Pure and Applied Functional Analysis
Volume 9, Number 1, 2024, 111–168

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

MEAN VALUE INEQUALITY AND GENERALIZED CAPACITY ON DOUBLING SPACES

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Abstract. We prove a mean value inequality for subharmonic functions of a regular Dirichlet form in a doubling metric measure space, assuming that the Dirichlet form satisfies the Faber-Krahn inequality, the tail estimate of jump measure outside balls, as well as the generalized capacity condition. We also prove the equivalence between different forms of the generalized capacity condition.

1. Introduction

The classical mean value theorem for harmonic functions says the following: if u is a harmonic function in an open domain $\Omega \subset \mathbb{R}^n$ then, for any ball $B(x_0, r) \Subset \Omega$,

$$u(x_0) = \fint_{B(x_0,r)} u\, dx.$$

(Here and in the sequel, the notation $A \Subset U$ means that $\overline{A}$ is compact and $\overline{A} \subset U$). This theorem implies all other essential properties of harmonic functions including convergence theorems and the Harnack inequality. J. Moser proved in [43] the Harnack inequality for solutions of the equation $Lu = 0$ where

$$Lu = \sum_{i,j=1}^{n} \partial_{x_i} \left(a_{ij}(x) \partial_{x_j} u \right) \tag{1.1}$$

is a uniformly elliptic operator with measurable coefficients. One of the main ingredients of Moser's proof was the *mean value inequality*:

$$\operatorname*{esup}_{B(x_0,\frac{1}{2}r)} |u| \le C \left(\fint_{B(x_0,r)} u^2 dx \right)^{1/2}$$

that he proved by means of an ingenious iteration argument that is nowadays referred to as Moser's iteration.

The mean value and Harnack inequalities play also an important role in Analysis on metric measure spaces where the operator L is replaced by the generator of a

AG was Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - SFB 1283/2 2021 - 317210226' and the Tsinghua Global Scholars Fellowship Program. EH was supported by the National Natural Science Foundation of China (No. 12171354). JH was supported by the National Natural Science Foundation of China (No. 11871296) and SFB 1283.

Dirichlet form. For example, for the operator (1.1) the corresponding Dirichlet form is

$$\mathcal{E}(f,f)=\int_{\mathbb{R}^n}\sum_{i,j=1}^{n}a_{ij}\partial_{x_i}f\,\partial_{x_j}f\,dx. \tag{1.2}$$

This Dirichlet form is *local* as it is determined by differential operators. However, of high interest are also non-local Dirichlet forms whose generators are integral operators. For example, the following Dirichlet form of *jump type*

$$\mathcal{E}(f,f)=C(n,\beta)\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(f(x)-f(y))^2}{|x-y|^{n+\beta}}dxdy \tag{1.3}$$

has the generator $\Delta^{\beta/2}$ provided $0<\beta<2$ where Δ is the positive definite Laplace operator and $C(n,\beta)$ is a positive constant depending only on n and β.

Our purpose is to develop Analysis on a general metric measure space (M,d,μ) with a Dirichlet form $(\mathcal{E},\mathcal{F})$ that is defined axiomatically in the spirit of [18] and can contain a local part $\mathcal{E}^{(L)}$ as well as a jump part

$$\mathcal{E}^{(J)}(f,f)=\iint_{M\times M}(f(x)-f(y))^2J(x,y)d\mu(x)d\mu(y), \tag{1.4}$$

where J is a jump kernel. A major motivation is to include fractal spaces where the existence of self-similar local regular Dirichlet forms and associated diffusions was proved in [3], [9], [20], [33], [34], [35], etc. There has been a number of works devoted to the mean value and Harnack inequalities as well as to heat kernel bounds. Various results in the case of local Dirichlet forms were obtained in [1], [4], [5], [7], [8], [27], [28], [30], [31], etc, while the jump type Dirichlet forms were considered in [6], [10], [13], [15], [16], [17], [23], [24], [25], [29], etc.

All the works in this area have encountered one major difficulty that was not present in similar research in $\mathbb{R}^n$ or on manifolds: a priori absence of suitable cutoff functions. Given a pair of concentric balls $B(x,R)$ and $B(x,R+r)$ in $\mathbb{R}^n$, a *bump function* ϕ of this pair is equal to 1 on the interior ball, vanishes outside the exterior ball and is linear in radius in the annulus between the balls so that

$$|\nabla\phi|\le\frac{1}{r}. \tag{1.5}$$

It follows that, for any measurable function u,

$$\int_{\mathbb{R}^n}u^2|\nabla\phi|^2\,dx\le\frac{1}{r^2}\int_{\mathbb{R}^n}u^2\,dx, \tag{1.6}$$

and this simple inequality is frequently used (in particular, in Moser's argument). Perhaps, in Analysis in $\mathbb{R}^n$ nobody would give a significance to (1.6) but when working on general metric measure spaces, one quickly realizes helplessness without such a function ϕ. More precisely, although a bump function can be still defined as above by using the distance function, but an analogue of (1.5) does not have to be true as the gradient is determined by the Dirichlet form, and the latter does not have to be related in any way to the distance function.

For applications one needs an analogue of (1.6), and the existence of such function ϕ was obtained in a tricky way in [1] and [4] assuming that the heat kernel of $(\mathcal{E},\mathcal{F})$ satisfies a certain sub-Gaussian upper bounds. This analogue of (1.6) was referred to in [1] and [4] as a *cutoff Sobolev inequality*. Different versions of this condition were used in [24] and [30] under the name *generalized capacity estimate*.

In the present paper we consider two versions of the generalized capacity estimate: a weak version shortly denoted by (Gcap) which claims the existence of a test function ϕ specific to u, and a strong version denoted by (GU) which claims the existence of a universal test function ϕ that serves all functions u (like in $\mathbb{R}^n$).

One of the main results of this paper is the equivalence (Gcap) $\Leftrightarrow$ (GU) that we prove under some other hypotheses about space and energy (Theorem 2.11).

Another main result is the *mean value inequality* for subharmonic functions in the general setting, assuming (Gcap) and some other hypotheses (Theorem 2.10). It is worth mentioning that the proof of the implication (Gcap) $\Rightarrow$ (GU) in Theorem 2.11 is done by using the mean value inequality of Theorem 2.10.

The "other hypotheses" mentioned above include the Faber-Krahn inequality (FK) and a tail estimate of the jump kernel J denoted by (TJ). They are explained in details below. Here we only mention that (FK) refers to the spectral properties of the generator of $\mathcal{E}$, while (TJ) is an upper bound in terms of x and r of the integral

$$\int_{M\setminus B(x,r)} J(x,y)d\mu(y),$$

that is called the tail of the jump kernel. As far as we know, these hypotheses are weakest possible among all considered in the literature as they do not require pointwise estimates of $J(x,y)$.

The results of this paper will be used in subsequent research for obtaining heat kernel estimates under weakest possible hypotheses about the jump kernel, and we plan to address these problems in forthcoming papers.

In conclusion of this introduction, let us mention that creation of tools for a direct derivation of (Gcap) remains one of the most important open problems in this area.

Structure of the paper. In Section 2 we give all necessary definitions and state the main results. In Sections 3 and 4 we give examples of the Faber-Krahn inequality and the generalized capacity condition, respectively.

In Section 5, we recall some properties of energy measures. In Sections 6 and 7 we discuss one more condition (ABB) that serves as a bridge between (Gcap) and the *energy product* property that is proved in Section 8.

In Section 9 we prove some elementary properties of subharmonic functions. The mean value inequality for subharmonic functions (Theorem 2.10) is proved in Section 10 as Theorem 10.1.

In Section 11 we prove a so called Lemma of Growth that is used then in Section 12 to obtain estimates of the mean exit time from balls, which in turn implies a *survival estimate* in Section 13; the latter yields then (GU). Finally, Theorem 2.11 is proved in Section 14 as Theorem 14.1 that contains all the results of this paper.

In Appendix we prove some auxiliary results.

NOTATION. Letters c, C, C', C_1, C_2, etc. are used to denote universal positive numbers, whose values may change at any occurrence but depend only on the constants in the hypotheses. In the double integral $\iint_{U\times V} F(x,y)dj(x,y)$, the variable x is taken in U and y in V. Moreover, we may write $\iint_{U\times V} F(x,y)dj(x,y)$ shortly as $\iint_{U\times V} F(x,y)dj$. For a measurable function u on M, the notation $\mathrm{supp}(u)$ means the support of u, that is, the complement of the maximal open set where $u=0$ a.e..

2. Main results

Metric measure space with energy. Let (M,d) be a locally compact separable metric space and let μ be a Radon measure on M with full support. The triple (M,d,μ) is referred to as a *metric measure space*. Let $(\mathcal{E},\mathcal{F})$ be a regular symmetric Dirichlet form in $L^2 := L^2(M,\mu)$. In this paper we always assume that the Dirichlet form $(\mathcal{E},\mathcal{F})$ has no killing part, which means that

$$\mathcal{E}(u,v) = \mathcal{E}^{(L)}(u,v) + \mathcal{E}^{(J)}(u,v) \tag{2.1}$$

where $\mathcal{E}^{(L)}$ is the *local part* (or *diffusion part*) and $\mathcal{E}^{(J)}$ is *the jump part* associated with a unique Radon measure j defined on $M\times M\setminus \mathrm{diag}$:

$$\mathcal{E}^{(J)}(u,v) = \iint_{M\times M\setminus \mathrm{diag}} (u(x)-u(y))(v(x)-v(y))dj(x,y), \tag{2.2}$$

For simplicity, we set $j=0$ on diag and will drop diag in expression $M\times M\setminus \mathrm{diag}$ in (2.2) when no confusion arises.

Denote by $\mathrm{diam}\, M$ the diameter of the metric space (M,d) and fix throughout the paper a value $\overline{R}\in(0,\mathrm{diam} M]$. Note that $\overline{R}$ can be finite or infinite when M is unbounded.

In order to state our main results, let us introduce some notations and hypotheses. Denote metric balls in (M,d) by

$$B(x,r) := \{y\in M : d(y,x) < r\}$$

and set

$$V(x,r) := \mu(B(x,r)).$$

We say that a measure μ satisfies the *volume doubling* condition (VD) (or μ is a *doubling measure*) if there exists a constant $C\geq 1$ such that, for all $x\in M$ and all $r>0$,

$$V(x,2r) \leq CV(x,r). \tag{2.3}$$

Condition (VD) implies that $0 < V(x,r) < \infty$ for all $r>0$. We set $V(x,0)=0$ for all $x\in M$. If μ is a doubling measure, then the space (M,d,μ) is called a *doubling space*.

It is known that (VD) implies (and hence, is equivalent to) the following condition: there exists a positive number α such that, for all $x,y\in M$ and all $0<r\leq R<\infty$,

$$\frac{V(x,R)}{V(y,r)} \leq C\left(\frac{d(x,y)+R}{r}\right)^{\alpha},$$

where constant C can be taken the same as in (VD).

Scaling function and generalized capacity. Let us fix another function $W(x,r)$ also defined for all $x \in M$ and $r > 0$. We refer to W as a *scaling function* as it will be used for describing connection of the energy $\mathcal{E}$ to the metric measure structure and, consequently, the space/time scaling for the Hunt process associated to $(\mathcal{E}, \mathcal{F})$. For example, if $M = \mathbb{R}^n$ with the Euclidean distance and Lebesgue measure and if $\mathcal{E}$ is the classical Dirichlet integral

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^n} |\nabla u|^2 dx$$

then $W(x,r) = r^2$. On typical fractal spaces with appropriate local Dirichlet form we have

$$W(x,r) = r^\beta \tag{2.4}$$

where $\beta > 2$ is the *walk dimension.* On the other hand, if $\mathcal{E}$ is the following jump type Dirichlet form in $\mathbb{R}^n$

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+\beta}} dxdy$$

then W has also the form (2.4) although this time $0 < \beta < 2$.

In general, let us impose the following restriction on function W:

(1) for any $x \in M$, the function $r \mapsto W(x,r)$ is strictly increasing, $W(x,0) = 0$ and $W(x,\infty) = \infty$;
(2) there exist positive numbers C, β such that, for all $0 < r \leq R < \infty$ and for all $x, y \in M$ with $d(x,y) \leq R$,

$$\frac{W(x,R)}{W(y,r)} \leq C \left(\frac{R}{r}\right)^\beta. \tag{2.5}$$

Let us give an example of the jump process on $\mathbb{R}^n$ whose the space/time scaling can be described by a function W that actually depends on the space variable x.

Example 2.1. On $\mathbb{R}^n$, for $0 < \varepsilon < \beta < 2$, set

$$W(x,r) = \left(\frac{|x| + r}{r}\right)^\varepsilon r^\beta, \quad x \in \mathbb{R}^n, \ r > 0.$$

It is easy to prove that $W(x, \cdot)$ is strictly increasing for any fixed $x \in \mathbb{R}^n$, and for any $0 < r < R < \infty$ and $x, y \in \mathbb{R}^n$ with $|x - y| \leq R$,

$$\frac{1}{2^\varepsilon} \left(\frac{R}{r}\right)^{\beta - \varepsilon} \leq \frac{W(x,R)}{W(y,r)} \leq 2^\varepsilon \left(\frac{R}{r}\right)^\beta.$$

In particular, (2.5) is satisfied. Consider the jump kernel J satisfying

$$J(x,y) \simeq \frac{1}{|x-y|^n W(x, |x-y|)},$$

and denote the energy form associated with the above jump kernel J as in (1.4) by $\mathcal{E}^{(W)}$. (Here and in the sequel the notation $\simeq$ means that the ratio of the functions on its both sides is bounded from above and below by two positive constants respectively.)

Since $W(x,r) \geq r^{\beta}$ for all $x \in \mathbb{R}^n$ and $r > 0$, it is easy to see that

$$\mathcal{E}^{(W)}(f,f) \leq c\mathcal{E}(f,f) < \infty, \quad f \in C_0^{\infty}(\mathbb{R}^n),$$

where $\mathcal{E}$ is the Dirichlet form (defined in (1.3)) with generator $\Delta^{\beta/2}$, $c > 0$ is a universal constant, and $C_0^{\infty}(\mathbb{R}^n)$ is the collection of smooth functions on $\mathbb{R}^n$ with compact supports. Hence, the form $\mathcal{E}^{(W)}$ can be extended to a regular Dirichlet form, say $(\mathcal{E}^{(W)}, \mathcal{F})$, on $L^2(\mathbb{R}^n)$, where

$$\mathcal{F} := \text{ the closure of } C_0^{\infty}(\mathbb{R}^n) \text{ with respect to the norm } \sqrt{\mathcal{E}_1^{(W)}(\cdot,\cdot)},$$

and $\mathcal{E}_1^{(W)}(u,v) = \mathcal{E}^{(W)}(u,v) + (u,v)_{L^2}$. In this case, the space/time scaling for the jump process associated to $(\mathcal{E}^{(W)}, \mathcal{F})$ can be described by the function W.

One can construct more examples of functions W, similar to that in Example 2.1, on an abstract metric space (M,d). For example, let $0 < \varepsilon < \beta$ and

$$W(x,r) = \left(\frac{d(o,x)+r}{r}\right)^{\varepsilon} \phi(r), \quad x \in M, \ r > 0,$$

where $o \in M$ is a fixed point and $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfies the following condition: for all $0 < r < R$,

$$\frac{\phi(R)}{\phi(r)} \leq C\left(\frac{R}{r}\right)^{\beta}.$$

For convenience, for any metric ball $B := B(x,r)$, we write

$$W(B) := W(x,r).$$

Note that in some metric spaces a ball as a subset of M may have different centers and radii, that is, it may be possible that $B(x_1,r_1) = B(x_2,r_2)$ whereas $x_1 \neq x_2$ or $r_1 \neq r_2$. To avoid ambiguities in the notation $W(B)$ and other similar notations, we always identify a ball as a pair of center and radius rather than as a subset of M.

Let us define the notion of κ-cutoff function where $\kappa \geq 1$ is a fixed real. Let $U \subset M$ be an open set and A be a Borel subset of U. A *κ-cutoff function* of the pair (A,U) is any function $\phi \in \mathcal{F}$ such that

- $0 \leq \phi \leq \kappa$ μ-a.e. in M;
- $\phi \geq 1$ μ-a.e. in A;
- $\phi = 0$ μ-a.e. in U^c.

We denote by κ-cutoff(A,U) the collection of all κ-cutoff functions of the pair (A,U). Any 1-cutoff function for $\kappa = 1$ will be simply referred to as a *cutoff function*. Clearly, $\phi \in \mathcal{F}$ is a cutoff function of (A,U) if and only if $0 \leq \phi \leq 1$, $\phi|_A = 1$ and $\phi|_{U^c} = 0$. Set also

$$\operatorname{cutoff}(A,U) := 1\text{-}\operatorname{cutoff}(A,U).$$

Note that for every $\kappa \geq 1$,

$$\operatorname{cutoff}(A,U) \subset \kappa\text{-}\operatorname{cutoff}(A,U),$$

and that

$$\phi \in \kappa\text{- cutoff}(A, U) \Rightarrow 1 \wedge \phi \in \text{cutoff}(A, U). \tag{2.6}$$

It is known that, for a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, the class cutoff$(A, U)$ is not empty for any nonempty precompact set $A \Subset U$ (recall that $A \Subset U$ means that $\overline{A}$ is compact and $\overline{A} \subset U$).

Let $\mathcal{F}'$ be a linear spaced defined by

$$\mathcal{F}' := \{u + a : u \in \mathcal{F},\ a \in \mathbb{R}\},$$

that is, $\mathcal{F}'$ is obtained from $\mathcal{F}$ by adding all constants. Since $(\mathcal{E}, \mathcal{F})$ has no killing part, the bilinear form $\mathcal{E}$ can be extended to functions from $\mathcal{F}'$ as follows:

$$\mathcal{E}(u + a, v + b) = \mathcal{E}(u, v)$$

for all $u, v \in \mathcal{F}$ and $a, b \in \mathbb{R}$.

For any $u \in \mathcal{F}' \cap L^\infty$ and for any $\kappa \geq 1$, define the *generalized capacity* of a pair (A, U) as follows:

$$\text{cap}_u^{(\kappa)}(A, U) := \inf\left\{\mathcal{E}(u^2\phi, \phi) : \phi \in \kappa\text{- cutoff}(A, U)\right\}$$

(the function $u^2\phi$ belongs to $\mathcal{F}$ by Proposition 15.1(ii)). If $u \equiv 1$ then replacing ϕ by $1 \wedge \phi$, we obtain the usual capacity

$$\text{cap}_1^{(1)}(A, U) = \text{cap}(A, U) := \inf\left\{\mathcal{E}(\phi, \phi) : \phi \in \text{cutoff}(A, U)\right\}. \tag{2.7}$$

The following definition plays a central role in this paper.

Definition 2.2. We say that $(\mathcal{E}, \mathcal{F})$ satisfies the *generalized capacity condition* (Gcap) if there exist numbers $\kappa \geq 1$, $C > 0$ such that, for all $u \in \mathcal{F}' \cap L^\infty$ and any pair of concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R + r)$ with $x_0 \in M$ and $0 < R < R + r < \overline{R}$,

$$\text{cap}_u^{(\kappa)}(B_0, B) \leq \sup_{x \in B} \frac{C}{W(x, r)} \int_B u^2 d\mu. \tag{2.8}$$

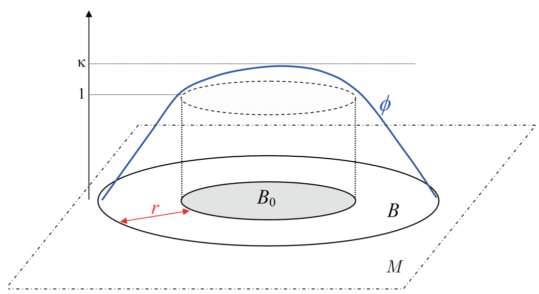

FIGURE 1. A function $\phi \in \kappa$-cutoff(B_0, B)

In other words, (Gcap) is satisfied if for all B_0, B as above and for any $u \in \mathcal{F}' \cap L^\infty$, there exists some $\phi \in \kappa$-cutoff(B_0, B) (as on Fig. 1) such that

$$\mathcal{E}(u^2\phi, \phi) \leq \sup_{x \in B} \frac{C}{W(x, r)} \int_B u^2 d\mu. \tag{2.9}$$

Let us emphasize that the function ϕ in (2.9) may depend on u, B_0, B but the constants κ, C are independent of u, B_0, B.

If the scaling function $W(x,r)$ is independent of the space variable x, say, $W(x,r) = W(r)$, then the inequality (2.9) becomes simpler

$$\mathcal{E}(u^2\phi, \phi) \leq \frac{C}{W(r)} \int_B u^2 d\mu.$$

Setting in (2.8) $u \equiv 1$ and using (2.6), we obtain

$$\operatorname{cap}(B_0, B) \leq \sup_{x \in B} \frac{C}{W(x,r)} \mu(B). \tag{2.10}$$

In particular, it follows from (2.10) and (2.5) that

$$\operatorname{cap}(\tfrac{1}{2}B, B) \leq \frac{C}{W(B)} \mu(B). \tag{2.11}$$

Definition 2.3. We say that $(\mathcal{E}, \mathcal{F})$ satisfies the capacity condition ($\text{Cap}_{\leq}$) if there exists a constant $C > 0$, such that (2.11) holds for all balls B of radius $R < \overline{R}$.

The above argument shows that

$$(\text{Gcap}) \Rightarrow (\text{Cap}_{\leq}). \tag{2.12}$$

Unlike (Gcap), the condition ($\text{Cap}_{\leq}$) can be effectively verified in many examples. We conjecture that in most (or even all) results about heat kernel estimates (Gcap) can be replaced by ($\text{Cap}_{\leq}$).

Now let us introduce a stronger condition (GU) that has a full title the *generalized capacity condition with universal cutoff function.*

Definition 2.4. We say that $(\mathcal{E}, \mathcal{F})$ satisfies the condition (GU) if there exist two numbers $\kappa \geq 1, C > 0$ such that, for any pair two concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R + r)$ with $x_0 \in M$ and $0 < R < R + r < \overline{R}$, there exists some $\phi \in \kappa\text{-cutoff}(B_0, B)$ such that (2.9) is satisfied for all $u \in \mathcal{F}' \cap L^\infty$.

Hence, in contrast to (Gcap), the test function ϕ in (2.9) is now independent of u, that is, universal (but, of course, ϕ depends on the balls). Clearly, (GU) $\Rightarrow$ (Gcap).

One of the results of this paper is that, under some mild assumptions, the opposite implication (Gcap) $\Rightarrow$ (GU) is also true (see Theorem 2.11 below).

Faber-Krahn inequality. For a non-empty open set $U \subset M$, denote by $C_0(U)$ the space of all continuous functions with compact supports in U. Let $\mathcal{F}(U)$ be a vector space defined by

$$\mathcal{F}(U) = \text{ the closure of } \mathcal{F} \cap C_0(U) \text{ with respect to the norm } \sqrt{\mathcal{E}_1(\cdot,\cdot)},$$

where $\mathcal{E}_\lambda(u,v) := \mathcal{E}(u,v) + \lambda(u,v)_{L^2}$ for $u, v \in \mathcal{F}$ and $\lambda > 0$. By the theory of Dirichlet form, $(\mathcal{E}, \mathcal{F}(U))$ is a regular Dirichlet form on $L^2(U, \mu)$ (see, for example, [19, Theorem 4.4.3]).

Let $\mathcal{L}^U$ be the (positive definite) generator of the Dirichlet form $(\mathcal{E}, \mathcal{F}(U))$. Denote by $\lambda_1(U)$ the *bottom* of the spectrum of $\mathcal{L}^U$ in $L^2(U, \mu)$. It is known that

$$\lambda_1(U) = \inf_{u \in \mathcal{F}(U) \setminus \{0\}} \frac{\mathcal{E}(u,u)}{\|u\|_2^2}. \tag{2.13}$$

Definition 2.5. We say that $(\mathcal{E}, \mathcal{F})$ satisfies the *Faber-Krahn inequality*, shortly (FK), if there exist $\sigma \in (0, 1]$ and $C, \nu > 0$ such that, for any ball $B = B(x, R)$ with $R < \sigma\overline{R}$ and for any non-empty open set $U \subset B$,

$$\lambda_1(U) \geq \frac{C^{-1}}{W(B)} \left(\frac{\mu(B)}{\mu(U)}\right)^{\nu}. \tag{2.14}$$

Sometimes we use notation (FK_ν) for (FK) in order to emphasize the exponent ν. Note that the value of ν can always be reduced without violating (2.14).

Remark 2.6. It is easy to see that (FK) and (VD) imply the following *lower* bound of capacity for any ball B of radius $R < \sigma\overline{R}$

$$\mathrm{cap}(\tfrac{1}{2}B, B) \geq C^{-1} \frac{\mu(B)}{W(B)}. \tag{2.15}$$

Indeed, for any $\phi \in \mathrm{cutoff}(U, B)$ we have by (2.13) and (2.14) with $U = B$

$$\mathcal{E}(\phi, \phi) \geq \lambda_1(B)\|\phi\|_2^2 \geq \frac{C^{-1}}{W(B)}\mu(\tfrac{1}{2}B),$$

whence (2.15) follows. This observation shows that, in some sense, the hypotheses (FK) and (Gcap) are complementary to each other. Nevertheless, they both are related to upper bounds of the heat kernel (see Sections 3 and 4).

Further results about deep relationships between eigenvalues and capacities can be found in [22], [37], [38], [39], [40], [41].

Tail estimate. We introduce here the condition (TJ) that provides a *tail estimate* of the jump measure in the exterior of balls.

Let $\mathcal{B}(M)$ be the sigma-algebra of Borel sets of M. Recall that a *transition kernel* $J : M \times \mathcal{B}(M) \mapsto \mathbb{R}_+$ is a map satisfying the following two properties:

- for every fixed x in M, the map $E \mapsto J(x, E)$ is a measure on $\mathcal{B}(M)$;
- for every fixed E in $\mathcal{B}(M)$, the map $x \mapsto J(x, E)$ is a non-negative measurable function on M.

Definition 2.7. We say that condition (TJ) is satisfied if there exists a transition kernel $J(x, E)$ on $M \times \mathcal{B}(M)$ such that

$$dj(x, y) = J(x, dy)d\mu(x) \quad \text{in } M \times M,$$

and, for any point x in M and any $R > 0$,

$$J(x, B(x, R)^c) = \int_{B(x,R)^c} J(x, dy) \leq \frac{C}{W(x, R)}, \tag{2.16}$$

where $C \in [0, \infty)$ is a constant independent of x, R.

If $B(x, R)^c$ is empty, the inequality (2.16) is automatically true. If $W(x, R) = R^\beta$ for any x in M and $R > 0$ then the inequality (2.16) becomes

$$J(x, B(x, R)^c) \leq \frac{C}{R^\beta} \quad \text{for all } x \text{ in } M \text{ and } R > 0.$$

The latter condition was introduced and studied in [10] in the setting of ultra-metric spaces.

Andres-Barlow-Bass condition. The local part $\mathcal{E}^{(L)}$ of the Dirichlet form $(\mathcal{E},\mathcal{F})$ determines for any $u\in\mathcal{F}'$ an *energy measure* $\Gamma^{(L)}(u)$ that, in particular, satisfies the identity

$$\mathcal{E}^{(L)}(u,u)=\int_M d\Gamma^{(L)}(u)$$

(see Section 5 for details). For example, for the Dirichlet form (1.2), we have

$$d\Gamma^{(L)}(u)(x)=\sum_{i,j=1}^{n}a_{ij}(x)\partial_{x_i}u\,\partial_{x_j}u\,dx.$$

The jump part $\mathcal{E}^{(J)}$ also gives rise to an energy measure as follows. For any open set $\Omega\subset M$ and any $u\in\mathcal{F}'$, define a measure $\Gamma_\Omega^{(J)}(u)$ by

$$d\Gamma_\Omega^{(J)}(u)(x)=\int_{y\in\Omega}(u(x)-u(y))^2dj(x,y),$$

which means that, for any non-negative measurable function v,

$$\int_M v\,d\Gamma_\Omega^{(J)}(u)=\int_{x\in M}\int_{y\in\Omega}v(x)(u(x)-u(y))^2dj(x,y).$$

In particular, we have

$$\mathcal{E}^{(J)}(u,u)=\int_M d\Gamma_M^{(J)}(u).$$

Define a measure $\Gamma_\Omega(u)$ by

$$d\Gamma_\Omega(u)=d\Gamma^{(L)}(u)+d\Gamma_\Omega^{(J)}(u). \tag{2.17}$$

Definition 2.8. We say that condition (ABB) is satisfied if there exist constants $C_1\geq 0$, $C_2>0$ such that, for any $u\in\mathcal{F}'\cap L^\infty$ and for any three concentric balls $B_0:=B(x_0,R)$, $B:=B(x_0,R+r)$ and $\Omega:=B(x_0,R')$ with $0<R<R+r<R'<\overline{R}$, there exists some $\phi\in\operatorname{cutoff}(B_0,B)$ such that

$$\int_\Omega u^2d\Gamma_\Omega(\phi)\leq C_1\int_B\phi^2d\Gamma_B(u)+\sup_{x\in\Omega}\frac{C_2}{W(x,r)}\int_\Omega u^2d\mu.$$

In Section 6 we will give a refined version of (ABB) tracing the value of C_1. The condition (ABB) is named after Andres, Barlow and Bass, who first introduced it in [1], [4] for local Dirichlet forms under the name *cut-off Sobolev inequality.* For jump type Dirichlet forms it was introduced and used in [10], [15], [16], [17], [24].

As (Gcap), the condition (ABB) is also meant to be a replacement of (1.6) in Analysis on metric measure spaces. Although the definition of (ABB) is more complicated than that of (Gcap), condition (ABB) is easier in applications. In fact, we prove in this paper that, under standing hypotheses (VD), (FK) and (TJ), the following equivalence holds:

$$\text{(Gcap)}\Leftrightarrow\text{(ABB)}+(\text{Cap}_{\leq}) \tag{2.18}$$

(see Theorem 2.11 below).

Subharmonic functions.

Definition 2.9. Let Ω be an open subset of M. We say that a function $u \in \mathcal{F}$ is *subharmonic* in Ω if, for any $0 \le \varphi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(u, \varphi) \le 0.$$

For any ball $B = B(x, r)$ and a positive number λ, denote $\lambda B := B(x, \lambda r)$. Here is our first main result: the *mean value inequality* for subharmonic functions.

Theorem 2.10. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part and with jump measure j. Assume that conditions* (VD), (Gcap), (FK) *and* (TJ) *hold. Let a function $u \in \mathcal{F} \cap L^\infty$ be non-negative and subharmonic in a ball $B := B(x_0, R)$ with $0 < R < \sigma \overline{R}$. Then, for any $\varepsilon > 0$,*

$$\operatorname*{esup}_{\frac{1}{2}B} u \le C(1 + \varepsilon^{-\frac{1}{2\nu}}) \left(\frac{1}{\mu(B)} \int_B u^2 d\mu \right)^{1/2} + \varepsilon K \|u_+\|_{L^\infty((\frac{1}{2}B)^c)}, \tag{2.19}$$

where the constant C depends only on the constants in the hypotheses (but does not depend on ε), the constants ν and σ come from (FK), *and*

$$K = \begin{cases} 1 & \text{if the measure } j \not\equiv 0, \\ 0 & \text{if the measure } j \equiv 0. \end{cases}$$

In the case when the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is strongly local (that is, when $K = 0$) the mean value inequality ((2.19) with $\varepsilon = 1$) for subharmonic functions was proved in [30, Theorem 6.3] although in the case when the scaling function W is independent of x and $\overline{R} = \infty$. The mean value inequality was one of the main ingredients for the proof of the Harnack inequality for harmonic functions in [30, Theorem 1.1].

Our contribution in Theorem 2.10 is therefore threefold:

(1) the mean value inequality is proved for the first time for general Dirichlet forms containing jump part;
(2) the scaling function $W(x, r)$ is of general form allowing dependence on x;
(3) the result is localized in space: if (FK) holds for balls with restricted radii then the mean value inequality is also satisfied for balls with restricted radii.

We mention that mean value inequalities in different shapes are also obtained in [12] for fractional Laplacian on $\mathbb{R}^n$ and in [14] for symmetric jump processes on general metric measure spaces. In both papers, the pointwise upper bounds of jump kernel are fully used, while, in Theorem 2.10, we only use the tail estimate of the jump measure (TJ) (which does not require the existence of the jump kernel, and then it is much weaker than the pointwise upper bound of jump kernel). However, under the stronger condition, it is proved in [12] that the essential supremum of a harmonic function u over a smaller ball can be bounded from above by the L^p ($p > 1$) average of u over a larger ball plus the tail of u outside some ball. Similar result is obtained in [14] but for $p \in [1, 2]$.

Moreover, we prove here the following theorem clarifying the relationships between aforementioned conditions.

Theorem 2.11. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form without killing part. Assume that* (VD)*,* (FK) *and* (TJ) *hold true. Then we have the following equivalences:*

$$\begin{aligned}(\mathrm{Gcap}) &\Leftrightarrow (\mathrm{ABB}) + (\mathrm{Cap}_{\leq})\\ &\Leftrightarrow \textit{mean value inequality (2.19)} + (\mathrm{Cap}_{\leq})\\ &\Leftrightarrow (\mathrm{GU}).\end{aligned}$$

Of course, Theorem 2.10 is contained in Theorem 2.11, but it is interesting to observe that the proof of the implication

$$(\mathrm{Gcap}) \Rightarrow (\mathrm{GU})$$

goes through the mean value inequality! In Section 14 we state and prove even more general Theorem 14.1 containing Theorem 2.11.

3. Examples of (FK)

Example 3.1. Let (M,d,μ) be a complete Riemannian manifold of dimension n and $\mathcal{E}$ be the Dirichlet integral

$$\mathcal{E}(u,u)=\int_M |\nabla u|^2 d\mu,$$

where $u\in\mathcal{F}=W^{1,2}(M)$. The generator of $(\mathcal{E},\mathcal{F})$ is the Laplace-Beltrami operator Δ, and, for a precompact open set U, $\lambda_1(U)$ is the bottom eigenvalue of Δ in U with the Dirichlet boundary condition.

If $M=\mathbb{R}^n$ then, by a theorem of Faber and Krahn,

$$\lambda_1(U)\geq\lambda_1(U^*)$$

where U^* is a ball of the same volume as U. Let ρ be the radius of U^*. Since

$$\lambda_1(U^*)=\frac{c'}{\rho^2}$$

and

$$\mu(U)=\mu(U^*)=c''\rho^n,$$

where c',c'' are positive constants depending on n, it follows that

$$\lambda_1(U)\geq c\mu(U)^{-2/n}.$$

If $U\subset B=B(x,r)$ then it follows that

$$\lambda_1(U)\geq\frac{c}{r^2}\left(\frac{\mu(B)}{\mu(U)}\right)^{2/n}, \tag{3.1}$$

that is, (FK) is satisfied with the scaling function $W(x,r)=r^2$ and $\nu=2/n$.

It is known that if M has non-negative Ricci curvature then it also satisfies (3.1), that is, (FK) holds with $W(x,r)=r^2$, too (see [21]).

Example 3.2. Let (M,d,μ) be *α-regular*, that is, for all $x\in M$ and $r>0$,

$$V(x,r)\simeq r^\alpha$$

for some $\alpha>0$. If the jump kernel of $(\mathcal{E},\mathcal{F})$ satisfies the lower bound

$$J(x,y)\geq\frac{c}{d(x,y)^{\alpha+\beta}}$$

with some $c, \beta > 0$, then (FK) holds with the scaling function $W(x, r) = r^\beta$ (see [24, Lemma 3.5]).

Example 3.3. Let (M, d, μ) satisfy not only (VD) but also the *reverse volume doubling*: for all $R \geq r > 0$ and $x \in M$,

$$\frac{V(x,R)}{V(x,r)} \geq C^{-1}\left(\frac{R}{r}\right)^{\alpha'},$$

for some constants $C, \alpha' > 0$. Assume also that the scaling function W satisfies a similar condition: for all $R \geq r > 0$ and $x, y \in M$ with $d(x, y) \leq R$,

$$\frac{W(x,R)}{W(y,r)} \geq C^{-1}\left(\frac{R}{r}\right)^{\beta'},$$

for some constants $C, \beta' > 0$. Let the jump kernel of $(\mathcal{E}, \mathcal{F})$ satisfy the following lower bound: for all distinct $x, y \in M$,

$$J(x,y) \geq \frac{c}{V(x,y)W(x,y)},$$

where $V(x, y) = V(x, r)$ with $r = d(x, y)$ and similarly $W(x, y) = W(x, r)$. Then it was proved in [26] that $(\mathcal{E}, \mathcal{F})$ satisfies (FK) with the scaling function $W(x, r)$.

Let $\mathcal{L}$ be the (positive definite) generator of $(\mathcal{E}, \mathcal{F})$. Denote by $\{P_t\}$ the associated semigroup in L^2, that is, $P_t = e^{-t\mathcal{L}}$, and by $p_t(x, y)$ the integral kernel of the operator P_t should it exists. The function $p_t(x, y)$ is called the *heat kernel* of $(\mathcal{E}, \mathcal{F})$.

Example 3.4. Let now (M, d, μ) satisfy (VD). Assume that the heat kernel $p_t(x, y)$ of $(\mathcal{E}, \mathcal{F})$ satisfies for all $t > 0$ and for almost all $x, y \in M$ the following inequality:

$$p_t(x,y) \leq \frac{C}{\sqrt{V(x,t^{1/\beta})V(y,t^{1/\beta})}}, \tag{3.2}$$

for some $\beta > 0$. For example, if M is α-regular then (3.2) becomes

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}},$$

and this estimate is known to be satisfied for self-similar local Dirichlet forms on many fractal spaces. It was proved in [27, Proof of Theorem 2.1] that (3.2) implies (FK) with the scaling function $W(x, r) = r^\beta$.

4. Examples of (Gcap) and (ABB)

Here we give some examples of spaces and Dirichlet forms satisfying (Gcap) and (ABB).

Example 4.1. Let (M, d) be an *ultra-metric* space, that is, d satisfies the ultra-metric triangle inequality

$$d(x,y) \leq \max(d(x,z), d(y,z)).$$

For example, for a prime p, a field $\mathbb{Q}_p$ of p-adic numbers with p-adic distance is an ultra-metric space. An ultra-metric space has remarkable metric properties. For example, any point x inside a ball $B(x_0, \rho)$ is also its center, that is, $B(x, \rho) = B(x_0, \rho)$ (see [11] for details).

Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form of jump type (in fact, an ultra-metric space cannot carry a local Dirichlet form). We claim that in this case

$$\text{(TJ)} \Rightarrow \text{(Gcap)}.$$

Indeed, given two concentric $B_0 = B(x_0, R)$ and $B = B(x_0, R + r)$, it suffices to find a function $\phi \in \mathcal{F}$ such that

$$0 \leq \phi \leq 1, \quad \phi|_{B_0} = 1, \quad \phi|_{B^c} = 0$$

and

$$\mathcal{E}(u^2\phi, \phi) \leq \sup_{x\in B} \frac{C}{W(x, r)} \int_B u^2 d\mu \tag{4.1}$$

for any $u \in \mathcal{F}' \cap L^\infty$. A key observation is that, on ultra-metric space, the indicator functions of balls belong to $\mathcal{F}$ so that we take

$$\phi = \mathbf{1}_B.$$

(see [10] for details). With this ϕ we have

$$\begin{aligned}
\mathcal{E}(u^2\phi, \phi) &= \iint_{M\times M} \left(u^2\varphi(x) - u^2\varphi(y)\right)(\varphi(x) - \varphi(y))\, J(x, dy)d\mu(x) \\
&= 2\int_{x\in B}\int_{y\in B^c} \left(u^2\varphi(x) - u^2\varphi(y)\right)(\varphi(x) - \varphi(y))\, J(x, dy)d\mu(x) \\
&= 2\int_{x\in B}\int_{y\in B(x_0,R+r)^c} u^2(x)J(x, dy)d\mu(x) \\
&= 2\int_B u^2(x)\left(\int_{B(x,R+r)^c} J(x, dy)\right) d\mu(x) \\
&\leq C\int_B \frac{u^2(x)}{W(x, R + r)} d\mu(x) \\
&\leq C\int_B \frac{u^2(x)}{W(x, r)} d\mu(x),
\end{aligned}$$

which implies (4.1).

Since (TJ) is assumed as hypothesis in most of our results, in the case of ultra-metric space, in these results, the condition (Gcap) can be dropped from the list of hypotheses.

Example 4.2. Let a metric measure space satisfy the following hypothesis: for all balls $B(x, r)$,

$$\mu(B(x, r)) \leq Cr^\alpha$$

for some $C, \alpha > 0$. Let $(\mathcal{E},\mathcal{F})$ be of pure jump type and be given by (1.4) with a jump kernel $J(x, y)$. Assume also that, for some $\beta > 0$, the jump kernel satisfies the upper bound

$$J(x, y) \leq \frac{C}{d(x, y)^{\alpha+\beta}}. \tag{4.2}$$

Then it is easy to verify that (TJ) is satisfied with the scale function $W(x, r) = r^\beta$.

We claim that if $\beta < 2$ then (ABB) and (Gcap) also hold with the same scaling function. Indeed, let ϕ be a bump function of the pair of balls $B_0 = B(x_0, R)$ and $B = B(x_0, R + r)$ (see Fig. 2) so that

$$|\phi(x) - \phi(y)| \leq \frac{d(x,y)}{r}. \tag{4.3}$$

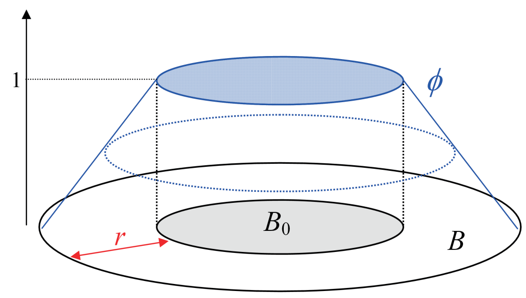

FIGURE 2. A bump function $\phi \in \text{cutoff}(B_0, B)$

Using (4.3) and $\beta < 2$, a computation in [24, Corollary 2.12] yields that, for any $x \in M$,

$$\int_M (\phi(x) - \phi(y))^2 J(x,y) d\mu(y) \leq \frac{C}{r^\beta}, \tag{4.4}$$

which implies that, for any open set $\Omega \supset B$,

$$\begin{aligned}\int_\Omega u^2 d\Gamma_\Omega(\phi) &= \int_\Omega \int_\Omega u^2(x)(\phi(x) - \phi(y))^2 J(x,y) d\mu(x) d\mu(y) \\ &\leq \frac{C}{r^\beta} \int_\Omega u^2(x) d\mu(x),\end{aligned}$$

whence (ABB) follows with $W(x,r) = r^\beta$ and $C_1 = 0$.

Let us verify ($\text{Cap}_\leq$). We have clearly

$$\begin{aligned}\mathcal{E}(\phi,\phi) &= \int_M \int_M (\phi(x) - \phi(y))^2 J(x,y) d\mu(y) \\ &= \int_B \int_M (\phi(x) - \phi(y))^2 J(x,y) d\mu(y) + \int_{B^c} \int_M (\phi(x) - \phi(y))^2 J(x,y) d\mu(y)\end{aligned}$$

and

$$\begin{aligned}\int_{B^c} \int_M (\phi(x) - \phi(y))^2 J(x,y) d\mu(y) &= \int_{B^c} \int_B (\phi(x) - \phi(y))^2 J(x,y) d\mu(y) \\ &\leq \int_M \int_B (\phi(x) - \phi(y))^2 J(x,y) d\mu(y).\end{aligned}$$

Hence, by the symmetry and (4.4),

$$\mathcal{E}(\phi,\phi) \leq 2 \int_B \int_M (\phi(x) - \phi(y))^2 J(x,y) d\mu(y) \leq \frac{C}{r^\beta} \mu(B),$$

which proves ($\text{Cap}_\leq$) with $W(x,r) = r^\beta$. By (2.18) we conclude that (Gcap) also holds with the same scaling function.

Note that on most fractal spaces there exist regular jump type Dirichlet forms with jump kernels

$$J(x,y) \simeq \frac{1}{d(x,y)^{\alpha+\beta}}$$

where $\beta \geq 2$. For this jump kernel there is no obvious cutoff function ϕ to ensure even ($\mathrm{Cap}_{\leq}$). Besides, in our main results $J(x,y)$ does not have to satisfy the upper bound (4.2), and the Dirichlet form may have also a local part.

Therefore, for the time being, (Gcap) and/or (ABB) should be accepted as hypotheses, leaving to the future the development of methods for proving them.

Example 4.3. Let (M,d,μ) be α-regular and $(\mathcal{E},\mathcal{F})$ be a jump type conservative Dirichlet form. Assume that its heat kernel satisfies the following *stable-like* upper estimate for some $\beta > 0$:

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}.$$

Then (Gcap) is satisfied with the scaling function $W(x,r) = r^\beta$ by a result of [29, Theorem 2.3]. A more general result of this type in the setting of doubling spaces was proved in [15, Proposition 3.5].

Example 4.4. Let (M,d,μ) satisfy (VD). Assume that $(\mathcal{E},\mathcal{F})$ is strongly local and conservative. Assume also that the heat kernel $p_t(x,y)$ of $(\mathcal{E},\mathcal{F})$ exists and satisfies for all $t > 0$ and for almost all $x, y \in M$ the following sub-Gaussian upper bounds:

$$p_t(x,y) \leq \frac{C}{V(x,t^{1/\beta})} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right), \tag{4.5}$$

with some $\beta > 1$. Then (Gcap) is satisfied with the scaling function $W(x,r) = r^\beta$ by results of [1, Theorem 1.12] and [30, Theorem 1.3]. Note that (4.5) holds on many fractal spaces with $\beta > 2$ (cf. [2]).

5. Energy measure

In this section we collect some elementary properties on energy measures, which will be used later on. Everywhere here and below, $(\mathcal{E},\mathcal{F})$ is any regular Dirichlet form in L^2 without killing part, that is, of the form (2.1). Set

$$\mathcal{F}_{\mathrm{loc}} := \{u : \forall\, U \Subset M, \text{ there exists } v \in \mathcal{F} \text{ so that } v = u \ \mu\text{-a.e. on } U\}.$$

Since $(\mathcal{E},\mathcal{F})$ is regular, the constant function 1 belongs to $\mathcal{F}_{\mathrm{loc}}$, so that

$$\mathcal{F}' \subset \mathcal{F}_{\mathrm{loc}}.$$

It is known that, for any $u \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty$, there exists a unique Radon measure $\Gamma^{(L)}(u) := \Gamma^{(L)}(u,u)$ such that, for any test function $f \in \mathcal{F} \cap C_0(M)$,

$$\int_M f d\Gamma^{(L)}(u) = \mathcal{E}^{(L)}(uf,u) - \frac{1}{2}\mathcal{E}^{(L)}(u^2,f).$$

Moreover, for any $u \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty$, we have

$$\mathcal{E}^{(L)}(u,u) = \int_M d\Gamma^{(L)}(u,u)$$

(see, for example, [19, Eq. (3.2.20), Lemma 3.2.3, and the first two paragraphs on p.130]).

The energy measures satisfy the following properties, for all $u, v, w \in \mathcal{F}_{\text{loc}} \cap L^\infty$:

- the *product rule* ([19, Lemma 3.2.5, and the second paragraph on p.130]):

$$d\Gamma^{(L)}(uv, w) = ud\Gamma^{(L)}(v, w) + vd\Gamma^{(L)}(u, w); \tag{5.1}$$

- the *chain rule* ([19, Theorem 3.2.2, and the second paragraph on p.130]):

$$d\Gamma^{(L)}(\Phi(u), v) = \Phi'(u)d\Gamma^{(L)}(v, w) \tag{5.2}$$

  for any $\Phi \in C^1(\mathbb{R})$ (one does not need to assume $\Phi(0) = 0$);
- the *strong locality*: if $u_1 \in \mathcal{F}_{\text{loc}}$ is constant in an open subset Ω of M and $u_2 \in \mathcal{F}_{\text{loc}}$ is arbitrary, then

$$\mathbf{1}_\Omega d\Gamma^{(L)}(u_1, u_2) = 0 \text{ on } M \tag{5.3}$$

  (cf. [19, Corollary 3.2.1 on p.128], or [42, Eq. (3.8) on p.387]), and

$$d\Gamma^{(L)}(u_+, v) = \mathbf{1}_{\{u>0\}}d\Gamma^{(L)}(u, v) \text{ on } M, \tag{5.4}$$

  where $u_+ = u \vee 0$ (cf. [42, formula (3.14) on p.390]);
- the *Cauchy-Schwarz inequality*: for any $f \in L^2(M, \Gamma^{(L)}(u))$, $g \in L^2(M, \Gamma^{(L)}(v))$

$$\int |fg| d\Gamma^{(L)}(u, v) \leq \left(\int f^2 d\Gamma^{(L)}(u)\right)^{1/2} \left(\int g^2 d\Gamma^{(L)}(v)\right)^{1/2} \tag{5.5}$$

  (cf. [42, on p. 390]).

Moreover, for any $u \in \mathcal{F}_{\text{loc}} \cap L^\infty$, we have

$$d\Gamma^{(L)}(|u|) = d\Gamma^{(L)}(u), \tag{5.6}$$

since $d\Gamma^{(L)}(u_+, u_-) = 0$ by using (5.4), (5.3), which gives that

$$\begin{aligned} d\Gamma^{(L)}(|u|) &= d\Gamma^{(L)}(u_+ + u_-, u_+ + u_-) \\ &= d\Gamma^{(L)}(u_+) + 2d\Gamma^{(L)}(u_+, u_-) + d\Gamma^{(L)}(u_-) \\ &= d\Gamma^{(L)}(u_+) + d\Gamma^{(L)}(u_-) \\ &= d\Gamma^{(L)}(u_+ - u_-, u_+ - u_-) \\ &= d\Gamma^{(L)}(u). \end{aligned}$$

Recall that for an open subset Ω of M and $u \in \mathcal{F}'$, the measure $\Gamma_\Omega(u)$ is defined in (2.17), that is,

$$d\Gamma_\Omega(u)(x) := d\Gamma^{(L)}(u)(x) + \int_M \mathbf{1}_\Omega(y)(u(x) - u(y))^2 dj(x, y).$$

Here the measure j vanishes on $\{x = y\}$ as a convention stated at the beginning of Section 2. Clearly, for any three open sets A, B, Ω with $A \subset B$, for any $u \in \mathcal{F}'$ and for any measurable function $f \geq 0$, the following inequalities hold:

$$\int_\Omega f d\Gamma_A(u) \leq \int_\Omega f d\Gamma_B(u), \tag{5.7}$$

and

$$\int_\Omega f d\Gamma_B(u\wedge 1) \le \int_\Omega f d\Gamma_B(u). \tag{5.8}$$

Proposition 5.1. *For any open set $\Omega \subset M$ and for any two functions $u \in \mathcal{F}'\cap L^\infty$, $\phi \in \mathcal{F}\cap L^\infty$ with* $\mathrm{supp}(\phi) \subset \Omega$*, we have*

$$\int_\Omega u^2 d\Gamma_\Omega(\phi) \le 4\int_\Omega \phi^2 d\Gamma_\Omega(u) + 2\mathcal{E}(u^2\phi,\phi). \tag{5.9}$$

A similar result was obtained in [32] but for $u \in \mathcal{F}\cap L^\infty$ (instead of $u \in \mathcal{F}'\cap L^\infty$ as here). We sketch the proof for the reader's convenience.

Proof. Since $u \in \mathcal{F}'\cap L^\infty$ and $\phi \in \mathcal{F}\cap L^\infty$, we have by Proposition 15.1

$$u^2 \in \mathcal{F}'\cap L^\infty \quad \text{and} \quad u^2\phi \in \mathcal{F}\cap L^\infty.$$

We first show that

$$\int_\Omega u^2 d\Gamma^{(L)}(\phi) \le 2\mathcal{E}^{(L)}(u^2\phi,\phi) + 4\int_\Omega \phi^2 d\Gamma^{(L)}(u). \tag{5.10}$$

Indeed, without loss of generality, we may assume that u,ϕ stand for their quasi continuous version (see Definition 15.2 and [19, Thorem 2.1.3, p. 71]). Since $\Gamma^{(L)}(\phi)$ charges no set of zero capacity (by [19, Lemma 3.2.4, p. 127]), by Proposition 15.3 in Appendix, we have

$$\int_M u^2 d\Gamma^{(L)}(\phi) \le \|u\|_\infty^2 \int_M d\Gamma^{(L)}(\phi) = \|u\|_\infty^2 \mathcal{E}^{(L)}(\phi) \le \|u\|_\infty^2 \mathcal{E}(\phi) < \infty,$$

which implies that $u \in L^2(M, d\Gamma^{(L)}(\phi))$. In a similar way, we have $\phi \in L^2(M, d\Gamma^{(L)}(u))$. Thus, using the chain rule (5.1) and the product rule (5.2) of $d\Gamma^{(L)}(\cdot)$ and using the Cauchy-Schwarz inequality (5.5), we have

$$\begin{aligned}\int_M u^2 d\Gamma^{(L)}(\phi) &= \int_M d\Gamma^{(L)}(u^2\phi,\phi) - 2\int_M u\phi d\Gamma^{(L)}(u,\phi)\\ &\le \mathcal{E}^{(L)}(u^2\phi,\phi) + \frac{1}{2}\int_M u^2 d\Gamma^{(L)}(\phi) + 2\int_M \phi^2 d\Gamma^{(L)}(u),\end{aligned}$$

which yields that, for any $u \in \mathcal{F}'\cap L^\infty$, $\phi \in \mathcal{F}\cap L^\infty$

$$\int_M u^2 d\Gamma^{(L)}(\phi) \le 2\mathcal{E}^{(L)}(u^2\phi,\phi) + 4\int_M \phi^2 d\Gamma^{(L)}(u). \tag{5.11}$$

Since ϕ is supported in Ω, we see by (5.3) that $d\Gamma^{(L)}(\phi) = 0$ outside Ω, and the two integrals in (5.11) are actually taken over Ω, thus proving (5.10).

We next show that

$$\begin{aligned}\iint_{\Omega\times\Omega} u^2(x)(\phi(x)-\phi(y))^2 dj \le{}& 2\mathcal{E}^{(J)}(u^2\phi,\phi)\\ &+ 4\iint_{\Omega\times\Omega} \phi^2(x)(u(x)-u(y))^2 dj.\end{aligned} \tag{5.12}$$

Indeed, note that

$$\frac{1}{2}(u^2(x)+u^2(y))(\phi(x)-\phi(y))^2 \le 2(\phi(x)-\phi(y))(u^2(x)\phi(x)-u^2(y)\phi(y))$$

$$+ 2(\phi^2(x) + \phi^2(y))(u(x) - u(y))^2,$$

see for example [24, the inequality on lines 3-4 on p. 447] with $f = u$, and $g = \phi$. Integrating over $\Omega \times \Omega$ against dj and using the symmetry of j, we have

$$\begin{aligned}\iint_{\Omega\times\Omega} u^2(x)(\phi(x) - \phi(y))^2 dj \leq 2 \iint_{\Omega\times\Omega} (\phi(x) - \phi(y))(u^2(x)\phi(x) - u^2(y)\phi(y))dj \\ + 4 \iint_{\Omega\times\Omega} \phi^2(x)(u(x) - u(y))^2 dj. \qquad (5.13)\end{aligned}$$

On the other hand, using the fact that $\operatorname{supp}(\phi) \subset \Omega$, we have

$$\begin{aligned}&\iint_{\Omega\times\Omega} (\phi(x) - \phi(y))(u^2(x)\phi(x) - u^2(y)\phi(y))dj \\ &= \left(\iint_{M\times M} - \iint_{\Omega\times\Omega^c} - \iint_{\Omega^c\times\Omega} - \iint_{\Omega^c\times\Omega^c}\right)\cdots \\ &= \mathcal{E}^{(J)}(u^2\phi, \phi) - \iint_{\Omega\times\Omega^c} \phi^2(x)u^2(x)dj - \iint_{\Omega^c\times\Omega} \phi^2(y)u^2(y)dj \\ &\leq \mathcal{E}^{(J)}(u^2\phi, \phi).\end{aligned}$$

Plugging this into (5.13), we obtain (5.12).

Finally, combining (5.10), (5.12), we conclude by definitions (2.1) and (2.17) that

$$\begin{aligned}\int_\Omega u^2 d\Gamma_\Omega(\phi) &= \int_\Omega u^2 d\Gamma^{(L)}(\phi) + \iint_{\Omega\times\Omega} u^2(x)(\phi(x) - \phi(y))^2 dj \\ &\leq 2\mathcal{E}^{(L)}(u^2\phi, \phi) + 4\int_\Omega \phi^2 d\Gamma^{(L)}(u) \\ &\quad + 2\mathcal{E}^{(J)}(u^2\phi, \phi) + 4\iint_{\Omega\times\Omega} \phi^2(x)(u(x) - u(y))^2 dj \\ &= 2\mathcal{E}(u^2\phi, \phi) + 4\int_\Omega \phi^2 d\Gamma_\Omega(u),\end{aligned}$$

thus proving (5.9). The proof is complete. □

Next, we need the following inequality.

Proposition 5.2. *For any open subset $\Omega \subset M$ and for any two functions $u \in \mathcal{F}' \cap L^\infty$, $\phi \in \mathcal{F} \cap L^\infty$ with* $\operatorname{supp}(\phi) \subset \Omega$, *we have*

$$\int_\Omega \phi^2 d\Gamma_\Omega(u) \leq 2\mathcal{E}(u, u\phi^2) + 4\int_\Omega u^2 d\Gamma_\Omega(\phi) + 4\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj. \qquad (5.14)$$

Proof. By definition (2.1), we have

$$\mathcal{E}(u, u\phi^2) = \mathcal{E}^{(L)}(u, u\phi^2) + \mathcal{E}^{(J)}(u, u\phi^2). \qquad (5.15)$$

For the local part $\mathcal{E}^{(L)}(u, u\phi^2)$, interchanging u and ϕ in (5.11) (at this stage the assumption $\operatorname{supp}(\phi) \subset \Omega$ has not yet been used) and then using (5.3) and $\operatorname{supp}(\phi) \subset \Omega$, we obtain

$$\int_\Omega \phi^2 d\Gamma^{(L)}(u) = \int_M \phi^2 d\Gamma^{(L)}(u) \leq 2\mathcal{E}^{(L)}(u, u\phi^2) + 4\int_\Omega u^2 d\Gamma^{(L)}(\phi). \qquad (5.16)$$

For the jump part $\mathcal{E}^{(J)}(u, u\phi^2)$, we have by (2.2) and the fact that $\mathrm{supp}(\phi) \subset \Omega$

$$\begin{aligned}
\mathcal{E}^{(J)}(u, u\phi^2) &= \iint_{M\times M} (u(x)-u(y))(u(x)\phi^2(x)-u(y)\phi^2(y))dj \\
&= \left(\iint_{\Omega\times\Omega} + \iint_{\Omega^c\times M} + \iint_{\Omega\times\Omega^c}\right) F(x,y)dj \\
&= \iint_{\Omega\times\Omega} F(x,y)dj - \iint_{\Omega^c\times M} (u(x)-u(y))u(y)\phi^2(y)dj \\
&\qquad + \iint_{\Omega\times\Omega^c} (u(x)-u(y))u(x)\phi^2(x)dj, \\
&\geq \iint_{\Omega\times\Omega} F(x,y)dj - \iint_{\Omega^c\times M} u(x)u(y)\phi^2(y)dj \\
&\qquad - \iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj, \qquad (5.17)
\end{aligned}$$

where F is defined by

$$F(x,y) := (u(x)-u(y))(u(x)\phi^2(x)-u(y)\phi^2(y)).$$

Since ϕ is supported in Ω,

$$\begin{aligned}
\iint_{\Omega^c\times M} u(x)u(y)\phi^2(y)dj &= \iint_{\Omega^c\times\Omega} u(x)u(y)\phi^2(y)dj \\
&= \iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj
\end{aligned}$$

by using the symmetry of j. It follows from (5.17) that

$$\iint_{\Omega\times\Omega} F(x,y)dj \leq \mathcal{E}^{(J)}(u, u\phi^2) + 2\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj. \qquad (5.18)$$

On the other hand, by using a general result in [24, Lemma 2.2] with $E = \Omega, f = \phi, g = u$ and noting that $\phi|_{\Omega^c} = 0$, we obtain

$$\begin{aligned}
\iint_{\Omega\times\Omega} (u(x)-u(y))^2\phi^2(x)dj &\leq 2\iint_{\Omega\times\Omega} F(x,y)dj \\
&\qquad + 4\iint_{\Omega\times\Omega} u^2(x)(\phi(x)-\phi(y))^2 dj \qquad (5.19)
\end{aligned}$$

(see also [24, Eq. (3.22) on p.473]). Combining (5.18) and (5.19), we obtain

$$\begin{aligned}
\iint_{\Omega\times\Omega} (u(x)-u(y))^2\phi^2(x)dj &\leq 2\mathcal{E}^{(J)}(u, u\phi^2) + 4\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj \\
&\qquad + 4\iint_{\Omega\times\Omega} u^2(x)(\phi(x)-\phi(y))^2 dj. \qquad (5.20)
\end{aligned}$$

Finally, it follows from (5.15), (5.16), (5.20) that

$$
\begin{aligned}
\int_\Omega \phi^2 d\Gamma_\Omega(u) = & \int_\Omega \phi^2 d\Gamma^{(L)}(u) + \iint_{\Omega\times\Omega} (u(x)-u(y))^2\phi^2(x)dj \\
\le & \; 2\mathcal{E}^{(L)}(u,u\phi^2) + 4\int_\Omega u^2 d\Gamma^{(L)}(\phi) \\
& + 2\mathcal{E}^{(J)}(u,u\phi^2) \\
& \quad + 4\iint_{\Omega\times\Omega} u^2(x)(\phi(x)-\phi(y))^2 dj + 4\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj \\
= & \; 2\mathcal{E}(u,u\phi^2) + 4\int_\Omega u^2 d\Gamma_\Omega(\phi) + 4\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj,
\end{aligned}
$$

thus proving (5.14). □

Proposition 5.3. *For any $u \in \mathcal{F}' \cap L^\infty$ and any $\phi \in \mathcal{F} \cap L^\infty$, we have*

$$
\mathcal{E}(u\phi) = \mathcal{E}(u,u\phi^2) + \int_M u^2 d\Gamma^{(L)}(\phi) + \iint_{M\times M} u(x)u(y)(\phi(x)-\phi(y))^2 dj. \tag{5.21}
$$

Proof. Since $u \in \mathcal{F}' \cap L^\infty$ and $\phi \in \mathcal{F} \cap L^\infty$, we see by Proposition 15.1 that $u^2 \in \mathcal{F}' \cap L^\infty$, and

$$
u\phi,\; u\phi^2 \in \mathcal{F} \cap L^\infty.
$$

By the product and chain rules ((5.1) and (5.2)), we have

$$
\begin{aligned}
\mathcal{E}^{(L)}(u\phi) &= \int_M d\Gamma^{(L)}(u\phi) \\
&= \int_M \phi^2 d\Gamma^{(L)}(u) + 2\int_M u\phi d\Gamma^{(L)}(u,\phi) + \int_M u^2 d\Gamma^{(L)}(\phi),
\end{aligned}
$$

while

$$
\begin{aligned}
\mathcal{E}^{(L)}(u,u\phi^2) &= \int_M d\Gamma^{(L)}(u,u\phi^2) \\
&= \int_M \phi^2 d\Gamma^{(L)}(u) + 2\int_M u\phi d\Gamma^{(L)}(u,\phi).
\end{aligned}
$$

Thus,

$$
\mathcal{E}^{(L)}(u\phi) = \mathcal{E}^{(L)}(u,u\phi^2) + \int_M u^2 d\Gamma^{(L)}(\phi). \tag{5.22}
$$

On the other hand, for the jump part we claim that

$$
\mathcal{E}^{(J)}(u\phi) = \mathcal{E}^{(J)}(u,u\phi^2) + \iint_{M\times M} u(x)u(y)(\phi(x)-\phi(y))^2 dj. \tag{5.23}
$$

Indeed, by a direct computation, we have for any points $x, y \in M$,

$$
\begin{aligned}
(u(x)\phi(x) - u(y)\phi(y))^2 = & \,(u(x)-u(y))\left(u(x)\phi^2(x) - u(y)\phi^2(y)\right) \\
& + u(x)u(y)(\phi(x)-\phi(y))^2.
\end{aligned}
$$

Integrating this against measure j over $M \times M$ and using definition (2.2), we obtain (5.23).

Therefore, it follows from (5.22), (5.23) that

$$
\mathcal{E}(u\phi) = \mathcal{E}^{(L)}(u\phi) + \mathcal{E}^{(J)}(u\phi)
$$

$$
\begin{aligned}
&= \mathcal{E}^{(L)}(u, u\phi^2) + \int_M u^2 d\Gamma^{(L)}(\phi) + \mathcal{E}^{(J)}(u, u\phi^2) \\
&\quad + \iint_{M\times M} u(x)u(y)(\phi(x)-\phi(y))^2 dj \\
&= \mathcal{E}(u, u\phi^2) + \int_M u^2 d\Gamma^{(L)}(\phi) + \iint_{M\times M} u(x)u(y)(\phi(x)-\phi(y))^2 dj,
\end{aligned}
$$

thus proving (5.21). The proof is complete. □

Proposition 5.4. *Let Ω be a measurable subset of M. Then, for any $u \in \mathcal{F}' \cap L^\infty$ and for any $\phi \in \mathcal{F} \cap L^\infty$ with* $\mathrm{supp}(\phi) \subset \Omega$*, we have*

$$
\mathcal{E}(u\phi) \le \mathcal{E}(u, u\phi^2) + \int_\Omega u^2 d\Gamma_\Omega(\phi) + 2\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x) dj. \tag{5.24}
$$

Proof. We will use (5.21) to derive (5.24). To do this, note that ϕ vanishes outside Ω. Then by the symmetry of dj,

$$
\begin{aligned}
\iint_{M\times M} u(x)u(y)(\phi(x)-\phi(y))^2 dj &= \left(\iint_{\Omega\times\Omega} + \iint_{\Omega\times\Omega^c} + \int_{\Omega^c\times\Omega} + \int_{\Omega^c\times\Omega^c}\right)\cdots \\
&= \iint_{\Omega\times\Omega} u(x)u(y)(\phi(x)-\phi(y))^2 dj \\
&\quad + 2\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x) dj.
\end{aligned}
$$

From this and using the fact that

$$
\int u^2 d\Gamma^{(L)}(\phi) = \int_\Omega u^2 d\Gamma^{(L)}(\phi),
$$

we conclude from (5.21) that

$$
\begin{aligned}
\mathcal{E}(u\phi) &= \mathcal{E}(u, u\phi^2) + \int_M u^2 d\Gamma^{(L)}(\phi) + \iint_{M\times M} u(x)u(y)(\phi(x)-\phi(y))^2 dj \\
&= \mathcal{E}(u, u\phi^2) + \int_\Omega u^2 d\Gamma^{(L)}(\phi) + \iint_{\Omega\times\Omega} u(x)u(y)(\phi(x)-\phi(y))^2 dj \\
&\qquad + 2\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x) dj \\
&\le \mathcal{E}(u, u\phi^2) + \int_\Omega u^2 d\Gamma^{(L)}(\phi) + \frac{1}{2}\iint_{\Omega\times\Omega} (u(x)^2+u(y)^2)(\phi(x)-\phi(y))^2 dj \\
&\qquad + 2\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x) dj \\
&= \mathcal{E}(u, u\phi^2) + \int_\Omega u^2 d\Gamma_\Omega(\phi) + 2\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x) dj,
\end{aligned}
$$

thus proving (5.24). □

6. Relations between (Gcap) and (ABB)

In this section we do not use condition (VD). Let us repeat the definition of (ABB) by paying more attention to constant coefficients.

Definition 6.1. Given $\zeta \geq 0$, we say that condition (ABB$_\zeta$) is satisfied if there exists $C > 0$ such that, for any $u \in \mathcal{F}' \cap L^\infty$ and for any three concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R+r)$ and $\Omega := B(x_0, R')$ with $0 < R < R+r < R' < \overline{R}$, there exists some $\phi \in \operatorname{cutoff}(B_0, B)$ such that

$$\int_\Omega u^2 d\Gamma_\Omega(\phi) \leq \zeta \int_B \phi^2 d\Gamma_B(u) + \sup_{x\in\Omega} \frac{C}{W(x,r)} \int_\Omega u^2 d\mu,$$

where $d\Gamma_\Omega$ is defined by (2.17).

Note that condition (ABB) holds if and only if condition (ABB$_\zeta$) holds for some $\zeta \geq 0$.

Lemma 6.2. *We have*

$$\text{(Gcap)} + \text{(TJ)} \Rightarrow (\text{ABB}_{4\kappa^2}),$$

where κ is the constant from condition (Gcap).

Proof. Fix a function $u \in \mathcal{F}' \cap L^\infty$. Let $B_0 := B(x_0, R)$, $B := B(x_0, R + r)$ and $\Omega := B(x_0, R')$ be any three concentric balls with $0 < R < R + r < R' < \overline{R}$. We will show that there exists some $\phi \in \operatorname{cutoff}(B_0, B)$ such that

$$\int_\Omega u^2 d\Gamma_\Omega(\phi) \leq 4\kappa^2 \int_B \phi^2 d\Gamma_B(u) + \sup_{x\in\Omega} \frac{C}{W(x,r)} \int_\Omega u^2 d\mu \tag{6.1}$$

for a constant $C > 0$ independent of u, B_0, B, Ω, which will exactly mean that (ABB$_\zeta$) holds with $\zeta = 4\kappa^2$.

Set $\widetilde{B} := B(x_0, R+r/2)$. By (Gcap), there exists a function $\widetilde{\phi}$ in κ-$\operatorname{cutoff}(B_0, \widetilde{B})$ such that

$$\mathcal{E}(u^2\widetilde{\phi}, \widetilde{\phi}) \leq \sup_{x\in\widetilde{B}} \frac{C}{W(x,r/2)} \int_{\widetilde{B}} u^2 d\mu \leq \sup_{x\in\Omega} \frac{C}{W(x,r/2)} \int_\Omega u^2 d\mu.$$

Applying (5.9), we obtain

$$\begin{aligned}\int_B u^2 d\Gamma_B(\widetilde{\phi}) &\leq 4 \int_B \widetilde{\phi}^2 d\Gamma_B(u) + 2\mathcal{E}(u^2\widetilde{\phi}, \widetilde{\phi}) \\ &\leq 4 \int_B \widetilde{\phi}^2 d\Gamma_B(u) + \sup_{x\in\Omega} \frac{2C}{W(x,r/2)} \int_\Omega u^2 d\mu.\end{aligned}$$

Define the function ϕ by

$$\phi := 1 \wedge \widetilde{\phi} \in \operatorname{cutoff}(B_0, \widetilde{B}).$$

Note that $\widetilde{\phi} \leq \kappa\phi$ in M. Using (5.8) and the previous inequality, we obtain

$$\begin{aligned}\int_B u^2 d\Gamma_B(\phi) \leq \int_B u^2 d\Gamma_B(\widetilde{\phi}) &\leq 4 \int_B \widetilde{\phi}^2 d\Gamma_B(u) + \sup_{x\in\Omega} \frac{2C}{W(x,r/2)} \int_\Omega u^2 d\mu \\ &\leq 4\kappa^2 \int_B \phi^2 d\Gamma_B(u) + \sup_{x\in\Omega} \frac{2C}{W(x,r/2)} \int_\Omega u^2 d\mu. \end{aligned} \tag{6.2}$$

On the other hand, as ϕ is supported in $\widetilde{B} \subset B \subset \Omega$, we have

$$\begin{aligned}
&\iint_{\Omega\times\Omega} u^2(x)(\phi(x)-\phi(y))^2 dj \\
&\quad = \left(\iint_{B\times B} + \iint_{B\times(\Omega\setminus B)} + \iint_{(\Omega\setminus B)\times B} + \iint_{(\Omega\setminus B)\times(\Omega\setminus B)}\right)\cdots \\
&\quad = \iint_{B\times B} u^2(x)(\phi(x)-\phi(y))^2 dj + \iint_{\widetilde{B}\times(\Omega\setminus B)} u^2(x)\phi^2(x)dj \\
&\qquad + \iint_{(\Omega\setminus B)\times\widetilde{B}} u^2(x)\phi^2(y)dj. \qquad (6.3)
\end{aligned}$$

Let us estimate the last two integrals in (6.3). Indeed, observe that $\mathrm{dist}(\widetilde{B}, \Omega\setminus B) \geq r/2$ so that $\Omega \setminus B \subset B(x, r/2)^c$ for any $x \in \widetilde{B}$. Hence, we have (TJ) and by (2.5) that

$$\operatorname*{esup}_{x\in\widetilde{B}} \int_{\Omega\setminus B} J(x,dy) \leq \operatorname*{esup}_{x\in\widetilde{B}} \int_{B(x,r/2)^c} J(x,dy) \leq \sup_{x\in\widetilde{B}} \frac{C'}{W(x,r/2)} \leq \sup_{x\in\Omega} \frac{C'}{W(x,r/2)}.$$

From this and using $0 \leq \phi \leq 1$ and that $\widetilde{B} \subset \Omega$, we obtain

$$\begin{aligned}
\iint_{\widetilde{B}\times(\Omega\setminus B)} u^2(x)\phi^2(x)dj &= \iint_{\widetilde{B}\times(\Omega\setminus B)} u^2(x)\phi^2(x)J(x,dy)d\mu(x) \\
&\leq \int_{\widetilde{B}} u^2(x)\phi^2(x)\left(\sup_{x\in\widetilde{B}}\int_{\Omega\setminus B} J(x,dy)\right)d\mu(x) \\
&\leq \sup_{x\in\Omega} \frac{C'}{W(x,r/2)} \int_\Omega u^2(x)d\mu(x). \qquad (6.4)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\iint_{(\Omega\setminus B)\times\widetilde{B}} u^2(x)\phi^2(y)dj &= \iint_{(\Omega\setminus B)\times\widetilde{B}} u^2(x)\phi^2(y)J(x,dy)d\mu(x) \\
&\leq \iint_{\Omega\setminus B} u^2(x)\left(\operatorname*{esup}_{x\in\Omega\setminus B}\int_{B(x,r/2)^c} J(x,dy)\right)d\mu(x) \\
&\leq \sup_{x\in\Omega} \frac{C'}{W(x,r/2)} \int_\Omega u^2(x)d\mu(x). \qquad (6.5)
\end{aligned}$$

Therefore, substituting (6.4), (6.5) into (6.3), we obtain

$$\iint_{\Omega\times\Omega} u^2(x)(\phi(x)-\phi(y))^2 dj \leq \iint_{B\times B} u^2(x)(\phi(x)-\phi(y))^2 dj + \sup_{x\in\Omega} \frac{2C'}{W(x,r/2)} \int_\Omega u^2 d\mu.$$

From this and using (5.3) and the fact that $\mathrm{supp}(\phi) \subset \widetilde{B} \subset B$, we obtain

$$\begin{aligned}
\int_\Omega u^2 d\Gamma_\Omega(\phi) &= \int_\Omega u^2 d\Gamma^{(L)}(\phi) + \iint_{\Omega\times\Omega} u^2(x)(\phi(x)-\phi(y))^2 dj \\
&\leq \int_B u^2 d\Gamma^{(L)}(\phi) + \iint_{B\times B} u^2(x)(\phi(x)-\phi(y))^2 dj
\end{aligned}$$

$$+\sup_{x\in\Omega}\frac{2C'}{W(x,r/2)}\int_\Omega u^2 d\mu$$

$$= \int_B u^2 d\Gamma_B(\phi) + \sup_{x\in\Omega}\frac{2C'}{W(x,r/2)}\int_\Omega u^2 d\mu. \tag{6.6}$$

Finally, substituting (6.2) into (6.6), we conclude that,

$$\begin{aligned}\int_\Omega u^2 d\Gamma_\Omega(\phi) &\le \int_B u^2 d\Gamma_B(\phi) + \sup_{x\in\Omega}\frac{2C'}{W(x,r/2)}\int_\Omega u^2 d\mu \\ &\le 4\kappa^2\int_B \phi^2 d\Gamma_B(u) + \sup_{x\in\Omega}\frac{2C}{W(x,r/2)}\int_\Omega u^2 d\mu + \sup_{x\in\Omega}\frac{2C'}{W(x,r/2)}\int_\Omega u^2 d\mu \\ &\le 4\kappa^2\int_B \phi^2 d\Gamma_B(u) + \sup_{x\in\Omega}\frac{C''}{W(x,r/2)}\int_\Omega u^2 d\mu.\end{aligned}$$

On the other hand, by (2.5), we have that

$$\frac{W(x,r)}{W(x,r/2)} \le C, \quad x \in M.$$

Finally, combining the above two inequalities, we obtain (6.1). □

7. Self-improvement of (ABB)

In the next lemma we show how (ABB) self-improves. The self-improvement property of (ABB) was first observed and proved in [1] for local Dirichlet forms, while for jump type Dirichlet form it was done in [15] and [24].

Lemma 7.1. *Assume that every metric ball of any radius $R < \overline{R}$ has finite measure. Then*

$$(\mathrm{ABB}) + (\mathrm{TJ}) \Rightarrow (\mathrm{ABB}_{1/8}).$$

Proof. Let $u \in \mathcal{F}' \cap L^\infty$ and let B_0, B, Ω be any three concentric balls given by

$$B_0 := B(x_0, R),\ B := B(x_0, R+r),\ \Omega := B(x_0, R')$$

with $0 < R < R+r < R' < \overline{R}$. We will show that there exists some $\phi \in \mathrm{cutoff}(B_0, B)$ such that

$$\int_\Omega u^2 d\Gamma_\Omega(\phi) \le \frac{1}{8}\int_B \phi^2 d\Gamma_B(u) + \sup_{x\in\Omega}\frac{C}{W(x,r)}\int_\Omega u^2 d\mu \tag{7.1}$$

for a universal constant $C > 0$ independent of B_0, B, Ω, u.

The idea of constructing such a cutoff function ϕ is as follows (which was motivated by [1, the proof of Lemma 5.1]): first dividing the ball B into infinitely many concentric balls $\{B_n\}_{n=0}^\infty$, then choosing ϕ_n to be a cutoff function for the triple (B_n, B_{n+1}, Ω) by using condition (ABB), and finally letting

$$\phi := \sum_{n=1}^\infty a_n\phi_n, \tag{7.2}$$

which is the desired cutoff function by choosing suitable $\{a_n\} \subset \mathbb{R}_+$. The proof here is motivated by that in [24, Lemma 2.9, pages 452-460] for the pure jump-type (non-local) Dirichlet form. The proof is quite technical.

If $u \equiv 0$ in Ω, then (7.1) holds for any $\phi \in \operatorname{cutoff}(B_0, B)$. Fix $u \in \mathcal{F}' \cap L^\infty$ with $\|u\|_{L^2(\Omega)} > 0$. Set $u_\varepsilon := |u| + \varepsilon$, where

$$\varepsilon := \left(\fint_\Omega u^2 d\mu \right)^{1/2} > 0. \tag{7.3}$$

Clearly, $u_\varepsilon \in \mathcal{F}' \cap L^\infty$.

Let $q > 1$ be a number to be chosen later. Define the sequences $\{r_n\}_{n=0}^\infty$ and $\{s_n\}_{n=1}^\infty$ by

$$r_n = \left(1 - q^{-n}\right) r, \quad s_n = r_n - r_{n-1} = (q-1)q^{-n} r.$$

Set $B_n := B(x_0, R + r_n)$ and $U_n := B_{n+1} \setminus B_n$. Obviously, $r_n \uparrow r$, $B_n \uparrow B$ as $n \to +\infty$, and $\cup_{n=1}^\infty U_n = B \setminus B_1$.

Applying (ABB) to the function u_ε and to each triple (B_n, B_{n+1}, Ω), we obtain that there exist some constants ζ, $C > 0$ and some $\phi_n \in \operatorname{cutoff}(B_n, B_{n+1})$ such that

$$\int_\Omega u_\varepsilon^2 d\Gamma_\Omega(\phi_n) \le \ \zeta \int_{B_{n+1}} \phi_n^2 d\Gamma_{B_{n+1}}(u_\varepsilon) + \sup_{x \in \Omega} \frac{C}{W(x, s_{n+1})} \int_\Omega u_\varepsilon^2 d\mu. \tag{7.4}$$

Note that ϕ_n may depend on u_ε, but if it does not, the proof would be simpler, as we will see below. Since

$$u_\varepsilon^2 = (|u| + \varepsilon)^2 \le 2(u^2 + \varepsilon^2),$$

and since by definition (2.17) and equality (5.6)

$$\begin{aligned}
\int_{B_{n+1}} \phi_n^2 d\Gamma_{B_{n+1}}(u_\varepsilon) &= \int_{B_{n+1}} \phi_n^2 d\Gamma^{(L)}(u_\varepsilon) + \iint_{B_{n+1} \times B_{n+1}} \phi_n^2(x) \left(u_\varepsilon(x) - u_\varepsilon(y)\right)^2 dj \\
&= \int_{B_{n+1}} \phi_n^2 d\Gamma^{(L)}(|u|) + \iint_{B_{n+1} \times B_{n+1}} \phi_n^2(x) \left(|u|(x) - |u|(y)\right)^2 dj \\
&\le \int_{B_{n+1}} \phi_n^2 d\Gamma^{(L)}(u) + \iint_{B_{n+1} \times B_{n+1}} \phi_n^2(x)(u(x) - u(y))^2 dj \\
&= \int_{B_{n+1}} \phi_n^2 d\Gamma_{B_{n+1}}(u),
\end{aligned}$$

it follows from (7.4), (7.3) and the fact that $0 \le \phi_n \le 1$ in M, that

$$\begin{aligned}
\int_\Omega u_\varepsilon^2 d\Gamma_\Omega(\phi_n) &\le \zeta \int_{B_{n+1}} \phi_n^2 d\Gamma_{B_{n+1}}(u) + \sup_{x \in \Omega} \frac{C}{W(x, s_{n+1})} \int_\Omega 2(u^2 + \varepsilon^2) d\mu \\
&\le \zeta \int_{B_{n+1}} d\Gamma_{B_{n+1}}(u) + \sup_{x \in \Omega} \frac{4C}{W(x, s_{n+1})} \int_\Omega u^2 d\mu.
\end{aligned} \tag{7.5}$$

Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=0}^\infty$ be two sequences of positive numbers given by

$$b_n = q^{-\beta n}, \quad a_n = b_{n-1} - b_n = \left(q^\beta - 1\right) q^{-\beta n},$$

where β is the constant in (2.5). Clearly,

$$\sum_{n=1}^\infty a_n = b_0 = 1.$$

Let ϕ be defined by (7.2) with this choice of $\{a_n\}$. We will prove the following two properties:

(*i*) $\phi \in \mathcal{F}$ (this will imply that $\phi \in \operatorname{cutoff}(B_0, B)$);
(*ii*) if q is close enough to 1, then ϕ satisfies (7.1) (this will prove condition $(\mathrm{ABB}_{1/8})$).

To verify (*i*), consider the partial sums of the series (7.2):

$$\Phi_N := \sum_{n=1}^{N} a_n \phi_n, \quad N > 0.$$

Clearly, $\Phi_N \uparrow \phi$ pointwise as $N \to \infty$. We will also show that the sequence $\{\Phi_N\}_{N=1}^{\infty}$ converges to ϕ in $\mathcal{E}_1$-norm. For this, it suffices to show that $\{\Phi_N\}$ is a Cauchy sequence in $\mathcal{F}$:

$$\|\Phi_{N+k} - \Phi_N\|_{\mathcal{E}_1}^2 = \mathcal{E}(\Phi_{N+k} - \Phi_N) + \|\Phi_{N+k} - \Phi_N\|_2^2 \to 0$$

as $N, k \to \infty$.

Indeed, note that every metric ball of radius smaller than $\overline{R}$ has finite measure, and observe that

$$\|\phi_n\|_{L^2} \leq \mu(B)^{1/2} < \infty. \tag{7.6}$$

Since $\varepsilon^2 \leq u_\varepsilon^2$ in M and

$$\begin{aligned}\int_{B_{n+1}} d\Gamma_{B_{n+1}}(u) &= \int_{B_{n+1}} d\Gamma^{(L)}(u) + \iint_{B_{n+1}\times B_{n+1}} (u(x)-u(y))^2 dj \\ &\leq \int_M d\Gamma^{(L)}(u) + \iint_{M\times M} (u(x)-u(y))^2 dj = \mathcal{E}(u),\end{aligned}$$

we have by (7.5) and (2.5),

$$\begin{aligned}\int_\Omega d\Gamma_\Omega(\phi_n) &\leq \zeta\varepsilon^{-2} \int_{B_{n+1}} d\Gamma_{B_{n+1}}(u) + \sup_{x\in\Omega} \frac{4C\varepsilon^{-2}}{W(x, s_{n+1})} \int_\Omega u^2 d\mu \\ &\leq \zeta\varepsilon^{-2}\mathcal{E}(u) + \sup_{x\in\Omega} \frac{C'\|u\|_{L^2(\Omega)}^2}{W(x,r)} \left(\frac{r}{s_{n+1}}\right)^\beta \leq Cq^{\beta n}\end{aligned}$$

for some positive constant C independent of n. From this and using the fact that ϕ_n is supported in $B \subset \Omega$, we see that

$$\begin{aligned}\mathcal{E}(\phi_n, \phi_n) =\ & \mathcal{E}^{(L)}(\phi_n, \phi_n) + \mathcal{E}^{(J)}(\phi_n, \phi_n) \\ =\ & \int_M d\Gamma^{(L)}(\phi_n) + \iint_{M\times M} (\phi_n(x) - \phi_n(y))^2 dj \\ =\ & \int_\Omega d\Gamma_\Omega(\phi_n) + 2\iint_{B\times\Omega^c} \phi_n^2(x) dj \leq Cq^{\beta n} + 2\iint_{B\times\Omega^c} \phi_n^2(x) dj.\end{aligned} \tag{7.7}$$

Since $d(B, \Omega^c) \geq R' - (R + r) := r_0 > 0$ so that $\Omega^c \subset B(x, r_0)^c$ for any x in B and since $0 \leq \phi_n \leq 1$ in M, we have by condition (TJ), (2.5) and (7.6) that

$$\begin{aligned}\iint_{B\times\Omega^c} \phi_n^2(x) dj &= \int_B \phi_n^2(x) d\mu(x) \int_{\Omega^c} J(x, dy) \\ &\leq \int_B \phi_n^2(x) \left(\operatorname*{esup}_{x\in B} \int_{B(x,r_0)^c} J(x, dy)\right) d\mu(x)\end{aligned}$$

$$\leq \left(\operatorname*{esup}_{x\in B}\frac{C}{W(x,r_0)}\right)\int_B \phi_n^2(x)d\mu(x) \leq \frac{C'\mu(B)}{W(x_0,R')}\left(\frac{R'}{r_0}\right)^{\beta} < \infty \tag{7.8}$$

uniformly in n. It follows from (7.7), (7.8) that

$$\mathcal{E}(\phi_n,\phi_n) \leq Cq^{\beta n}, \tag{7.9}$$

where the constant C depends on all variables in question except on n. Since $a_n = (q^\beta - 1)q^{-\beta n}$, we obtain from (7.6), (7.9) that for any $k \geq 1$

$$\begin{aligned}
\|\Phi_{N+k} - \Phi_N\|_{\mathcal{E}_1}^{1/2} &= \|\sum_{n=N+1}^{N+k} a_n\phi_n\|_{\mathcal{E}_1}^{1/2} \leq \sum_{n=N+1}^{N+k} a_n\|\phi_n\|_{\mathcal{E}_1}^{1/2} \\
&= \sum_{n=N+1}^{N+k} a_n\left(\|\phi_n\|_{L^2}^2 + \mathcal{E}(\phi_n,\phi_n)\right)^{1/2} \\
&\leq \sum_{n=N+1}^{N+k} a_n\left(\mu(B) + Cq^{\beta n}\right)^{1/2} \\
&\leq C\left(q^\beta - 1\right)\sum_{n=N+1}^{\infty} q^{-\beta n/2},
\end{aligned}$$

thus showing that property (i) is true.

To verify (ii), let us prove the following inequality

$$\int_\Omega u^2 d\Gamma_\Omega(\Phi_N) \leq \frac{1}{8}\int_B \phi^2 d\Gamma_B(u) + \sup_{x\in\Omega}\frac{C}{W(x,r)}\int_\Omega u^2 d\mu. \tag{7.10}$$

In this case, inequality (7.1) will follow from (7.10) by letting $N \to \infty$ and by using the fact that $\mathcal{E}_1(\Phi_N - \phi) \to 0$ that was already proved above.

To this end, note that by the bilinearity of $d\Gamma_\Omega$

$$\begin{aligned}
d\Gamma_\Omega(\Phi_N) &= d\Gamma_\Omega\left(\sum_{n=1}^N a_n\phi_n, \sum_{m=1}^N a_m\phi_m\right) \\
&= \sum_{n=1}^N a_n^2 d\Gamma_\Omega(\phi_n) + 2\sum_{n=1}^{N-1} a_n a_{n+1} d\Gamma_\Omega(\phi_n,\phi_{n+1}) \\
&\quad + 2\sum_{m=1}^{N-2}\sum_{n=m+2}^{N} a_n a_m d\Gamma_\Omega(\phi_n,\phi_m),
\end{aligned}$$

and, hence

$$\begin{aligned}
S_N(u) &:= \int_\Omega u^2 d\Gamma_\Omega(\Phi_N) \\
&= \sum_{n=1}^N a_n^2\int_\Omega u^2 d\Gamma_\Omega(\phi_n) + 2\sum_{n=1}^{N-1} a_n a_{n+1}\int_\Omega u^2 d\Gamma_\Omega(\phi_n,\phi_{n+1}) \\
&\quad + 2\sum_{m=1}^{N-2}\sum_{n=m+2}^{N} a_n a_m \int_\Omega u^2 d\Gamma_\Omega(\phi_n,\phi_m).
\end{aligned} \tag{7.11}$$

We will estimate the second term on the right-hand side of (7.11).

Indeed, notice that $d\Gamma^{(L)}(\phi_n, \phi_m) = 0$ for any $n > m$ by using (5.3), since $\phi_n = 1$ in $\mathrm{supp}(\phi_m)$. Thus by definition (2.17)

$$
\begin{aligned}
d\Gamma_\Omega(\phi_n, \phi_m) &= d\Gamma^{(L)}(\phi_n, \phi_m) + \iint_{\Omega\times\Omega} (\phi_n(x) - \phi_n(y))\,(\phi_m(x) - \phi_m(y))dj \\
&= \iint_{\Omega\times\Omega} (\phi_n(x) - \phi_n(y))\,(\phi_m(x) - \phi_m(y))dj \quad \text{for any } n > m. \qquad (7.12)
\end{aligned}
$$

From this, we have by using the elementary inequality $2ab \le a^2 + b^2$

$$
\begin{aligned}
&2\sum_{n=1}^{N-1} a_n a_{n+1} \int_\Omega u^2 d\Gamma_\Omega(\phi_n, \phi_{n+1}) \\
&= 2\sum_{n=1}^{N-1} a_n a_{n+1} \iint_{\Omega\times\Omega} u^2(x)(\phi_n(x) - \phi_n(y))\,(\phi_{n+1}(x) - \phi_{n+1}(y))\,dj \\
&\le \sum_{n=1}^{N-1} a_n^2 \iint_{\Omega\times\Omega} u^2(x)(\phi_n(x) - \phi_n(y))^2 dj \\
&\quad + \sum_{n=1}^{N-1} a_{n+1}^2 \iint_{\Omega\times\Omega} u^2(x)(\phi_{n+1}(x) - \phi_{n+1}(y))^2 dj \\
&\le 2\sum_{n=1}^{N} a_n^2 \iint_{\Omega\times\Omega} u^2(x)(\phi_n(x) - \phi_n(y))^2 dj \le 2\sum_{n=1}^{N} a_n^2 \int_\Omega u^2 d\Gamma_\Omega(\phi_n). \qquad (7.13)
\end{aligned}
$$

Therefore, plugging (7.13) into (7.11), we obtain that, using (7.12),

$$
\begin{aligned}
S_N(u) &\le 3\sum_{n=1}^{N} a_n^2 \int_\Omega u^2 d\Gamma_\Omega(\phi_n) + 2\sum_{m=1}^{N-2}\sum_{n=m+2}^{N} a_n a_m \int_\Omega u^2 d\Gamma_\Omega(\phi_n, \phi_m) \\
&= \underbrace{3\sum_{n=1}^{N} a_n^2 \int_\Omega u^2 d\Gamma_\Omega(\phi_n)}_{I_1} \\
&\quad + \underbrace{2\sum_{m=1}^{N-2}\sum_{n=m+2}^{N} a_n a_m \iint_{\Omega\times\Omega} u^2(x)(\phi_m(x) - \phi_m(y))\,(\phi_n(x) - \phi_n(y))\,dj}_{I_2} \\
&= 3I_1 + 2I_2. \qquad (7.14)
\end{aligned}
$$

We will estimate each term I_1, I_2. To estimate the term I_1, we will use condition (ABB), whilst to the term I_2 we will use condition (TJ).

To do this, for the term I_1, we have by (7.5),

$$
I_1 = \sum_{n=1}^{N} a_n^2 \int_\Omega u^2 d\Gamma_\Omega(\phi_n)
$$

$$
\begin{aligned}
&\leq \zeta \sum_{n=1}^{\infty} a_n^2 \int_{B_{n+1}} d\Gamma_{B_{n+1}}(u) + 4C \sum_{n=1}^{\infty} \sup_{x\in\Omega} \frac{a_n^2}{W(x,s_{n+1})} \int_\Omega u^2 d\mu \\
&= \underbrace{\zeta \sum_{n=1}^{\infty} a_n^2 \int_{B_1} d\Gamma_{B_{n+1}}(u)}_{I_{11}} + \underbrace{\zeta \sum_{n=1}^{\infty} a_n^2 \int_{B_{n+1}\setminus B_1} d\Gamma_{B_{n+1}}(u)}_{I_{12}} \\
&\qquad\qquad + \underbrace{4C \sum_{n=1}^{\infty} \sup_{x\in\Omega} \frac{a_n^2}{W(x,s_{n+1})} \int_\Omega u^2 d\mu}_{I_{13}} \\
&= \zeta I_{11} + \zeta I_{12} + 4C I_{13}. \qquad (7.15)
\end{aligned}
$$

We will estimate each term I_{11}, I_{12}, I_{13}.

Indeed, since $\phi = 1$ on B_1 and

$$
\sum_{n=1}^{\infty} a_n^2 = \frac{q^\beta - 1}{q^\beta + 1},
$$

we obtain by (5.7) that

$$
\begin{aligned}
I_{11} = &\sum_{n=1}^{\infty} a_n^2 \int_{B_1} d\Gamma_{B_{n+1}}(u) = \sum_{n=1}^{\infty} a_n^2 \int_{B_1} \phi^2 d\Gamma_{B_{n+1}}(u) \\
\leq &\sum_{n=1}^{\infty} a_n^2 \int_{B_1} \phi^2 d\Gamma_B(u) = \frac{q^\beta - 1}{q^\beta + 1} \int_{B_1} \phi^2 d\Gamma_B(u). \qquad (7.16)
\end{aligned}
$$

For I_{12}, using the facts that $a_m \leq (q^\beta - 1)\phi$ in $U_m = B_{m+1} \setminus B_m$ (cf. [24, formula (2.17) on p.458]) and $a_n = q^{-(n-m)\beta} a_m$, we have that

$$
\begin{aligned}
I_{12} &= \sum_{n=1}^{\infty} a_n^2 \int_{B_{n+1}\setminus B_1} d\Gamma_{B_{n+1}}(u) \leq \sum_{n=1}^{\infty} a_n^2 \int_{B_{n+1}\setminus B_1} d\Gamma_B(u) \\
&= \sum_{n=1}^{\infty} a_n^2 \sum_{m=1}^{n} \int_{U_m} d\Gamma_B(u) = \sum_{n=1}^{\infty} \sum_{m=1}^{n} q^{-2(n-m)\beta} \int_{U_m} a_m^2 d\Gamma_B(u) \\
&\leq \sum_{n=1}^{\infty} \sum_{m=1}^{n} q^{-2(n-m)\beta} \int_{U_m} (q^\beta - 1)^2 \phi^2 d\Gamma_B(u) \\
&= \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} q^{-2(n-m)\beta} \int_{U_m} (q^\beta - 1)^2 \phi^2 d\Gamma_B(u) \\
&\leq \frac{q^{2\beta}(q^\beta - 1)}{q^\beta + 1} \int_{B\setminus B_1} \phi^2 d\Gamma_B(u). \qquad (7.17)
\end{aligned}
$$

For I_{13}, since by (2.5),

$$
\frac{W(x,r)}{W(x,s_{n+1})} \leq C \left(\frac{r}{s_{n+1}} \right)^\beta, \quad x \in M,
$$

we have that, using $a_n = \left(q^\beta - 1\right) q^{-\beta n}$ and $s_{n+1} = (q-1)q^{-(n+1)}r$,

$$
\begin{aligned}
I_{13} &= \sum_{n=1}^{\infty} \sup_{x\in\Omega} \frac{a_n^2}{W(x,s_{n+1})} \int_\Omega u^2 d\mu \\
&\le \sup_{x\in\Omega} \frac{C}{W(x,r)} \sum_{n=1}^{\infty} a_n^2 \left(\frac{r}{s_{n+1}}\right)^\beta \int_\Omega u^2 d\mu \\
&= \sup_{x\in\Omega} \frac{C}{W(x,r)} \sum_{n=1}^{\infty} \frac{(q^\beta-1)^2 q^{-2\beta n}}{(q-1)^\beta q^{-\beta(n+1)}} \int_\Omega u^2 d\mu \\
&= \sup_{x\in\Omega} \frac{C}{W(x,r)} \frac{q^\beta(q^\beta-1)}{(q-1)^\beta} \int_\Omega u^2 d\mu. \qquad (7.18)
\end{aligned}
$$

Therefore, plugging (7.16), (7.17), and (7.18) into (7.15), we conclude

$$
\begin{aligned}
I_1 \le \zeta I_{11} + \zeta I_{12} + 4C I_{13} \le & \frac{\zeta(1+q^{2\beta})(q^\beta-1)}{q^\beta+1} \int_B \phi^2 d\Gamma_B(u) \\
& + \sup_{x\in\Omega} \frac{C(q)}{W(x,r)} \int_\Omega u^2 d\mu. \qquad (7.19)
\end{aligned}
$$

For the term I_2, we repeat the same argument in [24, formula (2.20), p.459]. The only difference is to use condition (TJ) here rather than the pointwise upper bound of the jump kernel therein. In fact, for any $m \ge 1$, $n \ge m+2$

$$
(\phi_m(x) - \phi_m(y))(\phi_n(x) - \phi_n(y)) = \phi_m(x)(1-\phi_n(y)) + \phi_m(y)(1-\phi_n(x))
$$

since $\phi_n\phi_m \equiv \phi_m$ in M by using $\phi_n = 1$ in $\mathrm{supp}(\phi_m)$. Thus,

$$
\begin{aligned}
I_2 &= \sum_{m=1}^{N-2} \sum_{n=m+2}^{N} a_n a_m \iint_{\Omega\times\Omega} u^2(x)(\phi_m(x)-\phi_m(y))(\phi_n(x)-\phi_n(y))dj \\
&\le \underbrace{\sum_{m=1}^{\infty} \sum_{n=m+2}^{\infty} a_m a_n \iint_{\Omega\times\Omega} u^2(x)\phi_m(x)(1-\phi_n(y))dj}_{I_{21}} \\
&\quad + \underbrace{\sum_{m=1}^{\infty} \sum_{n=m+2}^{\infty} a_m a_n \iint_{\Omega\times\Omega} u^2(x)\phi_m(y)(1-\phi_n(x))dj}_{I_{22}}. \qquad (7.20)
\end{aligned}
$$

To estimate I_{21}, noting that $d(B_{m+1}, B_n^c) \ge s_{m+2}$ for any $n \ge m+2$, we have by condition (TJ) and (2.5) that

$$
\begin{aligned}
\operatorname*{esup}_{x\in B_{m+1}} \iint_{\Omega\setminus B_n} J(x,dy) \le & \operatorname*{esup}_{x\in B_{m+1}} \int_{B(x,s_{m+2})^c} J(x,dy) \\
& \le \sup_{x\in B_{m+1}} \frac{C}{W(x,s_{m+2})} \le \sup_{x\in\Omega} \frac{C}{W(x,r)} \left(\frac{r}{s_{m+2}}\right)^\beta,
\end{aligned}
$$

and hence

$$\iint_{B_{m+1}\times(\Omega\setminus B_n)} u^2(x)dj \leq \operatorname*{esup}_{x\in B_{m+1}} \iint_{\Omega\setminus B_n} J(x,dy)\int_{B_{m+1}} u^2(x)d\mu(x)$$
$$\leq \sup_{x\in\Omega}\frac{C}{W(x,r)}\left(\frac{r}{s_{m+2}}\right)^{\beta}\int_\Omega u^2 d\mu.$$

Therefore, using the fact that ϕ_m is supported in B_{m+1} and $1-\phi_n$ supported in B_n^c, we have

$$\begin{aligned}
I_{21} &= \sum_{m=1}^{\infty}\sum_{n=m+2}^{\infty} a_m a_n \iint_{\Omega\times\Omega} u^2(x)\phi_m(x)(1-\phi_n(y))dj \\
&\leq \sum_{m=1}^{\infty}\sum_{n=m+2}^{\infty} a_m a_n \iint_{B_{m+1}\times(\Omega\setminus B_n)} u^2(x)dj \\
&\leq \sup_{x\in\Omega}\frac{C}{W(x,r)}\sum_{m=1}^{\infty} a_m\left(\frac{r}{s_{m+2}}\right)^{\beta}\sum_{n=m+2}^{\infty} a_n\int_\Omega u^2 d\mu \\
&= \sup_{x\in\Omega}\frac{C(q)}{W(x,r)}\int_\Omega u^2 d\mu. \qquad (7.21)
\end{aligned}$$

For I_{22}, we run the same argument as above and obtain

$$\begin{aligned}
I_{22} &= \sum_{m=1}^{\infty}\sum_{n=m+2}^{\infty} a_m a_n \iint_{\Omega\times\Omega} u^2(x)\phi_m(y)(1-\phi_n(x))dj \\
&\leq \sum_{m=1}^{\infty}\sum_{n=m+2}^{\infty} a_m a_n \iint_{(\Omega\setminus B_n)\times B_{m+1}} u^2(x)dj \\
&\leq \sum_{m=1}^{\infty}\sum_{n=m+2}^{\infty} a_m a_n \int_\Omega \sup_{x\in\Omega}\frac{Cu^2(x)}{W(x,s_{m+2})}d\mu(x) \leq \sup_{x\in\Omega}\frac{C(q)}{W(x,r)}\int_\Omega u^2 d\mu. \qquad (7.22)
\end{aligned}$$

Therefore, plugging (7.21) and (7.22) into (7.20), we obtain that

$$I_2 \leq I_{21}+I_{22} \leq \sup_{x\in\Omega}\frac{C(q)}{W(x,r)}\int_\Omega u^2 d\mu. \qquad (7.23)$$

Finally, substituting (7.23) and (7.19) into (7.14), and choosing $q>1$ close enough to 1, we obtain

$$\begin{aligned}
S_N(u) &\leq 3I_1 + 2I_2 \\
&\leq \frac{3\zeta(1+q^{2\beta})(q^\beta-1)}{q^\beta+1}\int_B \phi^2 d\Gamma_B(u) + \sup_{x\in\Omega}\frac{C(q)}{W(x,r)}\int_\Omega u^2 d\mu \\
&\quad + \sup_{x\in\Omega}\frac{C(q)}{W(x,r)}\int_\Omega u^2 d\mu \\
&\leq \frac{1}{8}\int_B \phi^2 d\Gamma_B(u) + \sup_{x\in\Omega}\frac{C(q)}{W(x,r)}\int_\Omega u^2 d\mu,
\end{aligned}$$

thus showing (7.10). □

8. Energy of product

Let us introduce a condition (EP), that is called the *energy of product*.

Definition 8.1 (Condition (EP))**.** We say that the condition (EP) is satisfied if there exists a constant $C > 0$ such that, for any three concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R+r)$ and $\Omega := B(x_0, R')$ with $0 < R < R + r < R' < \overline{R}$, and for any $u \in \mathcal{F}' \cap L^\infty$, there exists $\phi \in \mathrm{cutoff}(B_0, B)$ such that

(8.1)
$$\mathcal{E}(u\phi) := \mathcal{E}(u\phi, u\phi) \leq \frac{3}{2}\mathcal{E}(u, u\phi^2) + \sup_{x\in\Omega} \frac{C}{W(x,r)} \int_\Omega u^2 d\mu + 3 \iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj.$$

We remark that the coefficients "$\frac{3}{2}$" and "3" appearing in (8.1) are unimportant. Condition (EP) will play an important role in deriving the mean value inequality, as we will see in Section 10 below.

Lemma 8.2. *We have*

$$(\mathrm{ABB}_{1/8}) \Rightarrow (\mathrm{EP}). \tag{8.2}$$

Consequently, if every metric ball of radius smaller than $\overline{R}$ has finite measure, then we have the following implications:

$$(\mathrm{Gcap}) + (\mathrm{TJ}) \Rightarrow\ (\mathrm{ABB}) + (\mathrm{TJ}) \Rightarrow\ (\mathrm{ABB}_{1/8}) \Rightarrow (\mathrm{EP}). \tag{8.3}$$

Proof. Let $B_0 := B(x_0, R)$, $B := B(x_0, R+r)$ and $\Omega := B(x_0, R')$ be any three concentric balls with $0 < R < R+r < R' < \overline{R}$ as before. For $u \in \mathcal{F}' \cap L^\infty$, we have by condition $(\mathrm{ABB}_{1/8})$ that there exists some $\phi \in \mathrm{cutoff}(B_0, B)$ such that

$$\begin{aligned}\int_\Omega u^2 d\Gamma_\Omega(\phi) &\leq \frac{1}{8}\int_B \phi^2 d\Gamma_B(u) + \sup_{x\in\Omega}\frac{C}{W(x,r)}\int_\Omega u^2 d\mu \\ &\leq \frac{1}{8}\int_\Omega \phi^2 d\Gamma_\Omega(u) + \sup_{x\in\Omega}\frac{C}{W(x,r)}\int_\Omega u^2 d\mu.\end{aligned}$$

From this and using (5.14), we obtain

$$\begin{aligned}\int_\Omega u^2 d\Gamma_\Omega(\phi) &\leq \frac{1}{8}\left(2\mathcal{E}(u,u\phi^2) + 4\int_\Omega u^2 d\Gamma_\Omega(\phi) + 4\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj\right) \\ &\quad + \sup_{x\in\Omega}\frac{C}{W(x,r)}\int_\Omega u^2 d\mu \\ &= \frac{1}{4}\mathcal{E}(u,u\phi^2) + \frac{1}{2}\int_\Omega u^2 d\Gamma_\Omega(\phi) + \frac{1}{2}\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj \\ &\quad + \sup_{x\in\Omega}\frac{C}{W(x,r)}\int_\Omega u^2 d\mu.\end{aligned}$$

Rearranging the above inequality, we have

$$\int_\Omega u^2 d\Gamma_\Omega(\phi) \leq \frac{1}{2}\mathcal{E}(u,u\phi^2) + \iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj + \sup_{x\in\Omega}\frac{2C}{W(x,r)}\int_\Omega u^2 d\mu.$$

From this and using (5.24), we conclude that

$$\mathcal{E}(u\phi) \leq \mathcal{E}(u,u\phi^2) + \int_\Omega u^2 d\Gamma_\Omega(\phi) + 2\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj$$

$$
\begin{aligned}
&\le \mathcal{E}(u,u\phi^2)+\frac{1}{2}\mathcal{E}(u,u\phi^2)+\iint_{\Omega\times\Omega^c}u(x)u(y)\phi^2(x)dj\\
&\quad+\sup_{x\in\Omega}\frac{2C}{W(x,r)}\int_\Omega u^2d\mu+2\iint_{\Omega\times\Omega^c}u(x)u(y)\phi^2(x)dj\\
&=\frac{3}{2}\mathcal{E}(u,u\phi^2)+\sup_{x\in\Omega}\frac{2C}{W(x,r)}\int_\Omega u^2d\mu+3\iint_{\Omega\times\Omega^c}u(x)u(y)\phi^2(x)dj,
\end{aligned}
$$

thus proving condition (EP). This proves the implication (8.2).

Finally, the implications in (8.3) follow directly from Lemmas 6.2 and 7.1. The proof is complete. □

9. Subharmonic functions

In this section we will prove a simple property of subharmonic functions stated in Lemma 9.3 below. We start with the following observation.

Proposition 9.1. *Assume that a function $F\in C^2(\mathbb{R})$ satisfies*

$$\sup_{\mathbb{R}}|F'|<\infty,\ \ F''\ge 0,\ \ \sup_{\mathbb{R}}F''<\infty.$$

Then, for all $u,\varphi\in\mathcal{F}'\cap L^\infty$, both functions $F(u)$ and $F'(u)\varphi$ belong to the space $\mathcal{F}'\cap L^\infty$. Moreover, if in addition $\varphi\ge 0$ on M, then

$$\mathcal{E}(F(u),\varphi)\le\mathcal{E}(u,F'(u)\varphi). \tag{9.1}$$

Proof. Since F is Lipschitz in $\mathbb{R}$, we see by Proposition 15.1(i) that $F(u)\in\mathcal{F}'$. Since $u\in L^\infty$, we have also $F(u)\in L^\infty$ and, hence,

$$F(u)\in\mathcal{F}'\cap L^\infty.$$

Similarly, we obtain

$$F'(u)\in\mathcal{F}'\cap L^\infty.$$

Since $\varphi\in\mathcal{F}'\cap L^\infty$, it follows from Proposition 15.1(ii) that

$$F'(u)\varphi\in\mathcal{F}'\cap L^\infty.$$

Let us verify (9.1) assuming that $\varphi\ge 0$. Indeed, by the chain and product rules ((5.2) and (5.1)), and by the fact that $F''\ge 0$, we obtain

$$
\begin{aligned}
\mathcal{E}^{(L)}(F(u),\varphi)&=\int_M d\Gamma^{(L)}(F(u),\varphi)=\int_M F'(u)d\Gamma^{(L)}(u,\varphi)\\
&=\int_M d\Gamma^{(L)}(u,F'(u)\varphi)-\int_M F''(u)\varphi d\Gamma^{(L)}(u)\\
&\le\int_M d\Gamma^{(L)}(u,F'(u)\varphi)=\mathcal{E}^{(L)}(u,F'(u)\varphi). \qquad (9.2)
\end{aligned}
$$

On the other hand, the condition $F''\ge 0$ implies that, for all $X,Y,a,b\in\mathbb{R}_+$,

$$(F(X)-F(Y))(a-b)\le(X-Y)(F'(X)a-F'(Y)b)$$

(see for example [24, Eq. (3.3), p. 464]). Substituting here $X=u(x)$, $Y=u(y)$, $a=\phi(x)$, $b=\phi(y)$, we obtain

$$\mathcal{E}^{(J)}(F(u),\phi)=\iint_{M\times M}(F(u(x))-F(u(y)))(\phi(x)-\phi(y))dj$$

$$\le \iint_{M\times M} (u(x)-u(y))(F'(u(x))\phi(x) - F'(u(y))\phi(y))dj$$
$$= \mathcal{E}^{(J)}(u, F'(u)\phi),$$

that is,

$$\mathcal{E}^{(J)}(F(u),\varphi) \le \mathcal{E}^{(J)}(u, F'(u)\varphi).$$

Therefore, it follows from (9.2) and the above inequality that

$$\begin{aligned}\mathcal{E}(F(u),\varphi) &= \mathcal{E}^{(L)}(F(u),\varphi) + \mathcal{E}^{(J)}(F(u),\varphi)\\ &\le \mathcal{E}^{(L)}(u, F'(u)\varphi) + \mathcal{E}^{(J)}(u, F'(u)\varphi)\\ &= \mathcal{E}(u, F'(u)\varphi),\end{aligned}$$

thus proving (9.1). □

Let us extend Definition 2.9 of subharmonic functions to the space $\mathcal{F}'$.

Definition 9.2. Let Ω be an open subset of M. We say that a function $u \in \mathcal{F}'$ is *subharmonic* (resp. *superharmonic*) in Ω if, for any $0 \le \varphi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(u,\varphi) \le 0 \quad (\text{resp. } \mathcal{E}(u,\varphi) \ge 0).$$

A function $u \in \mathcal{F}'$ is called *harmonic* in Ω if it is both subharmonic and superharmonic in Ω.

Lemma 9.3. *If $u \in \mathcal{F}' \cap L^\infty$ is subharmonic in a non-empty open set $\Omega \subset M$, then u_+ belongs to $\mathcal{F}' \cap L^\infty$ and is also subharmonic in Ω.*

Proof. Clearly, $u_+ \in \mathcal{F}'$. Fix a function $0 \le \phi \in \mathcal{F}(\Omega)$ and prove that $\mathcal{E}(u_+,\phi) \le 0$. Since each function in $\mathcal{F}$ can be approximated by a sequence of functions in $\mathcal{F}\cap L^\infty$ in the norm of $\mathcal{F}$ (see for example [19, Thoerem 1.4.2(iii)]), we may assume in addition that $\phi \in L^\infty$. Let $\{F_k\}_{k=1}^\infty$ be a sequence of C^2-functions on $\mathbb{R}$ satisfying for all $n \ge 1$ the conditions

$$F_n|_{(-\infty,0]} = 0, \quad 0 \le F_n' \le 1, \quad F_n'' \ge 0, \quad \sup_{\mathbb{R}} F_n'' < \infty,$$

and

$$F_n(t) \rightrightarrows t_+ \text{ uniformly in } \mathbb{R} \text{ as } n \to \infty. \tag{9.3}$$

Such a sequence can be constructed as follows: first fix any function $0 \le f \in C_0[0,1]$ with

$$\int_0^1 f(t)dt = 0,$$

then set for any $n \ge 1$

$$f_n(t) = nf(nt) \in C_0[0,\tfrac{1}{n}],$$

and determine F_n from the equation $F_n'' = f_n$, that is,

$$F_n'(t) = \int_0^t f_n dt \quad \text{and} \quad F_n(t) = \int_0^t F_n' dt$$

(see Fig. 3). Then $F_n(t) = 0$ for all $t \le 0$ and $F_n'(t) \to 1$ as $n \to \infty$ for any $t > 0$ whence (9.3) follows.

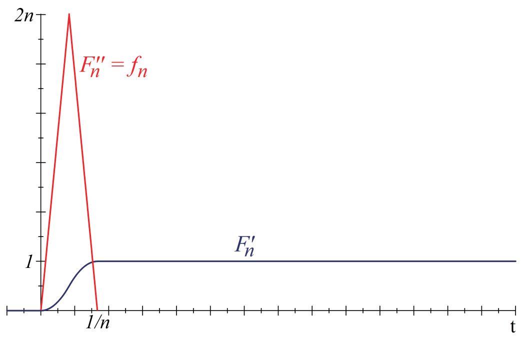

FIGURE 3. Functions F_n' and F_n''

By Proposition 9.1, the functions $F_n(u)$ and $F_n'(u)\phi$ belong to $\mathcal{F}' \cap L^\infty$ and

$$\mathcal{E}(F_n(u), \phi) \leq \mathcal{E}(u, F_n'(u)\phi).$$

Moreover, by Proposition 15.1$(i), (iii)$ we have also

$$F_n'(u)\phi \in \mathcal{F}(\Omega) \cap L^\infty$$

Since u is subharmonic in Ω and $F_n'(u)\phi \geq 0$, we have,

$$\mathcal{E}(u, F_n'(u)\phi) \leq 0,$$

which implies

$$\mathcal{E}(F_n(u), \phi) \leq 0.$$

Hence, it suffices to verify that

$$\mathcal{E}(u_+, \phi) = \lim_{n\to\infty} \mathcal{E}(F_n(u), \phi). \tag{9.4}$$

Since $u \in \mathcal{F}'$, there exists $w \in \mathcal{F}$ and $a \in \mathbb{R}$ such that $u = w + a$. Consider the functions

$$w_n = F_n(u) - F_n(a)$$

and observe that by (9.3) there is a pointwise convergence

$$w_n \to u_+ - a_+ \quad \text{as } n \to \infty.$$

Denote

$$L := \sup_n \sup_{\mathbb{R}} F_n' < \infty.$$

Since

$$|w_n| = |F_n(u) - F_n(a)| \leq L|u - a| = L|w| \in L^2,$$

we conclude by the dominated convergence theorem that

$$w_n \overset{L^2}{\to} u_+ - a_+ \quad \text{as} \quad n \to \infty. \tag{9.5}$$

On the other hand, since

$$\frac{1}{L} w_n = \frac{F_n(w + a) - F_n(a)}{L}$$

is a normal contraction of w, we obtain that, for any n,

$$\mathcal{E}(w_n, w_n) \leq L^2 \mathcal{E}(w, w). \tag{9.6}$$

By (9.5), (9.6) and Proposition 15.5, we conclude that

$$\lim_{n\to\infty} \mathcal{E}(F_n(u), \phi) = \lim_{n\to\infty} \mathcal{E}(w_n, \phi) = \mathcal{E}(u_+ - a_+, \phi) = \mathcal{E}(u_+, \phi),$$

which is exactly (9.4). $\square$

10. Mean value inequality

In this section, we prove the mean value inequality for subharmonic functions.

Theorem 10.1. *Assume that conditions* (EP), (VD), (FK), *and* (TJ) *hold. Let $u \in \mathcal{F}' \cap L^\infty$ be non-negative, subharmonic in a ball $B := B(x_0, R)$ with $0 < R < \sigma\overline{R}$. Then the mean value inequality* (2.19) *holds, that is, for any $\varepsilon > 0$,*

$$\operatorname*{esup}_{\frac{1}{2}B} u \le C(1 + \varepsilon^{-\frac{1}{2\nu}}) \left(\frac{1}{\mu(B)} \int_B u^2 d\mu \right)^{1/2} + \varepsilon K \|u_+\|_{L^\infty((\frac{1}{2}B)^c)}, \tag{10.1}$$

where the constant C depends only on the constants in the hypotheses (but does not depend on ε), the constants ν and σ come from (FK), *and*

$$K = \begin{cases} 1 & \text{if the measure } j \not\equiv 0, \\ 0 & \text{if the measure } j \equiv 0. \end{cases} \tag{10.2}$$

Consequently, we have the following implication:

$$(\mathrm{VD}) + (\mathrm{Gcap}) + (\mathrm{FK}) + (\mathrm{TJ}) \Rightarrow (2.19).$$

To prove Theorem 10.1, we need the following lemma. Fix a point $x_0 \in M$, some numbers $0 < r_2 < r_1$ and consider two balls $B_i := B(x_0, r_i)$, $i = 1, 2$, so that $B_2 \subset B_1$. Fix also some numbers $0 < b_1 < b_2$, a measurable function u and set

$$a_i := \int_{B_i} (u - b_i)_+^2 d\mu, \quad i = 1, 2 \tag{10.3}$$

(see Fig. 4). Clearly, we have $a_2 \le a_1$. In the next lemma we show that a_2 can be controlled by $a_1^{1+\nu}$ for some $\nu > 0$ when u is a subharmonic function.

Lemma 10.2. *Let the jump measure j be given by*

$$dj(x, y) = J(x, dy) d\mu(x),$$

where $J(\cdot, \cdot)$ is a kernel on $M \times \mathcal{B}(M)$. Assume that conditions (FK_ν) *and* (EP) *hold.*

Let $u \in \mathcal{F}' \cap L^\infty$ be subharmonic in $B(x_0, r_1)$ with $r_1 < \sigma\overline{R}$, where σ comes from (FK_ν)*, and let a_1, a_2 be defined by* (10.3) *for $0 < r_2 < r_1$. Then*

$$a_2 \le \frac{CW(B_1)}{(b_2 - b_1)^{2\nu} \mu(B_1)^\nu} \left(\sup_{x \in B_1} \frac{1}{W(x, r_1 - r_2)} + \frac{A}{b_2 - b_1} \right) a_1^{1+\nu}, \tag{10.4}$$

where the constant $C > 0$ depends only on the constants in hypotheses, and A is given by

$$A := \operatorname*{esup}_{x \in B(x_0, \frac{1}{2}(r_1 + r_2))} \int_{B_1^c} u_+(y) J(x, dy).$$

Proof. In this proof, for any function in $\mathcal{F}$, we always use its *quasi-continuous* version (cf. Definition 15.2 or [19, Theorem 2.1.3 on p.71]).

Denote
$$U := B(x_0, \tfrac{1}{2}(r_1 + r_2)) \quad \text{and} \quad E := \{u > b_2\} \cap U.$$
By the outer regularity of μ, for any $\varepsilon > 0$, there is an open set V such that
$$E \subset V \subset B_1$$
and
$$\mu(V) \leq \mu(E) + \varepsilon \tag{10.5}$$
(see Fig. 4).

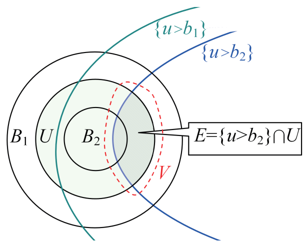

FIGURE 4. Sets E and V

Consider the function
$$v := (u - b_2)_+.$$
It follows from Lemma 9.3 that $v \in \mathcal{F}' \cap L^\infty$ and that v is subharmonic in B_1. By Proposition 15.1(ii),
$$\phi v \in \mathcal{F} \cap L^\infty \ \text{ for any } \phi \in \mathcal{F} \cap L^\infty.$$
Fix a function $\phi \in \text{cutoff}(B_2, U)$. Then
$$a_2 = \int_{B_2} (u - b_2)_+^2 d\mu = \int_{B_2} \phi^2 v^2 d\mu \leq \int_M \phi^2 v^2 d\mu. \tag{10.6}$$
Note that, for any $w \in \mathcal{F}$ and any open subset Ω of M, we have $w \in \mathcal{F}(\Omega)$ if and only if $\widetilde{w} = 0$ q.e. in Ω^c, where $\widetilde{w}$ is a quasi-continuous version of w and q.e. means *quasi-everywhere* (cf. [19, Corollary 2.3.1 on p.98]). Since $v = 0$ on $\{u \leq b_2\}$ and $\phi = 0$ q.e. on U^c, we see that
$$\phi v = 0 \ \text{ q.e. on } \ E^c = \{u \leq b_2\} \cup U^c.$$
Since $\phi v \in \mathcal{F}$ and $\phi v = 0$ q.e. in $V^c \subset E^c$, we conclude that
$$\phi v \in \mathcal{F}(V). \tag{10.7}$$
By the definition (2.13) of $\lambda_1(V)$ and by (10.6), we have
$$\lambda_1(V) \leq \frac{\mathcal{E}(\phi v, \phi v)}{\|\phi v\|_{L^2}^2} \leq \frac{\mathcal{E}(\phi v, \phi v)}{a_2},$$
and, hence,
$$a_2 \leq \frac{\mathcal{E}(\phi v, \phi v)}{\lambda_1(V)}.$$

On the other hand, by (FK_ν) and (10.7),

$$\frac{1}{\lambda_1(V)} \le CW(B_1)\left(\frac{\mu(V)}{\mu(B_1)}\right)^\nu.$$

Using also (10.5), we obtain

$$\begin{aligned} a_2 &\le CW(B_1)\left(\frac{\mu(V)}{\mu(B_1)}\right)^\nu \mathcal{E}(\phi v,\phi v) \\ &\le CW(B_1)\left(\frac{\mu(E)+\varepsilon}{\mu(B_1)}\right)^\nu \mathcal{E}(\phi v,\phi v). \qquad (10.8)\end{aligned}$$

Now let us estimate $\mathcal{E}(\phi v,\phi v)$ from above. By Proposition 15.1(*iii*) and using (10.7), we see that

$$0 \le v\phi^2 = v\phi\cdot\phi \in \mathcal{F}(V) \subset \mathcal{F}(B_1).$$

Since v is subharmonic in B_1, we have

$$\mathcal{E}(v, v\phi^2) \le 0. \qquad (10.9)$$

Applying (EP) to the triple B_2, U, B_1 and to the function v, we conclude that there exists $\phi \in \mathrm{cutoff}(B_2,U)$ such that

$$\mathcal{E}(\phi v,\phi v) \le \frac{3}{2}\mathcal{E}(v,v\phi^2) + \sup_{x\in B_1}\frac{C}{W(x,r)}\int_{B_1} v^2 d\mu + 3\iint_{B_1\times B_1^c} v(x)v(y)\phi^2(x)J(x,dy),$$

where $r := (r_2-r_1)/2$. Using (10.9) and the fact that $\phi = 0$ outside U, we obtain that

$$\mathcal{E}(v\phi,v\phi) \le \sup_{x\in B_1}\frac{C}{W(x,r)}\int_{B_1} v^2 d\mu + 3\int_U v(x)d\mu(x)\cdot\operatorname*{esup}_{x\in U}\int_{B_1^c} v(y)J(x,dy). \qquad (10.10)$$

Note that if $u \ge b_2$ then

$$(u-b_1)^2 \ge (u-b_1)(b_2-b_1) \ge (u-b_2)(b_2-b_1),$$

which implies that, for all values of u,

$$v = (u-b_2)_+ \le \frac{(u-b_1)_+^2}{b_2-b_1}.$$

Hence, we obtain from (10.10)

$$\begin{aligned} \mathcal{E}(v\phi,v\phi) &\le \sup_{x\in B_1}\frac{C}{W(x,r)}\int_{B_1}(u-b_1)_+^2 d\mu \\ &\quad + 3\int_{B_1}\frac{(u-b_1)_+^2}{b_2-b_1}d\mu\cdot\operatorname*{esup}_{x\in U}\int_{B_1^c} u_+(y)J(x,dy) \\ &= \left(\sup_{x\in B_1}\frac{C}{W(x,r)} + \frac{3A}{b_2-b_1}\right)a_1. \qquad (10.11)\end{aligned}$$

Next, let us estimate $\mu(E)$ from above as follows:

$$\begin{aligned} \mu(E) &= \int_{U\cap\{u>b_2\}} d\mu \\ &\le \int_{U\cap\{u>b_2\}} \frac{(u-b_1)_+^2}{(b_2-b_1)^2}d\mu \end{aligned}$$

$$\leq \frac{1}{(b_2-b_1)^2}\int_{B_1}(u-b_1)_+^2 d\mu$$
$$= \frac{a_1}{(b_2-b_1)^2}.$$

Substituting the last inequality and (10.11) into (10.8) and letting $\varepsilon \to 0$, we obtain

$$\begin{aligned} a_2 &\leq CW(B_1)\left(\frac{\mu(E)}{\mu(B_1)}\right)^{\nu}\mathcal{E}(\phi v, \phi v) \\ &\leq CW(B_1)\left(\frac{a_1}{(b_2-b_1)^2\mu(B_1)}\right)^{\nu}\left(\sup_{x\in B_1}\frac{C}{W(x,r)}+\frac{3A}{b_2-b_1}\right)a_1, \end{aligned}$$

which together with (2.5) implies (10.4). □

Proof of Theorem 10.1. Let a ball $B(x_0,R)$ and a subharmonic function u be as in the statement. Fix also some $\rho>0$ to be determined later. Let $\{R_k\}_{k=0}^{\infty}$, $\{\rho_k\}_{k=0}^{\infty}$ be two sequences of positive numbers defined for any $k\geq 0$ by

$$R_k = (2^{-1}+2^{-k-1})R \quad \text{and} \quad \rho_k = (1-2^{-k})\rho.$$

Then $\{R_k\}$ is decreasing with $R_0=R,\ R_k \downarrow \frac{1}{2}R$, and

$$R_{k-1}-R_k = 2^{-k-1}R < R_k, \tag{10.12}$$

while $\{\rho_k\}$ is increasing with $\rho_0=0,\ \rho_k\uparrow\rho$, and

$$\rho_k-\rho_{k-1} = 2^{-k}\rho. \tag{10.13}$$

Set also for all $k\geq 0$

$$B_k = B(x_0,R_k) \quad \text{and} \quad a_k = \int_{B_k}(u-\rho_k)_+^2 d\mu,$$

so that

$$B(x_0,R)=B_0\supset B_{k-1}\supset B_k\supset B_\infty := B(x_0,\tfrac{1}{2}R)$$

(see Fig. 5).

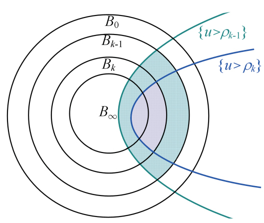

FIGURE 5. Sequence of balls $\{B_k\}$

Applying the inequality (10.4) of Lemma 10.2 for the pair of balls $B_k\subset B_{k-1}$ and for

$$b_1=\rho_{k-1},\ b_2=\rho_k,$$

we obtain, for all $k \geq 1$,
(10.14)
$$a_k \leq \frac{CW(B_{k-1})}{(\rho_k - \rho_{k-1})^{2\nu}\mu(B_{k-1})^\nu}\left(\sup_{x\in B_{k-1}}\frac{1}{W(x,R_{k-1}-R_k)} + \frac{A_k}{\rho_k - \rho_{k-1}}\right)a_{k-1}^{1+\nu},$$
where
$$A_k = \operatorname*{esup}_{x\in B(x_0,\frac{1}{2}(R_k+R_{k-1}))}\int_{B_{k-1}^c} u_+(y)J(x,dy).$$
Let us estimate every term on the right hand side of (10.14). Note that $B_{k-1}^c \subset B_\infty^c$, and, for all $x \in B(x_0, \frac{1}{2}(R_k + R_{k-1}))$, we have
$$B(x, 2^{-k-2}R) = B(x, \tfrac{1}{2}(R_{k-1} - R_k)) \subset B(x_0, R_{k-1}) = B_{k-1}.$$
Using this, (TJ) and (2.5), we obtain
$$\begin{aligned}
A_k &\leq \|u_+\|_{L^\infty(B_\infty^c)} \sup_{x\in B(x_0,\frac{1}{2}(R_k+R_{k-1}))}\int_{B(x,2^{-k-2}R)^c} J(x,dy)\\
&= \|u_+\|_{L^\infty(B_\infty^c)} \sup_{x\in B(x_0,\frac{1}{2}(R_k+R_{k-1}))}\frac{CK}{W(x,2^{-k-2}R)}\\
&\leq \frac{CK2^{\beta k}}{W(B_0)}\|u_+\|_{L^\infty(B_\infty^c)},
\end{aligned}$$
where the constant K is defined in (10.2).

By (VD) we have
$$\mu(B_{k-1}) \geq \mu(B_\infty) \geq c\mu(B_0).$$
Hence, substituting into (10.14) the above two inequalities as well as using (10.13) and (2.5), we obtain
$$\begin{aligned}
a_k &\leq \frac{CW(B_0)}{(2^{-k}\rho)^{2\nu}\mu(B_0)^\nu}\left(\left(\frac{R}{R_{k-1}-R_k}\right)^\beta \frac{1}{W(B_0)} + \frac{K2^{\beta k}}{W(B_0)}\frac{\|u_+\|_{L^\infty(B_\infty^c)}}{2^{-k}\rho}\right)a_{k-1}^{1+\nu}\\
&= \frac{C2^{2k\nu}W(B_0)}{\rho^{2\nu}\mu(B_0)^\nu}\left(\frac{2^{(k+1)\beta}}{W(B_0)} + \frac{K2^{(\beta+1)k}}{W(B_0)}\frac{\|u_+\|_{L^\infty(B_\infty^c)}}{\rho}\right)a_{k-1}^{1+\nu}\\
&\leq \frac{C}{\rho^{2\nu}\mu(B_0)^\nu}\left(1 + \frac{K\|u_+\|_{L^\infty(B_\infty^c)}}{\rho}\right)2^{(2\nu+\beta+1)k}a_{k-1}^{1+\nu}.
\end{aligned}$$
Setting
$$D := \frac{C}{\rho^{2\nu}\mu(B_0)^\nu}\left(1 + \frac{K\|u_+{}_-L^\infty(B_\infty^c)}{\rho}\right) \quad \text{and} \quad \lambda := 2^{(2\nu+\beta+1)},$$
we obtain, for all $k \geq 1$,
$$a_k \leq D\lambda^k a_{k-1}^{1+\nu}.$$
Then Proposition 15.4 from Appendix yields, for all $k \geq 1$,
$$a_k \leq D^{-\frac{1}{\nu}}\left(D^{\frac{1}{\nu}}\lambda^{\frac{1+\nu}{\nu^2}}a_0\right)^{(1+\nu)^k}.$$

Hence, if

$$D^{\frac{1}{\nu}}\lambda^{\frac{1+\nu}{\nu^2}}a_0 \leq \tfrac{1}{2}, \tag{10.15}$$

then $a_k \to 0$ as $k \to \infty$ and, hence,

$$\int_{B_\infty} (u-\rho)_+^2 d\mu = \lim_{k\to\infty} a_k = 0. \tag{10.16}$$

The inequality (10.15) is equivalent to

$$D \leq (\tfrac{1}{2}\lambda^{-\frac{1+\nu}{\nu^2}}a_0^{-1})^\nu =: ca_0^{-\nu}$$

where $c = (\frac{1}{2}\lambda^{-\frac{1+\nu}{\nu^2}})^\nu$, that is, to

$$\frac{C}{\rho^{2\nu}\mu(B_0)^\nu}\left(1 + \frac{K\|u_+\|_{L^\infty(B_\infty^c)}}{\rho}\right) \leq ca_0^{-\nu}. \tag{10.17}$$

Given $\varepsilon > 0$, (10.17) can be achieved if ρ satisfies the following conditions:

$$\rho \geq \varepsilon K\|u_+\|_{L^\infty(B_\infty^c)} \quad \text{and} \quad \frac{C(1+\varepsilon^{-1})}{\rho^{2\nu}\mu(B_0)^\nu} \leq ca_0^{-\nu}.$$

Clearly, the both inequalities here are satisfied for

$$\rho := \left(\frac{C\left(1+\varepsilon^{-1}\right)}{c}\right)^{\frac{1}{2\nu}}\left(\frac{a_0}{\mu(B_0)}\right)^{\frac{1}{2}} + \varepsilon K\|u_+\|_{L^\infty(B_\infty^c)}.$$

Choosing ρ as here we obtain by (10.16) that

$$\operatorname*{esup}_{B_\infty} u \leq \rho,$$

which is equivalent to (10.1). $\square$

The next statement provides a multiplicative form of the mean value inequality (2.19).

Corollary 10.3. *Under the hypotheses of Theorem* 10.1*, we have also*

$$\operatorname*{esup}_{\frac{1}{2}B} u \leq CS^\theta \max\{S,T\}^{1-\theta}, \tag{10.18}$$

where $\theta := \frac{2\nu}{1+2\nu}$ *and*

$$S := \left(\frac{1}{\mu(B)}\int_B u^2 d\mu\right)^{1/2} \quad \text{and} \quad T := K\|u_+\|_{L^\infty((\frac{1}{2}B)^c)}.$$

In particular, we have

$$(\mathrm{VD}) + (\mathrm{FK}_\nu) + (\mathrm{Gcap}) + (\mathrm{TJ}) \Rightarrow (10.18).$$

Proof. Applying (2.19), we have

$$\operatorname*{esup}_{\frac{1}{2}B} u \leq C(1+\varepsilon^{-\frac{1}{2\nu}})S + \varepsilon T.$$

Let us choose ε to satisfy the equation

$$C\varepsilon^{-\frac{1}{2\nu}}S = \varepsilon T,$$

that is,

$$\varepsilon=\left(\frac{CS}{T}\right)^{\frac{2\nu}{1+2\nu}}=\left(\frac{CS}{T}\right)^{\theta}.$$

Then we obtain

$$\operatorname*{esup}_{\frac12 B} u\le CS+2\varepsilon T=CS+2\left(\frac{CS}{T}\right)^{\theta}T\le C'S^{\theta}\max\{S^{1-\theta},T^{1-\theta}\},$$

thus proving (10.18). □

11. Lemma of Growth

Definition 11.1. We say that condition (LG) (*Lemma of Growth*) holds if there exist some numbers $\varepsilon_0,\sigma,\eta\in(0,1)$ such that, for any ball $B:=B(x_0,R)$ with $0<R<\sigma\overline{R}$ and for any function $u\in\mathcal{F}'\cap L^\infty$ that is superharmonic in B and is non-negative in M, the following is true: if, for some $a>0$,

$$\frac{\mu(B\cap\{u<a\})}{\mu(B)}\le\varepsilon_0,\tag{11.1}$$

then

$$\operatorname*{einf}_{\frac12 B}u\ge\eta a.\tag{11.2}$$

(see Fig. 6).

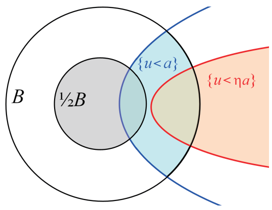

Figure 6. Illustration to Definition 11.1

We mention that all constants $\varepsilon_0,\sigma,\eta$ in (LG) must be independent of a,B,u. The following statement is the main result of this section.

Lemma 11.2 (Lemma of growth). *If the mean value inequality* (10.18) *holds, then* (LG) *is also satisfied. Consequently, we have*

$$(\text{VD})+(\text{Gcap})+(\text{FK})+(\text{TJ})\ \Rightarrow(\text{LG}).$$

Proof. The idea is to use the fact that the function $\frac{1}{u+\varepsilon}$ is subharmonic for $\varepsilon>0$ and to apply the mean value inequality (10.18) to this function.

Let us fix a constant $\varepsilon>0$ to be specified later on and choose a function $F\in C^2(\mathbb{R})$ such that

$$F(t)=\frac{1}{t+\varepsilon}\quad\text{for all } t\ge-\frac{\varepsilon}{2}$$

and

$$\sup_{\mathbb{R}}|F'|<\infty,\qquad\inf_{\mathbb{R}}F''\ge0,\qquad\sup_{\mathbb{R}}F''<\infty.$$

(see Fig. 7).

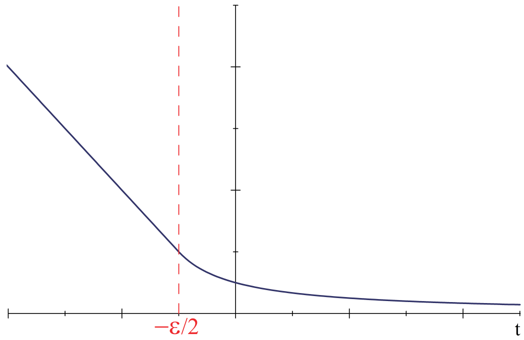

FIGURE 7. Function $F(t)$

Let us prove that $F(u)$ is subharmonic in B. By Proposition 9.1, $F(u) \in \mathcal{F}' \cap L^\infty$. We need to verify that, for any $0 \le \phi \in \mathcal{F}(B) \cap L^\infty$,

$$\mathcal{E}(F(u), \phi) \le 0. \tag{11.3}$$

By Proposition 9.1 we have also that $F'(u)\phi \in \mathcal{F}' \cap L^\infty$ and

$$\mathcal{E}(F(u), \phi) \le \mathcal{E}(u, F'(u)\phi). \tag{11.4}$$

By Proposition 15.1$(i), (iii)$ we have $F'(u) \in \mathcal{F}' \cap L^\infty$ and

$$F'(u)\phi \in \mathcal{F}(B).$$

Since u is superharmonic in B and $F'(u) \le 0$ (because $u \ge 0$), we obtain

$$\mathcal{E}(u, F'(u)\phi) \le 0,$$

which together with (11.4) yields (11.3).

Applying the mean value inequality (10.18) to subharmonic function $F(u)$, we obtain

$$\operatorname*{esup}_{\frac{1}{2}B} F(u) = \operatorname*{esup}_{\frac{1}{2}B} (u+\varepsilon)^{-1} \le C S_\varepsilon^\theta \max\left(S_\varepsilon, T_\varepsilon\right)^{1-\theta},$$

where

$$S_\varepsilon = \left(\fint_B F(u)^2 d\mu\right)^{1/2} = \left(\fint_B (u+\varepsilon)^{-2} d\mu\right)^{1/2},$$

$$T_\varepsilon = \|F(u)\|_{L^\infty((\frac{1}{2}B)^c)} = \|(u+\varepsilon)^{-1}\|_{L^\infty((\frac{1}{2}B)^c)}.$$

Hence, it follows that

$$\operatorname*{einf}_{\frac{1}{2}B} u \ge \frac{C^{-1}}{S_\varepsilon^\theta \max\left(S_\varepsilon, T_\varepsilon\right)^{1-\theta}} - \varepsilon. \tag{11.5}$$

Since $S_\varepsilon \le \varepsilon^{-1}$ and $T_\varepsilon \le \varepsilon^{-1}$, we have

$$\max\left(S_\varepsilon, T_\varepsilon\right)^{1-\theta} \le \varepsilon^{-(1-\theta)}. \tag{11.6}$$

On the other hand, by the hypothesis (11.1), we have

$$\begin{aligned} S_\varepsilon^2 &= \fint_B (u+\varepsilon)^{-2} d\mu \\ &= \frac{1}{\mu(B)} \left(\int_{B \cap u<a\}} + \int_{B\cap\{u \geq a\}} \right) (u+\varepsilon)^{-2} d\mu \\ &\leq \frac{1}{\varepsilon^2} \frac{\mu(B \cap \{u<a\})}{\mu(B)} + (a+\varepsilon)^{-2} \\ &\leq \frac{\varepsilon_0}{\varepsilon^2} + (a+\varepsilon)^{-2}. \end{aligned}$$

Now let us choose ε so that

$$\frac{\varepsilon_0}{\varepsilon^2} = (a+\varepsilon)^{-2},$$

that is,

$$\varepsilon := \frac{a}{\varepsilon_0^{-1/2} - 1} > 0. \tag{11.7}$$

With this choice of ε, we have

$$S_\varepsilon^2 \leq \frac{2\varepsilon_0}{\varepsilon^2}. \tag{11.8}$$

Therefore, plugging (11.8), (11.6) into (11.5) and using (11.7), we obtain

$$\begin{aligned} \operatorname*{einf}_{\frac{1}{2}B} u &\geq \frac{C^{-1}}{\left(\frac{2\varepsilon_0}{\varepsilon^2}\right)^{\theta/2} \varepsilon^{-(1-\theta)}} - \varepsilon \\ &= \left(\frac{C^{-1}}{(2\varepsilon_0)^{\theta/2}} - 1 \right) \varepsilon \\ &= \left(\frac{C^{-1}}{(2\varepsilon_0)^{\theta/2}} - 1 \right) \frac{a}{\varepsilon_0^{-1/2} - 1} = \eta a, \end{aligned}$$

where η is defined by

$$\eta = \left(\frac{C^{-1}}{(2\varepsilon_0)^{\theta/2}} - 1 \right) \frac{1}{\varepsilon_0^{-1/2} - 1} > 0,$$

assuming that $\varepsilon_0 > 0$ is sufficiently small. $\square$

12. Mean exit time

In this section we will obtain upper and lower estimates of *mean exit time* from a metric ball. Our approach is as follows: the upper estimate of the mean exit time follows directly from the Faber-Krahn inequality, while the lower bound follows from the Lemma of Growth, which is the most difficult part of this argument.

For any open set $\Omega \subset M$, let $\{P_t^\Omega\}$ be the heat semigroup of the Dirichlet form $(\mathcal{E}, \mathcal{F}(\Omega))$. For any $f \in L^2(\Omega)$, the function $t \mapsto P_t^\Omega f$ is continuous as a mapping from $[0, \infty)$ to $L^2(\Omega)$, which allows to integrate $P_t^\Omega f$ in t as an L^2-valued function. Define the *Green operator* G^Ω by

$$G^\Omega f := \int_0^\infty P_t^\Omega f \, dt$$

for any $0 \leq f \in L^2(\Omega)$. The function $G^\Omega f$ takes values in $[0, \infty]$. The monotonicity of $G^\Omega f$ in f allows us to extend this operator to any non-negative $f \in L^2_{\mathrm{loc}}(\Omega)$, in particular, to $f \equiv 1$.

For any non-empty subset Ω of M, denote

$$E^\Omega := G^\Omega 1 = \int_0^\infty P_t^\Omega \mathbf{1}_\Omega dt. \tag{12.1}$$

The function E^Ω is called the *mean exit time* from the set Ω. The value $E^\Omega(x)$ has the following probabilistic meaning: it is the expectation of the exit time from Ω of the Hunt process X_t, associated with $(\mathcal{E}, \mathcal{F})$, that starts at x (see Fig. 8).

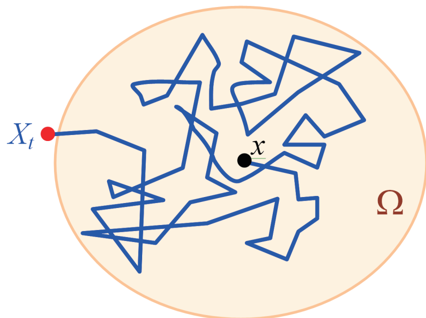

FIGURE 8. The probabilistic meaning: $E^\Omega(x) = \mathbb{E}_x \tau_\Omega$ where $\tau_\Omega = \inf\{t \geq 0 : X_t \notin \Omega\}$

Next, we introduce conditions $(E_\leq)$, $(E_\geq)$ and (E).

Definition 12.1. We say that condition $(E_\leq)$ holds, if there exist constants δ, $C > 0$ such that, for all balls $B \subset M$ of radius $< \delta\overline{R}$,

$$\operatorname*{esup}_B E^B \leq C W(B). \tag{12.2}$$

We say that condition $(E_\geq)$ holds, if there exists a constant $C > 0$ such that, for all balls $B \subset M$ of radius $< \overline{R}$,

$$\operatorname*{einf}_{\frac{1}{4}B} E^B \geq C^{-1} W(B). \tag{12.3}$$

We say that condition (E) holds if both conditions $(E_\leq)$ and $(E_\geq)$ are satisfied.

The following gives upper bound of E^B on any ball B by using the Faber-Krahn inequality only.

Lemma 12.2. *We have*

$$(\mathrm{FK}) \Rightarrow (E_\leq).$$

Proof. Let $B := B(x_0, R)$ with $R < \sigma\overline{R}$ where $\sigma \in (0,1)$ is the constant form condition (FK). We are to prove that

$$\operatorname*{esup}_B E^B \leq CW(B) \tag{12.4}$$

for constant some $C > 0$. This inequality was proved in [30, Theorem 9.4, p.1542] assuming that $\overline{R} = \infty$ and $W(x_0, R) = W(R)$. However, the same argument not only works for a general $W(x_0, R)$ when $\overline{R} = \infty$, but also allows to obtain (12.4) for balls of radius $R < \sigma\overline{R}$ when $\overline{R} < \infty$. Hence, ($E_{\leq}$) holds true with $\delta := \sigma$. □

In order to obtain a lower bound of the mean exit time, we use the following statement.

Proposition 12.3. *Let a function $u \in \mathcal{F}' \cap L^\infty$ be non-negative in an open set $B \subset M$ and $\phi \in \mathcal{F} \cap L^\infty$ be such that $\phi = 0$ in B^c. Fix any $\lambda > 0$ and set $u_\lambda := u + \lambda$. Then $\frac{\phi^2}{u_\lambda} \in \mathcal{F} \cap L^\infty$ and*

$$\mathcal{E}(u, \frac{\phi^2}{u_\lambda}) \leq \ 3\mathcal{E}(\phi, \phi). \tag{12.5}$$

Proof. Let us first show that $\frac{\phi^2}{u_\lambda} \in \mathcal{F} \cap L^\infty$. Indeed, as u is non-negative in B and $\phi = 0$ in B^c, the function $\frac{\phi^2}{u_\lambda}$ is well defined and $\frac{\phi^2}{u_\lambda} = F(u)\phi^2$ on M, where F is a function on $\mathbb{R}$ given by

$$F(t) := \frac{1}{|t| + \lambda}.$$

Since this function is Lipschitz (with Lipschitz constant λ^{-2}) and $u \in \mathcal{F}' \cap L^\infty$, we obtain by Proposition 15.1(i)

$$F(u) \in \mathcal{F}'.$$

Since $\phi \in \mathcal{F} \cap L^\infty$, we have

$$\phi^2 \in \mathcal{F} \cap L^\infty,$$

(see [19, Theorem 1.4.2(ii), p.28]). Since also $F(u) \in L^\infty$, it follows from Proposition 15.1(ii) that

$$F(u)\phi^2 \in \mathcal{F} \cap L^\infty.$$

Let us now prove (12.5). Indeed, it follows from [24, Lemma 3.7, p. 469] that

$$\mathcal{E}^{(J)}(u, \frac{\phi^2}{u_\lambda}) \leq \ 3\mathcal{E}^{(J)}(\phi, \phi), \tag{12.6}$$

On the other hand, by using the product and chain rules ((5.1) and (5.2)) as well as the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\mathcal{E}^{(L)}(u, \frac{\phi^2}{u_\lambda}) &= \int_M d\Gamma^{(L)}(u, \frac{\phi^2}{u_\lambda}) \\
&= \int \frac{2\phi}{u_\lambda} d\Gamma^{(L)}(u, \phi) - \int \frac{\phi^2}{u_\lambda^2} d\Gamma^{(L)}(u, u_\lambda) \\
&\leq \frac{1}{2}\int \frac{\phi^2}{u_\lambda^2} d\Gamma^{(L)}(u, u) + 2\int d\Gamma^{(L)}(\phi) - \int \frac{\phi^2}{u_\lambda^2} d\Gamma^{(L)}(u, u)
\end{aligned}$$

$$= -\frac{1}{2}\int \frac{\phi^2}{u_\lambda^2} d\Gamma^{(L)}(u) + 2\mathcal{E}^{(L)}(\phi,\phi)$$
$$\leq 2\mathcal{E}^{(L)}(\phi,\phi).$$

From this and (12.6), we conclude that

$$\mathcal{E}(u, \frac{\phi^2}{u_\lambda}) = \mathcal{E}^{(L)}(u, \frac{\phi^2}{u_\lambda}) + \mathcal{E}^{(J)}(u, \frac{\phi^2}{u_\lambda})$$
$$\leq 2\mathcal{E}^{(L)}(\phi,\phi) + 3\mathcal{E}^{(J)}(\phi,\phi) \leq 3\mathcal{E}(\phi,\phi),$$

thus proving (12.5). □

Let us recall the capacity condition ($\mathrm{Cap}_\leq$) from Definition 2.3: it is satisfied if there exists a constant $C > 0$ such that for all balls B of radius $R < \overline{R}$

$$\mathrm{cap}(\tfrac{1}{2}B, B) \leq C\frac{\mu(B)}{W(B)}, \tag{12.7}$$

where the capacity $\mathrm{cap}(A,U)$ is defined by (2.7). By (2.12), the condition ($\mathrm{Cap}_\leq$) follows from (Gcap).

Lemma 12.4. *We have*

$$(\mathrm{VD}) + (\mathrm{LG}) + (\mathrm{Cap}_\leq) \Rightarrow (\mathrm{E}_\geq).$$

Proof. Let $B := B(x_0, R)$ with $0 < R < \overline{R}$. Denote

$$u := E^B.$$

Note that u is superharmonic in B and is non-negative in M. We need to show that there exists a constant $C > 0$ such that

$$\operatorname*{einf}_{\frac{1}{4}B} u \geq C^{-1}W(B). \tag{12.8}$$

Let us first assume that $0 < R < \sigma\overline{R}$, where constant σ comes from condition (LG). For any $a > 0$ we have

$$\mu(\frac{1}{2}B \cap \{u < a\}) \leq a\int_{\frac{1}{2}B}\frac{1}{u}d\mu = a\mu(\frac{1}{2}B)\fint_{\frac{1}{2}B}\frac{1}{u}d\mu$$

where we use the fact that $u \geq 0$ in M. Choose a number a such that

$$a\fint_{\frac{1}{2}B}\frac{1}{u}d\mu = \varepsilon_0,$$

where $\varepsilon_0 > 0$ is the constant from Lemma of Growth (LG). It follows that

$$\frac{\mu(\frac{1}{2}B \cap \{u < a\})}{\mu(\frac{1}{2}B)} \leq a\fint_{\frac{1}{2}B}\frac{1}{u}d\mu = \varepsilon_0$$

so that (11.1) is satisfied, with B being replaced by $\frac{1}{2}B$. Applying Lemma 11.2, we obtain

$$\operatorname*{einf}_{\frac{1}{4}B} u \geq \eta a = \eta\varepsilon_0\left(\fint_{\frac{1}{2}B}\frac{1}{u}d\mu\right)^{-1}. \tag{12.9}$$

By $(\mathrm{Cap}_\leq)$ there is a cutoff function ϕ of the pair $(\frac{1}{2}B, B)$ such that

$$\mathcal{E}(\phi,\phi) \leq 2\mathrm{cap}(\frac{1}{2}B, B) \leq 2C\frac{\mu(B)}{W(B)}. \tag{12.10}$$

For any $\lambda > 0$ set

$$u_\lambda := u + \lambda.$$

We have by (12.5) and (12.10) that

$$\mathcal{E}(u, \frac{\phi^2}{u_\lambda}) \leq \ 3\mathcal{E}(\phi,\phi) \leq 6C\frac{\mu(B)}{W(B)}. \tag{12.11}$$

On the other hand, since $\frac{\phi^2}{u_\lambda} \in \mathcal{F}(B) \cap L^\infty$ and $\phi = 1$ in $\frac{1}{2}B$, we see that

$$\mathcal{E}(u, \frac{\phi^2}{u_\lambda}) = (1, \frac{\phi^2}{u_\lambda}) = \int_B \frac{\phi^2}{u_\lambda} d\mu \geq \int_{\frac{1}{2}B} \frac{1}{u_\lambda} d\mu.$$

From this and using (12.11), it follows that

$$\int_{\frac{1}{2}B} \frac{1}{u_\lambda} d\mu \leq 6C\frac{\mu(B)}{W(B)},$$

which yields as $\lambda \to 0$ that

$$\int_{\frac{1}{2}B} \frac{1}{u} d\mu \leq 6C\frac{\mu(B)}{W(B)}.$$

Therefore, combining (12.9) with the above inequality, we obtain

$$\operatorname*{einf}_{\frac{1}{4}B} u \geq \eta\varepsilon_0 \left(\fint_{\frac{1}{2}B} \frac{1}{u} d\mu \right)^{-1} \geq \eta\varepsilon_0 \frac{W(B)}{6C} =: c_0 W(B). \tag{12.12}$$

Now we extend the inequality (12.12) that was proved for balls B of radius $< \sigma\overline{R}$ to all balls B of radii $R < \overline{R}$. Indeed, assume that $\overline{R} < \infty$ and $R \in [\sigma\overline{R}, \overline{R})$. Then there exists an at most countable sequence of balls $\{B_i\}$ that covers B and such that each ball $B_i = B(x_i, r)$ has a center $x_i \in \frac{1}{4}B$ and the radius $r = \frac{1}{8}\sigma\overline{R}$. Applying (12.12) to balls $4B_i$ of radius $4r < \sigma\overline{R}$ and then using (2.5), we obtain

$$\operatorname*{einf}_{B_i} E^{4B_i} \geq c_0 W(4B_i) \geq c_0 C' \left(\frac{4r}{R}\right)^\beta W(B) \geq C^{-1} W(B).$$

Observing that

$$d(x_0, x_i) + 4r < \frac{1}{4}R + \frac{1}{2}\sigma\overline{R} < R,$$

we obtain that $4B_i \subset B$ whence it follows that

$$\operatorname*{einf}_{B_i} E^B \geq \operatorname*{einf}_{B_i} E^{4B_i} \geq C^{-1} W(B).$$

Since $\frac{1}{4}B$ is covered by $\{B_i\}$, we obtain (12.8). □

Combining Lemmas 12.2 and 12.4, we obtain the following.

Corollary 12.5. *We have*

$$(\mathrm{VD}) + (\mathrm{FK}) + (\mathrm{LG}) + (\mathrm{Cap}_\leq) \Rightarrow (\mathrm{E}).$$

13. Survival estimate and (GU)

We introduce condition (S), called the *survival estimate*.

Definition 13.1. We say that condition (S) holds if there exist two small constants $\varepsilon, \delta \in (0,1)$ such that, for all balls B of radius $< \overline{R}$,

$$\operatorname*{einf}_{\frac{1}{4}B} P_t^B \mathbf{1}_B \geq \varepsilon \quad \text{for all } 0 < t \leq \delta W(B) \tag{13.1}$$

The value $P_t^B \mathbf{1}_B(x)$ has the following probabilistic meaning: it is equal to the probability that the process X_t started at x stays inside B until time t; equivalently, assuming the killing conditions in B^c, this means the probability of survival up to time t.

Let us define also the following modification of (S).

Definition 13.2. We say that condition (S_+) holds if there exist two small constants ε, c in $(0,1)$ such that for all balls B of radius $< \overline{R}$ and for all $t > 0$,

$$\operatorname*{einf}_{\frac{1}{4}B} P_t^B \mathbf{1}_B \geq \varepsilon - \frac{ct}{W(B)} \quad \text{for all } t > 0. \tag{13.2}$$

Let us emphasize that in the condition (S_+) there is no restriction on the range of time t unlike that in (S).

Remark 13.3. By a standard covering arguments (see, for example, the second part in the proof of Lemma 12.4) and (2.5), one can extend (13.1) to all balls of radius $< C_0\overline{R}$ with any $C_0 \geq 1$ by adjusting the value of δ accordingly. The same observation is valid also for (S_+).

Proposition 13.4. *We have*

$$(\mathrm{E}) \Rightarrow (\mathrm{S}_+) \Rightarrow (\mathrm{S}).$$

Proof. Let B be a ball with radius $R < \delta\overline{R}$, where $\delta \in (0,1]$ is the constant from condition ($\mathrm{E}_\leq$). Note that the following inequality is true in general: for all $t > 0$ and μ-almost all $x \in B$,

$$P_t^B \mathbf{1}_B(x) \geq \frac{(E^B(x) - t)_+}{\|E^B\|_\infty}$$

where the function E^B is defined by (12.1) (see for example [10, formula (10.3)]). From this and (E) we have

$$\operatorname*{einf}_{\frac{1}{4}B} P_t^B \mathbf{1}_B(x) \geq \operatorname*{einf}_{\frac{1}{4}B} \frac{(E^B(x) - t)_+}{CW(B)} \geq \frac{C^{-1}W(B) - t}{CW(B)} = C^{-2} - \frac{C^{-1}t}{CW(B)},$$

thus showing that (13.2) holds with $\varepsilon = C^{-2} \in (0,1)$ and $c = C^{-1} \in (0,1)$. Moreover, using a standard covering arguments (see, for example, the second part in the proof of Lemma 12.4) and (2.5), one can extend (13.2) to all balls of radius $< \overline{R}$. Hence, we have proved the implication (E) $\Rightarrow$ (S_+).

Finally, the implication (S_+) $\Rightarrow$ (S) is trivial. □

The following result is an analogue of [24, Lemma 2.8, p. 451], which in turn was motivated by the argument in [1, Lemma 5.4]. However, the present proof has required some modifications due to the dependence of $W(x,r)$ on space variable x.

Lemma 13.5. *If every metric ball of radius smaller than $\overline{R}$ has finite measure then*

$$(\mathrm{S}) \Rightarrow (\mathrm{GU}).$$

Proof. We will prove that there exists a number $\kappa \geq 1$ such that, for any pair of balls $B_0 := B(x_0, R)$, $B := B(x_0, R+r)$ with $x_0 \in M$ and $0 < R < R+r < \overline{R}$, there exists a function $\phi \in \kappa$-cutoff(B_0, B) such that, for all $u \in \mathcal{F}' \cap L^\infty$,

$$\mathcal{E}(u^2\phi, \phi) \leq \frac{\kappa^2}{\inf_{x \in B_0} W(x,r)} \int_B u^2 \phi d\mu, \tag{13.3}$$

which will settle (GU) since $B_0 \subset B$. Fix $\lambda > 0$ to be determined later, and consider the function

$$G_\lambda^B 1_B := \int_0^\infty e^{-\lambda t} P_t^B 1_B dt.$$

Note that $G_\lambda^B 1_B \in \mathcal{F}(B)$ by [19, Theorem 4.4.1]. For any $0 \leq f \in L^2(B)$, we have

$$\begin{aligned}
(G_\lambda^B 1_B, f) &= \int_0^\infty e^{-\lambda t} \left(P_t^B 1_B, f\right) dt \\
&\leq \int_0^\infty e^{-\lambda t} dt \cdot \|f\|_1 \\
&= \lambda^{-1} \|f\|_1,
\end{aligned}$$

which implies that

$$G_\lambda^B 1_B \leq \lambda^{-1}, \quad \mu\text{-a.e. on } B.$$

Let us establish a lower bound of $G_\lambda^B 1_B$ in B_0. Fix a point $x \in B_0$ and consider a ball $\widetilde{B} := B(x,r) \subset B$. By condition (S), we have, for any $0 \leq f \in L^2(\frac{1}{4}\widetilde{B})$,

$$\begin{aligned}
(G_\lambda^B 1_B, f) &= \int_0^\infty e^{-\lambda t} \left(P_t^B 1_B, f\right) dt \\
&\geq \int_0^{\delta W(x,r)} e^{-\lambda t} \left(P_t^{\widetilde{B}} 1_{\widetilde{B}}, f\right) dt \\
&\geq \int_0^{\delta W(x,r)} e^{-\lambda t} dt \cdot \varepsilon \|f\|_1 \\
&= \lambda^{-1} \left(1 - e^{-\lambda \delta W(x,r)}\right) \varepsilon \|f\|_1 \\
&\geq \lambda^{-1} \left(1 - e^{-\lambda \delta \inf_{x \in B_0} W(x,r)}\right) \varepsilon \|f\|_1,
\end{aligned}$$

where the constants ε, δ are those from (S). Moreover, since B_0 can be covered by a family of countable balls like $\frac{1}{4}\widetilde{B}$ and f is arbitrary, we obtain that

$$G_\lambda^B 1_B \geq \lambda^{-1} \left(1 - e^{-\lambda \delta \inf_{x \in B_0} W(x,r)}\right) \varepsilon \quad \mu\text{-a.e. on } B_0.$$

Setting $\lambda := (\inf_{x\in B_0} W(x,r))^{-1}$ and $\kappa := (1-e^{-\delta})^{-1}\varepsilon^{-1}$, we see that

$$G_\lambda^B 1_B \begin{cases} \le \inf_{x\in B_0} W(x,r), & \mu\text{-a.e. on } B, \\ \ge \kappa^{-1}\inf_{x\in B_0} W(x,r), & \mu\text{-a.e. on } B_0. \end{cases}$$

Define the function

$$\phi := \frac{\kappa G_\lambda^B 1_B}{\inf_{x\in B_0} W(x,r)}$$

and observe that it satisfies $\phi \in \mathcal{F}(B)$, $0 \le \phi \le \kappa$, $\phi|_{B_0} \ge 1$ and $\phi|_{B^c} = 0$. That is, $\phi \in \kappa$-cutoff(B_0, B).

Let us prove that ϕ satisfied (13.3). By Proposition 15.1(*iii*), we have $u^2\phi \in \mathcal{F}(B)$ for any $u \in \mathcal{F}' \cap L^\infty$. Using the notation

$$\mathcal{E}_\lambda(w,v) := \mathcal{E}(w,v) + \lambda(w,v)$$

for $w, v \in \mathcal{F}$ and applying the identity

$$\mathcal{E}_\lambda(w, G_\lambda^B v) = (w,v)$$

for $w \in \mathcal{F}(B)$ and $v \in L^2(B)$ (see [19, Theorem 4.4.1]), we obtain that

$$\begin{aligned} \mathcal{E}(u^2\phi,\phi) &\le \mathcal{E}_\lambda(u^2\phi,\phi) \\ &= \frac{\kappa}{\inf_{x\in B_0} W(x,r)}\mathcal{E}_\lambda(u^2\phi, G_\lambda^B 1_B) \\ &= \frac{\kappa}{\inf_{x\in B_0} W(x,r)}(u^2\phi, 1_B) \\ &= \frac{\kappa}{\inf_{x\in B_0} W(x,r)}\int_B u^2\phi\, d\mu \\ &\le \frac{\kappa^2}{\inf_{x\in B_0} W(x,r)}\int_B u^2 d\mu, \end{aligned}$$

which finishes the proof. □

14. A full circle of equivalences

Finally, we can prove Theorem 2.11 that, in fact, is contained in the next Theorem 14.1 that combines together all the results of this paper.

Theorem 14.1. *Let* $(\mathcal{E}, \mathcal{F})$ *be a regular Dirichlet form without killing part. Assume that* (VD), (FK) *and* (TJ) *are satisfied. Then we have the following equivalences:*

$$\begin{aligned} \text{(Gcap)} &\Leftrightarrow \text{(ABB)} + (\text{Cap}_\le) \\ &\Leftrightarrow (\text{ABB}_{1/8}) + (\text{Cap}_\le) \\ &\Leftrightarrow \text{(EP)} + (\text{Cap}_\le) \\ &\Leftrightarrow \textit{mean value inequality (2.19)} + (\text{Cap}_\le) \\ &\Leftrightarrow \text{(LG)} + (\text{Cap}_\le) \\ &\Leftrightarrow \text{(E)} \\ &\Leftrightarrow (\text{S}_+) \Leftrightarrow \text{(S)} \\ &\Leftrightarrow \text{(GU)}. \end{aligned}$$

Proof. To prove the implications in the direction of "$\Rightarrow$", we use the following implications:

$$\begin{aligned}
(\mathrm{Gcap}) + (\mathrm{TJ}) &\Rightarrow (\mathrm{ABB}) \quad \text{(Lemma 6.2)}\\
(\mathrm{Gcap}) &\Rightarrow (\mathrm{Cap}_{\le}) \quad \text{(cf. (2.12))}\\
(\mathrm{ABB}) + (\mathrm{TJ}) &\Rightarrow (\mathrm{ABB}_{1/8}) \quad \text{(Lemma 7.1)}\\
(\mathrm{ABB}_{1/8}) &\Rightarrow (\mathrm{EP}) \quad \text{(Lemma 8.2)}\\
(\mathrm{EP}) + (\mathrm{VD}) + (\mathrm{FK}) + (\mathrm{TJ}) &\Rightarrow \text{mean value inequality (2.19)} \quad \text{(Theorem 10.1)}\\
\text{mean value inequality (2.19)} &\Rightarrow (\mathrm{LG}) \quad \text{(Corollary 10.3 and Lemma 11.2)}\\
(\mathrm{LG}) + (\mathrm{Cap}_{\le}) + (\mathrm{VD}) + (\mathrm{FK}) &\Rightarrow (\mathrm{E}) \quad \text{(Corollary 12.5)}\\
(\mathrm{E}) &\Rightarrow (\mathrm{S}_{+}) \Rightarrow (\mathrm{S}) \quad \text{(Proposition 13.4)}\\
(\mathrm{S}) + (\mathrm{VD}) &\Rightarrow (\mathrm{GU}) \quad \text{(Lemma 13.5)}.
\end{aligned}$$

Finally, the reverse implication

$$(\mathrm{Gcap}) \Leftarrow (\mathrm{GU})$$

is trivial. Combining all the above implications, we complete the circle and the proof. $\square$

Corollary 14.2. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form without killing part. Assume that* (VD)*,* (Gcap)*,* (FK) *and* (TJ) *are satisfied. Then, the cutoff function in the conditions* (ABB)*,* (ABB_ζ)*,* (EP) *can be universal, that is, the cutoff function can be independent of the function u in the above conditions.*

Proof. Under the conditions (VD), (FK) and (TJ), we have (Gcap) $\Leftrightarrow$ (GU) by Theorem 14.1. Note that the cutoff function ϕ in (GU) is universal. Using (GU) instead of (Gcap) in the proofs of (ABB), (ABB_ζ), and (EP), we obtain universal ϕ also in these conditions. $\square$

15. Appendix

In this appendix, we collect some facts that have used in this paper.

Proposition 15.1. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2. Then the following statements are true.*

(*i*) *If $u \in \mathcal{F}'$ and $F : \mathbb{R} \mapsto \mathbb{R}$ is a Lipschitz function, then $F(u) \in \mathcal{F}'$.*
(*ii*) *If $u \in \mathcal{F}' \cap L^\infty$ and $v \in \mathcal{F} \cap L^\infty$ then $uv \in \mathcal{F} \cap L^\infty$*
(*iii*) *Let Ω be an open subset of M. If $u \in \mathcal{F}' \cap L^\infty$ and $v \in \mathcal{F}(\Omega) \cap L^\infty$, then $uv \in \mathcal{F}(\Omega)$.*

Proof. We repeat the arguments of [24, Proposition A.2 in Appendix] with minor modifications. Since $u \in \mathcal{F}'$, we have $u = w + a \in \mathcal{F}'$, where $w \in \mathcal{F}$ and $a \in \mathbb{R}$.

(*i*) Denote by L the Lipschitz constant of F and consider the function

$$f(t) = \frac{F(t) - F(a)}{L}.$$

Since $f(w)$ is a normal contraction of w, we obtain by [19, formula $(\mathcal{E}.4)''$, p. 5] that $f(w) \in \mathcal{F}$. It follows that

$$F(u) = Lf(u) + F(a) \in \mathcal{F}'.$$

(ii) Clearly, we have $w \in \mathcal{F} \cap L^\infty$ which implies by [19, Theorem 1.4.2, p. 28] that $vw \in \mathcal{F} \cap L^\infty$. Consequently,

$$uv = vw + av \in \mathcal{F} \cap L^\infty.$$

(iii) Let $\widetilde{v}$ and $\widetilde{w}$ be quasi-continuous modifications of v and w, respectively. Then $\widetilde{v}\widetilde{w}$ is a quasi-continuous modification of vw. Since $v \in \mathcal{F}(\Omega)$, we obtain

$$\widetilde{v}\,\widetilde{w} = 0 \quad \text{q.e. in } \Omega^c.$$

It follows that $vw \in \mathcal{F}(\Omega)$ and

$$uv = vw + av \in \mathcal{F}(\Omega).$$

□

Recall the notion of a regular $\mathcal{E}$*-nest* (cf. [19, Section2.1, p. 66-69]). For an open set $U \subset M$, define 1-capacity of U by

(15.1)
$$\mathrm{Cap}_1(U) := \inf\left\{\mathcal{E}(u) + \|u\|_2^2 : u \in \mathcal{F} \text{ and } u \geq 1 \ \mu\text{-almost everywhere on } U\right\}$$

(noting that $\mathrm{Cap}_1(U) = \infty$ if the set $\{u \in \mathcal{F} : u \geq 1 \ \mu\text{-a.e. on } U\}$ is empty). An increasing sequence of closed subsets $\{F_k\}_{k=1}^\infty$ of M is called an $\mathcal{E}$*-nest* of M if

$$\lim_{k\to\infty} \mathrm{Cap}_1(M \setminus F_k) = 0.$$

An $\mathcal{E}$-nest $\{F_k\}$ is said to be *regular* with respect to μ if for each k,

$$\mu(U(x) \cap F_k) > 0 \ \text{ for any } x \in F_k \text{ and any open neighborhood } U(x) \text{ of } x.$$

For an $\mathcal{E}$-nest $\{F_k\}_{k=1}^\infty$, let

$$C(\{F_k\}) := \{u \text{ is a function on } M : u|_{F_k} \text{ is continuous for each } k\}.$$

Definition 15.2. A function $u : M \mapsto \mathbb{R} \cup \{\infty\}$ is said to be *quasi-continuous* if and only if $u \in C(\{F_k\})$ for some $\mathcal{E}$-nest $\{F_k\}_{k=1}^\infty$.

Proposition 15.3. *Let $\{F_k\}$ be a μ-regular $\mathcal{E}$-nest and $u \in C(\{F_k\})$. Then for any open set $U \subset M$*

$$\sup_{U\cap F} u = \operatorname*{esup}_U u$$

where $F := \bigcup_{k\geq 1} F_k$.

Proof. Note that $\mu(F^c) = 0$ since $\mathrm{Cap}_1(F^c) = 0$ where the 1-capacity Cap_1 is defined in (15.1). Hence,

$$M_0 := \operatorname*{esup}_U u = \operatorname*{esup}_{U\cap F} u \leq \sup_{U\cap F} u.$$

Let us prove that $\sup_{U\cap F} u \leq M_0$. Indeed, by definition of M_0, there is a measurable set $E \subset U \cap F$ with $\mu(E) = 0$ such that

$$M_0 = \operatorname*{esup}_{U\cap F} u = \sup_{(U\cap F)\setminus E} u$$

It suffices to show that

$$u(x) \leq M_0 \quad \text{for any } x \in E,$$

since if so, then

$$\sup_{U\cap F} u = \Big(\sup_{(U\cap F)\setminus E} u\Big) \vee \Big(\sup_{E} u\Big) \le M_0.$$

To do this, suppose that there was a point $x \in E \subset U \cap F$ such that $u(x) > M_0$. Then there would exist an integer $k \ge 1$ such that

$$x \in U \cap F_k.$$

Since $u|_{F_k}$ is continuous, one can find an open neighborhood $U(x)$ of x such that

$$u(y) > M_0 \quad \text{for every } y \in U(x) \cap F_k.$$

Without loss of generality, we assume that $U(x) \subset U$. Since $\{F_k\}$ is μ-regular, we have

$$\mu(U(x) \cap F_k) > 0,$$

which implies, together with the fact that $U(x) \cap F_k \subset U \cap F$, that

$$\operatorname*{esup}_{U\cap F} u > M_0 = \operatorname*{esup}_{U\cap F} u,$$

leading to a contradiction. The proof is complete. □

The following iteration is elementary.

Proposition 15.4. *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of non-negative numbers such that*

$$a_k \le D\lambda^k a_{k-1}^{1+\nu} \quad \text{for } k = 1, 2, \cdots \tag{15.2}$$

for some constants $D, \nu > 0$ and $\lambda \ge 1$. Then for any $k \ge 0$,

$$a_k \le D^{-\frac{1}{\nu}} \left(D^{\frac{1}{\nu}} \lambda^{\frac{1+\nu}{\nu^2}} a_0\right)^{(1+\nu)^k}. \tag{15.3}$$

Proof. Setting $q := 1 + \nu$, we obtain by iterating (15.2)

$$\begin{aligned}
a_k &\le D\lambda^k a_{k-1}^q \le (D\lambda^k)\left(D\lambda^{k-1} a_{k-2}^q\right)^q \le \cdots \\
&\le (D^{1+q+\cdots+q^{k-1}})(\lambda^{k+(k-1)q+\cdots+q^{k-1}})a_0^{q^k} \\
&= D^{\frac{q^k-1}{q-1}} \lambda^{\frac{q^{k+1}-(k+1)q+k}{(q-1)^2}} a_0^{q^k} \\
&\le D^{\frac{q^k-1}{q-1}} \lambda^{\frac{q^{k+1}}{(q-1)^2}} a_0^{q^k}
\end{aligned}$$

where we have used the elementary fact that

$$k + (k-1)q + \cdots + q^{k-1} = \frac{q^{k+1} - (k+1)q + k}{(q-1)^2} \le \frac{q^{k+1}}{(q-1)^2}.$$

Therefore,

$$a_k \le D^{\frac{-1}{q-1}} \left(D^{\frac{1}{q-1}} \lambda^{\frac{q}{(q-1)^2}} a_0\right)^{q^k},$$

thus proving (15.3). The proof is complete. □

The following was proved in [36, Lemma 2.12].

Proposition 15.5. *Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form in L^2. If*

$$f_n \xrightarrow{L^2} f \quad \text{and} \quad \sup_n \mathcal{E}(f_n) < \infty,$$

then $f \in \mathcal{F}$. Besides, there exists a subsequence, still denoted by $\{f_n\}$, such that $f_n \xrightarrow{\mathcal{E}} f$ weakly, that is,

$$\mathcal{E}(f_n,\varphi) \to \mathcal{E}(f,\varphi) \quad \text{as } n \to \infty$$

for any $\varphi \in \mathcal{F}$. Moreover, there exists a subsequence $\{f_{n_k}\}$ such that its Cesaro mean $\frac{1}{n}\sum_{k=1}^{n} f_{n_k}$ converges to f in $\mathcal{E}_1$-norm. Finally, we have

$$\mathcal{E}(f,f) \le \liminf_{n\to\infty} \mathcal{E}(f_n,f_n).$$

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Manuscript received September 30 2022
revised November 16 2022

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