Pure and Applied Functional Analysis
Volume 9, Number 1, 2024, 195–210

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

DISCRETE RIESZ TRANSFORMS ON REARRANGEMENT-INVARIANT BANACH SEQUENCE SPACES AND MAXIMALLY NONCOMPACT OPERATORS

OLEKSIY KARLOVYCH AND EUGENE SHARGORODSKY

Dedicated to Professor Vladimir Maz'ya on the occasion of his 85th birthday

Abstract. We say that an operator between Banach spaces is maximally noncompact if its operator norm coincides with its Hausdorff measure of noncompactness. We prove that a translation-invariant operator acting from a translation-invariant Banach sequence space $X(\mathbb{Z}^d)$ to a translation-invariant Banach sequence space $Y(\mathbb{Z}^d)$ is maximally noncompact whenever the target space $Y(\mathbb{Z}^d)$ satisfies mild additional conditions. As a consequence, we show that the discrete Riesz transforms R_j, $j = 1, \ldots, d$ on rearrangement-invariant Banach sequence spaces with non-trivial Boyd indices are maximally noncompact. We also observe that the same results are valid for translation-invariant operators between translation-invariant Banach function spaces $X(\mathbb{R}^d)$ and $Y(\mathbb{R}^d)$.

1. Introduction

For Banach spaces E, F, let $\mathcal{B}(E,F)$ and $\mathcal{K}(E,F)$ denote the sets of bounded linear and compact linear operators from E to F, respectively. We will abbreviate $\mathcal{B}(E) := \mathcal{B}(E,E)$ and $\mathcal{K}(E) := \mathcal{K}(E,E)$. The norm of an operator $A \in \mathcal{B}(E,F)$ is denoted by $\|A\|_{\mathcal{B}(E,F)}$. The essential norm of $A \in \mathcal{B}(E,F)$ is defined by

$$\|A\|_{\mathcal{B}(E,F),\mathrm{e}} := \inf\{\|A-K\|_{\mathcal{B}(E,F)} : \ K \in \mathcal{K}(E,F)\}.$$

For a bounded subset Ω of the space E, we denote by $\chi(\Omega)$ the greatest lower bound of the set of numbers r such that Ω can be covered by a finite family of open balls of radius r. For $A \in \mathcal{B}(E,F)$, set

$$\|A\|_{\mathcal{B}(E,F),\chi} := \chi\left(A(B_E)\right),$$

where B_E denotes the closed unit ball in E. The quantity $\|A\|_{\mathcal{B}(E,F),\chi}$ is called the (Hausdorff) measure of non-compactness of the operator A. It follows from the

2020 *Mathematics Subject Classification.* 42B15, 44A15, 46A45, 46B45, 46E30, 47B38.
Key words and phrases. Discrete Riesz transform, translation-invariant operator, Hausdorff measure of noncompactness, essential norm, translation-invariant space, rearrangement-invariant space, Boyd indices.
This work is funded by national funds through the FCT - Fundação para a Ciẽncia e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 and UIDP/00297/2020 (Center for Mathematics and Applications).

definition of the essential norm and [27, inequality (3.29)] that for every $A \in \mathcal{B}(E, F)$ one has

$$\|A\|_{\mathcal{B}(E,F),\chi} \le \|A\|_{\mathcal{B}(E,F),\mathrm{e}} \le \|A\|_{\mathcal{B}(E,F)}. \tag{1.1}$$

We refer to the monographs [1, 6, 10] for the general theory of measures of non-compactness. We will say that an operator $A \in \mathcal{B}(E, F)$ is *maximally noncompact* if $\|A\|_{\mathcal{B}(E,F),\chi} = \|A\|_{\mathcal{B}(E,F),\mathrm{e}} = \|A\|_{\mathcal{B}(E,F)}$.

For $h \in \mathbb{Z}^d$, let V_h denote the shift (translation) operator

$$(V_h f)(x) := f(x - h), \quad x \in \mathbb{Z}^d. \tag{1.2}$$

Let $X = X(\mathbb{Z}^d)$, $Y = Y(\mathbb{Z}^d)$ be Banach sequence spaces (see [11, Ch. 1] or Section 2 below for the definition of this notion). One says that the space $X(\mathbb{Z}^d)$ is translation-invariant if $\|V_h f\|_X = \|f\|_X$ for all $f \in X(\mathbb{Z}^d)$ and $h \in \mathbb{Z}^d$. If both $X(\mathbb{Z}^d)$ and $Y(\mathbb{Z}^d)$ are translation-invariant, then an operator $A \in \mathcal{B}(X, Y)$ is said to be translation-invariant if $AV_h = V_h A$ for all $h \in \mathbb{Z}^d$.

The discrete Hilbert transform H defined by

$$(Hf)(x) := \frac{1}{\pi} \sum_{y \in \mathbb{Z} \setminus \{x\}} \frac{f(y)}{y - x}, \quad x \in \mathbb{Z},$$

is one of the most important translation-invariant operators. Its boundedness on $\ell^p(\mathbb{Z})$ with $1 < p < \infty$ was established by Marcel Riesz [30]. Calderón and Zygmund [14, Theorem 14] extended this result to the multidimensional case of the discrete Riesz transforms R_j, $j = 1, \dots, d$ given by

$$(R_j f)(x) := \frac{1}{\pi^d} \sum_{y \in \mathbb{Z}^d \setminus \{x\}} \frac{y_j - x_j}{|y - x|^{d+1}} f(y), \quad x = (x_1, \dots, x_d) \in \mathbb{Z}^d,$$

where $|x| = \sqrt{x_1^2 + \dots + x_d^2}$. Andersen [3] generalized the above results to the setting of rearrangement-invariant Banach sequence spaces $X(\mathbb{Z}^d)$ and proved that the Hilbert transform H is bounded on $X(\mathbb{Z})$ (respectively, the Riesz transforms R_j, $j = 1, \dots, d$, are bounded on $X(\mathbb{Z}^d)$) if and only if the Boyd indices of $X(\mathbb{Z})$ (respectively, of the space $X(\mathbb{Z}^d)$) satisfy $0 < \alpha_X, \beta_X < 1$ (see Theorem 3.1).

Note that the norm and the essential norm of the continuous analogue $\mathcal{H}$ of the operator H on the space $L^p(\mathbb{R})$ for $1 < p < \infty$ are well known:

$$\|\mathcal{H}\|_{\mathcal{B}(L^p(\mathbb{R}))} = \|\mathcal{H}\|_{\mathcal{B}(L^p(\mathbb{R})),\mathrm{e}} = \cot(\pi/(2p^*)), \tag{1.3}$$

where

$$p^* := \max\{p, p/(p-1)\}$$

(see, e.g., [16, Ch. 13, Theorem 1.3] and [25, Ch. II, Example 4.2]). The lower estimate was obtained by Gohberg and Krupnik in 1968, and the upper estimate was proved four years later by Pichorides (we refer to the survey [26] for a more detailed history). Recently, Bañuelos and Kwaśnicki [8] proved the long standing conjecture on the norm of the discrete Hilbert transform:

$$\|H\|_{\mathcal{B}(\ell^p(\mathbb{Z}))} = \cot(\pi/(2p^*)), \quad 1 < p < \infty. \tag{1.4}$$

We also refer to [7, 9] for further results on sharp estimates for discrete singular integrals.

The values of the essential norm and of the Hausdroff measure of noncompactness of the discrete Hilbert transform on $\ell^p(\mathbb{Z})$ do not seem to be known. Our initial motivation was to fill in this gap, but we realised that one can study the problem of maximal noncompactness of more general translation-invariant operators in much more general settings of translation-invariant or rearrangement-invariant Banach sequence spaces.

Let $S_0(\mathbb{Z}^d)$ denote the set of all finitely supported sequences. If a Banach sequence space $Y(\mathbb{Z}^d)$ is non-separable, then $S_0(\mathbb{Z}^d)$ is not dense in it, but it may happen that every element of $Y(\mathbb{Z}^d)$ can be approximated by elements of $S_0(\mathbb{Z}^d)$ in a norm weaker than $\|\cdot\|_Y$. This possibility is described in the next theorem by introducing an auxiliary Banach sequence space $Z(\mathbb{Z}^d)$.

Theorem 1.1 (Main result). *Let $X = X(\mathbb{Z}^d)$, $Y = Y(\mathbb{Z}^d)$, and $Z = Z(\mathbb{Z}^d)$ be translation-invariant Banach sequence spaces such that Y is a subset of the closure of $S_0(\mathbb{Z}^d)$ in Z. If $A \in \mathcal{B}(X, Y)$ is a translation-invariant operator, then it is maximally noncompact.*

We will show that the assumption on the space $Y(\mathbb{Z}^d)$ is satisfied if $Y(\mathbb{Z}^d)$ is rearrangement-invariant and its lower Boyd index satisfies $\alpha_Y > 0$ (see Corollary 3.8). In particular, one can consider nonseparable target spaces, like the weak-ℓ^p space $\ell^{p,\infty}(\mathbb{Z}^d)$. Hence Theorem 1.1 and Andersen's results (see Theorem 3.1) on the boundedness of the Riesz transforms R_j, $1 \le j \le d$, on rearrangement-invariant Banach sequence spaces imply the following.

Corollary 1.2. *Let $X = X(\mathbb{Z}^d)$ be a rearrangement-invariant Banach sequence space with the Boyd indices satisfying $0 < \alpha_X$, $\beta_X < 1$. For all $a, b \in \mathbb{C}$ and $j = 1, \dots, d$, the operators $aI + bR_j$ are maximally noncompact on the space X.*

Combining this corollary with (1.4), we get

$$\|H\|_{\mathcal{B}(\ell^p(\mathbb{Z})),\chi} = \|H\|_{\mathcal{B}(\ell^p(\mathbb{Z})),\mathrm{e}} = \|H\|_{\mathcal{B}(\ell^p(\mathbb{Z}))} = \cot(\pi/(2p^*)), \quad 1 < p < \infty.$$

The paper is organized as follows. In Section 2, we recall the definitions and basic properties of Banach sequence spaces and their associate spaces. In Section 3, we deal with rearrangement-invariant Banach sequence spaces, their Boyd indices α_X, β_X (see [13]) and Zippin indices p_X, q_X (see [31]). We show that if the lower Zippin index of a rearrangement-invariant Banach sequence space $X(\mathbb{Z}^d)$ satisfies $p_X > 1/p$, where $1 < p < \infty$, then $X(\mathbb{Z}^d) \hookrightarrow \ell^p(\mathbb{Z}^d)$. This fact and the inequality $\alpha_X \le p_X$ imply that the assumptions of Theorem 1.1 are satisfied if $X(\mathbb{Z}^d) = Y(\mathbb{Z}^d)$ is a rearrangement-invariant Banach function space with nontrivial Boyd indices, which tells us that Corollary 1.2 follows from Theorem 1.1 and Andersen's Theorem 3.1. So, it remains to prove Theorem 1.1, which is done in Section 4. In Section 5, we present continuous analogues of Theorem 1.1 and Corollary 1.2.

2. Banach sequence spaces

2.1. **Definition of a Banach sequence space.** Let $\mathbb{S} \in \{\mathbb{N}, \mathbb{Z}^d\}$, let $\ell^0(\mathbb{S})$ be the linear space of all sequences $f : \mathbb{S} \to \mathbb{C}$, and let $\ell^0_+(\mathbb{S})$ be the cone of nonnegative sequences in $\ell^0(\mathbb{S})$. We equip $\mathbb{S}$ with the counting measure, i.e. the purely atomic measure with atoms having equal measure 1. According to [11, Ch. 1, Defintion 1.1], a Banach function norm $\varrho : \ell^0_+(\mathbb{S}) \to [0,\infty]$ is a mapping which satisfies the following axioms for all $f, g, \{f_n\}$ in $\ell^0_+(\mathbb{S})$, for all finite subsets $E \subset \mathbb{S}$, and all constants $a \geq 0$:

$$
\begin{array}{ll}
\text{(A1)} & \varrho(f) = 0 \Leftrightarrow f = 0,\ \varrho(af) = a\varrho(f),\ \varrho(f+g) \leq \varrho(f) + \rho(g), \\
\text{(A2)} & 0 \leq g \leq f \ \Rightarrow\ \varrho(g) \leq \varrho(f) \quad \text{(the lattice property)}, \\
\text{(A3)} & 0 \leq f_n \uparrow f \ \Rightarrow\ \varrho(f_n) \uparrow \varrho(f) \quad \text{(the Fatou property)}, \\
\text{(A4)} & \varrho(\chi_E) < \infty, \\
\text{(A5)} & \displaystyle\sum_{x \in E} f(x) \leq C_E \varrho(f),
\end{array}
$$

where χ_E is the characteristic (indicator) function of E, and the constant $C_E \in (0,\infty)$ may depend on ϱ and E, but is independent of f. The set $X(\mathbb{S})$ of all sequences $f \in \ell^0(\mathbb{S})$ for which $\varrho(|f|) < \infty$ is called a Banach sequence space. For each $f \in X(\mathbb{S})$, the norm of f is defined by $\|f\|_{X(\mathbb{S})} := \rho(|f|)$. The set $X(\mathbb{S})$ equipped with the natural linear space operations and this norm becomes a Banach space (see [11, Ch. 1, Theorems 1.4 and 1.6]). If ϱ is a Banach function norm, its associate norm ϱ' is defined on $\ell^0_+(\mathbb{S})$ by

$$
\varrho'(g) := \sup\left\{ \sum_{x \in \mathbb{S}} f(x)g(x) \ :\ f \in \ell^0_+(\mathbb{S}),\ \varrho(f) \leq 1 \right\}, \quad g \in \ell^0_+(\mathbb{S}).
$$

It is a Banach function norm itself [11, Ch. 1, Theorem 2.2]. The Banach sequence space $X'(\mathbb{S})$ determined by the Banach function norm ϱ' is called the associate space (Köthe dual) of $X(\mathbb{S})$. The associate space $X'(\mathbb{S})$ can be viewed as a subspace of the Banach dual space $X^*(\mathbb{S})$.

Recall that $S_0(\mathbb{Z}^d)$ denotes the set of all finitely supported sequences. The following lemma was proved in [22, Lemma 2.10] in the case of Banach function spaces. Its proof in the case of Banach sequence spaces is essentially the same.

Lemma 2.1. *Let $X(\mathbb{Z}^d)$ be a Banach sequence space and $X'(\mathbb{Z}^d)$ be its associate space. For every $f \in X(\mathbb{Z}^d)$,*

$$
\|f\|_{X(\mathbb{Z}^d)} = \sup\left\{ \left| \sum_{x \in \mathbb{Z}^d} f(x)s(x) \right| \ :\ s \in S_0(\mathbb{Z}^d),\ \|s\|_{X'(\mathbb{Z}^d)} \leq 1 \right\}.
$$

2.2. **Translation-invariant Banach sequence spaces and their associate spaces.** The next lemma was proved in [22, Lemma 2.1] in the case of translation-invariant Banach function spaces. Its proof in the case of Banach sequence spaces is essentially the same.

Lemma 2.2. *Let $X(\mathbb{Z}^d)$ be a Banach sequence space and $X'(\mathbb{Z}^d)$ be its associate space. Then $X(\mathbb{Z}^d)$ is translation-invariant if and only if $X'(\mathbb{Z}^d)$ is translation-invariant.*

3. Rearrangement-invariant Banach sequence spaces and their Boyd and Zippin indices

3.1. Rearrangement-invariant Banach sequence spaces. The distribution function of a sequence $f \in \ell^0(\mathbb{S})$ is defined by

$$d_f(\lambda) := \operatorname{card}\{x \in \mathbb{S} : |f(x)| > \lambda\}, \quad \lambda \geq 0.$$

For any sequence $f \in \ell^0(\mathbb{S})$, its decreasing rearrangement is given by

$$f^*(n) := \inf\{\lambda \geq 0 : d_f(\lambda) \leq n - 1\}, \quad n \in \mathbb{N}.$$

One says that sequences $f, g \in \ell^0(\mathbb{S})$ are equimeasurable if $d_f = d_g$. A Banach function norm $\varrho : \ell^0_+(\mathbb{S}) \to [0, \infty]$ is said to be rearrangement-invariant if $\varrho(f) = \varrho(g)$ for every pair of equimeasurable functions $f, g \in \ell^0_+(\mathbb{S})$. In that case, the Banach sequence space X generated by ρ is said to be a rearrangement-invariant Banach sequence space (cf. [11, Ch. 2, Definition 4.1]). It follows from [11, Ch. 2, Proposition 4.2] that if a Banach sequence space $X(\mathbb{S})$ is rearrangement-invariant, then its associate space $X'(\mathbb{S})$ is also a rearrangement-invariant Banach sequence space.

3.2. Boyd indices. Let $X(\mathbb{Z}^d)$ be a rearrangement-invariant Banach sequence space generated by a rearrangement-invariant Banach function norm $\varrho : \ell^0_+(\mathbb{Z}^d) \to [0, \infty]$. By the Luxemburg representation theorem (see [11, Ch. 2, Theorem 4.10]), there exists a unique rearrangement-invariant Banach function norm $\overline{\varrho} : \ell^0_+(\mathbb{N}) \to [0, \infty]$ such that

$$\varrho(f) = \overline{\varrho}(f^*), \quad f \in \ell^0_+(\mathbb{Z}^d).$$

The rearrangement-invariant Banach function space generated by $\overline{\rho}$ is denoted by $\overline{X}(\mathbb{N})$ and is called the Luxemburg representation of $X(\mathbb{Z}^d)$.

Let $t \geq 0$. We denote by $\lfloor t \rfloor$ the greatest integer less than or equal to t, and by $\lceil t \rceil$ the least integer greater than or equal to t. Thus $\lceil t \rceil = \lfloor t \rfloor = t$ if $t \in \mathbb{N} \cup \{0\}$ and $\lceil t \rceil = \lfloor t \rfloor + 1$ otherwise.

For $f : \mathbb{N} \to [0, \infty)$, consider the following operators:

$$(E_m f)(n) := f(mn), \quad (F_m f)(n) := f(\lfloor (n-1)/m \rfloor + 1), \quad m, n \in \mathbb{N}.$$

For $m \in \mathbb{N}$, let

$$H(m, X) := \sup\left\{\overline{\varrho}(E_m f^*) \ : \ f \in \overline{X}(\mathbb{N}),\ \overline{\varrho}(f) \leq 1\right\},$$
$$K(m, X) := \sup\left\{\overline{\varrho}(F_m f^*) \ : \ f \in \overline{X}(\mathbb{N}),\ \overline{\varrho}(f) \leq 1\right\}.$$

It follows from [13, Lemmas 3–5] that the following limits

$$\alpha_X := \lim_{m \to \infty} \frac{-\log H(m, X)}{\log m}, \quad \beta_X := \lim_{m \to \infty} \frac{\log K(m, X)}{\log m}$$

exist and satisfy

$$0 \leq \alpha_X \leq \beta_X \leq 1, \quad \alpha_{X'} = 1 - \beta_X, \quad \beta_{X'} = 1 - \alpha_X. \tag{3.1}$$

The numbers α_X and β_X are called the lower and the upper Boyd indices, respectively. One says that the Boyd indices are nontrivial if

$$0 < \alpha_X, \quad \beta_X < 1. \tag{3.2}$$

The importance of the concept of (nontrivial) Boyd indices for our work lies in the following result.

Theorem 3.1 ([2, Theorems 3.9 and 4.2], [3])**.** (a) *The discrete Hilbert transform H is bounded on a rearrangement-invariant Banach sequence space $X(\mathbb{Z})$ if and only if its Boyd indices satisfy* (3.2)*.*

(b) *The discrete Riesz transform R_j, $1 \le j \le d$ is bounded on a rearrangement-invariant Banach sequence space $X(\mathbb{Z}^d)$ if and only if its Boyd indices satisfy* (3.2)*.*

3.3. **Zippin indices.** The results of this subsection might be known to experts. They are analogous to the case of rearrangement-invariant Banach function spaces considered in [29, Section 4]. Since we were not able to find their proofs in the literature and since there are some technical differences between the case of rearrangement-invariant Banach function spaces (spaces over nonatomic measure spaces) and the case of Banach sequence spaces (spaces over purely atomic measure spaces) considered in this paper, we decided to provide detailed proofs.

Let $X(\mathbb{Z}^d)$ be a rearrangement-invariant Banach function space with the Luxemburg representation $\overline{X}(\mathbb{N})$. For $m \in \mathbb{N}$, let $\chi_m \in \overline{X}(\mathbb{N}), \overline{X}'(\mathbb{N})$ be the sequence given by

$$\chi_m := (\underbrace{1, \dots, 1}_{m}, 0, 0, \dots).$$

For $s \ge 1$, consider

$$L(s,X) := \inf_{n\in\mathbb{N}} \frac{\|\chi_{\lfloor sn\rfloor}\|_{\overline{X}(\mathbb{N})}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}}, \quad M(s,X) := \sup_{n\in\mathbb{N}} \frac{\|\chi_{\lceil sn\rceil}\|_{\overline{X}(\mathbb{N})}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}}.$$

Lemma 3.2. *For $s, t \ge 1$,*

$$L(st,X) \ge L(s,X)L(t,X), \tag{3.3}$$

$$M(st,X) \le M(s,X)M(t,X). \tag{3.4}$$

Proof. Let $n \in \mathbb{N}$ and $s,t \ge 1$. Then $\lfloor stn\rfloor \ge \lfloor s\lfloor tn\rfloor\rfloor$ and

$$\begin{aligned}\frac{\|\chi_{\lfloor stn\rfloor}\|_{\overline{X}(\mathbb{N})}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}} &\ge \frac{\|\chi_{\lfloor s\lfloor tn\rfloor\rfloor}\|_{\overline{X}(\mathbb{N})}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}} = \frac{\|\chi_{\lfloor s\lfloor tn\rfloor\rfloor}\|_{\overline{X}(\mathbb{N})}}{\|\chi_{\lfloor tn\rfloor}\|_{\overline{X}(\mathbb{N})}} \frac{\|\chi_{\lfloor tn\rfloor}\|_{\overline{X}(\mathbb{N})}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}} \\ &\ge L(x,X)L(t,X),\end{aligned}$$

which implies (3.3). Similarly, since $\lceil stn\rceil \le \lceil s\lceil tn\rceil\rceil$, we have

$$\begin{aligned}\frac{\|\chi_{\lceil stn\rceil}\|_{\overline{X}(\mathbb{N})}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}} &\le \frac{\|\chi_{\lceil s\lceil tn\rceil\rceil}\|_{\overline{X}(\mathbb{N})}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}} = \frac{\|\chi_{\lceil s\lceil tn\rceil\rceil}\|_{\overline{X}(\mathbb{N})}}{\|\chi_{\lceil tn\rceil}\|_{\overline{X}(\mathbb{N})}} \frac{\|\chi_{\lceil tn\rceil}\|_{\overline{X}(\mathbb{N})}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}} \\ &\le M(x,X)M(t,X),\end{aligned}$$

which implies (3.4). □

Lemma 3.3. *For $s \ge 1$, one has $1 \le L(s,X)$ and $M(s,X) \le \lceil s\rceil$.*

Proof. It is clear that $n \le \lfloor sn\rfloor$ for $s \ge 1$. Then

$$1 \le \frac{\|\chi_{\lfloor sn\rfloor}\|_{\overline{X}(\mathbb{N})}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}},$$

whence $1 \le L(s,X)$. On the other hand, since $n \le \lceil sn \rceil$, it follows from [11, Ch. 2, Corollary 5.3] that

$$\begin{aligned}\frac{\|\chi_{\lceil sn\rceil}\|_{\overline{X}(\mathbb{N})}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}} &= \frac{\|\chi_{\lceil sn\rceil}\|_{\overline{X}(\mathbb{N})}}{n}\frac{n}{\|\chi_n\|_{\overline{X}(\mathbb{N})}}\\ &\le \frac{\|\chi_{\lceil sn\rceil}\|_{\overline{X}(\mathbb{N})}}{n}\frac{\lceil sn\rceil}{\|\chi_{\lceil sn\rceil}\|_{\overline{X}(\mathbb{N})}} = \frac{\lceil sn\rceil}{n} \le \lceil s\rceil,\end{aligned}$$

which implies that $M(s,X) \le \lceil s \rceil$ and completes the proof. □

Consider the following quantities:

$$p_X := \sup_{s>1} \frac{\log L(s,X)}{\log s}, \quad q_X := \inf_{s>1} \frac{\log M(s,X)}{\log s}.$$

They were introduced by Zippin [31, pp. 283–284], who stated without proof the following result. For the convenience of the readers, we give its proof here.

Theorem 3.4. *We have*

$$p_X = \sup_{n\in\mathbb{N}} \frac{\log L(n,X)}{\log n} = \lim_{n\to\infty} \frac{\log L(n,X)}{\log n}, \tag{3.5}$$

$$q_X = \inf_{n\in\mathbb{N}} \frac{\log M(n,X)}{\log n} = \lim_{n\to\infty} \frac{\log M(n,X)}{\log n}. \tag{3.6}$$

Proof. The idea of the proof is borrowed from that of [2, Theorem 3.1(i)]. Consider the functions

$$\varphi(s) := -\log L(e^s,X), \quad \psi(s) := \log M(e^s,X), \quad s>0.$$

It follows from Lemmas 3.2 and 3.3 that for all $s,t>0$,

$$\varphi(s+t) \le \varphi(s)+\varphi(t),\ \psi(s+t) \le \psi(s)+\psi(t), \quad \varphi(s)<\infty,\ \psi(s)<\infty.$$

By [17, Theorem 7.6.1],

$$\inf_{t>0} \frac{\varphi(t)}{t} = \lim_{t\to\infty} \frac{\varphi(t)}{t}, \quad \inf_{t>0} \frac{\psi(t)}{t} = \lim_{t\to\infty} \frac{\psi(t)}{t}. \tag{3.7}$$

Hence, making the substitution $t = e^s$, we get

$$\begin{aligned} p_X &= \sup_{s>1} \frac{\log L(s,X)}{\log s} = -\inf_{s>1} \frac{-\log L(s,X)}{\log s} = -\inf_{t>0}\frac{\varphi(t)}{t}\\ &= -\lim_{t\to\infty}\frac{\varphi(t)}{t} = -\lim_{s\to\infty}\frac{-\log L(s,X)}{\log s} = \lim_{s\to\infty}\frac{\log L(s,X)}{\log s}\\ &= \lim_{n\to\infty}\frac{\log L(n,X)}{\log n} \le \sup_{n\in\mathbb{N}}\frac{\log L(n,X)}{\log n}. \end{aligned} \tag{3.8}$$

On the other hand, obviously,

$$p_X = \sup_{s>1} \frac{\log L(s,X)}{\log s} \ge \sup_{n\in\mathbb{N}} \frac{\log L(n,X)}{\log n}. \tag{3.9}$$

Combining (3.8) and (3.9), we arrive at (3.5).

Similarly, using the second equality in (3.7), we get

$$q_X = \inf_{s>1} \frac{\log M(s,X)}{\log s} = \inf_{t>0} \frac{\psi(t)}{t} = \lim_{t\to\infty} \frac{\psi(t)}{t} = \lim_{s\to\infty} \frac{\log M(s,X)}{\log s} \tag{3.10}$$
$$= \lim_{n\to\infty} \frac{\log M(n,X)}{\log n} \geq \inf_{n\in\mathbb{N}} \frac{\log M(n,X)}{\log n}.$$

On the other hand, clearly,

$$q_X = \inf_{s>1} \frac{\log M(s,X)}{\log s} \leq \inf_{n\in\mathbb{N}} \frac{\log M(n,X)}{\log n}. \tag{3.11}$$

Combining (3.10) and (3.11), we arrive at (3.6). □

Lemma 3.5. *If $X(\mathbb{Z}^d)$ is a rearrangement-invariant Banach sequence space and $X'(\mathbb{Z}^d)$ is its associate space, then*

$$p_{X'} = 1 - q_X, \quad q_{X'} = 1 - p_X. \tag{3.12}$$

Proof. By [11, Ch. 2, Theorem 5.2], for $m, n \in \mathbb{N}$,

$$\frac{\|\chi_{mn}\|_{\overline{X}(\mathbb{N})}}{\|\chi_n\|_{\overline{X}(\mathbb{X})}} = m \frac{\|\chi_n\|_{\overline{X'}(\mathbb{N})}}{\|\chi_{mn}\|_{\overline{X'}(\mathbb{X})}}.$$

Therefore, for $m \in \mathbb{N}$,

$$L(m,X) = \inf_{n\in\mathbb{N}} \frac{\|\chi_{mn}\|_{\overline{X}(\mathbb{N})}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}} = m \inf_{n\in\mathbb{N}} \left(\frac{\|\chi_{mn}\|_{\overline{X'}(\mathbb{N})}}{\|\chi_n\|_{\overline{X'}(\mathbb{N})}} \right)^{-1} \tag{3.13}$$
$$= m \left(\sup_{n\in\mathbb{N}} \frac{\|\chi_{mn}\|_{\overline{X'}(\mathbb{N})}}{\|\chi_n\|_{\overline{X'}(\mathbb{N})}} \right)^{-1} = \frac{m}{M(m,X')}$$

and

$$M(m,X) = \sup_{n\in\mathbb{N}} \frac{\|\chi_{mn}\|_{\overline{X}(\mathbb{N})}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}} = m \sup_{n\in\mathbb{N}} \left(\frac{\|\chi_{mn}\|_{\overline{X'}(\mathbb{N})}}{\|\chi_n\|_{\overline{X'}(\mathbb{N})}} \right)^{-1} \tag{3.14}$$
$$= m \left(\inf_{n\in\mathbb{N}} \frac{\|\chi_{mn}\|_{\overline{X'}(\mathbb{N})}}{\|\chi_n\|_{\overline{X'}(\mathbb{N})}} \right)^{-1} = \frac{m}{L(m,X')}.$$

It follows from Theorem 3.4 and formula (3.13) that

$$p_X = \lim_{m\to\infty} \frac{\log L(m,X)}{\log m} = 1 - \lim_{m\to\infty} \frac{\log M(m,X')}{\log m} = 1 - q_X.$$

The second equality in (3.12) can be proved analogously by using (3.14) instead of (3.13). □

Lemma 3.6. *If $X(\mathbb{Z}^d)$ is a rearrangement-invariant Banach sequence space, then its Boyd and Zippin indices satisfy*

$$0 \leq \alpha_X \leq p_X \leq q_X \leq \beta_X \leq 1.$$

Proof. We already know the inequalities $0 \le \alpha_X$ and $\beta_X \le 1$ (see (3.1)). Since $L(n,X) \le M(n,X)$ for $n \in \mathbb{N}$, it follows from Theorem 3.4 that

$$p_X = \lim_{n\to\infty} \frac{\log L(n,X)}{\log n} \le \lim_{n\to\infty} \frac{\log M(n,X)}{\log n} = q_X.$$

Let us prove that $q_X \le \beta_X$. It is not difficult to see that for $m, n \in \mathbb{N}$,

$$\frac{\chi_{mn}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}} = F_m\left(\frac{\chi_n}{\|\chi_n\|_{\overline{X}(\mathbb{N})}}\right), \quad m,n \in \mathbb{N}.$$

Then, for $m \in \mathbb{N}$,

$$\begin{aligned} M(m,X) &= \sup_{n\in\mathbb{N}} \frac{\|\chi_{nm}\|_{\overline{X}(\mathbb{N})}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}} = \sup_{n\in\mathbb{N}} \overline{\varrho}\left(\frac{\chi_{mn}}{\|\chi_n\|_{\overline{X}(\mathbb{N})}}\right) \\ &= \sup_{n\in\mathbb{N}} \overline{\varrho}\left(F_m\left(\frac{\chi_n}{\|\chi_n\|_{\overline{X}(\mathbb{N})}}\right)\right) \\ &\le \sup\left\{\overline{\varrho}(F_m f^*) \ : \ f \in \overline{X}(\mathbb{N}),\ \overline{\varrho}(f) \le 1\right\} = K(m,X). \end{aligned}$$

It follows from the definition of β_X, the above inequality and Theorem 3.4 that

$$q_X = \lim_{m\to\infty} \frac{\log M(m,X)}{\log m} \le \lim_{m\to\infty} \frac{\log K(m,X)}{\log m} = \beta_X.$$

Finally, the above inequality applied to the associate space and the duality relations given in (3.1) for the Boyd indices and in Lemma 3.5 for the Zippin indices imply that $\alpha_X = 1 - \beta_{X'} \le 1 - q_{X'} = p_X$, which completes the proof. □

3.4. **Continuous embedding of a rearrangement-invariant Banach sequence space into ℓ^p.** The idea of the proof of the following lemma is borrowed from the proof of [28, Lemma 4.2].

Lemma 3.7. *Let $1 < p < \infty$. If $X(\mathbb{Z}^d)$ is a rearrangement-invariant Banach sequence space with the lower Zippin index satisfying $p_X > 1/p$, then*

$$X(\mathbb{Z}^d) \hookrightarrow \ell^p(\mathbb{Z}^d).$$

Proof. If $p_X > 1/p$, then there exists $\varepsilon > 0$ such that

$$(p_X - \varepsilon)p > 1. \tag{3.15}$$

By Hölder's inequality [11, Ch. 1, Theorem 2.4], for $m \in \mathbb{N}$,

$$\begin{aligned} \sum_{j=1}^m f^*(j) &= \|\chi_m f^*\|_{\ell^1(\mathbb{N})} \le \|f^*\|_{\overline{X}(\mathbb{N})} \|\chi_m\|_{\overline{X}'(\mathbb{N})} \\ &= \|f\|_{X(\mathbb{Z}^d)} \|\chi_1\|_{\overline{X}'(\mathbb{N})} \frac{\|\chi_m\|_{\overline{X}'(\mathbb{N})}}{\|\chi_1\|_{\overline{X}'(\mathbb{N})}} \\ &\le \|f\|_{X(\mathbb{Z}^d)} \|\chi_1\|_{\overline{X}'(\mathbb{N})} \sup_{n\in\mathbb{N}} \frac{\|\chi_{nm}\|_{\overline{X}'(\mathbb{N})}}{\|\chi_n\|_{\overline{X}'(\mathbb{N})}} \\ &= c\|f\|_{X(\mathbb{Z}^d)} M(m,X'), \end{aligned}$$

where $c := \|\chi_1\|_{\overline{X'}(\mathbb{N})}$. This inequality and equality (3.14) imply that for $m \in \mathbb{N}$,

$$f^*(m) \le \frac{1}{m}\sum_{j=1}^{m} f^*(j) \le c\|f\|_{X(\mathbb{Z}^d)}\frac{M(m,X')}{m} = \frac{c\|f\|_{X(\mathbb{Z}^d)}}{L(m,X)}. \tag{3.16}$$

It follows from the second equality in (3.5) that there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$,

$$p_X - \varepsilon < \frac{\log L(m,X)}{\log m} < p_X + \varepsilon,$$

whence for $m > m_0$,

$$m^{p_X-\varepsilon} < L(m,X) < m^{p_X+\varepsilon}. \tag{3.17}$$

Combining inequality (3.16) with the first inequality in (3.17), we arrive at

$$\begin{aligned}\sum_{m=m_0+1}^{\infty} (f^*(m))^p &\le c^p\|f\|^p_{X(\mathbb{Z}^d)} \sum_{m=m_0+1}^{\infty}\left(\frac{1}{m}\right)^{(p_X-\varepsilon)p}\\ &= C_{\text{tail}}\|f\|^p_{X(\mathbb{Z}^d)},\end{aligned} \tag{3.18}$$

where

$$C_{\text{tail}} := c^p \sum_{m=m_0+1}^{\infty}\left(\frac{1}{m}\right)^{(p_X-\varepsilon)p} < \infty$$

in view of (3.15). On the other hand, inequality (3.16) and Lemma 3.3 imply that

$$\sum_{m=1}^{m_0} (f^*(m))^p \le C_{\text{head}}\|f\|^p_{X(\mathbb{Z}^d)}, \tag{3.19}$$

where

$$C_{\text{head}} := m_0 c^p < \infty.$$

Combining (3.18) and (3.19), we arrive at

$$\|f\|_{\ell^p(\mathbb{Z}^d)} = \left(\sum_{m=1}^{\infty}(f^*(m))^p\right)^{1/p} \le (C_{\text{head}} + C_{\text{tail}})^{1/p}\|f\|_{X(\mathbb{Z}^d)},$$

which completes the proof. □

Since $S_0(\mathbb{Z}^d)$ is dense in $\ell^p(\mathbb{Z}^d)$ whenever $1 < p < \infty$, the above lemma and the inequality $\alpha_Y \le p_Y$ (see Lemma 3.6) immediately imply the following.

Corollary 3.8. *If $Y(\mathbb{Z}^d)$ is a rearrangement-invariant Banach sequence space with the lower Boyd index satisfying $\alpha_Y > 0$, then $Y(\mathbb{Z}^d)$ is a subset of the closure of $S_0(\mathbb{Z}^d)$ in the space $\ell^p(\mathbb{Z}^d)$ for $1/\alpha_Y < p < \infty$.*

4. Proof of the main result

The idea of the proof is borrowed from the proofs of [23, Theorems 1.1–1.2]. In view of (1.1), it is sufficient to prove that

$$\|A\|_{\mathcal{B}(X,Y),\mathcal{X}} \ge \|A\|_{\mathcal{B}(X,Y)}. \tag{4.1}$$

Fix $\varepsilon > 0$. Then there exists $g \in X$ such that $\|g\|_X = 1$ and

$$\|Ag\|_Y > \|A\|_{\mathcal{B}(X,Y)} - \varepsilon. \tag{4.2}$$

It follows from Lemma 2.1 that there exists $s \in S_0(\mathbb{Z}^d) \setminus \{0\}$ such that $\|s\|_{Y'} \le 1$ and

$$\left|\sum_{x\in\mathbb{Z}^d}(Ag)(x)s(x)\right| \ge \|Ag\|_Y - \varepsilon. \tag{4.3}$$

Take any sequence $\{h_n\}_{n\in\mathbb{N}}$ in $\mathbb{Z}^d$ such that $|h_n| \to +\infty$ as $n \to \infty$, and set $V_n := V_{h_n}$, $s_n := V_n s$. Since Y is translation-invariant, in view of Lemma 2.2, so is Y'. Therefore,

$$\|s_n\|_{Y'} = \|s\|_{Y'} \le 1, \quad n \in \mathbb{N}. \tag{4.4}$$

Making a change of variables in the left-hand side of (4.3), we see that for all $n \in \mathbb{N}$,

$$\left|\sum_{x\in\mathbb{Z}^d}(V_nAg)(x)s_n(x)\right| = \left|\sum_{x\in\mathbb{Z}^d}(Ag)(x)s(x)\right| \ge \|Ag\|_Y - \varepsilon. \tag{4.5}$$

Take any finite set $\{\varphi_1, \dots, \varphi_m\} \subset Y$. Since Y is a subset of the closure of $S_0(\mathbb{Z}^d)$ in Z and $s \in S_0(\mathbb{Z}^d) \subset Z'$, there exists a set $\{\psi_1, \dots, \psi_m\} \subset S_0(\mathbb{Z}^d)$ such that

$$\|\varphi_j - \psi_j\|_Z < \frac{\varepsilon}{\|s\|_{Z'}}, \quad j \in \{1, \dots, m\}. \tag{4.6}$$

Taking into account that ψ_j and s are finitely supported and $|h_n| \to +\infty$ as $n \to \infty$, we see that there exists $N \in \mathbb{N}$ such that

$$\psi_j s_N = 0, \quad j \in \{1, \dots, m\}. \tag{4.7}$$

Since Z' is translation-invariant (see Lemma 2.2), we get $\|s_N\|_{Z'} = \|s\|_{Z'}$. Then, in view of equalities (4.7), Hölder's inequality for the space Z (see [11, Ch. 1, Theorem 2.4]), and inequalities (4.6), we get for $j \in \{1, \dots, m\}$,

$$\begin{aligned}\left|\sum_{x\in\mathbb{Z}^d}\varphi_j(x)s_N(x)\right| &= \left|\sum_{x\in\mathbb{Z}^d}\left(\varphi_j(x) - \psi_j(x)\right)s_N(x)\right| \\ &\le \|\varphi_j - \psi_j\|_Z\|s_N\|_{Z'} < \frac{\varepsilon}{\|s\|_{Z'}}\|s\|_{Z'} = \varepsilon.\end{aligned} \tag{4.8}$$

Combining (4.5) and (4.8), we see that for $j \in \{1, \dots, m\}$,

$$\left|\sum_{x\in\mathbb{Z}^d}\big((V_NAg)(x) - \varphi_j(x)\big)s_N(x)\right| \tag{4.9}$$

$$\geq \left|\sum_{x\in\mathbb{Z}^d}(V_NAg)(x)s_N(x)\right| - \left|\sum_{x\in\mathbb{Z}^d}\varphi_j(x)s_N(x)\right|$$
$$> \|Ag\|_Y - 2\varepsilon.$$

On the other hand, applying Hölder's inequality to the space Y (see [11, Ch. 1, Theorem 2.4]) and taking into account inequality (4.4), we get for $j \in \{1,\dots,m\}$,

$$\left|\sum_{x\in\mathbb{Z}^d}\big((V_NAg)(x) - \varphi_j(x)\big)s_N(x)\right| \leq \|V_NAg - \varphi_j\|_Y\|s_N\|_{Y'} \leq \|V_NAg - \varphi_j\|_Y. \tag{4.10}$$

Since A is a translation-invariant operator, we get $V_NA = AV_N$. Hence we deduce from (4.9), (4.10) and (4.2) that for $j \in \{1,\dots,m\}$,

$$\|AV_Ng - \varphi_j\|_Y = \|V_NAg - \varphi_j\|_Y > \|Ag\|_Y - 2\varepsilon > \|A\|_{\mathcal{B}(X,Y)} - 3\varepsilon.$$

Since X is a translation-invariant space, we have $\|V_Ng\|_X = \|g\|_X = 1$. So, for every finite set $\{\varphi_1,\dots,\varphi_m\} \subset Y$, there exist an element AV_Ng of the image of the unit ball $A\,(B_X)$ that lies at a distance greater than $\|A\|_{\mathcal{B}(X,Y)} - 3\varepsilon$ from every element of $\{\varphi_1,\dots,\varphi_m\}$. This means that $A(B_X)$ cannot be covered by a finite family of open balls of radius $\|A\|_{\mathcal{B}(X,Y)} - 3\varepsilon$. Hence, for all $\varepsilon > 0$,

$$\|A\|_{\mathcal{B}(X,Y),\chi} \geq \|A\|_{\mathcal{B}(X,Y)} - 3\varepsilon.$$

Passing to the limit as $\varepsilon \to 0+$, we arrive at (4.1), which completes the proof. ☐

5. Continuous case

In this section, we briefly describe analogues of the above results in the continuous case. For $h \in \mathbb{R}^d$, let $\mathcal{V}_h$ denote the shift (translation) operator $(\mathcal{V}_hf)(x) := f(x-h)$, $x \in \mathbb{R}^d$. Let $X = X(\mathbb{R}^d), Y = Y(\mathbb{R}^d)$ be Banach function spaces (see [11, Ch. 1, Definition 1.1]) over $\mathbb{R}^d$ equipped with the Lebesgue measure. As in the discrete case, one says that the space $X(\mathbb{R}^d)$ is translation-invariant if $\|\mathcal{V}_hf\|_X = \|f\|_X$ for all $f \in X(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$. If both $X(\mathbb{R}^d), Y(\mathbb{R}^d)$ are translation-invariant, then an operator $A \in \mathcal{B}(X,Y)$ is said to be translation-invariant if $A\mathcal{V}_h = \mathcal{V}_hA$ for all $h \in \mathbb{R}^d$. Let $S_0(\mathbb{R}^d)$ denote the set of all simple compactly supported functions. The following result is proved in the same way as Theorem 1.1.

Theorem 5.1. *Let $X = X(\mathbb{R}^d)$, $Y = Y(\mathbb{R}^d)$, and $Z = Z(\mathbb{R}^d)$ be translation-invariant Banach function spaces such that Y is a subset of the closure of $S_0(\mathbb{R}^d)$ in Z. If $A \in \mathcal{B}(X,Y)$ is a translation-invariant operator, then*

$$\|A\|_{\mathcal{B}(X,Y),\chi} = \|A\|_{\mathcal{B}(X,Y),\mathrm{e}} = \|A\|_{\mathcal{B}(X,Y)}.$$

We apply this result to the Hilbert and the Riesz transforms. Iwaniec and Martin [20, Theorem 1.1] (see also [21, Theorem 12.1.1]) calculated the L^p-norms of the Riesz transforms $\mathcal{R}_j$ with $j = 1,\dots,d$ defined by

$$(\mathcal{R}_jf)(x) := c_d\int_{\mathbb{R}^d}\frac{(x_j - y_j)f(y)}{|x-y|^{d+1}}dy,$$

where

$$c_d := \pi^{-(d+1)/2}\Gamma\left(\frac{d+1}{2}\right)$$

and the intergral is understood in the principal value sense. They extended (1.3) to the multidimensional case and proved that

$$\|\mathcal{R}_j\|_{\mathcal{B}(L^p(\mathbb{R}^d))} = \cot(\pi/(2p^*)), \quad j = 1, \dots, d, \quad 1 < p < \infty. \tag{5.1}$$

In the case $d = 1$ and $\mathcal{H} := \mathcal{R}_1$, more is known. First of all, for every $a, b \in \mathbb{C}$, one has

$$\|aI + b\mathcal{H}\|_{\mathcal{B}(L^p(\mathbb{R}))} = \|aI + b\mathcal{H}\|_{\mathcal{B}(L^p(\mathbb{R})),\mathrm{e}}$$

(see [25, Example 4.2]). Hollenbeck and Verbitsky [19] proved that

$$\|(I \pm i\mathcal{H})/2\|_{\mathcal{B}(L^p(\mathbb{R}))} = 1/\sin(\pi/p). \tag{5.2}$$

Further, Hollenbeck, Kalton, and Verbitsky [18, Corollary 4.4] showed that if $1 < p < \infty$ and $a, b \in \mathbb{R}$, then

$$\|aI + b\mathcal{H}\|_{\mathcal{B}(L^p(\mathbb{R}))} = (B_p(a,b))^{1/p}, \tag{5.3}$$

where

$$B_p(a,b) := \max_{x\in\mathbb{R}} \frac{|ax - b + (bx+a)\tan\gamma|^p + |ax - b - (bx+a)\tan\gamma|^p}{|x + \tan\gamma|^p + |x - \tan\gamma|^p}$$

and $\gamma = \pi/(2p)$ (see also [15] for an alternative proof of this result).

Let $X(\mathbb{R}^d)$ be a rearrangement-invariant Banach function space (see [11, Ch. 2, Section 4] or [13]) and let α_X, β_X be its Boyd indices [13] and p_X, q_X be its Zippin indices [31]. We refer to [29, Section 4] for their properties in the case of rearrangement-invariant Banach function spaces over nonatomic measure spaces. It follows from the Boyd interpolation theorem [13, Theorem 1] and (5.1) that the Riesz transforms $\mathcal{R}_j$ with $j = 1, \dots, d$ are bounded on $X(\mathbb{R}^d)$ whenever its Boyd indices satisfy $0 < \alpha_X, \beta_X < 1$. The fact that the converse is also true was mentioned by Boyd in [12, p. 219] without proof. It follows from a more general result for nondegenerate classical Calderón-Zygmund singular integral operators of convolution type [24, Corollary 1.2].

Let $1 < p < \infty$. Following [11, Ch. 3, Definition 1.2], let $L^1(\mathbb{R}^d) + L^p(\mathbb{R}^d)$ be the collection of all measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ that are representable as $f = g+h$ for some $g \in L^1(\mathbb{R}^d)$ and $h \in L^p(\mathbb{R}^d)$. For each function $f \in L^1(\mathbb{R}^d) + L^p(\mathbb{R}^d)$ its norm is defined as

$$\|f\|_{L^1(\mathbb{R}^d)+L^p(\mathbb{R}^d)} = \inf\{\|g\|_{L^1(\mathbb{R}^d)} + \|h\|_{L^p(\mathbb{R}^d)} : \ f = g+h\},$$

where the infimum is taken over all representations $f = g + h$ with $g \in L^1(\mathbb{R}^d)$ and $h \in L^p(\mathbb{R}^d)$.

Arguing similarly to the proof of [28, Lemma 4.2] (see also [4, Remark 4], [5, Remark 4], and Lemma 3.7), one can get the following.

Lemma 5.2. *Let $1 < p < \infty$. If $X(\mathbb{R}^d)$ is a rearrangement-invariant Banach function space with the lower Zippin index satisfying $p_X > 1/p$, then*

$$X(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d) + L^p(\mathbb{R}^d).$$

Remark 5.3. In fact, using techniques of the proof of [28, Lemma 4.2], one can prove a stronger result, substituting the Lebesgue space $L^p(\mathbb{R}^d)$ by the strictly smaller Lorentz space $L^{p,1}(\mathbb{R}^d)$.

Similarly to the discrete case, we obtain the following as a consequence of Theorem 5.1 and Lemma 5.2.

Corollary 5.4. *Let $X = X(\mathbb{R}^d)$ be a rearrangement-invariant Banach function space with the Boyd indices satisfying $0 < \alpha_X,\ \beta_X < 1$. For all $a, b \in \mathbb{C}$ and $j = 1, \dots, d$, one has*

$$\|aI + b\mathcal{R}_j\|_{\mathcal{B}(X),\chi} = \|aI + b\mathcal{R}_j\|_{\mathcal{B}(X),\mathrm{e}} = \|aI + b\mathcal{R}_j\|_{\mathcal{B}(X)}.$$

Proof. It follows from Lemma 5.2 and the inequality $\alpha_X \le p_X$ (see [29, inequalities (4.14)]) that $X(\mathbb{R}^d) \hookrightarrow Z(\mathbb{R}^d) = L^1(\mathbb{R}^d) + L^p(\mathbb{R}^d)$ for $p \in (1/\alpha_X, \infty)$. It is clear that $Z(\mathbb{R}^d)$ is translation-invariant because both $L^1(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$ are translation-invariant. If $f \in Z(\mathbb{R}^d)$, then there exist $g \in L^1(\mathbb{R}^d)$ and $h \in L^p(\mathbb{R}^d)$ such that $f = g + h$. Fix $\varepsilon > 0$. Since $S_0(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$ and in $L^p(\mathbb{R}^d)$, there exist $\varphi, \psi \in S_0(\mathbb{R}^d)$ such that

$$\|g - \varphi\|_{L^1(\mathbb{R}^d)} < \varepsilon/2, \quad \|h - \psi\|_{L^p(\mathbb{R}^d)} < \varepsilon/2.$$

Therefore,

$$\|f - (\varphi + \psi)\|_{Z(\mathbb{R}^d)} \le \|g - \varphi\|_{L^1(\mathbb{R}^d)} + \|h - \psi\|_{L^p(\mathbb{R}^d)} < \varepsilon.$$

Since $\varphi + \psi \in S_0(\mathbb{R}^d)$, we conclude that $S_0(\mathbb{R}^d)$ is dense in $Z(\mathbb{R}^d)$. It remains to apply Theorem 5.1 to the translation-invariant operators $T_j = aI + b\mathcal{R}_j$ with $j = 1, \dots, d$. ☐

Combining Corollary 5.4 with (5.1)–(5.3), we see that for $1 < p < \infty$,

$$\|(I \pm i\mathcal{H})/2\|_{\mathcal{B}(L^p(\mathbb{R})),\chi} = \|(I \pm i\mathcal{H})/2\|_{\mathcal{B}(L^p(\mathbb{R})),\mathrm{e}} = 1/\sin(\pi/p), \tag{5.4}$$

$$\|aI + b\mathcal{H}\|_{\mathcal{B}(L^p(\mathbb{R})),\chi} = \|aI + b\mathcal{H}\|_{\mathcal{B}(L^p(\mathbb{R})),\mathrm{e}} = \left(B_p(a,b)\right)^{1/p}, \quad a, b \in \mathbb{R}, \tag{5.5}$$

and

$$\|\mathcal{R}_j\|_{\mathcal{B}(L^p(\mathbb{R}^d)),\chi} = \|\mathcal{R}_j\|_{\mathcal{B}(L^p(\mathbb{R}^d)),\mathrm{e}} = \cot(\pi/(2p^*)), \quad j = 1, \dots, d. \tag{5.6}$$

It seems that none of the equalities in (5.4)–(5.6) was noticed before.

References

[1] R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina and B. N. Sadovskiĭ, *Measures of Noncompactness and Condensing Operators*, Operator Theory: Advances and Applications, vol. 55, Birkhäuser Verlag, Basel, 1992.

[2] K. F. Andersen, *Discrete Hilbert transformations on rearrangement-invariant sequence spaces*, ProQuest LLC, Ann Arbor, MI, 1970, Thesis (Ph.D.)–University of Toronto (Canada).

[3] K. F. Andersen, *Discrete Hilbert transforms and rearrangement invariant sequence spaces*, Applicable Anal. **5** (1975/76), 193–200.

[4] S. V. Astashkin and L. Maligranda, *On interpolation in L_p-spaces*, Mat. Zametki **74** (2003), 782–786.

[5] S. V. Astashkin and L. Maligranda, *Interpolation between L_1 and L_p, $1 < p < \infty$*, Proc. Amer. Math. Soc. **132** (2004), 2929–2938.

[6] J. M. Ayerbe Toledano, T. Domínguez Benavides and G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Operator Theory: Advances and Applications, vol. 99, Birkhäuser Verlag, Basel, 1997.

[7] R. Bañuelos, D. Kim and M. Kwaśnicki, *Sharp ℓ^p inequalities for discrete singular integrals*, *arXiv:2209.09737 [math.PR]*, 2022.

[8] R. Bañuelos and M. Kwaśnicki, *On the ℓ^p-norm of the discrete Hilbert transform*, Duke Math. J. **168** (2019), 471–504.

[9] R. Bañuelos and M. Kwaśnicki, *The ℓ^p norm of the Riesz–Titchmarsh transform for even integer p*, *arXiv:2210.00027 [math.CA]*, 2022.

[10] J. Banaś and M. Mursaleen, *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*, Springer, New Delhi, 2014.

[11] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics, vol.129, Academic Press, Inc., Boston, MA, 1988.

[12] D. W. Boyd, *Spaces between a pair of reflexive Lebesgue spaces*, Proc. Amer. Math. Soc. **18** (1967), 215–219.

[13] D. W. Boyd, *Indices of function spaces and their relationship to interpolation*, Canadian J. Math. **21** (1969), 1245–1254.

[14] A. P. Calderón and A. Zygmund, *Singular integrals and periodic functions*, Studia Math. **14** (1955), 249–271.

[15] Y. Ding, L. Grafakos and K. Zhu, *On the norm of the operator $aI + bH$ on $L^p(\mathbb{R})$*. Bull. Korean Math. Soc. **55** (2018), 1209–1219.

[16] I. Gohberg and N. Krupnik, *One-Dimensional Linear Singular Integral Equations. Vol. II. General theory and Applications*, Operator Theory: Advances and Applications. vol. 54, Birkhäuser Verlag, Basel, 1992.

[17] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, American Mathematical Society Colloquium Publications, Vol. 31. American Mathematical Society, Providence, R.I., 1957.

[18] B. Hollenbeck, N. J. Kalton and I. E. Verbitsky, *Best constants for some operators associated with the Fourier and Hilbert transforms*, Studia Math. **157** (2003), 237–278.

[19] B. Hollenbeck and I. E. Verbitsky, *Best constants for the Riesz projection*, J. Funct. Anal. **175** (2000), 370–392.

[20] T. Iwaniec and G. Martin, *Riesz transforms and related singular integrals*, J. Reine Angew. Math. **473** (1996), 25–57.

[21] T. Iwaniec and G. Martin, *Geometric Function Theory and Non-Linear Analysis*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2001.

[22] A. Karlovich and E. Shargorodsky, *When does the norm of a Fourier multiplier dominate its L^∞ norm?*, Proc. Lond. Math. Soc. (3) **118** (2019), 901–941.

[23] O. Karlovych and E. Shargorodsky, *On the essential norms of singular integral operators with constant coefficients and of the backward shift*, Proc. Amer. Math. Soc. Ser. B **9** (2022), 60–70.

[24] O. Karlovych and E. Shargorodsky, *Remark on singular integral operators of convolution type on rearrangement-invariant spaces Banach function spaces*, Real Anal. Exchange **48** (2023), 139–148.

[25] N. Krupnik, *Banach Algebras with Symbol and Singular Integral Operators*, Operator Theory: Advances and Applications, vol. 26, Birkhäuser Verlag, Basel, 1987.

[26] N. Krupnik, *Survey on the best constants in the theory of one-dimensional singular integral operators*, in: Topics in Operator theory. Volume 1. Operators, Matrices and Analytic Functions, Oper. Theory Adv. Appl., vol. 202, Birkhäuser Verlag, Basel, 2010, pp. 365–393.

[27] A. Lebow and M. Schechter, *Semigroups of operators and measures of noncompactness*, J. Functional Analysis **7** (1971), 1–26.

[28] L. Maligranda, *A generalization of the Shimogaki theorem*, Studia Math. **71** (1981/82), 69–83.

[29] L. Maligranda, *Indices and interpolation*, Dissertationes Math. (Rozprawy Mat.) **234** (1985): 49.

[30] M. Riesz, *Sur les fonctions conjuguées*, Math. Z. **27** (1927), 218–244.

[31] M. Zippin, *Interpolation of operators of weak type between rearrangement invariant function spaces*, J. Functional Analysis **7** (1971), 267–284.

Manuscript received August 9 2022
revised December 16 2022

O. Karlovych
Centro de Matemática e Aplicações, Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2829–516 Caparica, Portugal
E-mail address: oyk@fct.unl.pt

E. Shargorodsky
Department of Mathematics, King's College London, Strand, London WC2R 2LS, United Kingdom
E-mail address: eugene.shargorodsky@kcl.ac.uk