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UNIQUENESS FOR A CLASS OF NONLINEAR ELLIPTIC EQUATIONS WITH LOWER ORDER TERMS

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Abstract. In this paper we prove an uniqueness result for weak solution to a class of Dirichlet boundary value problems whose prototype is

$$\begin{cases} -\Delta_p u = \beta|\nabla u|^q + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N, N \geq 2$, $1 < p \leq 2$, $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $p-1+\frac{p}{N} < q < p$, β is a positive constant and f is a measurable function satisfying suitable summability conditions depending on q and a smallness condition. An existence result is also proved.

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, $N \geq 2$. Let us consider the following Dirichlet boundary value problem

$$\text{(1.1)} \qquad \begin{cases} -\operatorname{div}(\mathbf{a}(x,u,\nabla u)) = H(x,\nabla u) + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where

$$\mathbf{a} : (x,s,z) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbf{a}(x,s,z) \in \mathbb{R}^N$$

$$H : (x,z) \in \Omega \times \mathbb{R}^N \longrightarrow H(x,z) \in R$$

are Carathéodory functions which satisfy the following ellipticity condition

$$\text{(1.2)} \qquad \mathbf{a}(x,s,z)\cdot z \geq |z|^p ,$$

the monotonicity condition

$$\text{(1.3)} \qquad (\mathbf{a}(x,s,z) - \mathbf{a}(x,s,z'))\cdot(z-z') > 0 , \quad z, z' \neq 0 ,$$

and the following growth conditions

$$\text{(1.4)} \qquad |\mathbf{a}(x,s,z)| \leq a_0|z|^{p-1} + a_1|s|^{p-1} + a_2 , \quad a_0, a_1, a_2 > 0;$$

$$\text{(1.5)} \qquad |H(x,z)| \leq \beta|z|^q , \quad \beta > 0,$$

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with $1 < p < N$, $p-1+\frac{p}{N} \le q < p$. As regards the source term f, we suppose that it belongs to the Lorentz space $L^{\frac{N(q-p+1)}{q},\frac{p}{p-1}}(\Omega)$.

Our main interest is to investigate the uniqueness issue for solutions to problem (1.1). When dealing with the question of uniqueness, one has to consider the fact that for the model problem ($p = 2$)

$$\begin{cases} -\Delta u = |\nabla u|^q & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0), \end{cases} \tag{1.6}$$

uniqueness does not hold for solutions in $H_0^1(B_1(0))$, where $B_1(0)$ is the unitary ball. For instance, it is well known (see, for example [1]) that, in addition to the trivial solution $u = 0$, the function

$$u(x) = \mathcal{C}_\alpha(|x|^{-\alpha} - 1), \qquad \alpha = \frac{2-q}{q-1}, \, \mathcal{C}_\alpha = \frac{(N-\alpha-2)^{\frac{1}{q-1}}}{\alpha}, \tag{1.7}$$

solves problem (1.6) when $N > 2$, $1 + 2/N < q < 2$ and $u \in H_0^1(B_1(0))$.

Thus, uniqueness for problem (1.1) has been studied imposing some restrictions on the set to which the solution belongs. Actually existence which give further regularity results on the solution have been proved in literature depending on the summability of the source term f and the uniqueness has been established among the solutions which satisfy such further regularity property. Indeed, results have been proven for problem (1.1) which state the existence of a bounded solution u ([14], [16], [5], [18]), or the existence of a solution u is such that a certain function of u, $g(u(x))$, belongs to $H_0^1(\Omega)$([19], [20], [8]). In both cases corresponding uniqueness results are obtained, for example, in [27] for bounded solutions and in [6], [8], [27] under the condition $g(u) \in H_0^1(\Omega)$.

We explicitly remark that when dealing with the problem of existence of solutions for problems like (1.1), some necessary conditions are required on the data. Such necessary conditions are derived in a sharp way in [1] and [21]; the necessity of a size condition is a natural feature of the problem (1.1).

Let us finally recall that uniqueness issue in the case $q \le p - 1 + p/N$ has been completely settled ([28], [10], [12], see also [11]) and this explains why we consider the case $p - 1 + \frac{p}{N} < q < p$.

A slightly different approach to the existence, based also on symmetrization techniques ([3]), allows to prove the existence of a solution u, obtained as limit of approximation, for which an explicit estimate of its H_0^1−norm is derived in terms of the N, p, q, β, $|\Omega|$.

In the present paper we discuss the possibility to get a uniqueness result for solutions satisfying this condition. So the purpose of this article is twofold: firstly, we study the existence of a solution to problem (1.1) and we give an explicit bound on the gradient of such a solution, secondly, for such a solution we prove uniqueness in the case where some further assumptions on the structure of the problem are made.

In order to prove the existence of a weak solution u to problem (1.1) we consider the approximate problem

$$\text{(1.8)}\qquad \begin{cases} -\mathrm{div}\,(\mathbf{a}(x,u_n,\nabla u_n)) = T_n(H(x,\nabla u_n)) + T_n(f) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

where $T_n(s) = \min\{|s|, n\}\mathrm{sign}(s)$ denotes the usual truncation function. Since the right-hand side in the equation (1.8) is bounded, in view of (1.2)-(1.4), a classical result (see [24], [25]) implies the existence of a bounded weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, for any $n \in \mathbb{N}$, to problem (1.8). In order to prove the existence of a weak solution to problem (1.1) we use a standard strategy. Firstly, one proves that any bounded weak solution to problem (1.8) satisfies suitable a priori estimates. Making use of such estimates one can prove that, up to subsequence, $\{u_n\}_{n\in\mathbb{N}}$ converges to a measurable function u, in such a way that it is possible to pass to the limit in the weak formulation of (1.8), yielding that u is a weak solution to (1.1). Actually, the existence is proved under the additional assumption that the norm of f is sufficiently small and an explicit estimate of $\|\nabla u\|_p$ is proved (see Theorem 3.2 for details). We observe that it is well known that the "smallness" assumption on f in this context is necessary otherwise the existence can be lost as discussed, for example, in [19].

As regards the uniqueness we restrict our analysis to a restricted class of operators. We consider the case $1 < p \leq 2$ and we assume thet $\mathbf{a}$ is a Carathéodory function which does not depend on s

$$\text{(1.9)}\qquad \mathbf{a} \,:\, (x,z) \in \Omega \times \mathbb{R}^N \longrightarrow \mathbf{a}(x,z) \in \mathbb{R}^N$$

and satisfies a homogeneity condition

$$\text{(1.10)}\qquad \mathbf{a}(x,tz) = |t|^{p-1}\mathbf{a}(x,z)\,, \qquad t \in \mathbb{R}\,.$$

Moreover we substitute the monotonicity condition (1.3) with the "strong monotonicity" condition

$$\text{(1.11)}\qquad \big(\mathbf{a}\,(x,z) - \mathbf{a}\,(x,z')\big)\cdot\big(z - z'\big) \geq (|z| + |z'|)^{p-2}\,|z - z'|^2\,, \qquad z \neq z'\,.$$

Finally we consider a Carathéodory function

$$\text{(1.12)}\qquad H \,:\, (x,z) \in \Omega \times \mathbb{R}^N \longrightarrow H(x,z) \in R$$

such that

$$\text{(1.13)}\qquad H(x,\cdot) \qquad \text{is convex}\,,$$

and satisfies the homogeneity condition

$$\text{(1.14)}\qquad H(x,tz) = |t|^q H(x,z)\,, \qquad t \in \mathbb{R}\,.$$

We explicitly remark that the above assumptions are satisfied by the model case

$$-\mathrm{div}\,(\mathbf{a}(x,\nabla u)) = -\Delta_p u\,, \qquad H(x,\nabla u) = |\nabla u|^q\,.$$

Let us comment on the notion of weak solution $u \in W_0^{1,p}(\Omega)$ for which we prove uniqueness, that is,

$$\text{(1.15)}\qquad \int_\Omega \mathbf{a}(x,\nabla u)\cdot\nabla\phi\,dx = \int_\Omega H(x,\nabla u)\phi\,dx + \int_\Omega f(x)\phi\,dx\,,$$

for any $\phi \in W_0^{1,p}(\Omega) \cap L^\theta(\Omega)$, with $\theta \geq \frac{p}{p-q}$.

Let us explicitly remark that every term in (1.15) is meaningful. Indeed, since $H(x, \nabla u) \in L^{\frac{p}{q}}(\Omega)$, the first integral on the right-hand side is finite. Moreover $f(x)\phi$ belongs to $L^1(\Omega)$ since $\frac{p}{p-q} \geq (\frac{N(q-p+1)}{q})'$, that is $q \geq p - 1 + \frac{p}{N}$. Finally we note that $\frac{p}{p-q} \leq (N(q-p+1))^*$ since $q \geq p-1+\frac{p}{N}$, where usually t^* denotes the critical Sobolev exponent $Nt/(N-t)$, $t < N$.

Under the hypothesis that the $L^{\frac{N}{q},1}$-norm of f is sufficiently small, our main result on uniqueness states that there exists a constant $\mathcal{M}$ which depends on N, p, q, β, $|\Omega|$ such that, if u, v are two solutions to (1.15) with

$$\|\nabla u\|_p,\ \|\nabla v\|_p \leq \mathcal{M}, \tag{1.16}$$

then $u = v$. We remark that for the model problem (1.6) condition (1.16) selects only the trivial solution $u = 0$. The general computation is quite involved, here we consider only the case $N = 4$, $q = \frac{7}{4}$ where the constant in (1.16) reads as (see Theorem 4.1)

$$\mathcal{M} = \omega_4^{-\frac{17}{18}} \sqrt{6} \left(\frac{4}{21} \right)^{\frac{4}{3}}.$$

A straightforward calculation proves that the function u in (1.7) solves problem (1.6), but does not satisfy (1.16).

The paper is organized as follows. In Section 2, some preliminary results on rearrangements and the definition of some Lorentz spaces are recalled. Sections 3 and 4 are devoted to the proof of the existence and uniqueness results respectively.

2. Preliminary results

We begin by recalling some properties of rearrangements. If u is a measurable function defined in Ω and

$$\mu(t) = |\{x \in \Omega : |u(x)| \geq t\}|, \quad t \geq 0,$$

is its distribution function, then

$$u^*(s) = \sup\{t \geq 0 \,:\, \mu(t) > s\}, \quad s \in (0, |\Omega|),$$

is the decreasing rearrangement of u and $u_*(s) = u^*(|\Omega| - s)$ is the increasing rearrangement of u.

If ω_N is the measure of the unit ball of $\mathbb{R}^N$ and $\Omega^\#$ is the ball of $\mathbb{R}^N$ centered at the origin with the same measure as Ω,

$$u^\#(x) = u^*(\omega_N |x|^N), \quad u_\#(x) = u_*(\omega_N |x|^N), \qquad x \in \Omega^\#,$$

denote the spherically decreasing and increasing rearrangements of u, respectively. We recall the well-known Hardy-Littlewood inequality ([22])

$$\int_{\Omega^\#} u^\#(x)\, v_\#(x)\, dx \leq \int_\Omega |u(x)\, v(x)|\, dx \leq \int_{\Omega^\#} u^\#(x)\, v^\#(x)\, dx. \tag{2.1}$$

and the following result also due to Hardy ([22]).

Lemma 2.1. *Let f be a nonnegative function defined in $]0,+\infty[$. For $r\neq 1$, we denote*

$$F(s)=\begin{cases}\displaystyle\int_s^\infty f(t)dt\,, & \text{if } r<1\\ \displaystyle\int_0^s f(t)dt\,, & \text{if } r>1\,.\end{cases}$$

Then, the following inequality holds true

$$\int_0^\infty F(s)^q s^{-r}\,ds\le\left(\frac{q}{|1-r|}\right)^q\int_0^\infty f(s)^q s^{-r+q}\,ds, \tag{2.2}$$

for every $q>1$.

For any $q\in(1,+\infty)$, the Lorentz space $L^{q,r}(\Omega)$ is the collection of all measurable funtions u such that $\|u\|_{q,r}$ is finite, where we use the notation

$$\|u\|_{L^{q,r}}=\left(\int_0^{+\infty}\left[u^*(s)\,s^{1/q}\right]^r\frac{ds}{s}\right)^{1/r}$$

if $r\in]0,\infty[$,

$$\|u\|_{L^{q,\infty}}=\sup_{s>0}u^*(s)\,s^{1/q}=\sup_{t>0}t\,\mu(t)^{1/q} \tag{2.3}$$

if $r=\infty$.

These spaces give in some sense a refinement of the usual Lebesgue spaces. Indeed, $L^{q,q}(\Omega)=L^q(\Omega)$ and $L^{q,\infty}(\Omega)=M^q(\Omega)$ is the Marcinkiewicz space L^q-weak. The following embeddings hold true (see [23], [26])

$$L^{q,r_1}(\Omega)\subset L^{q,r_2}(\Omega),\quad \text{if}\quad r_1<r_2, \tag{2.4}$$

$$L^{t_1,r_1}(\Omega)\subset L^{t_2,r_2}(\Omega),\qquad \text{for}\quad t_1>t_2,\quad 0<r_1,r_2\le\infty, \tag{2.5}$$

and

$$L^{q_1,r}(\Omega)\subset L^q(\Omega)\,,\quad \text{if}\quad q<q_1. \tag{2.6}$$

Moreover the following inequalities hold:

$$\left(\frac{r_2}{q}\right)^{\frac{1}{r_2}}\|u\|_{q,r_2}\le\left(\frac{r_1}{q}\right)^{\frac{1}{r_1}}\|u\|_{q,r_1}\,,\quad \text{for } 0<r_1<r_2\le\infty, \tag{2.7}$$

and

$$\|u\|_{t_2,r_2}\le c_L\|u\|_{t_1,r_1},\quad \text{for}\quad t_1>t_2,\quad 0<r_1,r_2\le\infty, \tag{2.8}$$

where

$$c_L=\frac{t_1-t_2}{r_2t_1t_2}\left(\frac{r_1}{t_1}\right)^{\frac{1}{r_1}}|\Omega|^{\frac{1}{t_2}-\frac{1}{t_1}}. \tag{2.9}$$

Finally we recall the following Sobolev-type inequality (see [2], [4]).

Lemma 2.2. *Assume $u \in W_0^{1,p}(\Omega)$, $1 \le p < N$, $\alpha > 0$ and $\gamma > N$. If μ denotes the distribution function of u, the following inequalities hold true:*

$$\int_{\Omega^\#} \frac{\left[u^\#(x)\right]^p}{|x|^{p-N\alpha}} dx \le \omega_N^{-\alpha} \left(\frac{p}{N-p+N\alpha}\right)^p \int_\Omega [\mu(|u(x)|)]^\alpha \, |\nabla u|^p \, dx \tag{2.10}$$

$$\int_{\Omega^\#} \frac{\left[u^\#(x)\right]^p}{|x|^{p+N\alpha}} dx \le \omega_N^{-\alpha} \left(\frac{p}{N-p-N\alpha}\right)^p \int_\Omega \frac{|\nabla u|^p}{[\mu(|u(x)|)]^\alpha} \, dx \,. \tag{2.11}$$

Sketch of the proof. Without loss of generality we can assume that u is smooth enough.

By co-area formula, we get

$$\int_\Omega [\mu(|u(x)|)]^\alpha \, |\nabla u|^p \, dx = \int_0^{+\infty} [\mu(t)]^\alpha \; dt \int_{|u|=t} |\nabla u|^{p-1} \, d\sigma.$$

Moreover by classical isoperimetric inequality we get (cf. [29])

$$\int_{|u|=t} |\nabla u|^{p-1} \, d\sigma \ge N^p \omega_N^{\frac{p}{N}} |\mu'(t)|^{1-p} [\mu(t)]^{p-\frac{p}{N}} \,.$$

Therefore we obtain

$$\int_\Omega [\mu(|u(x)|)]^\alpha \, |\nabla u|^p \, dx \ge N^p \omega_N^{\frac{p}{N}} \int_0^{+\infty} [\mu(t)]^{\alpha+p-\frac{p}{N}} \, |\mu'(t)|^{1-p} \, dt. \tag{2.12}$$

Since u^* is an absolutely continuous function, by a change of variables, we get

$$\int_0^{+\infty} [\mu(t)]^{\alpha+p-\frac{p}{N}} \, |\mu'(t)|^{1-p} \, dt = \int_0^{|\Omega|} s^{\alpha+\frac{p(N-1)}{N}} \, |(u^*)'(s)|^p \, ds \,. \tag{2.13}$$

By Hardy inequality (2.2), (2.10) follows.

In analogous way we get (2.11).

3. Existence result

We begin this section by proving the following apriori estimate for weak solutions to the approximate problem (1.8)

Theorem 3.1. *Assume* (1.2)-(1.5) *with $1 < p < N$ and*

$$p - 1 + \frac{p}{N} \le q < p \,.$$

Let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution to the problem (1.1) *with $f \in C^\infty$. If the norm of f in $L^{\frac{N(q-p+1)}{q},\frac{p}{p-1}}(\Omega)$ satisfies the following smallness condition*

$$\|f\|_{\frac{N(q-p+1)}{q},\frac{p}{p-1}} \le \frac{p-q}{q\omega_N^{\frac{1}{N(q-p+1)}-\frac{1}{p}}} \left(\frac{p-1}{qc_0\beta}\right)^{\frac{p-1}{q-p+1}} , \tag{3.1}$$

then

$$\|\nabla u\|_{L^p} \le |\Omega|^{\frac{N(q-p+1)-p}{Np(q-p+1)}} \left(\frac{p-1}{qc_0\beta}\right)^{\frac{1}{q-p+1}} , \tag{3.2}$$

$$\|u\|_{(N(q-p+1))^*,p} \le \omega_N^{\frac{1}{N(q-p+1)}-\frac{1}{p}} \frac{q-p+1}{p-q}\left(\frac{p-1}{qc_0\beta}\right)^{\frac{1}{q-p+1}}, \tag{3.3}$$

where c_0 *is a positive constant depending only on* p, q, N *and defined in* (3.11).

Proof. For any $k>0$, define

$$u_k = (|u|-k)_+ \operatorname{sign}(u)$$

and consider the following test function

$$\varphi_k(x) = \operatorname{sign}(u_k)\int_0^{|u_k(x)|} \frac{1}{[\mu_k(t)]^\alpha}\,dt\,,$$

where μ_k is the distribution function of u_k and

$$\alpha = 1-\frac{p}{N(q-p+1)} > 0\,, \quad \text{if } q>p-1+\frac{p}{N}. \tag{3.4}$$

Since u is a bounded weak solution, $\varphi_k \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and therefore φ_k can be used as test function in (1.15). By assumptions (1.2) and (1.5), we get

$$\int_{|u|>k} \frac{|\nabla u_k|^p}{[\mu_k(|u_k(x)|)]^\alpha}\,dx \le \beta\int_\Omega |\nabla u|^q|\varphi_k(x)|\,dx + \int_\Omega |f\varphi_k(x)|\,dx\,, \tag{3.5}$$

where we have used the fact that $\nabla u = \nabla u_k$ on the set $\{|u|>k\}$.

Let us evaluate the first integral on the right-hand side. By Hölder inequality, we get

$$\int_\Omega |\nabla u|^q|\varphi_k(x)|\,dx = \int_{|u|>k} |\nabla u_k|^q|\varphi_k(x)|\,dx \le \tag{3.6}$$

$$\le \left(\int_{|u|>k}\frac{|\nabla u_k|^p}{[\mu_k(|u_k(x)|)]^\alpha}\,dx\right)^{\frac{q}{p}} \left(\int_\Omega |\varphi_k(x)|^{\frac{p}{p-q}}[\mu_k(|u_k(x)|)]^{\frac{\alpha q}{p-q}}\,dx\right)^{1-\frac{q}{p}}.$$

On the other hand co-area formula and classical properties of rearrangements imply

$$\int_\Omega |\varphi_k(x)|^{\frac{p}{p-q}}[\mu_k(|u_k(x)|)]^{\frac{\alpha q}{p-q}}\,dx \le \int_0^{|\Omega|} (u_k)^*(s)^{\frac{p}{p-q}}\frac{ds}{s^\alpha} = \|u_k\|^{\frac{p}{p-q}}_{\frac{p}{(1-\alpha)(p-q)},\frac{p}{p-q}}. \tag{3.7}$$

Since $\frac{p}{p-q}>p$, by the inclusions of Lorentz spaces (2.4) and (2.7), we get

$$\|u_k\|^{\frac{p}{p-q}}_{\frac{p}{(1-\alpha)(p-q)},\frac{p}{p-q}} \le (1-\alpha)^{\frac{q-p+1}{p-q}}\,(p-q)^{\frac{1}{p-q}}\,\|u_k\|^{\frac{p}{p-q}}_{\frac{p}{(1-\alpha)(p-q)},p}.$$

Moreover, by Sobolev-type inequality (2.11) and definition of α, since

$$\frac{p}{(1-\alpha)(p-q)} = \frac{Np}{N(1-\alpha)-p} = \frac{N(q-p+1)}{p-q}\,,$$

we get

$$\|u_k\|^{\frac{p}{p-q}}_{\frac{N(q-p+1)}{p-q},p} \le \omega_N^{-\frac{\alpha}{p-q}}\left(\frac{p}{N-p-N\alpha}\right)^{\frac{p}{p-q}}\left(\int_\Omega \frac{|\nabla u_k|^p}{[\mu_k(|u_k(x)|)]^\alpha}\,dx\right)^{\frac{1}{p-q}}. \tag{3.8}$$

Therefore by collecting (3.7) and (3.8), by definition of α, we deduce

$$\int_\Omega |\varphi_k(x)|^{\frac{p}{p-q}} \left[\mu_k(|u_k(x)|)\right]^{\alpha \frac{q}{p-q}} \, dx \leq \tag{3.9}$$

$$\leq (p-q)^{\frac{1}{p-q}} \left[\frac{p}{N(q-p+1)}\right]^{\frac{q-p+1}{p-q}} \omega_N^{-\frac{\alpha}{p-q}} \left(\frac{p}{N-p-N\alpha}\right)^{\frac{p}{p-q}} \left(\int_\Omega \frac{|\nabla u_k|^p}{[\mu_k(|u_k(x)|)]^\alpha} \, dx\right)^{\frac{1}{p-q}}$$

and by (3.6),

$$\int_\Omega |\nabla u|^q |\varphi_k(x)| \, dx \leq c_0 \left(\int_{|u|>k} \frac{|\nabla u_k|^p}{[\mu_k(|u_k(x)|)]^\alpha} \, dx\right)^{\frac{q+1}{p}}, \tag{3.10}$$

where

$$c_0 = (p-q)^{frac1p-1} \left(\frac{p}{N}\right)^{\frac{q-p+1}{p}} \omega_N^{\frac{1}{N(q-p+1)}-\frac{1}{p}} \left(\frac{1}{q-p+1}\right)^{\frac{q-2p+1}{p}}. \tag{3.11}$$

Now let us evaluate the integral

$$\int_\Omega |f\varphi_k| \, dx \, .$$

By definition of φ_k, it follows

$$|\varphi_k(x)| \leq \frac{|u_k(x)|}{(\mu_k(|u_k(x)|))^\alpha}.$$

Therefore by Hardy-Littlewood inequality, Hölder inequality and Sobolev-type inequality (2.11), we deduce

$$\int_\Omega |f||\varphi_k| \, dx \leq \int_\Omega \frac{|f(x)|\,|u_k(x)|}{(\mu_k(|u_k(x)|))^\alpha} \, dx \leq \int_\Omega \frac{f^\#(x)(u_k)^\#(x)}{|x|^{N\alpha}} \, dx \tag{3.12}$$

$$\leq \left(\int_\Omega \frac{|f^\#(x)|^{\frac{p}{p-1}}}{|x|^{N\alpha-p'}} \, dx\right)^{\frac{p-1}{p}} \left(\int_\Omega \frac{[(u_k)^\#(x)]^p}{|x|^{N\alpha+p}} \, dx\right)^{\frac{1}{p}}$$

$$\leq c_0' \|f\|_{\frac{Np}{p+N(1-\alpha)(p-1)},\frac{p}{p-1}} \left(\int_\Omega \frac{|\nabla u_k|^p}{[\mu_k(|u_k(x)|)]^\alpha} \, dx\right)^{\frac{1}{p}},$$

where c_0' is the constant defined by

$$c_0' = \omega_N^{-\frac{\alpha}{p}} \frac{p}{N-p-N\alpha} = \omega_N^{\frac{1}{N(q-p+1)}-\frac{1}{p}} \frac{q-p+1}{p-q} \, . \tag{3.13}$$

Collecting (3.5), (3.10) and (3.12), we deduce

$$\left(\int_\Omega \frac{|\nabla u_k|^p}{[\mu_k(|u_k(x)|)]^\alpha} \, dx\right)^{\frac{1}{p'}} \leq c_0\beta \left(\int_\Omega \frac{|\nabla u_k|^p}{[\mu_k(|u_k(x)|)]^\alpha} \, dx\right)^{\frac{q}{p}} + c_0' \|f\|_{\frac{N(q-p+1)}{q},\frac{p}{p-1}}, \tag{3.14}$$

where c_0, c_0' are the constants defined by (3.11) and (3.13) respectively.

If μ denotes the distribution function of u, in the set $\{x \in \Omega : |u(x)| > k\}$ it holds

$$\begin{aligned}\mu_k(|u_k(x)|) &= |\{y \in \Omega : |u_k(y)| > |u_k(x)|\}| \\ &= |\{y \in \Omega : |u(y)| > |u(x)|\}| = \mu(|u(x)|),\end{aligned}$$

and we put

$$X_k = \left(\int_\Omega \frac{|\nabla u_k|^p}{[\mu_k(|u_k(x)|)]^\alpha}\, dx\right)^{\frac{1}{p'}} = \left(\int_{|u|>k} \frac{|\nabla u|^p}{[\mu(|u(x)|)]^\alpha}\, dx\right)^{\frac{1}{p'}},$$

$$F = \|f\|_{\frac{N(q-p+1)}{q},\frac{p}{p-1}}\,.$$

Then (3.14) can be written in the following way

$$X_k \le c_0\beta X_k^{\frac{q}{p-1}} + c_0' F\,. \tag{3.15}$$

Now the proof proceeds as in [20]. Since F satisfies (3.1), by (3.15) it follows

$$X_k \le Z_1\,,$$

where Z_1 denotes the first positive zero of the function

$$G(\sigma) \equiv \sigma - c_o\beta\sigma^{\frac{q}{p-1}} - c_0' F, \quad \sigma > 0.$$

Moreover it is easy to verify that

$$Z_1 < \sigma_0 \equiv \left(\frac{p-1}{qc_0\beta}\right)^{\frac{p-1}{q-p+1}},$$

where σ_0 is the maximum point of the function $G(\sigma)$.

Now when k goes to $+\infty$, X_k tends to zero, therefore, by the continuity of the function

$$k \to \int_{|u|>k} \frac{|\nabla u|^p}{[\mu(|u(x)|)]^\alpha}\, dx,$$

we conclude that for any k,

$$X_k < \left(\frac{p-1}{qc_0\beta}\right)^{\frac{p-1}{q-p+1}}. \tag{3.16}$$

By Sobolev-type inequality (2.11), we deduce that, for any $k > 0$,

$$\|u_k\|_{(N(q-p+1))^*,p} \le c_0' \left(\frac{p-1}{qc_0\beta}\right)^{\frac{1}{q-p+1}}.$$

Therefore for $k = 0$, (3.3) is obtained. Finally Hölder inequality and (3.16) with $k = 0$ give the apriori estimate (3.2) in $W_0^{1,p}(\Omega)$.
Now we assume that $q = p - 1 + \frac{p}{N}$. Then $\alpha = 0$ in (3.4) and $\varphi_k(x) = u_k(x)$. It is easy to verify that the proof proceed in the same way. □

The previous apriori estimates allow to prove the following existence result for problem (1.1).

Theorem 3.2. *Assume* (1.2)-(1.5) *hold true with* $1 < p < N$ *and*

$$p - 1 + \frac{p}{N} \le q < p\,.$$

If $f \in L^{\frac{N(q-p+1)}{q}, \frac{p}{p-1}}(\Omega)$ *satisfies* (3.1)*, then there exists at least a weak solution to the problem* (1.1) *which satisfies* (3.2) *and* (3.3).

Proof. We consider a weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to the approximate problem (1.8). By the a priori estimates obtained in Theorem 3.1 we deduce that $|\nabla u_n|^q$ is bounded in $L^{\frac{p}{q}}(\Omega)$. Therefore by growth assumption (1.5) on H we deduce that $T_n(H(x, \nabla u_n)$ is bounded in $L^{\frac{p}{q}}(\Omega)$. Moreover, for every fixed $k > 0$, $T_k(u_n)$ can be used as test function in the usual weak formulation of (1.15) and we get

$$\int_\Omega |\nabla T_k(u_n)|^p\, dx \le k \int_\Omega [T_n(H(x, \nabla u_n)) + T_n(f)]\, dx\,. \tag{3.17}$$

This implies that $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, for every $k > 0$. Since the right-hand side in (1.8) is bounded in $L^1(\Omega)$, we can apply a well-known compactness result (see [7]), which implies that a function u exists such that, up to extracting a subsequence,

$$u_n \to u \qquad \text{and} \qquad \nabla u_n \to \nabla u \qquad \text{a.e. in } \Omega \tag{3.18}$$

with $u \in L^{(N(q-p+1))^*, p}(\Omega)$ and $|\nabla u| \in L^p(\Omega)$.

We deduce that $\mathbf{a}(x, u_n, \nabla u_n)$ converges pointwise to $\mathbf{a}(x, u, \nabla u)$ and $T_n(H(x, \nabla u_n)) + T_n(f)$ converges pointwise to $H(x, \nabla u) + f$. By Vitali's theorem we can pass to the limit in the weak formulation of the approximate problem (1.8), ie.

$$\int_\Omega \mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla \phi\, dx = \int_\Omega T_n(H(x, \nabla u_n))\phi\, dx + \int_\Omega T_n(f(x))\phi\, dx\,,$$

for any $\phi \in W_0^{1,p}(\Omega)$ and we get that u is a weak solution to (1.1), i.e.

$$\int_\Omega \mathbf{a}(x, u, \nabla u) \cdot \nabla \phi\, dx = \int_\Omega H(x, \nabla u))\phi\, dx + \int_\Omega f(x)\phi\, dx\,.$$

□

4. Uniqueness result

In this section we consider the following Dirichlet boundary value problem

$$\begin{cases} -\text{div}\,(\mathbf{a}(x, \nabla u)) = H(x, \nabla u) + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

Under the assumptions of Theorem 3.2, the existence of at least a weak solution to (4.1) is guaranteed.

Now we prove the uniqueness of such a solution under the stronger assumptions (1.10), (1.11), (1.13) and (1.14).

We prove the following result

Theorem 4.1. *Let $N \geq 2$ and*

$$\begin{cases} \dfrac{2N}{N+1} < p < 2, & \text{if} \quad N = 2 \\ \\ \dfrac{2N}{N+1} < p \leq 2, & \text{if} \quad N \geq 3. \end{cases} \tag{4.2}$$

Assume (1.10), (1.11), (1.13) *and* (1.14) *with*

$$p - 1 + \frac{p}{N} < q < \frac{(N+2)p - 2N}{p} \leq p \tag{4.3}$$

and

$$f \in L^{\frac{N}{q},1}(\Omega).$$

Denote by u, v two weak solutions belonging to $W_0^{1,p}(\Omega)$ to problem (4.1) *which satisfy estimate* (3.2). *If the norm of f in $L^{\frac{N}{q},1}(\Omega)$ is sufficiently small, that is,*

$$\|f\|_{\frac{N}{q},1} \leq \left(\frac{1}{2C_5^*}\right)^{\hat{s}} \left(\frac{1}{C_4^* q}\right)^{\frac{2}{\hat{s}(q-1)}} \left(\frac{q-1}{q}\right)^{\frac{2}{\hat{s}}}, \tag{4.4}$$

where the constants $C_4^*, C_5^*, \hat{s}$ are defined in (4.23), *then $u \equiv v$ a.e. in Ω.*

Remark 4.2. Let us explicitly remark that the uniqueness result given by Theorem 4.1 holds true under a more restrictive assumption on the summability of the datum f with respect to the existence result stated in Theorem 3.2. Indeed we prove uniqueness when f belongs to $L^{\frac{N}{q},1}(\Omega)$, while existence is proved when $f \in L^{\frac{N(q-p+1)}{q},\frac{p}{p-1}}(\Omega)$, with $\frac{N}{q} > \frac{N(q-p+1)}{q}$.

Proof. For any $t > 1$, denote

$$w = tu - (t-1)\gamma_t v\,,$$

where

$$\gamma_t = \left(\frac{t}{t-1}\right)^{\frac{2-p}{q-p+1}}. \tag{4.5}$$

For any $h > 0$, denote

$$w_h = (w - h)_+$$

and consider the following function

$$\varphi_h(x) = \int_0^{w_h(x)} \frac{1}{\mu_h(t)^\alpha}\,dt\,,$$

where μ_h is the distribution function of w_h and

$$\alpha = 1 - \frac{s}{N(q-s+1)} > 0\,,$$

with $q < s < p$ to be chosen.

Observe that $\alpha > 0$ since $s < p$ and by assumption

$$p - 1 + \frac{p}{N} < q < p\,.$$

By using φ_h as test function in problem (4.1) satisfied by u we get

$$t\int_\Omega \mathbf{a}(x,\nabla u)\nabla\varphi_h\,dx = t\int_\Omega H(x,\nabla u)\varphi_h\,dx + t\int_\Omega f\varphi_h\,dx\,. \tag{4.6}$$

Analogously by using φ_h as test function in problem (4.1) satisfied by v we get

$$(t-1)\int_\Omega \mathbf{a}(x,\nabla v)\nabla\varphi_h\,dx = (t-1)\int_\Omega H(x,\nabla v)\varphi_h\,dx + (t-1)\int_\Omega f\varphi_h\,dx\,. \tag{4.7}$$

Since $\mathbf{a}$ satisfies homogeneity condition (1.10), the equalities (4.6) and (4.7) can be written in the following way, respectively

$$t^{2-p}\int_\Omega \mathbf{a}(x,\nabla(tu))\nabla\varphi_h\,dx = t\int_\Omega H(x,\nabla u)\varphi_h\,dx + t\int_\Omega f\varphi_h\,dx\,, \tag{4.8}$$

$$(t-1)^{2-p}\int_\Omega \mathbf{a}(x,(t-1)\nabla v)\nabla\varphi_h\,dx = (t-1)\int_\Omega H(x,\nabla v)\varphi_h\,dx + (t-1)\int_\Omega f\varphi_h\,dx\,. \tag{4.9}$$

Moreover by homogeneity condition on $\mathbf{a}$ (1.10) and on H (1.14), equality (4.9) is equivalent to the following equality

$$t^{2-p}\int_\Omega \mathbf{a}(x,\nabla((t-1)\gamma_t v))\nabla\varphi_h\,dx = (t-1)\int_\Omega H(x,\nabla(\gamma_t v))\varphi_h\,dx + (t-1)\gamma_t^q\int_\Omega f\varphi_h\,dx\,, \tag{4.10}$$

where γ_t is defined in (4.5).

By subtracting equation (4.10) by equation (4.8) and by using convexity assumption on H (1.13), we have

$$\begin{aligned} t^{2-p}\int_\Omega &[\mathbf{a}(x,\nabla(tu)) - \mathbf{a}(x,\nabla((t-1)\gamma_t v))]\,\frac{\nabla w_h}{\mu_h(w_h(x))}\,dx \\ &= \int_\Omega [tH(x,\nabla u)\,dx + (1-t)H(x,\nabla(\gamma_t v))]\,\varphi_h\,dx \\ &\quad + \int_\Omega (tf + (1-t)\gamma_t^q f)\varphi_h\,dx \\ &\le \int_\Omega H(x,\nabla(tu) + \nabla((1-t)\gamma_t v))\varphi_h\,dx + K_t\int_\Omega f\varphi_h\,dx\,, \end{aligned} \tag{4.11}$$

where

$$K_t = |1 + (1-t)(\gamma_t^q - 1)|\,.$$

We note that

$$\lim_{t\to+\infty} K_t = \left|\frac{2q-(p-1)(q+1)}{q-p+1}\right|.$$

Hence, for a suitable $t_0 > 1$, we get

$$|K_t| < C_0\,, \qquad t \ge t_0\,, \tag{4.12}$$

where

$$C_0 = \left|\frac{2q-(p-1)(q+1)}{q-p+1}\right| + 1\,.$$

Now since **a** is a strongly monotone operator which satisfies (1.11) and since the following estimate holds true

$$\varphi_h(x) \le \frac{w_h(x)}{\mu_h(w_h(x))^\alpha},$$

we have

$$\int_\Omega \frac{|\nabla w_h|^2}{(|t\nabla u| + |\nabla(t-1)\gamma_t v|)^{2-p}[\mu_h(w_h(x))]^\alpha}\,dx \le \frac{\beta}{t^{2-p}}\int_\Omega |\nabla w_h|^q \frac{w_h}{[\mu(w_h(x))]^\alpha}\,dx + \frac{C_0}{t^{2-p}}\int_\Omega |f|\frac{w_h}{[\mu(w_h(x))]^\alpha}\,dx, \tag{4.13}$$

for any $t > t_0$. Moreover, since $s < p \le 2$, we have

$$\int_\Omega \frac{|\nabla w_h|^s}{[\mu_h(w_h(x))]^\alpha}\,dx \le \left(\int_\Omega \frac{|\nabla w_h|^2}{(|t\nabla u| + |\nabla(1-t)\gamma_t v|)^{2-p}[\mu_h(w_h(x))]^\alpha}\,dx\right)^{\frac{s}{2}} \times \left(\int_\Omega \frac{(|t\nabla u| + |\nabla(1-t)\gamma_t v|)^{\frac{(2-p)s}{2-s}}}{[\mu_h(w_h(x))]^\alpha}\,dx\right)^{1-\frac{s}{2}} \tag{4.14}$$

Denote

$$I_t = \left(\int_\Omega \frac{(|t\nabla u| + |\nabla(1-t)\gamma_t v|)^{\frac{(2-p)s}{2-s}}}{[\mu_h(w_h(x))]^\alpha}\,dx\right)^{1-\frac{s}{2}} = t^{\frac{(2-p)s}{2}}\left(\int_\Omega \frac{(|\nabla u| + |\nabla(\frac{1}{t}-1)\gamma_t v|)^{\frac{(2-p)s}{2-s}}}{[\mu_h(w_h(x))]^\alpha}\,dx\right)^{1-\frac{s}{2}}. \tag{4.15}$$

Let us estimate I_t. By Hölder inequality, since $s < p$ implies $\frac{(2-p)s}{2-s} < p$, we get

$$I_t \le t^{\frac{(2-p)s}{2}}\left(\int_\Omega (|\nabla u| + |\nabla(\tfrac{1}{t}-1)\gamma_t v|)^p\,dx\right)^{\frac{(2-p)s}{2p}} \times \left(\int_{\Omega^\#} \frac{1}{|x|^{\frac{N\alpha p(2-s)}{2(p-s)}}}\,dx\right)^{\frac{p-s}{p}} = C(s)t^{\frac{(2-p)s}{2}}\left(\int_\Omega (|\nabla u| + |\nabla(\tfrac{1}{t}-1)\gamma_t v|)^p\,dx\right)^{\frac{(2-p)s}{2p}}, \tag{4.16}$$

for any $t > t_0$. Here $C(s)$ is the following constant depending on s

$$C(s) = \left(\frac{2(p-s)}{2(p-s) - \alpha p(2-s)}\right)^{\frac{p-s}{p}} |\Omega|^{\frac{p-s}{p} - \alpha\frac{2-s}{2}}\,\omega_N^{\frac{\alpha(2-s)}{2}}$$

and s is chosen in such a way that

$$0<\alpha<\frac{2(p-s)}{p(2-s)}\,,$$

that is

$$q<s<\hat{s}=\frac{(N+2)p-2N-Nq(2-p)}{(N+1)p-2N}\,. \tag{4.17}$$

This choice of s is possible by the assumptions (4.3) on q and (4.2) on p. Indeed $q<\hat{s}$ is equivalent to $q<\frac{(N+2)p-2N}{p}$; $p-1+\frac{p}{N}<\frac{(N+2)p-2N}{p}$ is equivalent to $\frac{2N}{N+1}<p<N$ and $\hat{s}\le p$ is equivalent to $q\ge p-1+\frac{p}{N}$. Finally we note that $\frac{(N+2)p-2N}{p}>0$ requires the bound on p given by (4.2).

Moreover it is easy to verify that for a suitable $t_1>t_0$, the following estimate holds

$$|(\tfrac{1}{t}-1)\gamma_t|<2\,,\qquad t\ge t_1\,.$$

Therefore by (4.16), since u,v are weak solutions satisfying the estimate (3.2), we deduce for $t\ge t_1$

$$\begin{aligned}\frac{I_t}{t^{\frac{(2-p)s}{2}}} &\le C(s)2^{\frac{(2-p)s}{2}}\left(\|\nabla u\|_p+\|\nabla v\|_p\right)^{\frac{(2-p)s}{2}}\\ &\le C_1(s)\,,\end{aligned} \tag{4.18}$$

where

$$C_1(s)=C(s)2^{\frac{(2-p)s}{2}}\left[2|\Omega|^{\frac{N(q-p+1)-p}{Np(q-p+1)}}\left(\frac{p-1}{qc_0\beta}\right)^{\frac{1}{q-p+1}}\right]^{\frac{(2-p)s}{2}}\,.$$

By inequalities (4.13)-(4.16), since $q<s$, we get

$$\begin{aligned}&\int_\Omega \frac{|\nabla w_h|^s}{[\mu_h(w_h(x))]^\alpha}\,dx\\
&\le I_t\left(\frac{\beta}{t^{2-p}}\int_\Omega|\nabla w_h|^q\frac{w_h}{[\mu_h(w_h(x))]^\alpha}\,dx+\frac{C_0}{t^{2-p}}\int_\Omega|f|\frac{w_h}{[\mu_h(w_h(x))]^\alpha}\,dx\right)^{\frac{s}{2}}\\
&\le I_t\left[\frac{\beta}{t^{2-p}}\left(\int_\Omega\frac{|\nabla w_h|^s}{[\mu_h(w_h(x))]^\alpha}\,dx\right)^{\frac{q}{s}}\left(\int_\Omega\frac{w_h^{\frac{s}{s-q}}}{[\mu_h(w_h(x))]^\alpha}\,dx\right)^{1-\frac{q}{s}}\right.\\
&\qquad\left.+\frac{C_0}{t^{2-p}}\left(\int_\Omega\frac{|f|^{\frac{s}{q}}}{[\mu_h(w_h(x))]^\alpha}\,dx\right)^{\frac{q}{s}}\left(\int_\Omega\frac{w_h^{\frac{s}{s-q}}}{[\mu_h(w_h(x))]^\alpha}\,dx\right)^{1-\frac{q}{s}}\right]^{\frac{s}{2}}\,.\end{aligned} \tag{4.19}$$

Since $s < p$, the two weak solutions u, v belong to $W_0^{1,s}(\Omega)$. On the other hand co-area formula and classical properties of rearrangements imply

$$\left(\int_\Omega \frac{w_h^{\frac{s}{s-q}}}{[\mu_h(w_h(x))]^\alpha}\,dx\right)^{1-\frac{q}{s}} \le \left(\int_0^{|\Omega|} w_h^*(s)^{\frac{s}{s-q}}\,\frac{1}{s^\alpha}\,ds\right)^{1-\frac{q}{s}} = \|w_h\|_{\frac{s}{(1-\alpha)(s-q)},\frac{s}{s-q}}.$$

Since $s < p < q+1$ and therefore $\frac{s}{s-q} > s$, by the inclusions of Lorentz spaces (2.4) and inequality (2.7), we have

$$\|w_h\|_{\frac{s}{(1-\alpha)(s-q)},\frac{s}{s-q}} \le (1-\alpha)^{\frac{q-s+1}{s}}(s-q)^{\frac{1}{s}}\|w_h\|_{\frac{s}{(1-\alpha)(s-q)},s}.$$

Moreover, by Sobolev-type inequality (2.11) and definition of α, since

$$\frac{s}{(1-\alpha)(s-q)} = \frac{Ns}{N(1-\alpha)-s} = \frac{N(q-s+1)}{s-q},$$

we get

$$\|w_h\|_{\frac{N(q-s+1)}{s-q},s} \le \frac{\omega_N^{-\frac{\alpha}{s}}\,s}{N-s-N\alpha}\left(\int_\Omega \frac{|\nabla w_h|^s}{[\mu_h(w_h(x))]^\alpha}\,dx\right)^{\frac{1}{s}}.$$

Denote

$$C_2(s) = \omega_N^{-\frac{\alpha}{2}}\left(\frac{s}{N-s-N\alpha}\right)^{\frac{s}{2}}(1-\alpha)^{\frac{q-s+1}{2}}(s-q)^{\frac{1}{2}}.$$

Therefore by (4.19) and (4.18), since $s < p \le 2$, we get

$$\begin{aligned}(4.20)\quad &\int_\Omega \frac{|\nabla w_h|^s}{[\mu_h(w_h(x))]^\alpha}\,dx \\ &\le C_1(s)C_2(s)\beta^{\frac{s}{2}}\left(\int_\Omega \frac{|\nabla w_h|^s}{[\mu_h(w_h(x))]^\alpha}\,dx\right)^{\frac{q+1}{2}} \\ &\quad + C_0^{\frac{s}{2}}C_1(s)C_2(s)\left(\int_\Omega \frac{|f|^{\frac{s}{q}}}{[\mu_h(w_h(x))]^\alpha}\,dx\right)^{\frac{q}{2}}\left(\int_\Omega \frac{|\nabla w_h|^s}{[\mu_h(w_h(x))]^\alpha}\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

Moreover, by definition of α, Hardy-Littlewood inequality and inclusion (2.8), it follows

$$\left(\int_\Omega \frac{|f|^{\frac{s}{q}}}{[\mu(w_h(x))]^\alpha}\,dx\right)^{\frac{q}{2}} \le \|f\|^{\frac{s}{2}}_{\frac{s}{(1-\alpha)q},\frac{s}{q}} = \|f\|^{\frac{s}{2}}_{\frac{N(q-s+1)}{q},\frac{s}{q}} \le C_3(s)\|f\|^{\frac{s}{2}}_{\frac{N}{q},1},$$

where, in view of (2.8), (2.9),

$$C_3(s) = \left(\frac{s-q}{s(q-s+1)}\right)^{\frac{s}{2}}|\Omega|^{\frac{q}{N(q-s+1)}-\frac{q}{N}}.$$

Denote

$$X_h = \left(\int_\Omega \frac{|\nabla w_h|^s}{[\mu_h(w_h(x))]^\alpha}\,dx\right)^{\frac{1}{2}} = \left(\int_{w>h} \frac{|\nabla w|^s}{[\mu(|w(x)|)]^\alpha}\,dx\right)^{\frac{1}{2}},$$

where μ denotes the distribution function of w. By (4.25), we get

$$X_h \le C_4(s) X_h^q + C_5(s) \|f\|_{\frac{N}{q},1}^{\frac{s}{2}}, \tag{4.21}$$

for any fixed $t > t_1$, where

$$C_4(s) = C_1(s)C_2(s)\beta^{\frac{s}{2}}, \qquad C_5(s) = C_0^{\frac{s}{2}} C_1(s)C_2(s)C_3(s)$$

Now we choose s close to $\hat{s}$ such that (4.17) holds true. Therefore by (4.21) the following inequality holds:

$$X_h \le C_4^* X_h^q + 2C_5^* \|f\|_{\frac{N}{q},1}^{\frac{\hat{s}}{2}} \tag{4.22}$$

where

$$C_4^* = C_4(\hat{s})\beta^{\frac{\hat{s}}{2}} + 1, \quad C_5^* = C_5(\hat{s}) + 1, \quad \hat{s} = \frac{(N+2)p - 2N - Nq(2-p)}{(N+1)p - 2N}, \tag{4.23}$$

are positive constants which depend only on $p, q, N, |\Omega|, \beta$.

Since $\|f\|_{\frac{N}{q},1}$ satisfies the smallness condition (4.4), by (4.22), we deduce

$$X_h \le {Z'}_1, \tag{4.24}$$

where ${Z'}_1$ is the first positive zero of the function

$$F(\sigma) = C_4^* \sigma^q - \sigma + 2C_5^* \|f\|_{\frac{N}{q},1}^{\frac{\hat{s}}{2}}, \qquad \sigma > 0.$$

Denote $r < s$ such that

$$\left(\int_{w>h} |\nabla(\gamma_t v) + t\,\nabla(u - \gamma_t v)|^r \, dx\right)^{\frac{1}{r}} \le |\Omega|^{\frac{\alpha-1}{s} + \frac{1}{r}} X_h^{\frac{2}{s}}.$$

Therefore for any $h > 0$ and $t > t_1$, using (4.24), we have

$$\left(\int_{w>h} \left|\frac{\nabla(\gamma_t v)}{t} + \nabla(u - \gamma_t v)\right|^r dx\right)^{\frac{1}{r}} \le \frac{|\Omega|^{\frac{\alpha-1}{s}+\frac{1}{r}} {Z'}_1^{\frac{2}{s}}}{t}.$$

Now we firstly let h go to 0 and we deduce by Lebesgue dominated convergence theorem:

$$\left(\int_{w>0} \left|\frac{\nabla(\gamma_t v)}{t} + \nabla(u - \gamma_t v)\right|^r dx\right)^{\frac{1}{r}} \le \frac{|\Omega|^{\frac{\alpha-1}{s}+\frac{1}{r}} {Z'}_1^{\frac{2}{s}}}{t}. \tag{4.25}$$

Denote

$$E_t = \{x : w(x) = \gamma_t v(x) + t(u(x) - \gamma_t v(x)) > 0\}, \quad t \ge t_1$$

Then (4.25) can be written as

$$\left(\int_{\Omega} \left|\frac{\nabla(\gamma_t v)}{t} + \nabla(u - \gamma_t v)\right|^r \chi_{E_t}\, dx\right)^{\frac{1}{r}} \le \frac{|\Omega|^{\frac{\alpha-1}{s}+\frac{1}{r}} {Z'}_1^{\frac{2}{s}}}{t}, \tag{4.26}$$

for any $t > t_1$. Now let t go to $+\infty$. Then by Lebesgue dominated convergence theorem, since $\lim\limits_{t\to+\infty} \gamma_t = 1$ and

$$\chi_{E_t} \to \chi_{\{u-v\ge 0\}} \qquad \text{a.e.}$$

we get

$$\left(\int_{u-v\geq 0} |\nabla(u-v)|^r \, dx\right)^{\frac{1}{r}} \leq 0$$

and $u \leq v$ a.e. in Ω follows.

In analogous way we prove $v \leq u$ a.e. in Ω and therefore $u = v$ a.e. in Ω.

Remark 4.3. Let us explicitly remark that under that assumption $p > 2$ our approach can not applied since by dividing estimates (4.11) for t^{2-p}, the left-hand side goes to $+\infty$ when t tends to $+\infty$.

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