-1$ and $(n-1)q/(n-1-q) \leq p < \infty$ be fixed. Let ψ be a Lipschitz cuspidal function. We define a cylinder $\widehat{\mathsf{C}}_o$ by setting

$$\widehat{\mathsf{C}}_o := \{(t,x) \in [1,3) \times \mathbb{R}^{n-1} : |x| < 2\psi(1)\}. \tag{3.18}$$

Then we define two sub-cylinders $\widehat{\mathsf{C}}^1_o$ and $\widehat{\mathsf{C}}^2_o$ of $\widehat{\mathsf{C}}_o$ by setting

$$\widehat{\mathsf{C}}^1_o := \{(t,x) \in (1,2) \times \mathbb{R}^{n-1} : |x| < 2\psi(1)\}$$

and

$$\widehat{\mathsf{C}}^2_o := \{(t,x) \in (2,3) \times \mathbb{R}^{n-1} : |x| < 2\psi(1)\}.$$

We also define a sub-cylinder C^1_o of $\widehat{\mathsf{C}}^1_o$ by setting

$$\mathsf{C}^1_o := \{(t,x) \in (1,2) \times \mathbb{R}^{n-1} : |x| < \psi(1)\}.$$

Then $A_{\mathsf{C}^1_o} := \widehat{\mathsf{C}}^1_o \setminus \overline{\mathsf{C}}^1_o$ is an annular set. We define a reflection $\widetilde{\mathcal{R}_1} : A_{\mathsf{C}^1_o} \to \mathsf{C}^1_o$ by setting

$$\widetilde{\mathcal{R}_1}(t,x) := \left(t, \left(\frac{3}{2}\psi(1) - \frac{|x|}{2}\right)\frac{x}{|x|}\right). \tag{3.19}$$

There exists a positive constant C such that for every $(t,x) \in A_{\mathsf{C}^1_o}$, we have

$$|D\widetilde{\mathcal{R}_1}(t,x)| \leq C \text{ and } \frac{1}{C} \leq |J_{\widetilde{\mathcal{R}_1}}(t,x)| \leq C. \tag{3.20}$$

We also define a cut-off function $\widetilde{L}_1$ on the annular set $\overline{A_{\mathsf{C}^1_o}}$ with $\widetilde{L}_1 \equiv 0$ on $[1,2] \times \partial B^{n-1}(0, 2\psi(1))$ and $\widetilde{L}_1 \equiv 1$ on $[1,2] \times \partial B^{n-1}(0,\psi(1))$ by setting

$$\widetilde{L}_1(t,x) := 2 - \frac{|x|}{\psi(1)}. \tag{3.21}$$

There exists a positive constant C, such that for every $(t,x) \in A_{\mathsf{C}^1_o}$, we have

$$|\nabla \widetilde{L}_1(t,x)| \leq C \tag{3.22}$$

Next, we define a reflection $\widetilde{\mathcal{R}_2} : \widehat{\mathsf{C}}^2_o \to \widehat{\mathsf{C}}^1_o$ by setting

$$\widetilde{\mathcal{R}_2}(t,x) := (t-2, x). \tag{3.23}$$

There exists a positive constant C, such that for every $(t,x) \in \widehat{\mathsf{C}}_o^2$, we have

$$|D\widetilde{\mathcal{R}_2}(t,x)| \le C \text{ and } \frac{1}{C} \le |J_{\widetilde{\mathcal{R}_2}}(t,x)| \le C. \tag{3.24}$$

Moreover we define a cut-off function $\widetilde{L}_2$ on $\overline{\widehat{\mathsf{C}}_o^2}$ with $\widetilde{L}_2 \equiv 1$ on $\{2\} \times \overline{B^{n-1}(0, 2\psi(1))}$ and $\widetilde{L}_2 \equiv 0$ on $\{3\} \times \overline{B^{n-1}(0, 2\psi(1))}$ by setting

$$\widetilde{L}_2(t,x) := 3 - t. \tag{3.25}$$

Then, for every $(t,x) \in \widehat{\mathsf{C}}_o^2$, we have

$$|\nabla \widetilde{L}_2(t,x)| \le 2. \tag{3.26}$$

Next, we define a double outward cuspidal domain $\widehat{\Omega_\psi^n}$ by setting

$$\widehat{\Omega_\psi^n} := \{(t,x) \in (0,1] \times \mathbb{R}^{n-1} : |x| < 2\psi(t)\} \cup \widehat{\mathsf{C}}_o.$$

We will construct a bounded linear extension operator E from $W^{1,p}(\Omega_\psi^n)$ to $W^{1,q}(\mathbb{R}^n)$ such that for every function $u \in W^{1,p}(\Omega_\psi^n)$, we have $E(u) = 0$ on $\partial\widehat{\Omega_\psi^n} \setminus \{0\}$. We define an annular-type set by setting

$$A_\psi^n := \{(t,x) \in (0,1] \times \mathbb{R}^{n-1} : \psi(t) < |x| < 2\psi(t)\}.$$

Moreover we define a reflection $\mathcal{R} : A_\psi^n \to \Omega_\psi^n$ by setting

$$\mathcal{R}(z) = \mathcal{R}(t,x) := \left(t, \left(\frac{-|x|}{2} + \frac{3}{2}\psi(t)\right)\frac{x}{|x|}\right). \tag{3.27}$$

Since ψ is Lipschitz, there exists a positive constant C such that for every $z = (t,x) \in A_\psi^n$, we have

$$|D\mathcal{R}(z)| \le C \text{ and } \frac{1}{C} \le |J_{\mathcal{R}}(z)| \le C. \tag{3.28}$$

We define a cut-off function L on A_ψ^n by setting

$$L(t,x) := \frac{-|x|}{\psi(t)} + 2. \tag{3.29}$$

Since ψ is Lipschitz and

$$\psi(t) < |x| < 2\psi(t)$$

for every point $(t,x) \in A_\psi^n$, there exists a positive constant C such that

$$|\nabla L(t,x)| \le \frac{C}{\psi(t)} \text{ for almost every } (t,x) \in A_\psi^n. \tag{3.30}$$

By Lemma 2.4, $C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega_\psi^n)$ is dense in $W^{1,p}(\Omega_\psi^n)$. We first define a bounded linear extension operator from the dense subspace $C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega_\psi^n)$ to

$W^{1,q}(\mathbb{R}^n)$ and then extend it to the full space $W^{1,p}(\Omega^n_\psi)$. Let $u \in C^\infty_o(\mathbb{R}^n) \cap W^{1,p}(\Omega^n_\psi)$ be fixed. We define an extension $E(u)$ on $\widehat{\Omega^n_\psi}$ by setting

$$E(u)(z) := \begin{cases} u(z), & z \in \overline{\Omega^n_\psi}, \\ L(z)(u \circ \mathcal{R})(z), & z \in A^n_\psi, \\ \widetilde{L}_1(z)(u \circ \widetilde{\mathcal{R}}_1)(z), & z \in A_{\mathsf{C}^1_o}, \\ \widetilde{L}_2(z)(E(u) \circ \widetilde{\mathcal{R}}_2)(z), & z \in \widehat{\mathsf{C}}^2_o. \end{cases} \tag{3.31}$$

Then, $E(u)$ is continuous on $\widehat{\Omega^n_\psi}$ with $E(u) \equiv 0$ on $\partial\widehat{\Omega^n_\psi} \setminus \{0\}$. Hence, we can simply extend $E(u)$ to be zero outside the domain $\widehat{\Omega^n_\psi}$. First, let us estimate the L^q-norm of $E(u)$. By the definition of $E(u)$, the Hölder inequality implies

$$\left(\int_{\Omega^n_\psi} |E(u)(z)|^q dz\right)^{\frac{1}{q}} \leq C \left(\int_{\Omega^n_\psi} |u(z)|^p dz\right)^{\frac{1}{p}}. \tag{3.32}$$

By (3.24) and the fact that $0 \leq \widetilde{L}_2 \leq 1$ on $\widehat{\mathsf{C}}^2_o$, the change of variables formula implies

$$\int_{\widehat{\mathsf{C}}^2_o} |E(u)(z)|^q dz \leq C \int_{\widehat{\mathsf{C}}^2_o} \left|\left(E(u) \circ \widetilde{\mathcal{R}}_2\right)(z)\right|^q dz \leq C \int_{\widehat{\mathsf{C}}^1_o} |E(u)(z)|^q dz. \tag{3.33}$$

By (3.20) and the fact that $0 \leq \widetilde{L}_1 \leq 1$ on $A_{\mathsf{C}^1_o}$, the change of variables formula and the Hölder inequality imply

$$\begin{aligned} \left(\int_{\widehat{\mathsf{C}}^1_o} |E(u)(z)|^q dz\right)^{\frac{1}{q}} &\leq C \left(\int_{\mathsf{C}^1_o} |E(u)(z)|^q \, dz\right)^{\frac{1}{q}} \\ &+ C \left(\int_{A_{\mathsf{C}^1_o}} |E(u)(z)|^q \, dz\right)^{\frac{1}{q}} \leq C \left(\int_{\Omega^n_\psi} |u(z)|^p dz\right)^{\frac{1}{p}}. \end{aligned} \tag{3.34}$$

By combining (3.33) and (3.34), we obtain

$$\left(\int_{\widehat{\mathsf{C}}_o} |E(u)(z)|^q dz\right)^{\frac{1}{q}} \leq C \left(\int_{\Omega^n_\psi} |u(z)|^p dz\right)^{\frac{1}{p}}. \tag{3.35}$$

By (3.28) and the fact that $0 \leq L(z) \leq 1$ on A^n_ψ, the change of variables formula and the Hölder inequality imply

$$\left(\int_{A^n_\psi} |L(z)(u \circ \mathcal{R})(z)|^q dz\right)^{\frac{1}{q}} \leq C \left(\int_{\Omega^n_\psi} |u(z)|^p dz\right)^{\frac{1}{p}}. \tag{3.36}$$

Combining inequalities (3.32), (3.35) and (3.36), we obtain

$$\left(\int_{\widehat{\Omega^n_\psi}} |E(u)(z)|^q dz\right)^{\frac{1}{q}} \leq C \left(\int_{\Omega^n_\psi} |u(z)|^p dz\right)^{\frac{1}{p}}. \tag{3.37}$$

Now, let us estimate the L^q-norm of $|\nabla E(u)|$. First, the Hölder inequality implies

$$\left(\int_{\Omega_\psi^n} |\nabla E(u)(z)|^q dz\right)^{\frac{1}{q}} \leq C\left(\int_{\Omega_\psi^n} |\nabla u(z)|^p dz\right)^{\frac{1}{p}}. \tag{3.38}$$

The chain rule implies that for almost every $z \in A_{\mathsf{C}_o^1}$, we have

$$|\nabla E(u)(z)| \leq \left|\nabla \tilde{L}_1(z)(u \circ \widetilde{\mathcal{R}}_1)(z)\right| + \left|\tilde{L}_1(z)\nabla(u \circ \widetilde{\mathcal{R}}_1)(z)\right|. \tag{3.39}$$

By (3.20) and (3.22), the change of variables formula and the Hölder inequality imply

$$\left(\int_{A_{\mathsf{C}_o^1}} \left|\nabla \tilde{L}_1(z)(u \circ \widetilde{\mathcal{R}}_1)(z)\right|^q dz\right)^{\frac{1}{q}} \leq C\left(\int_{\Omega_\psi^n} |u(z)|^p\, dz\right)^{\frac{1}{p}}. \tag{3.40}$$

By (3.20) and the fact that $0 \leq \tilde{L}_1(z) \leq 1$ for every $z \in A_{\mathsf{C}_o^1}$, the change of variables formula and the Hölder inequality imply

$$\left(\int_{A_{\mathsf{C}_o^1}} \left|\tilde{L}_1(z)\nabla(u \circ \widetilde{\mathcal{R}}_1)(z)\right|^q dz\right)^{\frac{1}{q}} \leq C\left(\int_{\Omega_\psi^n} |\nabla u(z)|^p\, dz\right)^{\frac{1}{p}}. \tag{3.41}$$

By combining (3.39), (3.40) and (3.41), we obtain

$$\left(\int_{A_{\mathsf{C}_o^1}} |\nabla E(u)(z)|^q\, dz\right)^{\frac{1}{q}} \leq C\left(\int_{\Omega_\psi^n} |u(z)|^p + |\nabla u(z)|^p\, dz\right)^{\frac{1}{p}}. \tag{3.42}$$

The chain rule implies that for almost every $z \in \widehat{\mathsf{C}}_o^2$, we have

$$|\nabla E(u)(z)| \leq \left|\nabla \tilde{L}_2(z)\left(E(u) \circ \widetilde{\mathcal{R}}_2\right)(z)\right| + \left|\tilde{L}_2(z)\nabla\left(E(u) \circ \widetilde{\mathcal{R}}_2\right)(z)\right|. \tag{3.43}$$

By (3.24), (3.26) and (3.34), the change of variables formula and the Hölder inequality imply

$$\begin{aligned}\left(\int_{\widehat{\mathsf{C}}_o^2} \left|\nabla \tilde{L}_2(z)\left(E(u) \circ \widetilde{\mathcal{R}}_2\right)(z)\right|^q dz\right)^{\frac{1}{q}} &\leq C\left(\int_{\widehat{\mathsf{C}}_0^1} |E(u)(z)|^q\, dz\right)^{\frac{1}{q}} \\ &\leq C\left(\int_{\Omega_\psi^n} |u(z)|^p\, dz\right)^{\frac{1}{p}}.\end{aligned} \tag{3.44}$$

By (3.24) and the fact that $0 \leq \tilde{L}_2(z) \leq 1$ for almost every $z \in \widehat{\mathsf{C}}_o^2$, the change of variables formula, (3.38) and (3.42) imply

$$
\begin{aligned}
(3.45) \quad \left(\int_{\widehat{\mathsf{C}}_o^2} \left| \tilde{L}_2(z) \nabla \left(E(u) \circ \widetilde{R}_2 \right)(z) \right|^q dz \right)^{\frac{1}{q}} &\leq C \left(\int_{\widehat{\mathsf{C}}_o^1} |\nabla E(u)(z)|^q \, dz \right)^{\frac{1}{q}} \\
\leq C \left(\int_{A_{\mathsf{C}_o^1}} |\nabla E(u)(z)|^q \, dz \right)^{\frac{1}{q}} &+ \left(\int_{\mathsf{C}_o^1} |\nabla E(u)(z)|^q \, dz \right)^{\frac{1}{q}} \\
&\leq C \left(\int_{\Omega_\psi^n} |u(z)|^p + |\nabla u(z)|^p \, dz \right)^{\frac{1}{p}}.
\end{aligned}
$$

Hence, by combining (3.43), (3.44) and (3.45), we obtain

$$
(3.46) \qquad \left(\int_{\widehat{\mathsf{C}}_o^2} |\nabla E(u)(z)|^q \, dz \right)^{\frac{1}{q}} \leq C \left(\int_{\Omega_\psi^n} |u(z)|^p + |\nabla u(z)|^p \, dz \right)^{\frac{1}{p}}.
$$

Since $\mathsf{C}_o^1 \subset \Omega_\psi^n$, by combining (3.38), (3.42) and (3.46), we obtain

$$
(3.47) \qquad \left(\int_{\widehat{\mathsf{C}}_o} |\nabla E(u)(z)|^q \, dz \right)^{\frac{1}{q}} \leq C \left(\int_{\Omega_\psi^n} |u(z)|^p + |\nabla u(z)|^p \, dz \right)^{\frac{1}{p}}.
$$

The chain rule implies that for almost every $z \in A_\psi^n$, we have

$$
(3.48) \qquad |\nabla E(u)(z)| \leq |\nabla L(z)(u \circ \mathcal{R})(z)| + |L(z) \nabla (u \circ \mathcal{R})(z)|.
$$

By (3.28) and the fact $0 \leq L(z) \leq 1$ on A_ψ^n, the change of variables formula and the Hölder inequality yield

$$
(3.49) \qquad \left(\int_{A_\psi^n} |L(z) \nabla (u \circ \mathcal{R})(z)|^q \, dz \right)^{\frac{1}{q}} \leq \left(\int_{\Omega_\psi^n} |\nabla u(z)|^p dz \right)^{\frac{1}{p}}.
$$

The Hölder inequality implies
(3.50)

$$
\left(\int_{A_\psi^n} |\nabla L(z)(u \circ \mathcal{R})(z)|^q \, dz \right)^{\frac{1}{q}} \leq \left(\int_{A_\psi^n} |\nabla L(z)|^{\frac{pq}{p-q}} dz \right)^{\frac{p-q}{pq}} \cdot \left(\int_{A_\psi^n} |(u \circ \mathcal{R})(z)|^p dz \right)^{\frac{1}{p}}.
$$

By (3.28), the change of variables formula implies

$$
(3.51) \qquad \int_{A_\psi^n} |(u \circ \mathcal{R})(z)|^p dz \leq C \int_{\Omega_\psi^n} |u(z)|^p dz.
$$

By (3.30) and the fact that ψ is Lipschitz, we have

$$
(3.52) \qquad \int_{A_\psi^n} |\nabla L(z)|^{\frac{pq}{p-q}} \leq C \int_0^1 \psi(t)^{n-1-\frac{pq}{p-q}} dt < C \int_0^1 t^{n-1-\frac{pq}{p-q}} dt < \infty
$$

whenever $1 \leq q < n-1$ and $(n-1)q/(n-1-q) \leq p < \infty$. By combining inequalities (3.50), (3.51) and (3.52), we obtain

$$\left(\int_{A^n_\psi} |\nabla L(z)(u \circ \mathcal{R})(z)|^q \, dz\right)^{\frac{1}{q}} \leq C \left(\int_{\Omega^n_\psi} |u(z)|^p dz\right)^{\frac{1}{p}}. \tag{3.53}$$

By combining (3.48), (3.49) and (3.53), we obtain

$$\left(\int_{A^n_\psi} |\nabla E(u)(z)|^q \, dz\right)^{\frac{1}{q}} \leq C \left(\int_{\Omega^n_\psi} |u(z)|^p + |\nabla u(z)|^p dz\right)^{\frac{1}{p}}. \tag{3.54}$$

By combining inequalities (3.38), (3.46) and (3.54), we obtain

$$\left(\int_{\widehat{\Omega^n_\psi}} |\nabla E(u)(z)|^q dz\right)^{\frac{1}{q}} \leq C \left(\int_{\Omega^n_\psi} |u(z)|^p + |\nabla u(z)|^p dz\right)^{\frac{1}{p}}. \tag{3.55}$$

Finally, by combining (3.37) and (3.55), we obtain the desired norm inequality

$$\left(\int_{\widehat{\Omega^n_\psi}} |E(u)(z)|^q + |\nabla E(u)(z)|^q \, dz\right)^{\frac{1}{q}} \leq C \left(\int_{\Omega^n_\psi} |u(z)|^p + |\nabla u(z)|^p dz\right)^{\frac{1}{p}}. \tag{3.56}$$

Hence, the extension operator E defined in (3.31) is a bounded linear extension operator from the dense subspace $C^\infty_o(\mathbb{R}^n) \cap W^{1,p}(\Omega^n_\psi)$ to $W^{1,q}(\mathbb{R}^n)$ whenever $1 \leq q < n-1$ and $(n-1)q/(n-1-q) \leq p < \infty$. For an arbitrary $u \in W^{1,p}(\Omega^n_\psi)$, there exists a Cauchy sequence $\{u_m\}_{m=1}^\infty \subset C^\infty_o(\mathbb{R}^n) \cap W^{1,p}(\Omega^n_\psi)$ which converges to u with respect to the $W^{1,p}$-norm. Since E is a bounded linear extension operator from $C^\infty_o(\mathbb{R}^n) \cap W^{1,p}(\Omega^n_\psi)$ to $W^{1,q}(\mathbb{R}^n)$, there exists a subsequence of $\{u_m\}$ which converges to u almost everywhere on Ω^n_ψ and $\{E(u_m)\}$ is also a Cauchy sequence in $W^{1,q}(\mathbb{R}^n)$ which converges to some function $v \in W^{1,q}(\mathbb{R}^n)$. To simplify the notation, we still denote this subsequence by $\{u_m\}$. Then, we have $v\big|_{\Omega^n_\psi}(z) = u(z)$ for almost every $z \in \Omega^n_\psi$ and

$$\|v\|_{W^{1,q}(\mathbb{R}^n)} \leq \lim_{m\to\infty} \|E(u_m)\|_{W^{1,q}(\mathbb{R}^n)} \leq C \lim_{m\to\infty} \|u_m\|_{W^{1,p}(\Omega^n_\psi)} \leq C\|u\|_{W^{1,p}(\Omega^n_\psi)}.$$

Furthermore, by picking an extra subsequence if necessary, it follows that $v(z) = E(u)(z)$ for almost every $z \in \mathbb{R}^n$. Hence, E is a bounded linear extension operator from $W^{1,p}(\Omega^n_\psi)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $1 \leq q < n-1$ and $(n-1)q/(n-1-q) \leq p < \infty$. In conclusion, we have proved that a Lipschitz outward cuspidal domain Ω^n_ψ is a Sobolev (p,q)-extension domain, whenever $1 \leq q < n-1$ and $(n-1)q/(n-1-q) \leq p < \infty$.

Next, we extend the result to an arbitrary outward cuspidal domain by using global bi-Lipschitz transformations. Let $\psi : (0,1] \to (0,\infty)$ be an arbitrary cuspidal domain. By Proposition 3.5, there exists a Lipschitz cuspidal function $\hat{\psi}$ and a global bi-Lipschitz homeomorphism $\mathcal{O} : \mathbb{R}^n \to \mathbb{R}^n$ with $\mathcal{O}(\Omega^n_\psi) = \Omega^n_{\hat{\psi}}$. Fix $1 \leq q < n-1$ and $(n-1)q/(n-1-q) \leq p < \infty$. Let $u \in W^{1,p}(\Omega^n_\psi)$ be arbitrary. Since $\mathcal{O}$ is bi-Lipschitz with $\mathcal{O}(\Omega^n_\psi) = \Omega^n_{\hat{\psi}}$, we have $u \circ \mathcal{O}^{-1} \in W^{1,p}(\Omega^n_{\hat{\psi}})$ with

$$\|u \circ \mathcal{O}^{-1}\|_{W^{1,p}(\Omega^n_{\hat{\psi}})} \leq C\|u\|_{W^{1,p}(\Omega^n_\psi)}. \tag{3.57}$$

By the argument above, the Lipschitz outward cuspidal domain $\Omega^n_{\hat{\psi}}$ is a Sobolev (p,q)-extension domain, there exists a function $E(u \circ \mathcal{O}^{-1}) \in W^{1,q}(\mathbb{R}^n)$ with $E(u \circ \mathcal{O}^{-1})\big|_{\Omega^n_{\hat{\psi}}} \equiv u \circ \mathcal{O}^{-1}$ and

$$\|E(u \circ \mathcal{O}^{-1})\|_{W^{1,q}(\mathbb{R}^n)} \leq C\|u \circ \mathcal{O}^{-1}\|_{W^{1,p}(\Omega^n_{\hat{\psi}})}. \tag{3.58}$$

By the fact that $\mathcal{O}$ is bi-Lipschitz, we have $(E(u \circ \mathcal{O}^{-1})) \circ \mathcal{O} \in W^{1,q}(\mathbb{R}^n)$ with

$$\|(E(u \circ \mathcal{O}^{-1})) \circ \mathcal{O}\|_{W^{1,q}(\mathbb{R}^n)} \leq C\|E(u \circ \mathcal{O}^{-1})\|_{W^{1,q}(\mathbb{R}^n)}. \tag{3.59}$$

By the definitions, we have $(E(u \circ \mathcal{O}^{-1})) \circ \mathcal{O}\big|_{\Omega^n_\psi} \equiv u$. By combining (3.57), (3.58) and (3.59), we obtain the desired norm inequality

$$\|(E(u \circ \mathcal{O}^{-1})) \circ \mathcal{O}\|_{W^{1,q}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega^n_\psi)}.$$

Hence, Ω^n_ψ is a Sobolev (p,q)-extension domain whenever $1 \leq q < n-1$ and $(n-1)q/(n-1-q) \leq p < \infty$. □

Let us prove Proposition 1.7 which yields the sharpness of Theorem 1.6.

Proof of Proposition 1.7. Let $n-1 \leq p < \infty$. Set

$$s_1 := \frac{np-(n-1)}{(n-1)^2}.$$

By Theorem 1.2, $\Omega^n_{t^s}$ is a Sobolev $(p, n-1)$-extension domain if and only if $1 \leq s \leq s_1$. Let $1 \leq q < n-1$ and $q \leq p < \frac{(n-1)q}{n-1-q}$. Set

$$s_2 := \frac{pq+p-q}{pq+(n-1)(q-p)}.$$

By Theorem 1.2, $\Omega^n_{t^s}$ is a Sobolev (p,q)-extension domain if and only if $1 \leq s < s_2$. □

4. Further comments

In the monograph [19] and their papers referred to therein, Maz'ya and Poborchi also dealt with generalized outward cuspidal domains with Lipschitz base domains. To be more precise, for a bounded Lipschitz domain $U \in \mathbb{R}^{n-1}$ with $0 \in U$ and a cuspidal function $\psi : (0,1] \to (0,\infty)$, the corresponding generalized outward cuspidal domain with the base domain U is defined by setting

$$U^n_\psi := \left\{(t,x) \in (0,1] \times \mathbb{R}^{n-1} : x \in \psi(t)U\right\} \cup \left\{(t,x) \in (1,2) \times \mathbb{R}^{n-1} : x \in \psi(1)U\right\}. \tag{4.1}$$

We have only discussed outward cuspidal domains whose base domains are the unit ball. Maz'ya and Poborchi also established results for generalized Lipschitz outward cuspidal domains U^n_ψ with Lipschitz base domains $U \subset \mathbb{R}^{n-1}$. Hence, our results in Theorem 1.2 only extend Maz'ya and Poborchi's results in a special case. For a full extension, one would need to establish a global bi-Lipschitz equivalence analog of Proposition 3.5 for general outward cuspidal domains U^n_ψ. This appears to be technically challenging but we expect it to be doable.

Conjecture 4.2. *For every generalized outward cuspidal domain U^n_ψ, there exists a Lipschitz cuspidal function $\hat{\psi}$, a Lipschitz base domain $\hat{U}$ and a global bi-Lipschitz transformation $\mathcal{O} : \mathbb{R}^n \to \mathbb{R}^n$ with $\mathcal{O}(U^n_\psi) = \hat{U}^n_{\hat{\psi}}$.*

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Pekka Koskela
Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014, Jyväskylä, Finland
E-mail address: `pekka.j.koskela@jyu.fi`

Zheng Zhu
School of Mathematical science, Beihang University, Changping District Shahe Higher Education Park South Third Street No. 9, Beijing 102206, P. R. China
E-mail address: `zhzhu@buaa.edu.cn`