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HÖLDER REGULARITY FOR DOMAINS OF FRACTIONAL POWERS OF ELLIPTIC OPERATORS WITH MIXED BOUNDARY CONDITIONS

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Abstract. This work is about global Hölder regularity for solutions to elliptic partial differential equations subject to mixed boundary conditions on irregular domains. There are two main results. In the first, we show that if the domain of the realization of an elliptic differential operator in a negative Sobolev space with integrability $q > d$ embeds into a space of Hölder continuous functions, then so do the domains of suitable fractional powers of this operator. The second main result then establishes that the premise of the first is indeed satisfied. The proof goes along the classical techniques of localization, transformation and reflection which allows to fall back to the classical results of Ladyzhenskaya and Kinderlehrer-Stampacchia. One of the main features of our approach is that we do not require Lipschitz charts for the Dirichlet boundary part, but only an intriguing metric/measure-theoretic condition on the interface of Dirichlet- and Neumann boundary parts. A similar condition was posed in a related work by ter Elst and Rehberg in 2015 [10], but the present proof is much simpler, if only restricted to space dimension up to 4.

1. Introduction

In this paper, we consider global Hölder regularity for solutions to elliptic partial differential equations subject to mixed boundary conditions on irregular domains, in the exemplary form

$$\left.\begin{aligned} -\operatorname{div}(\mu\nabla u) + u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } D \subseteq \partial\Omega, \\ \nabla u \cdot \nu &= g \quad \text{on } N \coloneqq \partial\Omega \setminus D \end{aligned}\right\} \tag{1.1}$$

for a bounded open set $\Omega \subseteq \mathbb{R}^d$ with the unit outer normal ν at N, a bounded and elliptic coefficient function μ taking its values in $\mathbb{R}^{d\times d}$, and integrable functions f on Ω and g on N. It is well known that Hölder continuity is a natural regularity class for solutions to elliptic problems such as (1) and Hölder-equicontinuous sets of functions are precompact in the space of uniformly continuous functions by the Arzelà-Ascoli theorem. Such properties are, aside from intrinsic value, invaluable in the treatment of nonlinear problems. It is thus not surprising that this is a well researched subject and affirmative results are known even in the case of irregular

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domains and mixed boundary conditions with very weak compatibility conditions as established for example in [10].

The intention of this paper is essentially twofold: Firstly, we prove that if the domain $\mathrm{dom}(\mathcal{A}_q + 1)$ of the functional-analytic realization $\mathcal{A} + 1$ of the elliptic differential operator in (1.1) in a negative Sobolev space $W^{-1,q}_D(\Omega)$ embeds into a space of Hölder-continuous functions, then so does the domain $\mathrm{dom}((\mathcal{A}_q + 1)^\sigma)$ of a fractional power of $\mathcal{A} + 1$ when $\sigma > \frac{1}{2} + \frac{d}{2q}$. (We will introduce all objects properly in the main text below.) It is well known that $q > d$ is the expected condition in this context. This is done under the quite general assumption that $\overline{N}$ admits bi-Lipschitzian boundary charts and D is Ahlfors regular; the coefficient function μ is not supposed to be more than measurable, bounded and elliptic. (See Assumption 2.1 below.) The main motivation for this result are semilinear parabolic problems, since it is well known that since the semigroup associated to $\mathcal{A}_q + 1$ will be analytic, the domain $\mathrm{dom}((\mathcal{A}_q + 1)^\sigma)$ will be a natural phase space, see e.g. [23, Ch. 6.3]. We will come back to this below in a bit more detail.

Secondly, we consider a framework where the assumption of the first part is in fact satisfied; that is, we show that $\mathrm{dom}(\mathcal{A}_q + 1)$ indeed embeds into a Hölder space. This framework will essentially encapsulate the geometric assumptions from the first part, together with a classical assumption preventing outward cusps for D, and an intriguing metric/measure-theoretic condition for the interface of D and N, the Dirichlet- and Neumann boundary parts, which will ultimately allow to show that also at this interface, we can transform the problem under consideration to one which satisfies the foregoing classical assumption. (See Assumption 4.1 below.) To this end, we revisit [10] where the associated result was already established by means of Morrey-Campanato spaces and De Giorgi estimates for all spatial dimensions d including uniformity in the data, see [10, Thm. 6.8]. These instruments are both quite natural and powerful, but also rather involved. However, for spatial dimensions d up to 4 one can avoid this machinery and rely on the classical results on Hölder continuity for solutions of the pure Dirichlet problem by Ladyzhenskaya and Stampacchia by much simpler technical means. We carry out this simplified approach here. It is a drawback of this method that we cannot reproduce the uniformity of the Hölder estimates with respect to the given geometry class. Nevertheless, for many purposes this is not needed, like for example for the results in the first part of this paper.

Motivation. It was already mentioned above that one of the main motivations to consider Hölder regularity for $\mathrm{dom}(\mathcal{A}_q + 1)$ and associated domains of fractional powers comes from semilinear parabolic equations. Indeed, consider the following abstract one, posed in some Banach space X:

$$u'(t) + \mathcal{A}u(t) + u(t) = F(t, u(t)), \quad u(0) = u_0, \tag{1.2}$$

where $\mathcal{A} + 1$ is the realization of an elliptic operator such as the one in (1.1) in X. The way to treat such a problem by means of analytic semigroups is well established by now under weak assumptions on F, which require that the coordinate mappings $t \mapsto F(t, v)$ for fixed v and $v \mapsto F(t, v)$ for fixed t are reasonably well behaved, cf. [23, Ch. 6.3], the latter usually referring to Lipschitz continuity on bounded sets of the domain of a fractional power of $\mathcal{A}+1$. A particular interesting and relevant case is that of Nemytskii operators induced by scalar functions; these for example occur naturally in the form of polynomials in reaction-diffusion problems. For this general framework, the choice of the Banach space X is crucial. In fact, in the most prominent case $X = L^2(\Omega)$ and space dimensions up to 3, one can show that not only the domain of the elliptic operator $\mathcal{A}+1$ in $L^2(\Omega)$ embeds into $L^\infty(\Omega)$, but already the domain of a fractional power does so. This is established in an even more general geometric context than the present one, but only for a symmetric coefficient matrix μ, in [11]; for the general case, see also [22, Ch. 6.1] and Corollary 3.4 below. Since bounded functions are, essentially, ignorant of growth induced by a Nemytskii operator, such an embedding allows to consider very rough nonlinearities F induced by Nemytskii operators.

However, this strong property comes at a price, namely that a realization of $\mathcal{A}+1$ in $L^2(\Omega)$ implicitly restricts the considered problem to a strong interpretation with homogeneous Neumann boundary conditions. But this setup is in general insufficient for more sophisticated problems arising in real world applications. This already concerns nonhomogeneous Neumann boundary data. But also, consider for example a (two-dimensional) surface S in the (closure of the) domain $\Omega \subset \mathbb{R}^3$ and let $\mathcal{H}_2|_S$ be the induced two-dimensional surface measure. Let ϕ be a scalar and locally Lipschitz function and let Φ be the associated Nemytskii operator. Suppose that F in (1.2) is given by $v \mapsto \Phi(v)\mathcal{H}_2|_S$. Such a term would correspond to a nonlinear modulation for a jump-type condition for the solution $u(t)$ along S in a strong problem formulation, and, indeed, such conditions appear for example in the analysis of the semiconductor equations if surface charge densities, concentrated on S, are involved, see [7, 21] for a recent analytical treatment; see also [25, 26] for more physical background. (In this particular example, there are also nonlinear modulations on the boundary.)

Clearly, in such a setup, it is not sufficient to have $\operatorname{dom}((\mathcal{A}+1)^\sigma) \hookrightarrow L^\infty(\Omega)$ only, since this will in general not be enough to interpret, much less control, $\Phi(v)$ on the lower-dimensional surface S in dependence of $v \in \operatorname{dom}((\mathcal{A}+1)^\sigma)$. Alternatively, one could try to rely on trace operators to have a good control on $v \in L^r(S;\mathcal{H}_2)$ and then $\Phi(v)$ for r large enough in dependence on the growth conditions of ϕ. But this in turn would require to pass through a Sobolev space $W^{s,p}_D(\Omega)$ with $s > 1/p$ and justifying such a setup might be quite hard if one goes away from $(s,p) = (1,2)$, whereas the latter is rather limited, at least for $d = 3$.

From our point of view, it is thus preferable to rely on Hölder continuity for the domain of a fractional power of $\mathcal{A}+1$. Then elements from such a domain are well defined on any subset of $\overline{\Omega}$ and, as mentioned above, there are even compactness properties to exploit. It turns out that the negative Sobolev space $W^{-1,q}_D(\Omega)$, which is the (anti-) dual of $W^{1,q'}_D(\Omega)$, with $q > d$, provides the adequate functional-analytic framework X to obtain this Hölder continuity for the domain of a fractional power of the X-realization of $\mathcal{A}+1$, and then treat problems such as (1.2) with inhomogeneous data on lower-dimensional surfaces in $\overline{\Omega}$, be that $\partial\Omega$ or S. Indeed, negative Sobolev spaces are capable of representing distributional objects such as induced by inhomogenenous data on lower-dimensional surfaces, and as already mentioned above, it is well known that $q > d$ is the natural threshold for which one can obtain bounded or even continuous functions as elements of the domain of the associated realization $\mathcal{A}_q + 1$, that is, for solutions u to the abstract problem $(\mathcal{A}_q + 1)u = f$ with $f \in W_D^{-1,q}(\Omega)$.

Context. As explained above, Hölder regularity for elliptic problems such as (1.1) is a classical and ubiquitous subject in the regularity theory for partial differential equations. We locate our work between [10] with essentially the same, extremely general geometric setup but a much more sophisticated and involved machinery to achieve the desired result with uniformity of constants in the data in *any* space dimension, and [17], where the less general framework of Gröger regularity is used for dimensions up to $d = 4$. The technique of the present work, in terms of localization of an elliptic problem (1.1) and associated transformation to regular sets plus a possible reflection argument, is similar to the one employed in [17], but deviates from there along the different assumptions on D. We note also that while there is no uniformity statement in [17], there is the recent preprint [8] in which the authors there trace the constants in [17] to obtain a uniform result, which is then even transferred to solutions of parabolic problems. In all mentioned works, the coefficient function is also only assumed to be measurable, bounded and elliptic, as in the present one.

Overview. We set the stage with notation and the introduction of function spaces and differential operators with some associated properties in Section 2. Section 3 then deals with the first main result, Theorem 3.1: if the domain of the $W_D^{-1,q}(\Omega)$-realization of $\mathcal{A} + 1$ embeds into a Hölder space, then so does the domain of a fractional power. The proof is based on ultracontractivity of the semigroups associated to the $L^p(\Omega)$-realization of $\mathcal{A} + 1$, which we transfer to the negative Sobolev scale via the Kato square root property. Section 4 then deals with showing that the premise of the foregoing part is in fact satisfied in a wide geometric setting in Theorem 4.2. For this result, the proof is somewhat extensive. We thus prepare it with a series of preliminary results on the techniques of localization, transformation and reflection in Section 4.1 before proceeding to the actual meat of the proof in Section 4.2.

2. Preliminaries

We first clarify some basic notation. The spatial dimension will be $d > 1$. For $\mathrm{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $r > 0$ we denote the open ball around x with radius r by $B_r(\mathrm{x})$. The d-dimensional Lebesgue measure in $\mathbb{R}^d$ will be written as λ_d and $\omega_d = \lambda_d(B_1(0))$ means the volume of the unit ball. Given a normed vector space V, we denote by V^* the Banach space of *antilinear* continuous functionals on V. Finally, we use the convention of a generic constant c that may vary from occurence to occurence but never depends on the free variables in the actual context. All other notation will be standard.

2.1. **Function spaces.**

Let Λ be a nonempty, bounded open subset of $\mathbb{R}^d$ and let $F \subseteq \partial\Lambda$ be a closed subset of its boundary. Then, for $q \in [1, \infty]$, the first-order

Sobolev space $W^{1,q}(\Lambda)$ is given by the set of $L^q(\Lambda)$ functions with weak first-order derivatives in $L^q(\Lambda)$. We set

$$C_F^\infty(\Lambda) := \Big\{u|_\Lambda \colon u \in C_c^\infty(\mathbb{R}^d) \text{ with } \operatorname{supp}(u) \cap F = \emptyset\Big\}$$

and we use this space to define the first-order *Sobolev space with mixed boundary conditions* $W_F^{1,q}(\Lambda)$ as the closure of $C_F^\infty(\Lambda)$ in $W^{1,q}(\Lambda)$. Furthermore, by $W_F^{-1,q}(\Lambda) := W_F^{1,q'}(\Lambda)^*$ we denote the space of continuous *antilinear* functionals on $W_F^{1,q'}(\Lambda)$, where (here and in all what follows) $1/q + 1/q' = 1$. Finally, as commonly used we write $W_0^{1,q}(\Lambda)$ for $W_{\partial\Lambda}^{1,q}(\Lambda)$ and $W^{-1,q}(\Lambda)$ for $W_{\partial\Lambda}^{-1,q}(\Lambda)$.

For $\alpha \in (0,1)$, let $C^\alpha(\Lambda)$ denote the usual spaces of bounded and α-Hölder continuous functions on Λ with their norm given by the sum of the supremum norm and the Hölder seminorm. Of course, every function in $C^\alpha(\Lambda)$ admits a unique α-Hölder continuous extension to $\overline{\Lambda}$, so we will not discriminate between a Hölder-function on Λ and $\overline{\Lambda}$.

2.2. **Geometric setup.** We next introduce some geometric assumptions on the spatial domain Ω. Throughout the article, Ω denotes a given nonempty bounded open subset of $\mathbb{R}^d$ and $D \subseteq \partial\Omega$ is a closed portion of its boundary, the designated Dirichlet boundary part. We do not exclude that $\mathcal{H}_{d-1}(D) = 0$, the $(d-1)$-dimensional Hausdorff measure. The Neumann boundary part shall be denoted by $N := \partial\Omega \setminus D$.

Assumption 2.1. We consider the following geometric assumptions for Ω and D:

(a) For all $\mathrm{x} \in \overline{N}$, there is an open neighbourhood V_x and a bi-Lipschitz mapping ϕ_x from a neighbourhood of $\overline{V_\mathrm{x}}$ into $\mathbb{R}^d$ such that $\phi_\mathrm{x}(V_\mathrm{x}) = (-1,1)^d$, $\phi_\mathrm{x}(\Omega \cap V_\mathrm{x}) = \{\mathrm{x} \in (-1,1)^d \colon x_d < 0\}$, $\phi_\mathrm{x}(\partial\Omega \cap V_\mathrm{x}) = \{\mathrm{x} \in (-1,1)^d \colon x_d = 0\}$ and $\phi_\mathrm{x}(\mathrm{x}) = 0$.

(b) D is a $(d-1)$-set, i.e., there are constants $c_1, c_2 > 0$ such that for all $r \in (0,1]$ and all $\mathrm{x} \in D$ there holds

$$c_1 r^{d-1} \le \mathcal{H}_{d-1}\big(B_r(\mathrm{x}) \cap D\big) \le c_2 r^{d-1}$$

where $\mathcal{H}_{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure.

Remark 2.2. In Assumption 2.1 (a), for $\mathrm{x} \in N = \partial\Omega \setminus D$ one may assume without loss of generality that the local Neumann boundary part around x is transformed to the full midplate of the cube, that is, $\phi_\mathrm{x}(N \cap V_\mathrm{x}) = \{\mathrm{x} \in (-1,1)^d \colon x_d = 0\}$. In fact, since N is a (relatively) open subset of $\partial\Omega$, the image $\phi_\mathrm{x}(N \cap V_\mathrm{x})$ is a (relatively) open subset of $\{\mathrm{x} \in (-1,1)^d \colon x_d = 0\}$ that contains 0. Thus, one may shrink V_x to a suitable set $\phi_\mathrm{x}^{-1}((-\varepsilon,\varepsilon)^d)$ and afterwards rescale ϕ_x to $\frac{1}{\varepsilon}\phi_\mathrm{x}$.

Already the geometric setup of Assumption 2.1 (a) allows to construct a continuous linear extension operator for first-order Sobolev spaces with mixed boundary conditions. Indeed, the following result can be found in [4, Thm. 1.2 and Prop. 3.4]:

Proposition 2.3. *Suppose that Ω and D meet Assumption 2.1 (a). Then there exists a continuous extension operator from $W_D^{1,1}(\Omega)$ to $W^{1,1}(\mathbb{R}^d)$ that restricts to a continuous operator from $W_D^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^d)$ for all $p \in [1,\infty)$.*

Remark 2.4. Proposition 2.3 allows to establish the usual Sobolev embeddings, that is, $W^{1,q}_D(\Omega) \hookrightarrow L^p(\Omega)$ for $\frac{1}{p} \geq \frac{1}{q} - \frac{1}{d}$ if $q < d$ and $W^{1,q}_D(\Omega) \hookrightarrow C^\alpha(\Omega)$ for $\alpha \leq 1 - \frac{d}{q}$ if $q > d$, in a straightforward manner, including compactness for the strict cases.

In particular, for $d > 2$ the form domain $V = W^{1,2}_D(\Omega)$ is embedded into $L^{\frac{2d}{d-2}}(\Omega)$, and in the case $d = 2$ it embeds into $L^p(\Omega)$ for every $p < \infty$.

2.3. **Elliptic operators.** We define elliptic operators via the form $\mathfrak{t}$ on $V := W^{1,2}_D(\Omega)$ given by

$$\mathfrak{t}(u, v) := \int_\Omega \mu \nabla u \cdot \nabla \overline{v}, \qquad u, v \in V.$$

Here, μ is a real, measurable, bounded and uniformly elliptic coefficient function in the sense that there exists some $\kappa_{\mathrm{ell}} > 0$ such that $(\mu(\mathrm{x})\xi, \xi)_{\mathbb{R}^d} \geq \kappa_{\mathrm{ell}}|\xi|^2$ for all $\xi \in \mathbb{R}^d$ and almost all $\mathrm{x} \in \Omega$. Clearly, the form $\mathfrak{t}$ induces a natural operator $\mathcal{A}\colon V \to V^*$. For $q > 2$, let $\mathcal{A}_q$ be the part of $\mathcal{A} = \mathcal{A}_2$ in $W^{-1,q}_D(\Omega) \subset V^*$. By the Lax-Milgram lemma, $\mathcal{A} + \lambda$ is a topological isomorphism between V and V^* for every λ with $\operatorname{Re}\lambda > 0$.

On the other hand, $\mathfrak{t}$ also induces an operator A on $L^2(\Omega)$ by

$$\begin{aligned} \operatorname{dom} A &:= \Big\{ u \in V \colon \text{there is } f \in L^2(\Omega) \colon \mathfrak{t}(u, v) = (f, v)_{L^2(\Omega)} \text{ for all } v \in V \Big\} \\ Au &:= f, \text{ for } u \in \operatorname{dom} A. \end{aligned}$$

Since $\mathfrak{t}$ is $L^2(\Omega)$-elliptic, it is nowadays classical (e.g. [22, Thms. 1.54, 4.2 and 4.9]) that $-A$ is the generator of a contractive analytic C_0-semigroup (e^{-At}) on $L^2(\Omega)$ which is both sub-Markovian and substochastic, that is, positivity preserving and $L^\infty(\Omega)$- and $L^1(\Omega)$-contractive, from which we obtain the semigroup on every $L^p(\Omega)$ for $p \in [1, \infty]$ by interpolation.

These semigroups are contractive for all $p \in [1, \infty]$, they are strongly continuous for $p \in [1, \infty)$, and they are analytic for $p \in (1, \infty)$, see [22, Prop. 3.12, p.56/57&96]. We denote the respective (negative) generators on $L^p(\Omega)$ by A_p. Note that $\sigma(A_p) \cap [\operatorname{Re} z < 0] = \emptyset$ for every $p \in [1, \infty)$ by the Hille-Yosida theorem, and that the operators admit a bounded H^∞ functional calculus ([6, Cor. 3.9]) for $p > 1$; in particular, their fractional powers are well defined. Moreover, for $p > 2$, the operators A_p are the part of $A = A_2$ in $L^p(\Omega)$.

All the properties mentioned so far do not require *any* regularity assumption on Ω. Under the geometric assumptions from Assumption 2.1, however, we can say a bit more about the operators $\mathcal{A}_q$. Indeed, with Assumption 2.1 (a), we have $\sigma(\mathcal{A}_q) \cap [\operatorname{Re} z < 0] = \emptyset$ for $q \geq 2$ due to Sobolev embeddings via Proposition 2.3 and the restriction of the space dimension $d \leq 4$. Moreover, several of the good properties of A_q can be transferred to $\mathcal{A}_q$ by means of the square root in this setting, which we do next.

Proposition 2.5. *Let $q \in [2, \infty)$ and adopt Assumption 2.1. Then the following hold true.*

(a) The inverse square root operator $(\mathcal{A}_q + 1)^{-1/2}$ provides a topological isomorphism between $W^{-1,q}_D(\Omega)$ and $L^q(\Omega)$.

(b) The negative of the operator $\mathcal{A}_q$ generates an analytic semigroup on $W^{-1,q}_D(\Omega)$.

(c) For $s \in [0, \frac{1}{2})$, we have $\operatorname{dom}\big((\mathcal{A}_q + 1)^{1/2+s}\big) = \operatorname{dom}\big((A_q + 1)^s\big)$.

Proof. In [5, Thm. 1.1] it is proved that $A + 1$ has the Kato square root property in the present geometric setting. (And even beyond that.) Using this fundamental property, the claim (a) is one of the main results in [2], see Theorem 5.1 there. Further, since $(\mathcal{A}_q + 1)^{-1}$ and $(A_q + 1)^{-1}$ coincide on $L^q(\Omega)$, so do the inverse square roots, and we have the similarity

$$(\mathcal{A}_q + \lambda)^{-1} = (\mathcal{A}_q + 1)^{1/2}(A_q + \lambda)^{-1}(\mathcal{A}_q + 1)^{-1/2}.$$

Hence, we can transfer the generator property for an analytic semigroup from $-A_q$ to $-\mathcal{A}_q$ by means of resolvent estimates, see the characterization in [12, Thm. II.4.6]. (Note that we do not claim the semigroups generated by $-\mathcal{A}_q$ to be contractive.) This implies (b). Finally, the fractional powers of $\mathcal{A}_q$ are well defined since the bounded H^∞ calculus also transfers from A_q to $\mathcal{A}_q$ by means of the square root ([2, Thm. 11.5]). Then, (c) follows immediately from (a) since

$$\begin{aligned}\operatorname{dom}\big((\mathcal{A}_q + 1)^{1/2+s}\big) &= \big(\mathcal{A}_q + 1\big)^{-1/2-s} W_D^{-1,q}(\Omega)\\ &= \big(\mathcal{A}_q + 1\big)^{-s} L^q(\Omega)\\ &= \big(A_q + 1\big)^{-s} L^q(\Omega) = \operatorname{dom}\big((A_q + 1)^s\big).\end{aligned}$$

□

3. Embeddings for domains of fractional powers of $\mathcal{A}_q + 1$

In this section we show that if the domain of $\mathcal{A}_q + 1$ embeds into a Hölder space, so do suitable fractional powers of this operator. We remark on the domain of A_q after the proof of Theorem 3.1. The question of *when* the domain of $\mathcal{A}_q + 1$ actually embeds into a Hölder space will be considered in Section 4.

Theorem 3.1. *Let Assumption 2.1 be satisfied and let $q > d$. Suppose that $\operatorname{dom}(\mathcal{A}_q+1) \hookrightarrow C^\alpha(\Omega)$ for some $\alpha > 0$. Let $\kappa \in (0, \alpha)$ and $\sigma \in \big(\frac{1}{2} + \frac{d}{2q} + \frac{\kappa}{\alpha}(\frac{1}{2} - \frac{d}{2q}), 1\big)$. Then we have*

$$\big(W_D^{-1,q}(\Omega), \operatorname{dom}(\mathcal{A}_q + 1)\big)_{\sigma,1} \hookrightarrow C^\kappa(\Omega) \quad \textit{and} \quad \operatorname{dom}\big((\mathcal{A}_q + 1)^\sigma\big) \hookrightarrow C^\kappa(\Omega).$$

Before we start with the proof, a short remark:

Remark 3.2. Via Proposition 2.5, we also obtain from Theorem 3.1 that

$$\operatorname{dom}((A_q + 1)^\varsigma) \hookrightarrow C^\kappa(\Omega)$$

for $\varsigma \in \big(\frac{d}{2q} + \frac{\kappa}{\alpha}(\frac{1}{2} - \frac{d}{2q}), \frac{1}{2}\big)$. This is interesting because there is a natural connection between embeddings of the domain of a fractional power of $A_q + 1$ into a Hölder space and the Hölder continuity of the heat kernel associated to the semigroup generated by the negative of $A_q + 1$. We refer to [22, Ch. 6.2] and leave the details to the interested reader.

Our proof of Theorem 3.1 is based on ultracontractivity of semigroups generated by $-A_q$. We use ultracontractivity to derive a precise regularizing property for inverse fractional powers of $A_q + 1$ and then in turn transfer this to the $\mathcal{A}_q$ operator by means of Proposition 2.3.

The semigroups $(e^{-A_p t})$ are said to be *ultracontractive* if there exists a constant $c > 0$ and some $\gamma > 0$ such that

$$\big\|e^{-A_p t}\big\|_{L^p(\Omega)\to L^\infty(\Omega)} \le c t^{-\frac{\gamma}{2p}} \quad \text{for all } t \in (0,1],\ p \in [1,\infty). \tag{3.1}$$

In fact, if $\gamma > 2$, then this property is equivalent to $V \hookrightarrow L^{\frac{2\gamma}{\gamma-2}}(\Omega)$; we refer to [1, Ch. 7.3]. We concentrate on this case, since under the geometric assumptions of Assumption 2.1 (a), Proposition 2.3 provides a Sobolev extension operator from which the foregoing Sobolev embedding for V with $\gamma = d$ if $d > 2$ and any $\gamma > 2$ if $d = 2$ follows immediately as noted in Remark 2.4. This is already the proof of the next proposition:

Proposition 3.3 (Ultracontractivity)**.** *Adopt Assumption 2.1 (a). Then the semigroups* $(e^{-A_p t})$ *are* ultracontractive*, that is, there exists* $c > 0$ *such that* (3.1) *holds true for* $\gamma = d$ *if* $d > 2$ *and* $\gamma > 2$ *arbitrary if* $d = 2$*.*

We infer the following regularizing property for the inverse fractional powers of $A_p + 1$ for $p > d/2$:

Corollary 3.4. *Adopt Assumption 2.1 (a) and let* $p > d/2$*. Then, for every* $\tau \in (\frac{d}{2p}, 1]$*, we find* $(A_p + 1)^{-\tau} \in \mathcal{L}(L^p(\Omega) \to L^\infty(\Omega))$*. In particular,* $\mathrm{dom}((A_p + 1)^\tau) \hookrightarrow L^\infty(\Omega)$*.*

Proof. Consider the well-known Balakrishnan formula

$$(A_p + 1)^{-\tau} = \frac{1}{\Gamma(\tau)} \int_0^\infty t^{\tau-1} e^{-A_p t} e^{-t}\, \mathrm{d}t.$$

We split the integral $\int_0^\infty = \int_0^1 + \int_1^\infty$. From Proposition 3.3 and the growth bound (3.1) for $(e^{-A_p t})$, one observes immediately that the condition $\tau > \frac{d}{2p}$ is sufficient to have the $\int_0^1$ integral converge in $\mathcal{L}(L^p(\Omega) \to L^\infty(\Omega))$. (For $d = 2$, squeeze $\frac{\gamma}{2p}$ between $\frac{d}{2p}$ and τ by picking γ close enough to $d = 2$.) For the $\int_1^\infty$ integral, we note that for $t > 1$, we have $t^{\tau-1} \le 1$ and moreover, due to both ultra- and 'regular' contractivity of the semigroup generated by $-A_p$,

$$\big\|e^{-A_p t}\big\|_{L^p(\Omega)\to L^\infty(\Omega)} \le \big\|e^{-A_p}\big\|_{L^p(\Omega)\to L^\infty(\Omega)} \big\|e^{-A_p(t-1)}\big\|_{L^p(\Omega)\to L^p(\Omega)} \le c$$

with the constant c as in (3.1). Thus the $\int_1^\infty$ integral also converges in $\mathcal{L}(L^p(\Omega) \to L^\infty(\Omega))$ and the claim follows. □

As a last auxiliary result of potentially independent interest, we note the following remarkably simple embedding which holds true for any bounded open set without further assumptions on its geometry:

Lemma 3.5. *Let* $\alpha > 0$*. Then* $(L^\infty(\Omega), C^\alpha(\Omega))_{\theta,1} \hookrightarrow C^{\alpha\theta}(\Omega)$ *for any* $\theta \in (0,1)$*.*

Proof. Let $u \in C^\alpha(\Omega)$ and estimate

$$\begin{aligned} \sup_{\substack{\mathrm{x,y}\in\Omega \\ \mathrm{x}\neq\mathrm{y}}} \frac{|u(\mathrm{x}) - u(\mathrm{y})|}{|\mathrm{x}-\mathrm{y}|^{\alpha\theta}} &\le \sup_{\substack{\mathrm{x,y}\in\Omega \\ \mathrm{x}\neq\mathrm{y}}} |u(\mathrm{x}) - u(\mathrm{y})|^{1-\theta} \sup_{\substack{\mathrm{x,y}\in\Omega \\ \mathrm{x}\neq\mathrm{y}}} \frac{|u(\mathrm{x}) - u(\mathrm{y})|^\theta}{|\mathrm{x}-\mathrm{y}|^{\alpha\theta}} \\ &\le \big(2\|u\|_{L^\infty(\Omega)}\big)^{1-\theta} \|u\|_{C^\alpha(\Omega)}^\theta. \end{aligned}$$

Together with an obvious estimate for $\sup_{\mathrm{x}\in\Omega}|u(\mathrm{x})|$ one gets, for every $u \in C^\alpha(\Omega)$,

$$\|u\|_{C^{\alpha\theta}(\Omega)} \le 3\|u\|_{L^\infty(\Omega)}^{1-\theta}\|u\|_{C^\alpha(\Omega)}^{\theta}.$$

Thus, referring to [24, Lem. 1.10.1], $C^{\alpha\theta}(\Omega)$ is of class $J(\theta)$ with respect to $L^\infty(\Omega)$ and $C^\alpha(\Omega)$ from which we obtain the desired embedding. □

Proof of Theorem 3.1. Set $\theta := \kappa/\alpha \in (0,1)$ and $\sigma \in \bigl(\frac{1}{2} + \frac{d}{2q} + \theta(\frac{1}{2} - \frac{d}{2q}), 1\bigr)$ as in the theorem. A short computation shows that we can write $\sigma = (1-\theta)(\frac{1}{2}+\tau)+\theta$ with some $\tau \in (\frac{d}{2q}, \frac{1}{2})$. Thus, the reiteration theorem ([24, Thm. 1.10.2]) implies that

$$\begin{aligned}\bigl(W_D^{-1,q}(\Omega), \operatorname{dom}(\mathcal{A}_q+1)\bigr)_{\sigma,1} \\ &= \Bigl(\bigl(W_D^{-1,q}(\Omega), \operatorname{dom}(\mathcal{A}_q+1)\bigr)_{\frac{1}{2}+\tau,1}, \operatorname{dom}(\mathcal{A}_q+1)\Bigr)_{\theta,1}.\end{aligned}$$

We show that the first space on the right embeds continuously into $L^\infty(\Omega)$. Indeed, by interpolation for fractional power domains of so-called positive operators as in [24, Thm. 1.15.2], we have

$$\bigl(W_D^{-1,q}(\Omega), \operatorname{dom}(\mathcal{A}_q+1)\bigr)_{\frac{1}{2}+\tau,1} \hookrightarrow \operatorname{dom}((\mathcal{A}_q+1)^{1/2+\tau}).$$

But for $\tau \in (\frac{d}{2q}, \frac{1}{2})$, by combining Proposition 2.5 (c)—this is the point where we need Assumption 2.1 (b)—and Corollary 3.4, we find

$$\operatorname{dom}\bigl((\mathcal{A}_q+1)^{1/2+\tau}\bigr) = \operatorname{dom}\bigl((A_q+1)^{\tau}\bigr) \hookrightarrow L^\infty(\Omega).$$

By assumption, the restriction of the foregoing embedding to $\operatorname{dom}(\mathcal{A}_q+1)$ is precisely $\operatorname{dom}(\mathcal{A}_q+1) \hookrightarrow C^\alpha(\Omega)$. Interpolating these and using Lemma 3.5, we find

$$\bigl(W_D^{-1,q}(\Omega), \operatorname{dom}(\mathcal{A}_q+1)\bigr)_{\sigma,1} \hookrightarrow \bigl(L^\infty(\Omega), C^\alpha(\Omega)\bigr)_{\theta,1} \hookrightarrow C^{\alpha\theta}(\Omega)$$

and this was the claim, since $\alpha\theta = \kappa$.

Now the embedding for $\operatorname{dom}((\mathcal{A}_q+1)^\sigma)$ itself follows easily by squeezing s between $\frac{1}{2}+\frac{d}{2q}+\frac{\kappa}{\alpha}(\frac{1}{2}-\frac{d}{2q})$ and σ and using the previous part via [24, Thms. 1.3.3 and 1.15.2]:

$$\begin{aligned}\operatorname{dom}((\mathcal{A}_q+1)^\sigma) &\hookrightarrow \bigl(W_D^{-1,q}(\Omega), \operatorname{dom}(\mathcal{A}_q+1)\bigr)_{\sigma,\infty} \\ &\hookrightarrow \bigl(W_D^{-1,q}(\Omega), \operatorname{dom}(\mathcal{A}_q+1)\bigr)_{s,1} \hookrightarrow C^\kappa(\Omega).\end{aligned}$$ □

The domain of $\mathcal{A}_q$. In the above proof, we have worked only with $\mathcal{A}_q+1$ to have an invertible operator at hand which is much more convenient. The Banach spaces $\operatorname{dom}(\mathcal{A}_q)$ and $\operatorname{dom}(\mathcal{A}_q+1)$, each with the respective graph norm, are always equivalent as Banach spaces, so Theorem 3.1 is also valid when substituting $\mathcal{A}_q$ for $\mathcal{A}_q+1$. On the other hand, it is often also interesting to consider the operators with the respective 'reduced' graph norm, that is, $\|(\mathcal{A}_q+1)\cdot\|_{W_D^{-1,q}(\Omega)}$ and $\|\mathcal{A}_q\cdot\|_{W_D^{-1,q}(\Omega)}$. (For example, the main result in the upcoming Section 4 involves this norm in a natural way.) These 'reduced' graph norms are equivalent to the full graph norms if the operator in question admits a continuous linear inverse. This is the case for $\mathcal{A}_q+1$, but $\mathcal{A}_q$ might be non-injective; it is injective if $\mathcal{A}$ is.

By the Lax-Milgram lemma, the operator $\mathcal{A}$ in turn is injective whenever we have a Poincaré inequality for V at hand. For the latter it is enough to establish that nonzero constant functions do not belong to V. Within our geometric setup of Assumption 2.1, this is already guaranteed by either $D \cap \overline{N} \neq \emptyset$, so the Dirichlet- and Neumann boundary parts share a common interface, or by D containing at least one (relatively) inner point. See for instance [9, Lemma 7.3]. (In fact, in the former case it is already enough to have Lipschitz charts for all points in the relative boundary ∂D within $\partial\Omega$ at hand; cf. [6, Sect. 6].)

In this sense, the statement for $\mathcal{A}_q + 1$ in Theorem 3.1 can be immediately transferred to $\mathcal{A}_q$ also for the 'reduced' graph norm whenever the geometry assumptions admit a Poincaré inequality for V.

4. Hölder properties for $\mathrm{dom}(\mathcal{A}_q + 1)$

In the main result of Section 3 the embedding of $\mathrm{dom}(\mathcal{A}_q + 1)$ into some Hölder space was a given. We now turn to the question when such an embedding is true. A very general answer was given in [10, Theorem 1.1], where the result in Theorem 4.2 below was proved for all space dimensions d. This proof is extremely involved, the natural instruments being Morrey-Campanato spaces and De Giorgi estimates.

However, for dimensions up to 4 one can avoid this machinery and base the arguments only on the classical Ladyshenskaya result on Hölder continuity for solutions of the pure Dirichlet problem, see Proposition 4.4 below, and some more elementary yet intricate technical means. This is what we will carry out here. The present approach also allows to obtain a uniform result with respect to the $L^\infty(\Omega)$-bound and ellipticity constant κ_{ell} of the coefficient function μ.

In order to formulate our main result of this section, we introduce two more geometric conditions; the first one relies on the rather classical notion with a twist of saying that an open subset Λ of $\mathbb{R}^d$ is *of class* (A_γ) (at $\Upsilon \subseteq \partial\Lambda$) with a constant $\gamma > 0$, if

$$\lambda_d\bigl(B_r(\mathrm{x}) \setminus \Lambda\bigr) \geq \gamma \lambda_d\bigl(B_r(\mathrm{x})\bigr) \qquad \text{for all } \mathrm{x} \in \Upsilon,\ r \in (0,1].$$

Of course, necessarily $\gamma < 1$. This condition prevents inwards cusps of Λ at Υ. If $\Upsilon = \partial\Lambda$, we just refer to Λ being of class (A_γ). The second condition, rather intriguing, concerns the interface between the Dirichlet boundary part D and the Neumann boundary part $N = \partial\Omega \setminus D$ in the boundary of Ω:

Assumption 4.1. We consider the following further geometric assumptions for Ω and D:

(a) There is some $\gamma \in (0,1)$ such that Ω is of class (A_γ) at D.
(b) Using the notation of Assumption 2.1 (a), there are two constants $c_0 \in (0,1)$ and $c_1 > 0$ such that for any point $\mathrm{x} \in E := D \cap \overline{N}$, every $y \in \mathbb{R}^{d-1}$ such that $(\mathrm{y}, 0) \in \phi_{\mathrm{x}}(E \cap V_{\mathrm{x}})$ and every $s \in (0,1]$ it holds

$$\lambda_{d-1}\Bigl(\bigl\{\mathrm{z} \in \mathbf{B}_s(\mathrm{y}) \colon\ \mathrm{dist}\bigl(\mathrm{z}, \phi_{\mathrm{x}}(N \cap V_{\mathrm{x}})\bigr) > c_0 s\bigr\}\Bigr) \geq c_1 s^{d-1}.$$

Here and in the sequel, $\mathbf{B}_r(\mathrm{y})$ denotes the open ball of radius r in $\mathbb{R}^{d-1}$ with its center at $\mathrm{y} \in \mathbb{R}^{d-1}$, and in the distance function we tacitly consider $\phi_x(N \cap V_{\mathrm{x}}) \subset [z_d = 0]$ as a subset of $\mathbb{R}^{d-1}$ in the obvious manner.

We can now formulate the main theorem of this section.

Theorem 4.2. *Suppose that Ω and D satisfy Assumption 2.1 (a) and Assumption 4.1, and let $q > d$ with $d \in \{2,3,4\}$. If $d = 4$, suppose also that Assumption 2.1 (b) is satisfied. Then there is an $\alpha > 0$ such that for every $f \in W_D^{-1,q}(\Omega)$ the equation*

$$(\mathcal{A}_q + 1)v = f \tag{4.1}$$

has a unique solution $v \in W_D^{1,2}(\Omega)$ that belongs to the Hölder space $C^\alpha(\Omega)$. Moreover, the mapping $W_D^{-1,q}(\Omega) \ni f \mapsto v \in C^\alpha(\Omega)$ is continuous and its norm is uniform with respect to the $L^\infty(\Omega)$-bound and ellipticity constant κ_{ell} of μ.

Let us mention that in [10], Assumption 2.1 (b) is not needed at all for the proof of the Hölder continuity ([10, Thm. 6.8]). Here, it is required for a technical step in the localization of (4.1) in space dimension $d = 4$.

Remark 4.3. We comment on Theorem 4.2.

(a) Assumption 2.1 (a) (Lipschitz charts at $\overline{N}$) implies that Ω is also of class (A_γ) at $\overline{N}$. This follows from the fact that bi-Lipschitz mappings preserve the Lebesgue measure up to a constant, see [13, Thm. 2.5 and Thm. 2.8], and compactness of $\overline{N}$. Together with Assumption 4.1 (a), Ω is thus of class (A_γ).

(b) It is well known that, in general, the condition $q > d$ is already necessary for the boundedness of the solution, see [19, Ch. I.2].

(c) It is easily seen that if $f \in L^p(\Omega)$ with $p > d/2$, then also $f \in W_D^{-1,q}(\Omega)$ where $q = 2p > d$ with continuous embedding thanks to Remark 2.4. In this sense, Theorem 4.2 is also a result on Hölder regularity for the operators $A_p + 1$ for $p > d/2$. (Note that so far we had only seen that the $L^p(\Omega)$-solution to (4.1) is in $L^\infty(\Omega)$ via ultracontractivity as in Corollary 3.4—but this was already true for a fractional power of A_p+1 and so some opportunity for improvement for $A_p + 1$ itself was expected.)

Let us sketch an outline for the proof of Theorem 4.2. We will rely on the classical techniques of localization, transformation and reflection to tackle (4.1) in the form of a finite number of similar problems on model sets with a very particular geometry. For these we will rely on classical Hölder regularity results of Ladyzhenskaja or Stampacchia which base on variants of Assumption 4.1 (a). The treatment of local problems at the pure Dirichlet part $D \setminus \overline{N}$ will be quite immediate due to Assumption 4.1 (a), and we will also be able to transfer the Neumann boundary part $N = \partial\Omega \setminus D$ to the pure Dirichlet situation via Assumption 2.1 (a) and reflection techniques. Of course, the most interesting part will be the interface $D \cap \overline{N}$ with Assumption 4.1 (b). The intriguing idea here is that Assumption 4.1 (b) will allow to transform the localized problem once more in a particular way such that the resulting set will in fact be amendable by Assumption 4.1 (a).

4.1. **Localization and transformation techniques.** In this subsection we recall, for the reader's convenience, some technical results on localization and transformation techniques for (4.1) which are needed later on. For all the following considerations the coefficient function μ is considered as in Section 2; in particular it is elliptic with constant κ_{ell}.

We start by quoting a classical theorem (see [18, Ch. II Appendix B/C]) on the Hölder continuity for the solution of the Dirichlet problem. The result is formulated for a generic bounded domain $\Lambda \subset \mathbb{R}^d$ since we will use it for several local model sets in the proof of Theorem 4.2; the definitions of μ and $\mathcal{A}$ are to be understood *mutatis mutandis.*

Proposition 4.4. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain and let $v \in W_0^{1,2}(\Lambda)$ be the solution of*

$$\mathcal{A}v = f_0 + \sum_{j=1}^{d} \frac{\partial f_j}{\partial x_j}, \tag{4.2}$$

where $f_0, f_1, \ldots, f_d \in L^q(\Lambda)$ with $q > d$ and $\frac{\partial}{\partial x_j}$ denotes the distributional derivative. Then the following holds true.

(a) The function v admits a bound

$$\|v\|_{L^\infty(\Lambda)} \leq c \sum_{j=0}^{d} \|f_j\|_{L^q(\Lambda)}. \tag{4.3}$$

(b) Suppose that there exists $\gamma \in (0,1)$ such that Λ is of class (A_γ). Then v is Hölder-continuous, more precisely: there is an $\alpha \in (0,1)$ independent of $f_0, f_1, \ldots, f_d$ such that

$$\sup_{\mathrm{x},\mathrm{y} \in B_r(\mathrm{z}) \cap \Lambda} |v(\mathrm{x}) - v(\mathrm{y})| \leq c \sum_{j=0}^{d} \|f_j\|_{L^q(\Lambda)}\, r^\alpha \tag{4.4}$$

holds true for all $\mathrm{z} \in \mathbb{R}^d$ and $r > 0$.

In both estimates (4.3) and (4.4), the respective constant is uniform in the $L^\infty(\Lambda)$-bound and ellipticity constant of μ.

Remark 4.5. The right hand side of (4.2) is to be understood as the antilinear form

$$W_0^{1,q'}(\Lambda) \ni \psi \mapsto \int_\Lambda f_0 \overline{\psi} - \sum_{j=1}^{d} f_j \frac{\partial \overline{\psi}}{\partial x_j}$$

which clearly belongs to $W^{-1,q}(\Lambda) \hookrightarrow W^{-1,2}(\Lambda)$. Thus, the uniqueness of the solution v follows from the ellipticity of $\mathfrak{t}$ and the Lax-Milgram lemma.

On the other hand, while every antilinear form in $W^{-1,q}(\Lambda)$ can be represented in the foregoing form, this representation is in general non-unique. But it is in fact well known that $W^{-1,q}(\Lambda)$ is isometrically isomorphic to the quotient space with respect to such representations; see [20, Ch. 1.1.14]. Hence, taking the infimum over all representing families in the estimates (4.3) and (4.4), in the setting of Proposition 4.4 one obtains the continuity of

$$\mathcal{A}_q^{-1} \colon W^{-1,q}(\Lambda) \to C^\alpha(\Lambda).$$

The norm of this mapping is uniform in the $L^\infty(\Lambda)$-bound and ellipticity constant of μ.

The following extrapolation of the Lax-Milgram isomorphism will give us the small ε in regularity that allows us to treat also the case of dimension four.

Proposition 4.6 ([15, Thm 5.6]). *Let Assumptions 2.1 (a) and (b) be satisfied. Then there is an $\varepsilon > 0$ such that $\operatorname{dom}(\mathcal{A}_q + 1) = W^{1,q}_D(\Omega)$ for all $q \in [2, 2+\varepsilon)$, that is, the operator*

$$\mathcal{A}_q + 1 \colon W^{1,q}_D(\Omega) \to W^{-1,q}_D(\Omega)$$

is a topological isomorphism. The norms of $(\mathcal{A}_q + 1)^{-1}$ are uniform with respect to ε and the $L^\infty(\Omega)$-bound and ellipticity constant κ_{ell} of μ.

The plan how we aim to prove Theorem 4.2 was already sketched above. We now have seen the main tool with which we leverage Hölder-continuity for the localized and transformed problems in the form of Proposition 4.4. It remains to make sure that the localization, transformation and possibly reflection techniques are compatible with Proposition 4.4; this concerns continuity for the associated mappings between the function spaces involved and of course in particular the assumption in the domain in Proposition 4.4 for the actual Hölder estimate.

This we will do in the following series of technical lemmas. We start with three of them that deal with the localization. Recall the notation $N = \partial\Omega \setminus D$ for the Neumann boundary part. First, we deal with localized Sobolev functions with partially vanishing trace.

Lemma 4.7 ([17, Ch. 4.2]). *Let $U \subseteq \mathbb{R}^d$ be open and set $\Omega_\bullet := \Omega \cap U$ as well as $D_\bullet := \partial\Omega_\bullet \setminus (U \cap N)$. Fix an arbitrary function $\eta \in C^\infty_c(\mathbb{R}^d)$ with $\operatorname{supp}(\eta) \subseteq U$. Then for any $q \in (1, \infty)$ we have the following assertions:*

(a) If $v \in W^{1,q}_D(\Omega)$, then $\eta v|_{\Omega_\bullet} \in W^{1,q}_{D_\bullet}(\Omega_\bullet)$.

(b) Denote by E_0 the zero extension operator and let $f \in W^{-1,q}_D(\Omega)$. Then $f \mapsto f_\bullet$ with

$$f_\bullet \colon w \mapsto \big\langle f, E_0(\eta w)\big\rangle, \quad w \in W^{1,q'}_{D_\bullet}(\Omega_\bullet) \tag{4.5}$$

defines a continuous linear operator $W^{-1,q}_D(\Omega) \to W^{-1,q}_{D_\bullet}(\Omega_\bullet)$.

The next lemma is about the localization of a solution v to the elliptic equation $(\mathcal{A} + 1)v = f$ and the 'localized' equation. Here and also in the following, we will need several versions of the divergence-gradient type operators $\mathcal{A}$ with different underlying spatial sets, coefficient functions and associated Sobolev spaces respecting partially vanishing trace conditions. We will use the notation $-\nabla \cdot \eta\nabla$ with the coefficient function η for these. It will always be clear from the context which precise incarnation is meant.

Lemma 4.8 ([17, Lem. 4.7]). *Let U, η, $\Omega_\bullet$ and $D_\bullet$ be as in the foregoing lemma. Set $\mu_\bullet := \mu|_{\Omega_\bullet}$ and consider the operator $-\nabla \cdot \mu_\bullet \nabla \colon W^{1,2}_{D_\bullet}(\Omega_\bullet) \to W^{-1,2}_{D_\bullet}(\Omega_\bullet)$. Let $f \in W^{-1,2}_D(\Omega)$ and let $v \in W^{1,2}_D(\Omega)$ be the solution of $(\mathcal{A}+1)v = f$. Then $u := \eta v|_{\Omega_\bullet}$ satisfies*

$$-\nabla \cdot \mu_\bullet \nabla u = f^\bullet := f_\bullet - \nabla \cdot v\mu_\bullet\nabla\eta - \mu_\bullet\nabla v|_{\Omega_\bullet} \cdot \nabla\eta|_{\Omega_\bullet} - \eta v|_{\Omega_\bullet} \tag{4.6}$$

in $W^{-1,2}_{D_\bullet}(\Omega_\bullet)$ with $f_\bullet$ as in (4.5).

Note that although we do not rule out the case $D_\bullet = \emptyset$, in fact, in the actual proof of Theorem 4.2, $D_\bullet$ will always be a nontrivial boundary part of $\Omega_\bullet$. It is thus convenient to consider the localized problem without a zero-order term as in (4.6) which will also tie in with Proposition 4.4 smoothly.

Lemma 4.9. *Let Assumption 2.1 (a) be satisfied; if $d = 4$, let also Assumption 2.1 (b) hold true. Take U, η, $\Omega_\bullet$ and $D_\bullet$ as in Lemma 4.7 and assume that there is a linear extension operator which acts continuously from $W^{1,r}_{D_\bullet}(\Omega_\bullet)$ into $W^{1,r}(\mathbb{R}^d)$ for every $r \in (1,\infty)$. Let further $q > d$.*

Then there exists $p > d$ such that $f \in W^{-1,q}_D(\Omega)$ implies $f^\bullet \in W^{-1,p}_{D_\bullet}(\Omega_\bullet)$, where $f^\bullet$ is defined as in Lemma 4.8 via $(\mathcal{A}+1)v = f$, and the mapping $W^{-1,q}_D(\Omega) \ni f \mapsto f^\bullet \in W^{-1,p}_{D_\bullet}(\Omega_\bullet)$ is continuous.

Proof. We first recall that the usual Sobolev embeddings hold for Ω—cf. Remark 2.4—and $\Omega_\bullet$. Now, let us consider the terms in the right hand side of (4.6), so the definition of $f^\bullet$, from left to right. We have $f_\bullet \in W^{-1,q}_{D_\bullet}(\Omega_\bullet)$ depending continuously on $f \in W^{-1,q}_D(\Omega)$ thanks to Lemma 4.7 (b), so this term is fine without further ado. For the remaining terms, we distinguish between $d = 2,3$ and $d = 4$, starting with the former. We note that the proof of the continuity of $f \mapsto f^\bullet$ is implicitly contained in the following estimates.

Let first $d = 2,3$. Due to the Lax-Milgram lemma and Sobolev embedding, we have

$$\|v\|_{W^{1,2}_D(\Omega)} \le c\|f\|_{W^{-1,2}_D(\Omega)} \le c\|f\|_{W^{-1,q}_D(\Omega)} \tag{4.7}$$

where c only depends on geometry and the ellipticity constant of μ. Concerning $-\nabla\cdot v\mu_\bullet\nabla\eta$, for any $p \in [1,\infty]$ we have the estimate

$$\left|\langle -\nabla\cdot v\mu_\bullet\nabla\eta, w\rangle\right| \le \|v\|_{L^p(\Omega_\bullet)}\,\|\mu\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\,\|\nabla\eta\|_{L^\infty(\Omega_\bullet)}\,\|w\|_{W^{1,p'}_{D_\bullet}(\Omega_\bullet)} \tag{4.8}$$

In particular, for $p = \min(q,6) > d$, we find

$$\left|\langle -\nabla\cdot v\mu_\bullet\nabla\eta, w\rangle\right| \le c\|f\|_{W^{-1,q}_D(\Omega)}\,\|\mu\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\,\|\nabla\eta\|_{L^\infty(\Omega_\bullet)}\,\|w\|_{W^{1,p'}_{D_\bullet}(\Omega_\bullet)}$$

thanks to the Sobolev embedding $W^{1,2}_D(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^p(\Omega)$ and estimate (4.7). Thus, $-\nabla\cdot v\mu_\bullet\nabla\eta \in W^{-1,p}_{D_\bullet}(\Omega_\bullet)$. The same argument and (4.7) moreover shows that $\eta v|_{\Omega_\bullet} \in L^p(\Omega_\bullet) \hookrightarrow W^{-1,p}_{D_\bullet}(\Omega_\bullet)$.

Concerning the term $\mu_\bullet\nabla v|_{\Omega_\bullet}\cdot\nabla\eta|_{\Omega_\bullet}$, it is easily observed that if $v \in W^{1,r}_D(\Omega)$, then the term belongs to $L^r(\Omega)$ with the estimate

$$\|\mu_\bullet\nabla v|_{\Omega_\bullet}\cdot\nabla\eta|_{\Omega_\bullet}\|_{L^r(\Omega)} \le \|\mu\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\|\nabla\eta\|_{L^\infty(\Omega)}\|v\|_{W^{1,r}_D(\Omega)}. \tag{4.9}$$

In particular, for $r = 2$, we obtain via (4.7):

$$\|\mu_\bullet\nabla v|_{\Omega_\bullet}\cdot\nabla\eta|_{\Omega_\bullet}\|_{L^2(\Omega)} \le c\|\mu\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\|\nabla\eta\|_{L^\infty(\Omega)}\|f\|_{W^{-1,q}_D(\Omega)}.$$

Thus, with the same choice for p as before, $\mu_\bullet\nabla v|_{\Omega_\bullet}\cdot\nabla\eta|_{\Omega_\bullet} \in W^{-1,p}_D(\Omega_\bullet)$ due to the embedding $L^2(\Omega_\bullet) \hookrightarrow W^{-1,p}_{D_\bullet}(\Omega_\bullet)$.

Now let $d = 4$. Thanks to Proposition 4.6, there is an $\varepsilon > 0$ such that $v \in W^{1,2+\varepsilon}_D(\Omega)$ with the estimate

$$\|v\|_{W^{1,2+\varepsilon}_D(\Omega)} \le c\|f\|_{W^{-1,2+\varepsilon}_D(\Omega)} \le c\|f\|_{W^{-1,q}_D(\Omega)}. \tag{4.10}$$

Having this at hand, for the estimate of the term $-\nabla \cdot v\mu_\bullet \nabla\eta$ we again exploit (4.8), this time taking $p = 4 \cdot \frac{2+\varepsilon}{2-\varepsilon}$ such that precisely $W^{1,2+\varepsilon}_D(\Omega) \hookrightarrow L^p(\Omega)$. Note that $p > 4 = d$. Again, it follows analogously, this time via (4.10), that $\eta v|_{\Omega_\bullet} \in L^p(\Omega_\bullet) \hookrightarrow W^{-1,p}_{D_\bullet}(\Omega_\bullet)$.

Finally, we estimate again as in (4.9) but pick $r = 2 + \varepsilon$ and consider (4.10) to observe $\mu_\bullet \nabla v|_{\Omega_\bullet} \cdot \nabla\eta|_{\Omega_\bullet} \in L^{2+\varepsilon}(\Omega_\bullet)$ together with the estimate

$$\|\mu_\bullet \nabla v|_{\Omega_\bullet} \cdot \nabla\eta|_{\Omega_\bullet}\|_{L^{2+\varepsilon}(\Omega)} \le c\|\mu\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\|\nabla\eta\|_{L^\infty(\Omega)}\|f\|_{W^{-1,q}_D(\Omega)}.$$

With $p = \min(q, 4\frac{2+\epsilon}{2-\epsilon})$, one has the embedding $L^{2+\varepsilon}(\Omega_\bullet) \hookrightarrow W^{-1,p}_{D_\bullet}(\Omega_\bullet)$ and the claim follows. □

We now consider bi-Lipschitz transformations of the geometric setting.

Proposition 4.10. *Let $\Lambda \subseteq \mathbb{R}^d$ be a bounded, open set that is a Lipschitz domain, i.e., Λ satisfies Assumption 2.1 (a) in every point $\mathrm{x} \in \partial\Lambda$. Let Σ be a closed subset of its boundary. Assume that ϕ is a mapping from a neighbourhood of $\overline{\Lambda}$ into $\mathbb{R}^d$ that is bi-Lipschitz. Let us denote $\Lambda^\# := \phi(\Lambda)$ and $\Sigma^\# := \phi(\Sigma)$. Then the following holds true.*

(a) For every $p \in (1,\infty)$ and every $\alpha \in (0,1)$, the mapping ϕ induces a linear, topological isomorphism $\Phi f := f \circ \phi$ acting between

$$\Phi\colon W^{1,p}_{\Sigma^\#}(\Lambda^\#) \to W^{1,p}_{\Sigma}(\Lambda) \quad \text{and} \quad C^\alpha(\Lambda^\#) \to C^\alpha(\Lambda).$$

(b) Let ω be an essentially bounded, measurable function on Λ, taking its values in the set of $(d \times d)$-matrices. Then

$$\Phi^*\Big[-\nabla \cdot \omega\nabla\Big]\Phi = -\nabla \cdot \omega^\#\nabla$$

with

$$\omega^\#(\mathrm{y}) := \frac{(D\phi)\big(\phi^{-1}(\mathrm{y})\big)\ \omega\big(\phi^{-1}(\mathrm{y})\big)\ (D\phi)^T\big(\phi^{-1}(\mathrm{y})\big)}{\big|\det(D\phi)\big(\phi^{-1}(\mathrm{y})\big)\big|} \tag{4.11}$$

for almost all $\mathrm{y} \in \Lambda^\#$. Here, $D\phi$ denotes the Fréchet derivative of ϕ and $\det(D\phi)$ the corresponding determinant.

(c) If ω is real and uniformly elliptic almost everywhere on Λ, then so is $\omega^\#$ on $\Lambda^\#$.

Proof. The proof of (a) for the Sobolev spaces is contained in [14, Thm 2.10]; for the Hölder spaces it is easy to verify. Part (b) is well known, see [16] for an explicit verification, or [3, Ch. 0.8]. Finally, (c) is implied by (4.11) and the fact that for a bi-Lipschitz function ϕ the derivative $D\phi$ and its inverse $(D\phi)^{-1}$ are essentially bounded, see [13, Ch. 3.1]. □

It will be very useful that the class (A_γ) as in Assumption 4.1 (a) is preserved under bi-Lipschitz transformations, precisely:

Lemma 4.11. *Let $\phi\colon \mathbb{R}^d \to \mathbb{R}^d$ be a bi-Lipschitz map and assume that Ω and D satisfy Assumption 4.1 (a), so Ω is of class (A_γ) at D. Then $\phi(\Omega)$ is of class (A_{γ_ϕ}) at $\phi(D)$, that is, there is a constant $\gamma_\phi > 0$ such that for all $\mathrm{y} \in D$ and all $r \in (0,1]$:*

$$\lambda_d\big(B_r(\phi(\mathrm{y})) \setminus \phi(\Omega)\big) \geq \gamma_\phi \lambda_d\big(B_r(\phi(\mathrm{y}))\big).$$

Proof. For every Lebesgue measurable set $B \subseteq \mathbb{R}^d$ one has the estimate $\lambda_d(B) \geq \frac{1}{\ell^d}\lambda_d(\phi^{-1}(B))$, where ℓ is a Lipschitz constant of ϕ^{-1}, cf. [13, Thm. 2.5 and Thm. 2.8]. This entails for every $\mathrm{y} \in D$ and all $r \in (0,1]$ that

$$\begin{aligned}\lambda_d\big(B_r(\phi(\mathrm{y})) \setminus \phi(\Omega)\big) &\geq \frac{1}{\ell^d}\lambda_d\big(\phi^{-1}\big(B_r(\phi(\mathrm{y})) \setminus \phi(\Omega)\big)\big) \\ &= \frac{1}{\ell^d}\lambda_d\big(\phi^{-1}\big(B_r(\phi(\mathrm{y}))\big) \setminus \Omega\big).\end{aligned}$$

But $\phi^{-1}(B_r(\phi(\mathrm{y})))$ contains the ball $B_{\frac{r}{L}}(\mathrm{y})$, where $L \geq 1$ is a Lipschitz constant of ϕ. Using this and Assumption 4.1 (a) we may continue to estimate by

$$\begin{aligned}&\geq \frac{1}{\ell^d}\lambda_d\big(B_{\frac{r}{L}}(\mathrm{y}) \setminus \Omega\big) \\ &\geq \frac{\gamma}{\ell^d}\lambda_d\big(B_{\frac{r}{L}}(\mathrm{y})\big) = \frac{\gamma}{\ell^d L^d}\lambda_d\big(B_r(\phi(\mathrm{y}))\big)\end{aligned}$$

and we are done. □

As a final step in this preparatory subsection, we prepare the reflection argument in the proof of Theorem 4.2. For this we consider the matrix $R \coloneqq \operatorname{diag}(1,1,\dots,1,-1) \in \mathbb{R}^{d\times d}$ and define the bi-Lipschitz map $\phi_R(\mathrm{x}) = R\mathrm{x}$ for $\mathrm{x} \in \mathbb{R}^d$ that reflects at the plane $[x_d = 0]$.

Lemma 4.12. *Let $\Lambda \subseteq [x_d < 0]$ be open and bounded and define Γ as the (relative) interior of $\partial\Lambda \cap [x_d = 0]$ in the plane $[x_d = 0]$. Furthermore, set $\Sigma \coloneqq \partial\Lambda \setminus \Gamma$ and $\widehat{\Lambda} \coloneqq \Lambda \cup \Gamma \cup \phi_R(\Lambda)$, and consider for $v \in W^{1,2}_\Sigma(\Lambda)$ the reflected function $\hat{v}$ on $\widehat{\Lambda}$ with*

$$\hat{v}(\mathrm{y}) \coloneqq \begin{cases} v(\mathrm{y}) & \text{if } \mathrm{y} \in \Lambda, \\ v(R\mathrm{y}) & \text{if } R\mathrm{y} \in \Lambda. \end{cases}$$

Then the following holds:

(a) If $v \in W^{1,2}_\Sigma(\Lambda)$, then $\hat{v} \in W^{1,2}_0(\widehat{\Lambda})$.

(b) Consider Φ defined as in Proposition 4.10 for $\phi = \phi_R$. Let $f \in W^{-1,2}_\Sigma(\Lambda)$ and set

$$\langle \hat{f}, \psi\rangle \coloneqq \langle f, \psi|_\Lambda\rangle + \langle \Phi^* f, \psi|_{\phi_R(\Lambda)}\rangle, \qquad \psi \in C_c^\infty(\widehat{\Lambda}).$$

Then $f \mapsto \hat{f}$ is continuous from $W^{-1,p}_\Sigma(\Lambda)$ to $W^{-1,p}_0(\widehat{\Lambda})$ for every $p \geq 2$.

(c) Let $\eta\colon \Lambda \to \mathbb{R}^{d\times d}$. Define the reflected coefficient function $\hat{\eta}$ on $\widehat{\Lambda}$ by

$$\hat{\eta}(\mathrm{y}) \coloneqq \begin{cases} \eta(\mathrm{y}) & \text{if } \mathrm{y} \in \Lambda, \\ R\,\eta(R\mathrm{y})\,R & \text{if } R\mathrm{y} \in \Lambda. \end{cases}$$

Let v and f as before. Then we have

$$-\nabla\cdot\eta\nabla v = f \qquad \Longrightarrow \qquad -\nabla\cdot\hat{\eta}\nabla\hat{v} = \hat{f}.$$

Proof. In order to prove (a), note first that—thanks to the special geometric constellation—every $\psi \in C_\Sigma^\infty(\Lambda)$ can be extended by zero to the whole half space $H_- := [x_d < 0]$, resulting in a function in $W^{1,2}(H_-)$. By the density of $C_\Sigma^\infty(\Lambda)$ in $W_\Sigma^{1,2}(\Lambda)$ it follows that this extending procedure provides an isometry E_0 from $W_\Sigma^{1,2}(\Lambda)$ into $W^{1,2}(H_-)$. Now let $v \in W_\Sigma^{1,2}(\Lambda)$. We consider $E_0 v$ and reflect this function across the boundary of H_- to obtain a function $v_\pm \in W^{1,2}(\mathbb{R}^d)$ on all of $\mathbb{R}^d$ that satisfies

$$\|v_\pm\|_{W^{1,2}(\mathbb{R}^d)} = 2\|E_0 v\|_{W^{1,2}(H_-)} = 2\|v\|_{W_\Sigma^{1,2}(\Lambda)}.$$

This is easily verified by direct calculations. So, summing up, the mapping

$$v \mapsto E_0 v \mapsto v_\pm \mapsto v_\pm|_{\widehat{\Lambda}} = \hat{v}$$

is continuous from $W_\Sigma^{1,2}(\Lambda)$ to $W^{1,2}(\widehat{\Lambda})$. It remains to show that indeed $\hat{v} \in W_0^{1,2}(\widehat{\Lambda})$. To this end, let $(v_k) \subset C_\Sigma^\infty(\Lambda)$ be an approximating sequence for v in $W_\Sigma^{1,2}(\Lambda)$. Note that it is clear that $(v_k)_\pm|_{\widehat{\Lambda}}$ approximates $\hat{v}$ in $W^{1,2}(\widehat{\Lambda})$ and the supports of $(v_k)_\pm|_{\widehat{\Lambda}}$ have a positive distance to $\partial\widehat{\Lambda}$, but the functions are not smooth any more in general. But this can be rectified by mollifying each $(v_k)_\pm|_{\widehat{\Lambda}}$ with a suitable regularizing kernel such that the resulting smooth functions' supports still have a positive distance to $\partial\widehat{\Lambda}$, and it is easily shown that these functions still approximate $\hat{v}$ in $W^{1,2}(\widehat{\Lambda})$, so $\hat{v} \in W_0^{1,2}(\widehat{\Lambda})$.

The proof of (b) and (c) is concluded from a straightforward calculation and application of the definitions of the operators $-\nabla \cdot \mu \nabla$ and $-\nabla \cdot \hat{\mu} \nabla$ together with Proposition 4.10. □

Remark 4.13. From the proofs of the foregoing framework for localization, transformation and reflection it is easily seen that each step preserves uniform bounds in the data of an elliptic equation for a fixed geometry, that is, the right-hand side and the coefficient function. In this sense, whenever a result on elliptic regularity on the localized, transformed or reflected level yields a uniform estimate on the solution in the aforementioned data, this uniform estimate carries over to the original situation immediately. Of course, this is exactly the case for our main tool, Proposition 4.4.

4.2. **Proof of Theorem 4.2.** We now start the proof of the Hölder continuity following the program sketched in the preceding subsection, cf. page 179. According to the hypotheses of Theorem 4.2, from now on we suppose that Ω and D satisfy the Assumptions 2.1 (a) and, if $d = 4$, also (b), as well as (always) Assumption 4.1.

In order to start the localisation procedure, we fix some notation. For the Neumann boundary part we use again the shorthand $N = \partial\Omega \setminus D$. Now, based on Assumption 2.1 (a), choose for every $\mathrm{x} \in \overline{N}$ an associated open neighbourhood V_x and let $\{V_{\mathrm{x}_1}, \ldots, V_{\mathrm{x}_m}\}$ be a finite subcovering of $\overline{N}$.

Furthermore, choose a bounded open neighbourhood W of $\overline{\Omega}$ and put $U_0 := W \setminus \overline{N}$. Then U_0 is open and one has

$$U_0 \cap \Omega = \Omega \quad \text{and} \quad U_0 \cap \overline{N} = \emptyset.$$

The system $\mathcal{U} := \{U_0, V_{\mathrm{x}_1}, V_{\mathrm{x}_2}, \ldots V_{\mathrm{x}_m}\}$ forms an open covering of $\overline{\Omega}$. Moreover, all sets in $\mathcal{U}$ give rise to extension domains; this will come in handy in view of Lemma 4.9:

Lemma 4.14. *Let $U \in \mathcal{U}$ and put $\Omega_\bullet := \Omega \cap U$ and $D_\bullet := \partial\Omega_\bullet \setminus (U \cap N)$. Then for all $r \in (1,\infty)$ the space $W^{1,r}_{D_\bullet}(\Omega_\bullet)$ admits again the continuation property, i.e., there is a continuous extension operator $\mathfrak{E}_U \colon W^{1,r}_{D_\bullet}(\Omega_\bullet) \to W^{1,r}(\mathbb{R}^d)$ which is even universal in r.*

Proof. In the case $U = U_0$ one has $D_\bullet = \partial\Omega_\bullet$ by construction. Thus, $W^{1,r}_{D_\bullet}(\Omega_\bullet) = W^{1,r}_0(\Omega_\bullet)$ and the trivial extension by zero does the trick even without any condition on the boundary. If $U = V_{\mathrm{x}_j}$, then $\Omega_\bullet = \Omega \cap V_{\mathrm{x}_j}$ is mapped onto the lower half cube $\{\mathrm{x} \in (-1,1)^d \colon x_d < 0\}$ by the bi-Lipschitz map ϕ_{x_j} that is defined on a neighbourhood of $\overline{\Omega_\bullet}$, cf. Assumption 2.1 (a). The lower half cube is a Lipschitz domain. Thus, $\Omega_\bullet$ is also a Lipschitz domain and there is even an extension operator from $W^{1,r}(\Omega_\bullet)$ into $W^{1,r}(\mathbb{R}^d)$ which is also universal in r. See Proposition 2.3 *mutatis mutandis* for $\Omega = \Omega_\bullet$ and $D = \emptyset$. □

Corresponding to the open covering $\mathcal{U}$ of $\overline{\Omega}$ we choose a smooth partition of unity $\{\eta_0, \eta_1, \dots, \eta_m\} \subset C^\infty_c(\mathbb{R}^d)$ such that $\mathrm{supp}(\eta_0) \subseteq U_0$ and $\mathrm{supp}(\eta_j) \subseteq V_{\mathrm{x}_j}$ for $j \in \{1, \dots, m\}$.

Let from now on $q > d$ be fixed, let $f \in W^{-1,q}_D(\Omega)$, and let $v \in W^{1,2}_D(\Omega)$ be the solution to (4.1), so $(\mathcal{A} + 1)v = f$. We write $v = \sum_{j=0}^m \eta_j v$ and aim to show the Hölder continuity of every function $\eta_j v$ seperately. The easiest case is $j = 0$:

Lemma 4.15. *There exists an $\alpha_0 > 0$ independent of f such that $\eta_0 v \in C^{\alpha_0}(\Omega)$ and the estimate*

$$\|\eta_0 v\|_{C^{\alpha_0}(\Omega)} \le c\|f\|_{W^{-1,q}_D(\Omega)}$$

holds true. The constant c is uniform with respect to the $L^\infty(\Omega)$-bound and ellipticity constant κ_{ell} of the given coefficient function μ.

Proof. Since $\overline{N}$ does not intersect U_0, the function $\eta_0 v$ belongs to $W^{1,2}_0(\Omega)$, cf. Lemma 4.7. Moreover, by Lemma 4.8 and Lemma 4.9 there is a $p > d$, only depending on q and the geometry, and $f_0 \in W^{-1,p}(\Omega)$ such that the function $\eta_0 v$ satisfies the equation

$$-\nabla \cdot \mu \nabla(\eta_0 v) = f_0. \tag{4.12}$$

Recall that as mentioned in Remark 4.3, Ω is of class (A_γ). So we are now in the setting of a pure, homogeneous Dirichlet problem and can apply Proposition 4.4; note also Remark 4.5. This yields that the solution $\eta_0 v$ of (4.12) is Hölderian of some degree α_0 with the estimate

$$\|\eta_0 v\|_{C^{\alpha_0}(\Omega)} \le c\|f_0\|_{W^{-1,p}(\Omega)}.$$

Finally, combining Lemma 4.14 and Lemma 4.9, we conclude that

$$\|\eta_0 v\|_{C^{\alpha_0}(\Omega)} \le c\|f\|_{W^{-1,q}_D(\Omega)},$$

where α_0 does not depend on f. For the uniformity claim, see Remark 4.13. □

We turn to the Hölder continuity of the functions $\eta_j v$ for $j \in \{1, \dots, m\}$. For these, there will be a part of the Neumann boundary N present. To make do with this, we transform the localized problems via the diffeomorphisms ϕ_{x_j} to the model constellation on the unit cube as in Assumption 2.1 (a), which enables us to use a reflection argument to end up in a situation with a pure Dirichlet boundary condition. Then we can conclude by Proposition 4.4. For this we introduce the notation $Q := (-1,1)^d$ for the unit cube, $Q_- := \{\mathrm{x} \in Q : x_d < 0\}$ for its lower half and $P := \{\mathrm{x} \in Q : x_d = 0\}$ for its midplate.

Due to Lemma 4.8, there is $p > d$ such that each of the functions $\eta_j v$, $j = 1, \dots, m$ satisfies an equation like

$$-\nabla \cdot \mu \nabla(\eta_j v) = f_j \in W^{-1,p}_{D_j}(\Omega \cap V_{\mathrm{x}_j}),$$

with $D_j = \partial(\Omega \cap V_{\mathrm{x}_j}) \setminus N$. Note that the right hand sides f_j continuously depend on f, see Lemma 4.9. According to Proposition 4.10, one may transform these equations under the bi-Lipschitz diffeomorphisms ϕ_{x_j} and pass to the equation

$$-\nabla \cdot \mu_j^{\#} \nabla w_j = g_j \in W^{-1,p}_{\Sigma_j}(Q_-), \tag{4.13}$$

where $\Sigma_j = \phi_{\mathrm{x}_j}(D_j) \subseteq \partial Q_-$ is the transformed Dirichlet boundary part, $w_j \in W^{1,2}_{\Sigma_j}(Q_-)$ is the transformed version of the function $\eta_j v|_{\Omega \cap V_{\mathrm{x}_j}}$ and g_j is the transformation of f_j. Note that the whole 'lower mantle' boundary $\partial Q_- \setminus P$ belongs to Σ_j, since $\phi_{\mathrm{x}_j}(N \cap V_{\mathrm{x}_j}) \subseteq P$.

From now on we distinguish whether $\mathrm{x}_j \in N$ or $\mathrm{x}_j \in D \cap \overline{N}$, starting with the former.

Lemma 4.16. *Let $j \in \{1, 2, \dots, m\}$ with $\mathrm{x}_j \in N$. Then there is some $\alpha_j > 0$ independent of f such that $\eta_j v \in C^{\alpha_j}(\Omega)$ and we have*

$$\|\eta_j v\|_{C^{\alpha_j}(\Omega)} \le c\|f\|_{W^{-1,q}_D(\Omega)}.$$

The constant c is uniform with respect to the $L^\infty(\Omega)$-bound and ellipticity constant κ_{ell} of the given coefficient function μ.

Proof. Thanks to Remark 2.2 we can assume that $\Sigma_j = \partial Q_- \setminus P$. Thus, exploiting Lemma 4.12 with $\Lambda = Q_-$ and $\Gamma = P$, the symmetrically reflected function $\widehat{w_j}$ belongs to the space $W^{1,2}_0(Q)$ and obeys an elliptic equation on the cube Q with the right hand side $\widehat{g_j} \in W^{-1,p}(Q)$. The cube Q is obviously convex and satisfies the regularity condition in Proposition 4.4 with $\gamma = 1/2$. Thus, said Proposition 4.4 applies and gives us Hölder continuity of $\widehat{w_j}$ of degree, say, α_j, with an estimate in $\widehat{g_j} \in W^{-1,p}(Q)$. By Proposition 4.10, Lemma 4.12 and Lemma 4.9 via Lemma 4.14 we then have

$$\begin{aligned}\|\eta_j v\|_{C^{\alpha_j}(\Omega \cap V_{\mathrm{x}_j})} &\le c\|w_j\|_{C^{\alpha_j}(Q_-)} \le c\|\widehat{w_j}\|_{C^{\alpha_j}(Q)} \\ &\le c\|\widehat{g_j}\|_{W^{-1,p}(Q)} \le c\|g_j\|_{W^{-1,p}_{\Sigma_j}(Q_-)} \le c\|f_j\|_{W^{-1,p}_{D_j}(\Omega \cap V_{\mathrm{x}_j})} \le c\|f\|_{W^{-1,q}_D(\Omega)}.\end{aligned}$$

Since the support of η_j has a positive distance to $\Omega \setminus V_{\mathrm{x}_j}$, the α_j-Hölder continuity and norm estimate is preserved for $\eta_j v$ on the whole Ω. For the uniformity claim, see again Remark 4.13. ☐

It remains to treat the patches with $\mathrm{x}_j \in D \cap \overline{N}$ and it is here that Assumption 4.1 (b) comes into play. In order to reformulate this condition in our current notation, for some set $M \subseteq \partial Q_-$, we denote its relative boundary inside ∂Q_- by

$\mathrm{bd}_{\partial Q_-}(M)$ and inside P by $\mathrm{bd}_P(M)$. Then Assumption 4.1 (b) reads as follows: There are two constants $c_0 \in (0,1)$ and $c_1 > 0$, such that for all $(\mathrm{y},0) \in \mathrm{bd}_P(\Sigma_j)$ and all $s \in (0,1]$ we have

$$\lambda_{d-1}\big(\{\mathrm{z} \in \mathbf{B}_s(\mathrm{y})\colon\ \mathrm{dist}(\mathrm{z}, P \setminus \Sigma_j) > c_0 s\}\big) \geq c_1 s^{d-1}. \tag{4.14}$$

Later on it will be convenient to have this condition not only for the points in the interface $\mathrm{bd}_P(\Sigma_j)$, but for all points of Σ_j inside P. It is an interesting fact that this comes for free, once we suppose it on the interface. This will be elaborated in the next two lemmas.

Lemma 4.17. *Condition* (4.14) *carries over to all points* $(\mathrm{y},0) \in \mathrm{bd}_{\partial Q_-}(\Sigma_j)$ *with possibly different constants* $c_0, c_1 > 0$.

Proof. Since $\partial Q_- \setminus P \subseteq \Sigma_j$, we have the inclusion

$$\mathrm{bd}_{\partial Q_-}(\Sigma_j) = \mathrm{bd}_{\partial Q_-}(\Sigma_j \cap P) \subseteq \mathrm{bd}_P(\Sigma_j) \cup \mathrm{bd}_{\partial Q_-}(P).$$

For $(\mathrm{y},0) \in \mathrm{bd}_{\partial Q_-}(P)$ we estimate

$$\lambda_{d-1}\Big(\Big\{\mathrm{z} \in \mathbf{B}_s(\mathrm{y})\colon\ \mathrm{dist}(\mathrm{z}, P \setminus \Sigma_j) > \frac{s}{2}\Big\}\Big) \\ \geq \lambda_{d-1}\Big(\Big\{\mathrm{z} \in \mathbf{B}_s(\mathrm{y})\colon\ \mathrm{dist}(\mathrm{z}, P) > \frac{s}{2}\Big\}\Big) \geq \frac{\omega_{d-1}}{2^{d-1}} s^{d-1}.$$

So, (4.14) is true for all points $(\mathrm{y},0)$ in $\mathrm{bd}_{Q_-}(P)$ and it is true for all $(\mathrm{y},0)$ in $\mathrm{bd}_P(\Sigma_j)$ by hypotheses, with possibly different constants c_0 and c_1. In order to conclude, it suffices to observe the following: If for a point y and a number $s > 0$ the inequality (4.14) holds, then this remains true if the constants c_0, c_1 are replaced by smaller ones. ☐

The following lemma is already contained in [10] (see Lemma 5.4 there); we repeat it, including the proof, for the convenience of the reader.

Lemma 4.18. *We have for all* $(\mathrm{y},0) \in \Sigma_j \cap P$ *and all* $s \in (0,1]$

$$\lambda_{d-1}\big(\{\mathrm{z} \in \mathbf{B}_s(\mathrm{y})\colon\ \mathrm{dist}(\mathrm{z}, P \setminus \Sigma_j) > \hat{c}_0 s\}\big) \geq \hat{c}_1 s^{d-1},$$

for $\hat{c}_0 := \min\{\frac{1}{4}, \frac{c_0}{2}\}$ *and* $\hat{c}_1 := \min\{\frac{\omega_{d-1}}{4^{d-1}}, \frac{c_1}{2^{d-1}}\}$, *where* c_0 *and* c_1 *are from Lemma 4.17.*

Proof. For all $(\mathrm{y},0) \in \mathrm{bd}_P(\Sigma_j)$ the assertion is true by Assumption 4.1 (b) and, using again the observation made in the end of the proof of Lemma 4.17, it suffices to treat the case where $(\mathrm{y},0)$ is a relatively inner point of Σ_j in P. Since $\overline{P \setminus \Sigma_j}$ is compact, we then have

$$\varepsilon := \mathrm{dist}(\mathrm{y}, P \setminus \Sigma_j) = \mathrm{dist}(\mathrm{y}, \overline{P \setminus \Sigma_j}) > 0.$$

We distinguish three cases:
First case, $0 < s \leq \varepsilon/2$**:** In this case one finds

$$\big\{\mathrm{z} \in \mathbf{B}_s(\mathrm{y})\colon\ \mathrm{dist}(\mathrm{z}, P \setminus \Sigma_j) > s\big\} = \mathbf{B}_s(\mathrm{y}),$$

so

$$\lambda_{d-1}\big(\{\mathrm{z} \in \mathbf{B}_s(\mathrm{y})\colon\ \mathrm{dist}(\mathrm{z}, P \setminus \Sigma_j) > s\}\big) = \lambda_{d-1}\big(\mathbf{B}_s(\mathrm{y})\big) = \omega_{d-1} s^{d-1}.$$

Second case, $\varepsilon/2 < s \le 2\varepsilon$: Since $s/4 \le \varepsilon/2$, we infer from the first case

$$\lambda_{d-1}\Big(\Big\{\mathrm{z} \in \mathbf{B}_s(\mathrm{y})\colon \operatorname{dist}(\mathrm{z}, P \setminus \Sigma_j) > \frac{s}{4}\Big)\Big\}$$
$$\ge \lambda_{d-1}\Big(\Big\{\mathrm{z} \in \mathbf{B}_{\frac{s}{4}}(\mathrm{y})\colon \operatorname{dist}(\mathrm{z}, P \setminus \Sigma_j) > \frac{s}{4}\Big\}\Big) \ge \omega_{d-1}\frac{s^{d-1}}{4^{d-1}}.$$

Third case, $2\varepsilon < s \le 1$: From the fact that $\overline{P \setminus \Sigma_j}$ is compact, we not only get that $\varepsilon > 0$, but we also obtain the existence of a point $(\mathrm{y}^*, 0) \in \mathrm{bd}_{\partial Q_-}(\Sigma_j)$ with $\|\mathrm{y} - \mathrm{y}^*\|_{\mathbb{R}^{d-1}} = \varepsilon$. Since $\mathbf{B}_{s-\varepsilon}(\mathrm{y}^*) \subseteq \mathbf{B}_s(\mathrm{y})$, this yields

$$\lambda_{d-1}\Big(\Big\{\mathrm{z} \in \mathbf{B}_s(\mathrm{y})\colon \operatorname{dist}(\mathrm{z}, P \setminus \Sigma_j) > \frac{c_0}{2}s\Big\}\Big)$$
$$\ge \lambda_{d-1}\Big(\Big\{\mathrm{z} \in \mathbf{B}_{s-\varepsilon}(\mathrm{y}^*)\colon \operatorname{dist}(\mathrm{z}, P \setminus \Sigma_j) > \frac{c_0}{2}s\Big\}\Big).$$

The condition $2\varepsilon < s$ implies $\frac{c_0}{2}s < c_0(s-\varepsilon)$. Using this and Lemma 4.17, we continue to estimate

$$\cdots \ge \lambda_{d-1}\big(\big\{\mathrm{z} \in \mathbf{B}_{s-\varepsilon}(\mathrm{y}^*)\colon \operatorname{dist}(\mathrm{z}, P \setminus \Sigma_j) > c_0(s-\varepsilon\big\}\big)$$
$$\ge c_1(s-\varepsilon)^{d-1} \ge \frac{c_1}{2^{d-1}}s^{d-1}.$$

Invoking once more the observation from the end of the proof of Lemma 4.17, we deduce the claim. ☐

Let, in all what follows, $\hat{c}_0, \hat{c}_1$ be the constants from Lemma 4.18. Also, we will often use the decomposition $\mathbb{R}^d \ni \mathrm{x} = (\bar{\mathrm{x}}, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$.

For $t \in \mathbb{R}$, we define the mapping $\psi_t\colon \mathbb{R}^d \to \mathbb{R}^d$ by

$$\psi_t(\mathrm{x}) = \psi_t\big((\bar{\mathrm{x}}, x_d)\big) := \big(\bar{\mathrm{x}}, x_d - t\operatorname{dist}(\bar{\mathrm{x}}, P \setminus \Sigma_j)\big). \tag{4.15}$$

Later on we will transform our problem again under the mapping ψ_t for a suitably chosen value of t and afterwards reflect it in correspondence with Lemma 4.12. In order to justify this transformation, we first prove a little lemma.

Lemma 4.19. *Consider ψ_t be as in* (4.15). *Then the following holds true:*

(a) The function $\mathbb{R}^d \ni \mathrm{x} = (\bar{\mathrm{x}}, x_d) \mapsto \operatorname{dist}(\bar{\mathrm{x}}, P \setminus \Sigma_j)$ is a Lipschitz contraction.
(b) For every $t \in \mathbb{R}$, the function ψ_t is Lipschitz continuous and bijective with inverse ψ_{-t}. In particular, the inverse is also Lipschitz continuous.
(c) For every $t \in \mathbb{R}$, the function ψ_t is volume preserving.

Proof. The function under consideration in (a) is the concatenation of the projection $\mathbb{R}^d \ni \mathrm{x} \mapsto (\bar{\mathrm{x}}, 0)$ onto $[x_d = 0]$ and the restriction of the function $\mathbb{R}^d \ni \mathrm{x} \mapsto \operatorname{dist}(\mathrm{x}, P \setminus \Sigma_j)$ to $\mathbb{R}^{d-1} \times \{0\}$. Both of these functions are Lipschitz continuous contractions, thus so is the considered concatenation.

For (b), the first assertion follows from (a), and the second is easy to verify.

Finally, it is clear that the determinant of the Jacobian of ψ_t (cf. [13, Ch. 3.2.2]) is identically 1 a.e., thus the assertion (c) follows from [13, Ch. 3.3.3 Thm. 3.9]. ☐

In the following we choose $t = \frac{3}{\hat{c}_0}$ and abbreviate $\psi := \psi_{3/\hat{c}_0}$. We transform (4.13) under ψ to a problem

$$-\nabla \cdot \omega \nabla w = h \in W^{-1,p}_{\Sigma_\Delta}(Q_\Delta), \tag{4.16}$$

where the resulting domain is $Q_\Delta \coloneqq \psi(Q_-)$ and the new Dirichlet boundary part is $\Sigma_\Delta \coloneqq \psi(\Sigma_j)$. We suppress the dependence on j here, so w is the transformation of w_j by slight abuse of notation, and h is the transformed g_j. Furthermore, the resulting coefficient function ω is again real, elliptic and bounded thanks to Proposition 4.10.

The crucial effect of the transformation ψ is that the new Neumann boundary part $N_\Delta \coloneqq \partial Q_\Delta \setminus \Sigma_\Delta$ is identical to the old Neumann part $P \setminus \Sigma_j$ and that $\overline{N_\Delta} = \overline{P \setminus \Sigma_j} = \partial Q_\Delta \cap \overline{P}$. In particular, $\partial Q_\Delta \cap P$ consists of Neumann boundary only. Thus, the geometry of the problem (4.16) is now exactly of the shape needed to reflect the problem across the plane $[x_d = 0]$, according to Lemma 4.12. We end up with the domain

$$\Lambda \coloneqq Q_\Delta \cup N_\Delta \cup \bigl\{\mathrm{z} = (\bar{\mathrm{z}}, z_d) \in \mathbb{R}^d \colon (\bar{\mathrm{z}}, -z_d) \in Q_\Delta\bigr\},$$

while the new coefficient function $\hat{\omega}$ is again real, bounded and elliptic and the resulting right hand side $\hat{h}$ belongs to the space $W^{-1,p}(\Lambda)$ with $p > d$. Lemma 4.12 already tells us that the solution $\hat{w}$ of the equation on Λ belongs to $W^{1,2}_0(\Lambda)$. Thus, in order to infer Hölder continuity for $\hat{w}$ by Proposition 4.4, the only thing that is left to verify is that our final geometry satisfies Assumption 4.1 (a). This will be the main part of the proof.

Lemma 4.20. *The domain Λ is of class (A_γ) for some $\gamma \in (0, 1)$.*

Proof. The boundary of Λ is the union of the sets $\psi(\Sigma_j \cap P)$ and $\psi(\partial Q_- \setminus P)$ and their reflected counterparts. We show the assertion for the points from $\psi(\Sigma_j \cap P)$ and from $\psi(\partial Q_- \setminus P)$, the proof for points from the reflected parts is then analogous.

Let $r \in (0, 1]$ and assume $\mathrm{y} = (\bar{\mathrm{y}}, y_d) \in \psi(\Sigma_j \cap P)$. Then y is necessarily of the form $\bigl(\bar{\mathrm{y}}, -3/\hat{c}_0 \cdot \operatorname{dist}(\bar{\mathrm{y}}, P \setminus \Sigma_j)\bigr)$ with $(\bar{\mathrm{y}}, 0) \in P$. Now let first $r < 3/\hat{c}_0 \cdot \operatorname{dist}(\bar{\mathrm{y}}, P \setminus \Sigma_j)$. Then the ball $B_r(\mathrm{y})$ lies completely in the half space $[x_d < 0]$. This gives

$$B_r(\mathrm{y}) \setminus \Lambda = B_r(\mathrm{y}) \setminus Q_\Delta = B_r(\mathrm{y}) \setminus \psi(Q_-).$$

Applying the volume-preserving map ψ^{-1}, cf. Lemma 4.19, and using that $\psi^{-1}(\mathrm{y}) \in P$, one deduces the inequality

$$\begin{aligned} \lambda_d\bigl(B_r(\mathrm{y}) \setminus \Lambda\bigr) &= \lambda_d\bigl(\psi^{-1}(B_r(\mathrm{y})) \setminus Q_-\bigr) \\ &\geq \lambda_d\bigl(\psi^{-1}(B_r(\mathrm{y})) \cap [x_d > 0]\bigr) = \frac{1}{2}\lambda_d\bigl(\psi^{-1}(B_r(\mathrm{y}))\bigr). \end{aligned}$$

Since ψ^{-1} is Lipschitz continuous by Lemma 4.19, the set $\psi^{-1}\bigl(B_r(\mathrm{y})\bigr)$ contains the ball $B_{r/\ell}\bigl((\bar{\mathrm{y}}, 0)\bigr)$, where ℓ is the Lipschitz constant of ψ^{-1}. Thus, we can continue to estimate

$$\geq \frac{1}{2}\ell^d \omega_d r^d = \frac{1}{2}\ell^d \lambda_d\bigl(B_r(\mathrm{y})\bigr).$$

Now we consider the second case $r \geq 3/\hat{c}_0 \cdot \operatorname{dist}(\mathrm{y}, P \setminus \Sigma_j)$. Let

$$B_r^-(\mathrm{y}) \coloneqq B_r(\mathrm{y}) \cap \Bigl\{\mathrm{z} \in \mathbb{R}^d \colon z_d \leq -3/\hat{c}_0 \cdot \operatorname{dist}(\bar{\mathrm{y}}, P \setminus \Sigma_j)\Bigr\}.$$

Since $y_d = -3/\hat{c}_0 \cdot \operatorname{dist}(\bar{\mathrm{y}}, P \setminus \Sigma_j)$, this is exactly the 'lower' half of $B_r(\mathrm{y})$. By construction of Λ, one has

$$B_r(\mathrm{y}) \setminus \Lambda \supseteq B_r^-(\mathrm{y}) \setminus \Lambda = B_r^-(\mathrm{y}) \setminus Q_\Delta.$$

Due to the choice of ψ, we have $Q_\Delta \subseteq \{(\bar{z}, z_d) \in \mathbb{R}^d : z_d \leq -3/\hat{c}_0 \cdot \operatorname{dist}(\bar{z}, P \setminus \Sigma_j)\}$. Thus, we may continue

$$
\begin{aligned}
B_r(\mathrm{y}) \setminus \Lambda &\supseteq B_r^-(\mathrm{y}) \setminus \Big\{(\bar{z}, z_d) \in \mathbb{R}^d : z_d \leq -\frac{3}{\hat{c}_0} \operatorname{dist}(\bar{z}, P \setminus \Sigma_j)\Big\} \\
&= B_r^-(\mathrm{y}) \cap \Big\{(\bar{z}, z_d) \in \mathbb{R}^d : \frac{3}{\hat{c}_0} \operatorname{dist}(\bar{z}, P \setminus \Sigma_j) > -z_d\Big\}.
\end{aligned}
$$

We aim to parametrize the last set by layers along the z_d-direction. To this end, for $s \in [0, r]$ we denote by H_s the hyperplane $\{(\bar{z}, z_d) \in \mathbb{R}^d : z_d = -3/\hat{c}_0 \cdot \operatorname{dist}(\bar{y}, P \setminus \Sigma_j) - s\}$. Then we obtain

$$
\begin{aligned}
B_r(\mathrm{y}) \setminus \Lambda &\supseteq B_r^-(\mathrm{y}) \cap \Big(\bigcup_{s \in [0,r]} H_s\Big) \cap \Big\{(\bar{z}, z_d) \in \mathbb{R}^d : \frac{3}{\hat{c}_0} \operatorname{dist}(\bar{z}, P \setminus \Sigma_j) > -z_d\Big\} \\
&= \bigcup_{s \in [0,r]} \big(B_r^-(\mathrm{y}) \cap H_s\big) \cap \Big\{(\bar{z}, z_d) \in \mathbb{R}^d : \frac{3}{\hat{c}_0} \operatorname{dist}(\bar{z}, P \setminus \Sigma_j) > -z_d\Big\} \\
&=: \bigcup_{s \in [0,r]} G_s
\end{aligned}
$$

with

$$
\begin{aligned}
G_s := \Big\{(\bar{z}, z_d) \in \mathbb{R}^d : z_d = -\frac{3}{\hat{c}_0} \operatorname{dist}(\bar{y}, P \setminus \Sigma_j) - s,\ \bar{z} \in \mathbf{B}_{\sqrt{r^2 - s^2}}(\bar{y}),& \\
\frac{3}{\hat{c}_0} \operatorname{dist}(\bar{z}, P \setminus \Sigma_j) > \frac{3}{\hat{c}_0} \operatorname{dist}(\bar{y}, P \setminus \Sigma_j) + s\Big\}&.
\end{aligned}
$$

We now note the representation $G_s = \mathbf{G}_s \times \{-\frac{3}{\hat{c}_0} \operatorname{dist}(\bar{y}, P \setminus \Sigma_j) - s\}$ with

$$
\mathbf{G}_s = \mathbf{B}_{\sqrt{r^2 - s^2}}(\bar{y}) \cap \Big\{\bar{z} \in \mathbb{R}^{d-1} : \frac{3}{\hat{c}_0} \operatorname{dist}(\bar{z}, P \setminus \Sigma_j) > \frac{3}{\hat{c}_0} \operatorname{dist}(\bar{y}, P \setminus \Sigma_j) + s\Big\}.
$$

Thus, applying Cavalieri's principle,

$$
\lambda_d\big(B_r(\mathrm{y}) \setminus \Lambda\big) \geq \int_0^r \lambda_{d-1}(G_s) \, \mathrm{d}s = \int_0^r \lambda_{d-1}(\mathbf{G}_s) \, \mathrm{d}s \geq \int_{\frac{r}{2}}^{\frac{r}{\sqrt{2}}} \lambda_{d-1}(\mathbf{G}_s) \, \mathrm{d}s.
$$

For $s \in [0, \frac{r}{\sqrt{2}}]$ we have $\mathbf{B}_s(\bar{y}) \subseteq \mathbf{B}_{\sqrt{r^2 - s^2}}(\bar{y})$. On the other hand, for $s \geq \frac{r}{2}$ the supposition $r \geq 3/\hat{c}_0 \cdot \operatorname{dist}(\bar{y}, P \setminus \Sigma_j)$ yields $3s \geq r + s \geq 3/\hat{c}_0 \cdot \operatorname{dist}(\bar{y}, P \setminus \Sigma_j) + s$. So, for $\frac{r}{2} \leq s \leq \frac{r}{\sqrt{2}}$,

$$
\mathbf{G}_s \supseteq \mathbf{B}_s(\bar{y}) \cap \Big\{\bar{z} \in \mathbb{R}^{d-1} : \frac{3}{\hat{c}_0} \operatorname{dist}(\bar{z}, P \setminus \Sigma_j) > 3s\Big\},
$$

and using Lemma 4.18 we can continue to estimate:

$$\begin{aligned}\lambda_d\big(B_r(\mathrm{y}) \setminus \Lambda\big) &\geq \int_{\frac{r}{2}}^{\frac{r}{\sqrt{2}}} \lambda_{d-1}\big(\{\bar{\mathrm{z}} \in \mathbf{B}_s(\bar{\mathrm{y}})\colon \operatorname{dist}(\bar{\mathrm{z}}, P \setminus \Sigma_j) > \hat{c}_0 s\}\big)\,\mathrm{d}s \\ &\geq \hat{c}_1 \int_{\frac{r}{2}}^{\frac{r}{\sqrt{2}}} s^{d-1}\,\mathrm{d}s \\ &= \frac{\hat{c}_1}{d}\Big[\Big(\frac{1}{2}\Big)^{\frac{d}{2}} - \Big(\frac{1}{2}\Big)^{d}\Big] r^d = \frac{\hat{c}_1(2^{\frac{d}{2}} - 1)}{d\omega_d 2^d}\lambda_d\big(B_r(\mathrm{y})\big).\end{aligned}$$

This was the claim for $\mathrm{y} \in \Sigma_j \cap P$.

It remains to discuss the points $\mathrm{y} \in \psi(\partial Q_- \setminus P)$. Clearly, Λ is contained in the 'strip' $(-1,1)^{d-1} \times \mathbb{R}$, and ψ maps the lateral faces $M := \{-1,1\}^{d-1} \times [-1,0]$ of Q_- into $\{-1,1\}^{d-1} \times (-\infty, 0]$ which are exactly (the 'lower' half of) the faces of $(-1,1)^{d-1} \times \mathbb{R}$. Thus, for $\mathrm{y} \in \psi(M)$, the set $B_r(\mathrm{y}) \setminus \Lambda$ contains at least half of the ball $B_r(\mathrm{y})$ and we have

$$\lambda_d\big(B_r(\mathrm{y}) \setminus \Lambda\big) \geq \lambda_d\big(B_r(\mathrm{y}) \setminus ((-1,1)^d \times \mathbb{R})\big) \geq \frac{1}{2}\lambda_d(B_r(\mathrm{y})).$$

The only case left is $\mathrm{y} \in \psi\big(\{-1\} \times (-1,1)^{d-1}\big)$, i.e., y is in the image of the 'bottom' of the half cube. Then the ball $B_r(\mathrm{y})$ lies completely inside the 'lower' halfspace $[z_d \leq 0]$ for all $r \in (0,1]$. Thus, since ψ was volume-preserving,

$$B_r(\mathrm{y}) \setminus \Lambda = B_r(\mathrm{y}) \setminus \psi(Q_-) = B_r\big(\psi(\psi^{-1}(\mathrm{y}))\big) \setminus \psi(Q_-).$$

By Lemma 4.11 we get the desired estimate once we can prove it for the untransformed geometry $B_r(\psi^{-1}(\mathrm{y})) \setminus Q_-$ where $\psi^{-1}(\mathrm{y})$ is in the bottom face of the unit cube. But this is straightforward since Q_- is convex. □

Let us mention that in [10] there is an even sharper version of the foregoing result (see Lemma 5.7 there), giving an explicit estimate of γ in terms of the input data c_0, c_1 in Assumption 4.1. With Lemma 4.20 at hand, we complete the proof of Theorem 4.2 easily with the pendant to Lemma 4.16; its proof is completely analogous to the one of Lemma 4.16 up to the additional transformation ψ.

Lemma 4.21. *Let $j \in \{1,2,\dots,m\}$ with $\mathrm{x}_j \in D \cap \overline{N}$. Then there is some $\alpha_j > 0$ independent of f such that $\eta_j v \in C^{\alpha_j}(\Omega)$ and we have*

$$\|\eta_j v\|_{C^{\alpha_j}(\Omega)} \leq c\|f\|_{W_D^{-1,q}(\Omega)}.$$

The constant c is uniform with respect to the $L^\infty(\Omega)$-bound and ellipticity constant κ_{ell} of the given coefficient function μ.

We have shown in Lemmata 4.15, 4.16 and 4.21 that all localized functions ηv_j for $j = 0, \dots, m$ are Hölder continuous of (possibly different) degree α_j with an estimate against $f \in W_D^{-1,q}(\Omega)$ which is uniform with respect to the $L^\infty(\Omega)$-bound and ellipticity constant κ_{ell} of the given coefficient function μ. Thus, if we choose α to be the minimum of the α_j, the claim of Theorem 4.2 follows and we are done. □

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