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ON THE STRUCTURAL DECOMPOSITION OF PLANAR LIPSCHITZ QUOTIENT MAPPINGS

RICKY HUTCHINS AND OLGA MALEVA

Abstract. We show that for each fixed non-constant complex polynomial P of the plane there exists a homeomorphism h such that $P \circ h$ is a Lipschitz quotient mapping. This corrects errors in the construction given earlier in [7]. Further we introduce a stronger notion of pointwise co-Lipschitzness and characterise its equivalence to the standard pointwise definition whilst also highlighting its relevance to a long-standing conjecture concerning Lipschitz quotient mappings $\mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$.

1. Introduction

The motivation for this paper follows from a desire to understand how much planar Lipschitz quotient mappings are tied to the underlying structure of such mappings as discovered in [7], see also Theorem 1.1 below. For a pair of metric spaces X and Y mappings $f : X \to Y$ are called Lipschitz quotient mappings provided they are Lipschitz and additionally satisfy a 'dual' property of being co-Lipschitz. Namely, a mapping f is Lipschitz quotient if there exist constants $0 < c \leq L < +\infty$ such that

$$B^Y_{cr}(f(x)) \overset{(1)}{\subseteq} f\left(B^X_r(x)\right) \overset{(2)}{\subseteq} B^Y_{Lr}(f(x))$$

for any $x \in X$ and all $r > 0$. Here $B^Z_s(z)$ denotes the open ball of radius $s > 0$ centred at $z \in Z$ where $Z = X, Y$. If only inclusion (2) is satisfied for each $x \in X$ and for every $r > 0$, we say f is $(L-)$Lipschitz. Similarly, if only inclusion (1) holds for each $x \in X$ and for every $r > 0$, we say f is $(c-)$co-Lipschitz.

If f is a Lipschitz mapping we define $\mathrm{Lip}(f)$, the Lipschitz constant of f, to be the infimum over all such $L > 0$ for which inclusion (2) holds. Similarly, if f is a continuous co-Lipschitz mapping we define $\text{co-Lip}(f)$, the co-Lipschitz constant of f, to be the supremum over all such $c > 0$ for which inclusion (1) holds. We remark that to guarantee the supremum for the co-Lipschitz constant exists we impose the continuity restriction since by assuming the axiom of choice there exist functions, for example from $\mathbb{R}$ to $\mathbb{R}$, which are surjective to $\mathbb{R}$ on every non-empty open subset, cf. [2].

Pointwise notions of co- and Lipschitz mappings have been considered in different texts, for example [3, 11]. Here if (2) is satisfied for a fixed $x \in X$ and all $r < r_0$, for some $r_0 = r_0(x) > 0$, then we say f is pointwise L-Lipschitz at x. We

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define the notion of pointwise co-Lipschitzness similarly. Other local notions of co-Lipschitzness have also been considered, for example in [8]. Another local notion we are going to use is local injectivity. We say a mapping $f : X \to Y$ between two metric spaces is locally injective at $x \in X$ if there exists $r > 0$ such that the restriction of f to $B_r^X(x)$ is injective.

Co-Lipschitz mappings were first introduced in [5, 6, 12] but first systematically studied in [1, 7]. The results in [1] support the intuitive notion that Lipschitz quotient mappings are non-linear analogues for linear quotient mappings between Banach spaces. When considering linear quotient mappings $X \to Y$ the point preimage of each $x \in X$ is an affine subspace of X with dimension $d := \dim(X) - \dim(Y)$. In [10] it is shown that provided the constants c and L are sufficiently close then point preimages, under Lipschitz quotient mappings, cannot be $(d+1)$-dimensional. However, if no condition is imposed on the constants, it is shown in [3] that there exist Lipschitz quotient mappings $\mathbb{R}^3 \to \mathbb{R}^2$ which collapse a subset containing a 2-dimensional plane to a single point.

Such a result is not possible for planar Lipschitz quotient mappings, no matter how far the constants c and L are. It is shown in [7] that such mappings have a very specific structure:

Theorem 1.1 ([7, Theorem 2.8(i)]). *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a Lipschitz quotient mapping. Then $f = P \circ h$, where $h : \mathbb{C} \to \mathbb{C}$ is a homeomorphism and P is a complex polynomial of one complex variable.*

Note that in Theorem 1.1 we did not specify the norms associated to the domain and co-domain. This is justified since passing to an equivalent norm preserves whether a mapping is a Lipschitz quotient; hence in the finite dimensional setting there is no need to specify the norm considered. We also highlight here that the original statement of Theorem 1.1 in [7] is given for Lipschitz quotient mappings from $\mathbb{R}^2$ to itself. The restatement in Theorem 1.1 follows due to the natural identification of $\mathbb{R}^2$ with $\mathbb{C}$, which is required in the definition of the polynomial P in any case.

Surprisingly little is known for higher dimensional analogues of Theorem 1.1. It is not even known if Lipschitz quotient mappings $\mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$ are discrete, see Conjecture 2.28.

In light of Theorem 1.1, and in the search of converses to such a statement, the authors of [7] pose questions regarding the uniqueness of the homeomorphism h obtained from the decomposition of a Lipschitz quotient mapping and whether a converse statement to Theorem 1.1 holds also. It is shown that, up to a linear transformation, the homeomorphism obtained via the decomposition of a Lipschitz quotient mapping is unique, see [7, p. 22].

In connection to the structural decomposition of planar Lipschitz quotient mappings, we ask the following questions concerning converse statements to Theorem 1.1.

Question 1.2. (a) Can every planar homeomorphism $h : \mathbb{C} \to \mathbb{C}$ be obtained via a decomposition of a Lipschitz quotient mapping? In other words, is it true that for every homeomorphism $h : \mathbb{C} \to \mathbb{C}$ there exists a non-constant complex polynomial P such that $P \circ h$ is a Lipschitz quotient mapping?

(b) Can every non-constant complex polynomial P be obtained via a decomposition of a planar Lipschitz quotient mapping? In other words, is it true that for every non-constant polynomial P there exists a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $P \circ h$ is a Lipschitz quotient mapping?

We begin by considering Question 1.2 (a). We provide a planar homeomorphism h such that $P \circ h$ is not Lipschitz quotient for every non-constant complex polynomial P. Indeed, consider the homeomorphism $h : \mathbb{C} \to \mathbb{C}$ given by $h(z) = |z|^2 e^{i \arg(z)}$. Observe that $P \circ h$ is not Lipschitz for every non-constant complex polynomial P. This follows simply as

$$\lim_{R \to +\infty} \frac{|P \circ h(R) - P \circ h(0)|}{R} = +\infty.$$

The main motivation of this paper is to consider Question 1.2 (b), as the authors of [7] do. The authors claim to answer this in [7, Proposition 2.9] in the positive, and provide a sketch proof of the following statement.

Proposition 1.3. *Let P be a non-constant polynomial in one complex variable with complex coefficients. Then there exists a homeomorphism h of the plane such that $f = P \circ h$ is a Lipschitz quotient mapping.*

However, as we show in Section 3, the construction of their mapping h is not in fact a homeomorphism of the plane. In this paper we prove Proposition 1.3. To do so we follow the framework provided in [7] but correct oversights in the original sketch and in doing so introduce a stronger (pointwise) notion of co-Lipschitzness, namely strongly co-Lipschitz.

With this new notion, we pose a question regarding the existence of Lipschitz quotient mappings $\mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$ which are strongly co-Lipschitz at some point. We explain the logical equivalence between this question and the long-standing conjecture of [7] whether such mappings are necessarily discrete. Moreover, with this new notion, we consider the following question.

Question 1.4. For a fixed homeomorphism $h : \mathbb{C} \to \mathbb{C}$ does there exist a non-constant complex polynomial P such that $P \circ h$ is not a Lipschitz quotient mapping?

We answer Question 1.4 in the positive in Lemma 2.30.

2. Preliminaries

Throughout this paper, for a metric space X and $S \subseteq X$, $\mathrm{Int}(S)$ denotes the topological interior of S and ∂S represents the boundary of S.

Notation 2.1. For any $z \in \mathbb{C} \setminus \{0\}$ we take $\arg(z) \in (-\pi, \pi]$ to denote the *principal argument* of z. Further, for any $a > 0$, $b \in \mathbb{R}$ we define $|z|^a e^{ib \arg(z)} = 0$ when $z = 0$.

For any non-constant complex polynomial P in one complex variable and $a > 0$ we define the closed set

$$V_a^P = \bigcup_{z_j \in S(P')} \overline{B}_a(z_j), \tag{2.1}$$

where P' is the derivative of P and $S(P') = \{z \in \mathbb{C} : P'(z) = 0\}$.

We now state properties of particular functions which are important in the judicious choose of $r > 0$ which we are making in Claim 3.5. First, let P be a fixed non-constant complex polynomial of one complex variable, P' be its derivative and $z_j \in S(P')$. Of course if P is non-zero and linear, then $S(P') = \emptyset$. Define the polynomial

$$Q_j(z) := \frac{P(z) - P(z_j)}{(z - z_j)^{m_j}}, \tag{2.2}$$

where $m_j \geq 1$ is the multiplicity of z_j as a root of the polynomial $P(z) - P(z_j)$. Note, for future reference, that $P(z) = (z - z_j)^{m_j} Q_j(z) + P(z_j)$. Further, by the maximality of m_j,

$$Q_j(z_j) \neq 0. \tag{2.3}$$

We define the expansion of the polynomial Q_j about z_j by

$$Q_j(z) = \sum_{l=0}^{n-m_j} c_{l,j}(z - z_j)^l \tag{2.4}$$

where $n = \deg(P)$ and $c_{l,j} \in \mathbb{C}$. Thus (2.3) implies $c_{0,j} = Q_j(z_j) \neq 0$ for each j such that $z_j \in S(P')$.

We now define a function which proves useful in the construction of the Lipschitz quotient mapping in Section 3. For each $m \geq 1$, let $A_m \subseteq \mathbb{C} \times \mathbb{C}$ be defined by

$$A_m := \left\{(z, w) : |z|e^{im\arg(z)} \neq |w|e^{im\arg(w)}\right\} \cup \{(w, w) : w \in \mathbb{C} \setminus \{0\}\}.$$

Now, for each $m \geq 1$ and $l \in \{1, \dots, m\}$ we define the mapping $\Phi_{l,m} : A_m \to \mathbb{C}$ by

$$\Phi_{l,m}(z, w) = \begin{cases} \dfrac{|z|^{\frac{l+m}{m}} e^{i(l+m)\arg(z)} - |w|^{\frac{l+m}{m}} e^{i(l+m)\arg(w)}}{|z|e^{im\arg(z)} - |w|e^{im\arg(w)}}, & \text{if } z \neq w; \\[2ex] \dfrac{l+m}{m}|w|^{\frac{l}{m}} e^{il\arg(w)}, & \text{if } z = w. \end{cases} \tag{2.5}$$

Lemma 2.2. *Let $m \geq 1$ and $1 \leq l \leq m$. For each $w \in \mathbb{C} \setminus \{0\}$, there exists $\rho > 0$ such that $B_\rho(w) \times \{w\} \subseteq A_m$ and*

$$\lim_{\substack{z \to w \\ z \in B_\rho(w)}} \Phi_{l,m}(z, w) = \Phi_{l,m}(w, w).$$

Proof. Note for $w \in \mathbb{C} \setminus \{0\}$ fixed that there exist finitely many points $z \in \mathbb{C}$ such that $(z, w) \notin A_m$; namely this happens exactly when $z \neq w$ but $|z| = |w|$ and $e^{im\arg(z)} = e^{im\arg(w)}$. Hence, there exists $\rho > 0$ such that $B_\rho(w) \times \{w\} \subseteq A_m$.

If $z \in B_\rho(w) \setminus \{w\}$, then $\Phi_{l,m}(z, w) = (g(f(z)) - g(f(w)))/(f(z) - f(w))$ where $f, g : \mathbb{C} \to \mathbb{C}$ are given by $f(z) = |z|e^{im\arg(z)}$ and $g(z) = z^{(l+m)/m}$. As $w \neq 0$ is fixed, f is continuous at w and g is differentiable at $f(w)$, observe that

$$\lim_{\substack{z \to w \\ z \in B_\rho(w)}} \Phi_{l,m}(z, w) = \lim_{\substack{z \to w \\ z \in B_\rho(w)}} \frac{g(f(z)) - g(f(w))}{f(z) - f(w)} = g'(f(w)) = \Phi_{l,m}(w, w).$$

□

Corollary 2.3. *Let $m \geq 1$ and $1 \leq l \leq m$. For each $w \in \mathbb{C} \setminus \{0\}$ and $\varepsilon > 0$ there exists $\rho > 0$ such that $B_\rho(w) \times \{w\} \subseteq A_m$ and whenever $z \in B_\rho(w)$,*

$$|\Phi_{l,m}(z, w)| < \varepsilon + |\Phi_{l,m}(w, w)|. \tag{2.6}$$

The following result concerns planar mappings which have the inherent structure of a Lipschitz quotient mapping. The below identifies a finite set E such that mappings of the form $P \circ h$ are locally injective on $\mathbb{C} \setminus E$. In the following proof $\operatorname{card}(S)$ represents the cardinality of the set S.

Proposition 2.4. *Let $f : \mathbb{C} \to \mathbb{C}$ be a mapping such that $f = P \circ h$ where P is a non-constant complex polynomial in one complex variable and $h : \mathbb{C} \to \mathbb{C}$ is a homeomorphism. Then there exists a finite subset $E \subseteq \mathbb{C}$ such that f is locally injective at each $x \in \mathbb{C} \setminus E$. Moreover, $E = h^{-1}(S(P'))$.*

Proof. Fix $x_0 \in \mathbb{C}$ such that $h(x_0) \notin S(P')$. We claim f is locally injective at x_0. Since $P'(h(x_0)) \neq 0$, by [4, Theorem 7.5], there exists an open neighbourhood $N_{h(x_0)}$ of $h(x_0)$ such that $P\big|_{N_{h(x_0)}}$ is injective. Therefore $f = P \circ h$ is injective on the open neighbourhood $G = h^{-1}(N_{h(x_0)})$ of x_0.

As this holds for any $x_0 \in \mathbb{C}$ such that $h(x_0) \notin S(P')$, f is locally injective outside of $E = h^{-1}(S(P'))$. Since P' is a non-zero polynomial, $\operatorname{card}(S(P')) \leq \deg(P) - 1$. As h is bijective, $\operatorname{card}(E) = \operatorname{card}(S(P'))$. □

We state a couple of standard results regarding Lipschitz mappings.

Lemma 2.5. *Let X, Y be metric spaces, $A \subseteq X$ dense in X and $L > 0$. If $f : X \to Y$ is a continuous mapping such that $f\big|_A$ is L-Lipschitz, then f is L-Lipschitz.*

The following lemma ensures that a mapping which is pointwise Lipschitz everywhere, with a uniform constant, is necessarily Lipschitz, with the same constant. However, for this we need the linear structure induced by normed spaces.

Lemma 2.6. *Let X, Y be normed spaces, $U \subseteq X$ be open and convex and $L > 0$. If $f : X \to Y$ is pointwise L-Lipschitz at each $x \in U$, then $f\big|_U$ is L-Lipschitz.*

Recall [3, Section 4] and [11, Lemma 2.3] which introduce a result analogous to Lemma 2.6 for co-Lipschitz mappings in the case $U = X = Y = \mathbb{C}$.

Lemma 2.7. *Let $c > 0$. If $f : (\mathbb{C}, \|\cdot\|) \to (\mathbb{C}, |||\cdot|||)$ is continuous and is pointwise c-co-Lipschitz at every $x \in \mathbb{C}$, then f is (globally) c-co-Lipschitz.*

Homeomorphisms between two metric spaces preserve pointwise co- and Lipschitzness of such mappings and their inverses in the following manner.

Lemma 2.8. *Let X and Y be metric spaces, $h : X \to Y$ be a homeomorphism, $x_0 \in X$ and $c > 0$. Then h is pointwise c-co-Lipschitz at x_0 if and only if h^{-1} is pointwise $(1/c)$-Lipschitz at $h(x_0)$.*

Proof. If h is pointwise c-co-Lipschitz at x_0 there exists $r_0 > 0$ such that $B^Y_{cr}(h(x_0)) \subseteq h\left(B^X_r(x_0)\right)$ for each $r \in (0, r_0)$. Therefore

$$h^{-1}\left(B^Y_{cr}(h(x_0))\right) \subseteq h^{-1}\left(h\left(B^X_r(x_0)\right)\right) = B^X_r(x_0) = B^X_r\left(h^{-1}\left(h\left(x_0\right)\right)\right)$$

for each $r \in (0, r_0)$. Hence h^{-1} is pointwise $(1/c)$-Lipschitz at $h(x_0)$. The reverse direction follows similarly. □

The traditional examples of planar Lipschitz quotient mappings f_n, as defined in Lemma 2.9, possess sharp constants, in the sense that the ratios of constants c/L for such mappings are maximal, cf. [10, Theorem 2].

Lemma 2.9. *For each $n \in \mathbb{N}$ define $f_n : (\mathbb{C}, |\cdot|) \to (\mathbb{C}, |\cdot|)$ to be given by $f_n(z) = |z|e^{in\arg(z)}$. Then f_n is a Lipschitz quotient mapping; namely f_n is n-Lipschitz and 1-co-Lipschitz with respect to the Euclidean norm.*

Remark 2.10. We highlight here that in Corollary 2.26, which we prove later, we show that f_n satisfy properties which are stronger than 1-co-Lipschitzness.

The following lemma concerns the Lipschitz property of variants of the standard Lipschitz quotient mappings f_n introduced in Lemma 2.9.

Lemma 2.11. *Let $n \in \mathbb{N}$ and $k \in \{1, \dots, n-1\}$. For each $\varepsilon > 0$ there exists $D = D(\varepsilon, k, n) > 0$ such that $g_{k,n} : \mathbb{C} \setminus B_D(0) \to \mathbb{C}$ defined by $g_{k,n}(z) = |z|^{k/n} e^{ik\arg(z)}$ is ε-Lipschitz on $\mathbb{C} \setminus B_D(0)$.*

Proof. Fix $\varepsilon > 0$. Define $f_k(z) = |z|e^{ik\arg(z)}$ for $z \in \mathbb{C}$ as in Lemma 2.9. Further, define $h_k(t) = t^{k/n}$ for $t > 0$. Let $T > 0$ be such that h_k is $(\varepsilon/2)$-Lipschitz on $[T, +\infty)$ and let $R > 0$ be such that $\frac{k+1}{R^{1-k/n}} < \frac{\varepsilon}{2}$. Define $D := \max\{T, R\}$. Fix $z_1, z_2 \in \mathbb{C} \setminus B_D(0)$. Now

$$\begin{aligned} &|g_{k,n}(z_1) - g_{k,n}(z_2)| \leq \\ &\qquad \left| g_{k,n}(z_1) - |z_2|^{k/n} e^{ik\arg(z_1)} \right| + |z_2|^{k/n} \left| e^{ik\arg(z_1)} - e^{ik\arg(z_2)} \right|. \end{aligned} \tag{2.7}$$

As $|z_1|, |z_2| \geq D \geq T$ and as h_k is $(\varepsilon/2)$-Lipschitz on $[T, +\infty)$,

$$\begin{aligned} \left| g_{k,n}(z_1) - |z_2|^{k/n} e^{ik\arg(z_1)} \right| = |h_k(|z_1|) - h_k(|z_2|)| &\leq \frac{\varepsilon}{2} \Big| |z_1| - |z_2| \Big| \\ &\leq \frac{\varepsilon}{2} |z_1 - z_2|. \end{aligned} \tag{2.8}$$

Further, since $|z_2| \geq D \geq R$,

$$\begin{aligned} &|z_2|^{k/n} \left| e^{ik\arg(z_1)} - e^{ik\arg(z_2)} \right| \\ &\leq \left| |z_2|^{k/n} - |z_1| \cdot |z_2|^{\frac{k}{n}-1} \right| + \left| |z_1| \cdot |z_2|^{\frac{k}{n}-1} e^{ik\arg(z_1)} - |z_2|^{\frac{k}{n}} e^{ik\arg(z_2)} \right| \\ &= \frac{1}{|z_2|^{1-\frac{k}{n}}} \left(\Big| |z_1| - |z_2| \Big| + |f_k(z_1) - f_k(z_2)| \right) \leq \frac{\varepsilon}{2} |z_1 - z_2|, \end{aligned}$$

where the last inequality follows by the choice of $R > 0$ and Lemma 2.9. Substituting this and (2.8) into (2.7) we obtain

$$|g_{k,n}(z_1) - g_{k,n}(z_2)| \leq \varepsilon |z_1 - z_2|.$$

By the arbitrariness of $z_1, z_2 \in \mathbb{C} \setminus B_D(0)$ we conclude the required Lipschitzness of $g_{k,n}$. □

We now introduce a quick lemma regarding the composition of pointwise co-Lipschitz functions.

Lemma 2.12. *Suppose X, Y and Z are metric spaces and $f : X \to Y$, $g : Y \to Z$ are functions. Suppose f is pointwise a-co-Lipschitz at $x \in X$ and g is pointwise b-co-Lipschitz at $f(x) \in Y$ for some constants $a, b > 0$. Then $g \circ f$ is pointwise (ab)-co-Lipschitz at x.*

Proof. As f is pointwise a-co-Lipschitz at $x \in X$, there exists $\rho_f > 0$ such that $f(B_r^X(x)) \supseteq B_{ar}^Y(f(x))$ for each $r \in (0, \rho_f)$. Similarly, there exists $\rho_g > 0$ such that $g(B_r^Y(f(x))) \supseteq B_{br}^Z(g(f(x)))$ for each $r \in (0, \rho_g)$. Define $\rho := \min(\rho_f, \rho_g/a)$. Then, for each $r \in (0, \rho)$,

$$(g \circ f)\left(B_r^X(x)\right) \supseteq g\left(B_{ar}^Y(f(x))\right) \supseteq B_{abr}^Z\left((g \circ f)(x)\right).$$

Hence, $g \circ f$ is pointwise (ab)-co-Lipschitz at $x \in X$. □

The next lemma provides a sufficient property for a mapping between metric spaces to be pointwise co-Lipschitz at a given point. To be able to conveniently refer to this property, we first give the following definition.

Definition 2.13. Suppose (X, d_X) and (Y, d_Y) are metric spaces and $c > 0$. We say a function $f : X \to Y$ is *strongly c-co-Lipschitz* at $x_0 \in X$ if there exists $\rho > 0$ such that:

(i) $f(x_0) \in \mathrm{Int}\left(f\left(B_\rho^X(x_0)\right)\right)$;
(ii) $d_Y(f(x), f(x_0)) \geq c d_X(x, x_0)$ for all $x \in B_\rho^X(x_0)$.

If we do not need to specify c, we shall simply write f is strongly co-Lipschitz at x_0.

Lemma 2.14. *Let (X, d_X) and (Y, d_Y) be metric spaces and $c > 0$. If $f : X \to Y$ is strongly c-co-Lipschitz at $x_0 \in X$, then f is pointwise c-co-Lipschitz at x_0.*

Proof. Let $\rho > 0$ be as in Definition 2.13. By property (i) of Definition 2.13 there exists a positive constant $R < \rho$ such that

$$B_R^Y\left(f(x_0)\right) \subseteq \mathrm{Int}\left(f\left(B_\rho^X(x_0)\right)\right) \subseteq f\left(B_\rho^X(x_0)\right). \tag{2.9}$$

Define $r := \frac{R}{2c} > 0$, let $0 < s < r$ and fix $y \in B_{cs}^Y\left(f(x_0)\right)$. By the choice of r, note $cs < cr < R$. Thus (2.9) implies $y \in f\left(B_\rho^X(x_0)\right)$. Hence there exists $x \in B_\rho^X(x_0)$ such that $y = f(x)$. We claim $x \in B_s^X(x_0)$ follows by (ii) of Definition 2.13. Indeed, since $x \in B_\rho^X(x_0)$ and $y \in B_{cs}^Y(f(x_0))$,

$$cs > d_Y(y, f(x_0)) = d_Y(f(x), f(x_0)) \geq c d_X(x, x_0),$$

so $x \in B_s^X(x_0)$. Therefore $y = f(x) \in f\left(B_s^X(x_0)\right)$ and since $y \in B_{cs}^Y\left(f(x_0)\right)$ was arbitrary we deduce $B_{cs}^Y\left(f(x_0)\right) \subseteq f\left(B_s^X(x_0)\right)$. Finally, since $s \in (0, r)$ was arbitrary we conclude f is pointwise c-co-Lipschitz at x_0. □

Corollary 2.15. *Let (X, d_X), (Y, d_Y) be metric spaces. Suppose $f : X \to Y$ is an open map, $x_0 \in X$ and there exist positive constants c and r_0 such that $d_Y(f(x), f(x_0)) \geq c d_X(x, x_0)$ for each $x \in B_{r_0}^X(x_0)$. Then f is pointwise c-co-Lipschitz at x_0.*

Remark 2.16. When proving pointwise or strong co-Lipschitzness of mappings defined in Section 3, we will often consider X to be an open subset of $\mathbb{C}$. In such cases, instead of $B_r^X(x)$, we will consider balls centred at $x \in X$ and open in the Euclidean metric. To be able to use the definition of a co-Lipschitz mapping or Definition 2.13 and subsequent results about strongly co-Lipschitz mappings, it is enough to ensure r is sufficiently small so that the Euclidean ball of radius r around x coincides with $B_r^X(x)$.

Remark 2.17. Using the notion introduced in Definition 2.13, the following implication follows by Lemma 2.14:

$$\text{strongly } c\text{-co-Lipschitz at } x_0 \implies \text{pointwise } c\text{-co-Lipschitz at } x_0. \tag{2.10}$$

One may naturally ask the question of whether a reverse implication holds. In Lemma 2.18 below, we show that only property (ii) of Definition 2.13 needs to be verified for a pointwise co-Lipschitz mapping to be strongly co-Lipschitz.

Lemma 2.18. *Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \to Y$, $x_0 \in X$ and $c > 0$. Suppose f is pointwise c-co-Lipschitz at x_0. If there exists $\rho_0 > 0$ such that $d_Y(f(x_0), f(z)) \geq c d_X(x_0, z)$ for each $z \in B_{\rho_0}^X(x_0)$, then f is strongly c-co-Lipschitz at x_0.*

Proof. It is enough to prove (i) of Definition 2.13 is satisfied for some $0 < \rho < \rho_0$. Indeed, as f is pointwise c-co-Lipschitz at x_0, there exists a positive r_0 such that $f(B_r^X(x_0)) \supseteq B_{cr}^Y(f(x_0))$ for each $r \in (0, r_0)$. Define $\rho := \frac{1}{2}\min(r_0, \rho_0)$. Then $f(x_0) \in B_{c\rho}^Y(f(x_0)) \subseteq f(B_\rho^X(x_0))$. Hence as $B_{c\rho}^Y(f(x_0))$ is open, we deduce (i) is satisfied. Thus f is strongly c-co-Lipschitz at x_0. ☐

The reverse implication of (2.10) can easily be seen in the case when the function is locally injective, as we show in the following lemma.

Lemma 2.19. *Let (X, d_X), (Y, d_Y) be metric spaces, $x_0 \in X$ and $c > 0$. Suppose a mapping $f : X \to Y$ is both pointwise c-co-Lipschitz and locally injective at x_0. Then f is strongly c-co-Lipschitz at x_0.*

Proof. Since f is pointwise c-co-Lipschitz at x_0, by definition, there exists $r_0 > 0$ such that

$$B_{cr}^Y(f(x_0)) \subseteq f\left(B_r^X(x_0)\right) \quad \text{for each } r \in (0, r_0). \tag{2.11}$$

Since f is locally injective at x_0 there exists $r_1 > 0$ such that $f\big|_{B_{r_1}^X(x_0)}$ is injective. Define $\rho := \frac{1}{2}\min(r_0, r_1)$. Recall Lemma 2.18. Thus it suffices to show

$$d_Y(f(x), f(x_0)) \geq c d_X(x, x_0) \quad \text{for all } x \in B_\rho^X(x_0). \tag{2.12}$$

This is trivially satisfied for $x = x_0$. Suppose, for a contradiction, that (2.12) is not satisfied, i.e. there exists $x \in B_\rho^X(x_0) \setminus \{x_0\}$ such that $d_Y(f(x), f(x_0)) < c d_X(x, x_0)$. Define $r := d_X(x, x_0)$, so $0 < r < \rho < r_0$. Hence, $f(x) \in B_{cr}^Y(f(x_0)) \subseteq f\left(B_r^X(x_0)\right)$ where the inclusion follows by (2.11). Therefore, as $f\big|_{B_\rho^X(x_0)}$ is injective, $x \in B_\rho^X(x_0)$ and $B_r^X(x_0) \subseteq B_\rho^X(x_0)$, it follows $x \in B_r^X(x_0)$ and so $r = d_X(x, x_0) < r$, providing contradiction. Hence (2.12) is satisfied. ☐

Corollary 2.20. *Suppose X and Y are metric spaces, $f : X \to Y$ is a mapping which is locally injective at $x_0 \in X$ and $c > 0$. Then*

$$f \text{ is strongly } c\text{-co-Lipschitz at } x_0 \iff f \text{ is pointwise } c\text{-co-Lipschitz at } x_0.$$

Remark 2.21. We highlight the relevance of Corollary 2.20 in the context of mappings with the inherent structure of planar Lipschitz quotient mappings. Indeed Proposition 2.4 identifies at which points of the plane a composition $P \circ h$ of a polynomial P and a homeomorphism h is locally injective, hence where the notions of strongly co-Lipschitzness and pointwise co-Lipschitzness agree. In Corollary 2.25 below, we show that these two notions automatically agree everywhere for any Lipschitz quotient mapping. However, as mentioned in Section 1, not all mappings with this underlying structure $P \circ h$ are Lipschitz quotient.

Further, we are able to show the equivalence between the two notions of pointwise co-Lipschitz and strongly co-Lipschitz for discrete co-Lipschitz mappings. To see this we follow the method presented in [9, p. 2091]. Let us first recall the definition of a discrete mapping.

Definition 2.22. Let X, Y be topological spaces and $S \subseteq X$. We say:

- S is a *discrete set* if for each $x \in S$ there exists a neighbourhood U of x such that $U \cap S = \{x\}$;
- $f : X \to Y$ is a *discrete mapping* if $f^{-1}(y)$ is a discrete set for each $y \in Y$.

Lemma 2.23. *Suppose (X, d_X), (Y, d_Y) are metric spaces and $f : X \to Y$ is a discrete c-co-Lipschitz mapping for some $c > 0$. Then f is strongly c-co-Lipschitz at each $x \in X$.*

Proof. Fix $x \in X$ and define $\mathcal{A}_x = f^{-1}(f(x))$. Since f is a discrete mapping there exists $r_0 > 0$ such that $B(x, 2r_0) \cap \mathcal{A}_x = \{x\}$. Fix $z \in B^X_{r_0}(x) \setminus \{x\}$ and let $r := d_X(z, x)$. Then $B^X_r(z) \cap \mathcal{A}_x = \emptyset$, so $f(x) \notin f\left(B^X_r(z)\right)$. Since f is c-co-Lipschitz, $f\left(B^X_r(z)\right) \supseteq B^Y_{cr}(f(z))$. As $f(x) \notin f(B^X_r(z))$ this implies $d_Y(f(x), f(z)) \geq cr = cd_X(x, z)$.

Observe that $d_Y(f(x), f(z)) \geq cd_X(x, z)$ is trivially satisfied when $z = x$. Therefore, by Lemma 2.18, we conclude f is strongly c-co-Lipschitz at x. ☐

We highlight that Lemma 2.19 and Lemma 2.23 are the strongest possible, in the sense that there exist Lipschitz quotient mappings which are 1-co-Lipschitz but not locally injective, not discrete and are not strongly co-Lipschitz at any point. We show this in the following example.

Example 2.24. Let $n, k \geq 1$ be integers and $f : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be the standard projection, where both spaces are equipped with the Euclidean norm. Then f is 1-Lipschitz and 1-co-Lipschitz. This trivially follows since $f(B_r(x)) = B_r(f(x))$ for each $r > 0$ and $x \in \mathbb{R}^{n+k}$. Further, it is clear that f is not discrete. Moreover, f is neither injective nor strongly c-co-Lipschitz, for any $c > 0$, at any $x_0 \in \mathbb{R}^{n+k}$ as $f^{-1}(x_0)$ is a k-dimensional hyperplane.

Using Lemma 2.23, we deduce the following two corollaries. First we show that planar Lipschitz quotient mappings, or any continuous co-Lipschitz planar mappings, are necessarily strongly co-Lipschitz at every point.

Corollary 2.25. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a continuous c-co-Lipschitz mapping for some $c > 0$. Then f is strongly c-co-Lipschitz at each $x \in \mathbb{C}$.*

Proof. By [1, Proposition 4.3], or equivalently [7, Proposition 2.1], f is discrete and so Lemma 2.23 yields the result. □

Corollary 2.26. *For every $n \in \mathbb{N}$ let the function $f_n : \mathbb{C} \to \mathbb{C}$ be defined by $f_n(z) = |z|e^{in\arg(z)}$ as in Lemma 2.9. Then f_n is strongly 1-co-Lipschitz at each $z \in \mathbb{C}$.*

Following Corollary 2.25 one may ask the following question.

Question 2.27. For $n \geq 3$ do there exist Lipschitz quotient mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ which are not strongly co-Lipschitz at some $x_0 \in \mathbb{R}^n$?

We highlight the logical equivalence between Question 2.27 and a long-standing conjecture from [1, p. 1096]. Namely:

Conjecture 2.28. Suppose $n \geq 3$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz quotient mapping. Then f is a discrete mapping.

First we note that a positive answer to Conjecture 2.28 implies, via an application of Lemma 2.23, that every Lipschitz quotient mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$ is strongly c-co-Lipschitz everywhere, where $c = \text{co-Lip}(f)$, providing a negative answer to Question 2.27.

Conversely a negative answer to Question 2.27, i.e. every Lipschitz quotient mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is strongly co-Lipschitz everywhere, implies Conjecture 2.28. This implication is proved in the following simple lemma.

Lemma 2.29. *Let (X, d_X), (Y, d_Y) be metric spaces and $y \in Y$. If $f : X \to Y$ is strongly co-Lipschitz at each element of $f^{-1}(y)$, then $f^{-1}(y)$ is a discrete set.*

In particular, if f is strongly co-Lipschitz at every $x \in X$, then f is a discrete mapping.

Proof. To show $f^{-1}(y)$ is discrete we require to show for each $x \in f^{-1}(y)$ that there exists a neighbourhood U_x of x such that $U_x \cap f^{-1}(y) = \{x\}$. Fix $x \in f^{-1}(y)$. Since f is strongly co-Lipschitz at x, there exist positive constants c_x and ρ_x such that

$$d_Y(f(w), f(x)) \geq c_x d_X(w, x) \quad \text{for each } w \in B^X_{\rho_x}(x). \tag{2.13}$$

Define $U_x := B^X_{\rho_x}(x)$. Let $z \in U_x \cap f^{-1}(y)$. Then since $z \in B^X_{\rho_x}(x)$ and $f(z) = y$, by (2.13) it follows that $0 = d_Y(f(z), f(x)) \geq c_x d_X(z, x)$. Thus $z = x$ since $c_x > 0$ and so $U_x \cap f^{-1}(y) = \{x\}$. Since $x \in f^{-1}(y)$ was arbitrary, we conclude $f^{-1}(y)$ is a discrete set. □

With the introduction of the notion of strong co-Lipschitzness, we are in a position to answer Question 1.4 affirmatively. Formally, we prove the following.

Lemma 2.30. *Let $h : \mathbb{C} \to \mathbb{C}$ be a homeomorphism. Then there exists a complex polynomial P in one complex variable such that $P \circ h$ is not Lipschitz quotient.*

Naturally Lemma 2.30 is a consequence that squaring Lipschitz quotient mappings of the plane never produces a Lipschitz mapping, also. We prove this in the following lemma.

Lemma 2.31. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a Lipschitz quotient mapping. Then $g(z) = (f(z))^2$ is not Lipschitz.*

Proof. Suppose f is c_f-co-Lipschitz and L_f-Lipschitz and, for a contradiction, suppose g is Lipschitz. Let us assume, without loss of generality, that both g/f are Lipschitz/Lipschitz quotient with respect to the Euclidean norm. Now [11, Theorem 2.8 (1)] provides the existence of a positive constant R such that

$$|f(x)| \geq c_f\,(|x| - M)\,, \tag{2.14}$$

whenever $|x| > R$. Here $M := \max\{|z| : f(z) = 0\}$. Let $L_g > 0$ be such that g is L_g-Lipschitz and fix $z_0 \in \mathbb{C}$ such that $|z_0| > R + M + L_g/(2c_f^2)$. Since f is strongly c_f-co-Lipschitz at z_0, by Corollary 2.25, there exists $r_0 \in (0,1)$ such that $|f(z_0) - f(w)| \geq c_f\,|z_0 - w|$ for all $w \in B_{r_0}(z_0)$. As g is L_g-Lipschitz,

$$\begin{aligned} c_f\,|z_0 - w|\,|f(z_0) + f(w)| \leq \left|(f(z_0))^2 - (f(w))^2\right| &= |g(z_0) - g(w)| \\ &\leq L_g\,|z_0 - w|\,, \end{aligned}$$

for all $w \in B_{r_0}(z_0)$. Hence, for any $w \in B_{r_0}(z_0) \setminus \{z_0\}$, $|f(z_0) + f(w)| \leq L_g/c_f$. Thus, by the continuity of f, $|f(z_0)| \leq L_g/(2c_f)$. However, by our choice of z_0 and (2.14), $|f(z_0)| > L_g/(2c_f)$, providing contradiction. Hence g is not Lipschitz. □

3. Construction of the Lipschitz quotient mapping

Recall the function $h : \mathbb{C} \to \mathbb{C}$ given in [7, Proposition 2.9] (for some large $R > 0$):

$$h(z) = \begin{cases} z, & \text{if } |z| \leq R, \\ \left(\dfrac{2R - |z|}{R}|z| + \dfrac{|z| - R}{R}|z|^{1/n}\right) e^{i\arg(z)}, & \text{if } R \leq |z| \leq 2R, \\ |z|^{1/n} e^{i\arg(z)}, & \text{if } |z| \geq 2R. \end{cases} \tag{3.1}$$

The authors of [7] claim first this is a homeomorphism from $\mathbb{C}$ to itself and go on to provide a sketch for a proof of Proposition 1.3. However it is clear that h is not injective by observing that, for $R > 2^{1/(n-1)}$ if $n > 1$, the curve $\partial B_{2R}(0)$ is mapped under h inside the open ball $B_R(0)$ where the mapping remains fixed. Further, the authors introduce an amendment to the function h which may further provide points at which h is not injective. They describe how to change the function h defined by (3.1) on a finite collection of open balls. However they neglect the fact the prescribed radii of these balls are potentially very small and hence will require a 'scaling' to ensure the function is necessarily injective, as indicated by the $r^{1-(1/m_j)}$ term in (3.11). Finally, the authors state the co-Lipschitzness of the function h outside of the union of these balls, but do not verify the co-Lipschitzness on their boundaries, which is intricate.

Below we give a correct construction, for a fixed polynomial P, of a homeomorphism h of the plane to itself such that $P \circ h$ is a Lipschitz quotient mapping. The proof of Proposition 1.3 will be split into many claims, which verify the pointwise co- and Lipschitz property of the required functions, and remarks, which utilise earlier lemmata to conclude co- and Lipschitzness on specific regions. To highlight the end of the proof of a claim we use the symbol ♢, whereas the end of the proof of the proposition is highlighted by the usual □.

Proof of Proposition 1.3. Fix $n \in \mathbb{N}$. We may assume without loss of generality that P is a monic polynomial of degree n. Indeed if P is not monic, let $a \neq 0$ denote the leading coefficient of P. One can apply the present Proposition to the monic polynomial $Q := P/a$ to find the homeomorphism h such that $f(z) = (Q \circ h)(z)$ is a Lipschitz quotient mapping. Then $(P \circ h)(z) = af(z)$ is a Lipschitz quotient mapping.

Therefore, assume $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. If $n = 1$ define $h(z) := z$ and then $f(z) = (P \circ h)(z) = z + a_0$ is 1-co-Lipschitz and 1-Lipschitz.

Suppose $n \geq 2$. The structure of the proof is as follows: we begin by defining a homeomorphism h_1 of the plane, let $F_1 = P \circ h_1$ and show that F_1 is Lipschitz on $\mathbb{C}$ and pointwise co-Lipschitz on $\mathbb{C}$ with the exception of a small neighbourhood W of finitely many points. Namely, W contains a neighbourhood of the set of roots of the polynomial P', the derivative of P. We use this to show F_1 is strongly co-Lipschitz at each $z \in \mathbb{C} \setminus V$, where $V \supseteq W$. We then proceed by defining an amended homeomorphism h_2 which coincides with h_1 everywhere outside of V, define the new function $F_2 = P \circ h_2$ and prove F_2 is pointwise co- and Lipschitz at the remaining points. Let us introduce some notation which will be important in the construction.

Notation 3.1. If $a_k \neq 0$ and $1 \leq k \leq n-1$, let $D_k = D\left(1/(2n|a_k|), k, n\right)$ be provided by Lemma 2.11, such that $g_{k,n}(z) = |z|^{k/n} e^{ik \arg(z)}$ is $1/(2n|a_k|)$-Lipschitz on $\mathbb{R}^2 \setminus B_{D_k}(0)$; otherwise if $a_k = 0$, let $D_k = 0$.

Let $R > 1$ be such that

(a) the roots of the derivative P' lie inside the open ball of radius R centred at the origin;
(b) $R \geq \max\{D_k : 1 \leq k \leq n-1\}$.

Define $h_1 : \mathbb{C} \to \mathbb{C}$ by

$$h_1(z) = \phi(|z|) e^{i \arg(z)},$$

where

$$\phi(t) = \begin{cases} t^{1/n}, & \text{if } t \geq 2^n R^n; \\ \left(\dfrac{t - R}{2^n R^{n-1} - 1} + R \right), & \text{if } R \leq t \leq 2^n R^n; \\ t, & \text{if } 0 \leq t \leq R. \end{cases}$$

Since $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous, piecewise C^∞ strictly increasing homeomorphism, h_1 is bijective and continuous. Further we note $h_1^{-1}(z) = \phi^{-1}(|z|) e^{i \arg(z)}$ which is continuous. Hence h_1 is indeed a homeomorphism of $\mathbb{C}$ to itself. Finally, let $U_j := B_{2^n R^n + j}(0)$ for $j = 1, 2$. Define $F_1 = P \circ h_1$.

Claim 3.2. F_1 is Lipschitz on U_2.

Proof. We first show that h_1 is Lipschitz on U_2. Note that h_1 is pointwise 1-Lipschitz at each $z_0 \in B_R(0)$, since if $r > 0$ if sufficiently small such that $B_r(z_0) \subseteq B_R(0)$, then $h_1\left(B_r\left(z_0\right)\right) = B_r\left(z_0\right) = B_r\left(h_1(z_0)\right)$.

To see that h_1 is pointwise Lipschitz at each $z_0 \in U_2 \setminus \overline{B}_{R/2}(0)$, first note that ϕ is Lipschitz on $[R/2, 2^n R^n + 2]$. Moreover observe that $e^{i \arg(z)} = z/|z|$ is Lipschitz on $\mathbb{C} \setminus B_{R/2}(0)$, as if $z, w \in \mathbb{C} \setminus B_{R/2}(0)$, then

$$\left|\frac{z}{|z|} - \frac{w}{|w|}\right| \leq \frac{1}{|z| \cdot |w|}\left(|w| \cdot |z-w| + |w| \cdot \Big||w| - |z|\Big|\right) \leq \frac{4}{R}|z-w|.$$

Thus, $h_1(z) = \phi(|z|)e^{i\arg(z)}$ is the product of two bounded Lipschitz functions on the bounded domain $A = \{z \in \mathbb{C} : R/2 \leq |z| \leq 2^n R^n + 2\}$. Therefore, $h_1\big|_A$ is L-Lipschitz for some $L > 0$. In particular, we conclude that h_1 is pointwise L-Lipschitz at each $z \in U_2 \setminus \overline{B}_{R/2}(0)$.

Therefore Lemma 2.6 implies h_1 is $\max(1, L)$-Lipschitz on the convex, open set U_2. Now, $F_1 = P \circ h_1$ is the composition of P, a polynomial, which is Lipschitz on the bounded set $h_1(U_2)$ and h_1, which is Lipschitz on U_2. Therefore, F_1 is Lipschitz on U_2. ♢

Claim 3.3. F_1 is Lipschitz on $\mathbb{C} \setminus \overline{U_1}$.

Proof. To see F_1 is Lipschitz outside of $\overline{U_1}$ note for $z \notin \overline{U_1}$ that $F_1(z)$ takes the specific form

$$F_1(z) = a_0 + f_n(z) + \sum_{k=1}^{n-1} a_k g_{k,n}(z), \tag{3.2}$$

where f_n is defined as in Lemma 2.9 and $g_{k,n}$ as in Lemma 2.11 for each $k \in \{1, \ldots, n-1\}$.

Hence, as f_n is n-Lipschitz on $\mathbb{C}$ by Lemma 2.9, to show F_1 is Lipschitz on $\mathbb{C} \setminus \overline{U_1}$ it suffices to show for each $k \in \{1, \ldots, n-1\}$ that $a_k g_{k,n}$ is Lipschitz on $\mathbb{C} \setminus \overline{U_1}$; this follows by Lemma 2.11 and the choice of R and D_k in Notation 3.1 (b). Hence F_1 is Lipschitz on $\mathbb{C} \setminus \overline{U_1}$. ♢

Remark 3.4. Recall by Claims 3.2, 3.3 that F_1 is Lipschitz on both $\mathbb{C} \setminus \overline{U_1}$ and U_2. Therefore Lemma 2.6 yields that there exists $L_1 > 0$ such that F_1 is L_1-Lipschitz on $\mathbb{C}$.

Claim 3.5. Recall (2.1)-(2.4) from Notation 2.1 and the choice of R from Notation 3.1. There exists $r \in (0, 1)$ such that:

(i) the balls $\overline{B}_{2r}(z_j)$ around roots $z_j \in S(P')$ of P', are pairwise disjoint;
(ii) $V_{2r}^P \subseteq B_R(0)$;
(iii) $r \leq \min\limits_{j: z_j \in S(P')} \varepsilon_j^{m_j}$, where for each $z_j \in S(P')$ we define $\varepsilon_j > 0$ by

$$\varepsilon_j := \begin{cases} \dfrac{|Q_j(z_j)|}{2(1+n)\sum\limits_{k=1}^{n-m_j} |c_{k,j}|}, & \text{if } n > m_j \text{ and } \sum\limits_{k=1}^{n-m_j} |c_{k,j}| \neq 0, \\ 1, & \text{otherwise.} \end{cases}$$

(iv) $|Q_j(z_j)|/2 \leq |Q_j(y)| \leq 2|Q_j(z_j)|$ for each $y \in B_r(z_j)$ such that $z_j \in S(P')$.

Proof. Property (i) is easy to satisfy as there are only finitely many distinct roots in $S(P')$. Next, property (ii) is satisfied for sufficiently small $r > 0$ since $S(P') \subseteq B_R(0)$ and $B_R(0)$ is open. Property (iii) follows naturally by (2.3) since each ε_j is positive and there are only finitely many of these terms. Finally, it is possible to satisfy property (iv) since each polynomial Q_j is continuous on $\mathbb{C}$ and $Q_j(z_j) \neq 0$ by (2.3). $\diamondsuit$

For the rest of the proof of Proposition 1.3, we fix $r \in (0,1)$ provided by Claim 3.5. Recall (2.1), and define the closed sets W and V to be the following:

$$W = V^P_{r/2}, \quad V = V^P_r. \tag{3.3}$$

Claim 3.6. There exists $c_0 > 0$ such that F_1 is pointwise c_0-co-Lipschitz at each $z \in U_2 \setminus W$.

Proof. We first show that there exist positive constants L and ξ such that h_1 is pointwise $(1/L)$-co-Lipschitz at each $z \in U_2$ and the polynomial P is pointwise ξ-co-Lipschitz at each $z \in h_1(U_2 \setminus W)$. Then we appeal to Lemma 2.12 to conclude that F_1 is pointwise $c_0 := \left(\frac{\xi}{L}\right)$-co-Lipschitz at each $z \in U_2 \setminus W$.

By arguing similarly to the proof of Claim 3.2, namely as $h_1^{-1}(z) = \phi^{-1}(|z|)e^{i\arg(z)}$ is the product of two bounded Lipschitz functions, there exists $L > 0$ such that h_1^{-1} is pointwise L-Lipschitz at $h_1(z)$ for each $z \in U_2$. Thus Lemma 2.8 and the arbitrariness of $z \in U_2$ implies h_1 is pointwise $(1/L)$-co-Lipschitz at each $z \in U_2$.

Observe by Claim 3.5 (ii) that $S(P') \subseteq W \subseteq B_R(0)$. Therefore, as h_1 is the identity on $B_R(0)$ and since $|h_1(z)| \geq R$ for $|z| \geq R$, we conclude that $\overline{h_1(U_2 \setminus W)} \cap S(P') = \emptyset$. As P' is a polynomial, hence continuous, $|P'|$ assumes its minimal value $2\xi > 0$ on the compact set $\overline{h_1(U_2 \setminus W)}$. In particular for each $z \in h_1(U_2 \setminus W)$ note $P'(z) \neq 0$ and thus, by [4, Theorem 7.5], there exist open neighbourhoods $N_{P(z)} \subseteq F_1(U_2 \setminus W)$ and $N_z \subseteq h_1(U_2 \setminus W)$ of $P(z)$ and z respectively such that $P : N_z \to N_{P(z)}$ is a continuous bijective open mapping, hence a homeomorphism. Further, $(P^{-1})'(P(z)) = 1/P'(z)$. Therefore for each $z \in h_1(U_2 \setminus W)$ it follows that $|(P^{-1})'(P(z))| \leq 1/(2\xi)$. Hence P^{-1} is pointwise $\frac{1}{\xi}$-Lipschitz at $P(z)$. By Lemma 2.8 and Remark 2.16 we hence conclude P is pointwise ξ-co-Lipschitz at z since $P : N_z \to N_{P(z)}$ is a homeomorphism, N_z and $N_{P(z)}$ are open subsets of $\mathbb{C}$ and $z \in N_z$. We conclude P is pointwise ξ-co-Lipschitz at each $z \in h_1(U_2 \setminus W)$.

Now h_1 is pointwise $\frac{1}{L}$-co-Lipschitz at each $z \in U_2 \setminus W$ and P is pointwise ξ-co-Lipschitz at each $h_1(z) \in h_1(U_2 \setminus W)$. Therefore by Lemma 2.12 we conclude F_1 is pointwise c_0-co-Lipschitz at each $z \in U_2 \setminus W$ where $c_0 = \xi/L > 0$. $\diamondsuit$

Remark 3.7. Since $(U_2 \setminus W) \cap h_1^{-1}(S(P')) = \emptyset$, by Proposition 2.4, F_1 is locally injective at each $z \in U_2 \setminus W$. Further, $U_2 \setminus W$ is open. Therefore Remark 2.16, Corollary 2.20 and Claim 3.6 imply F_1 is strongly c_0-co-Lipschitz at each $z \in U_2 \setminus W$. In particular, for each $z \in U_2 \setminus \mathrm{Int}(V)$ there exists $\rho = \rho(z) > 0$ such that $B_\rho(z) \subseteq U_2 \setminus W$ and

$$|F_1(z) - F_1(x)| \geq c_0 |z - x| \quad \text{for all } x \in B_\rho(z). \tag{3.4}$$

Claim 3.8. F_1 is $\frac{1}{2}$-pointwise co-Lipschitz at each $z \in \mathbb{C} \setminus \overline{U_1}$.

Proof. Fix any $z_0 \in \mathbb{C} \setminus \overline{U_1}$. Recall $F_1 = P \circ h_1$ where P is a non-constant polynomial of one variable, so is an open map, and h_1 is a homeomorphism. Therefore F_1 is open. By Corollary 2.15 and Remark 2.16, as $\mathbb{C} \setminus \overline{U_1}$ is open, to check that F_1

is pointwise (1/2)-co-Lipschitz at z_0, it is enough to verify property (ii) of Definition 2.13 is satisfied; that is, to show that there exists $\rho = \rho(z_0) > 0$ such that

$$|F_1(x) - F_1(z_0)| \geq \frac{|x - z_0|}{2} \quad \text{for each } x \in B_\rho(z_0). \tag{3.5}$$

Recall by Corollary 2.26 that f_n is strongly 1-co-Lipschitz at z_0. Hence there exists $\rho_1 = \rho_1(z_0) > 0$ such that

$$|f_n(z_0) - f_n(x)| \geq |z_0 - x| \quad \text{for each } x \in B_{\rho_1}(z_0). \tag{3.6}$$

Choose $\rho = \rho(z_0) > 0$ sufficiently small such that $\rho < \rho_1$ and $B_\rho(z_0) \subseteq \mathbb{C} \setminus \overline{U_1}$. Let $x \in B_\rho(z_0)$ and put $s = |x - z_0| < \rho$. Recall (3.2), that is $F_1 = a_0 + f_n + \sum_{k=1}^{n-1} a_k g_{k,n}$, and so

$$\begin{aligned} |F_1(x) - F_1(z_0)| &= \left| (f_n(z_0) - f_n(x)) + \sum_{k=1}^{n-1} a_k \left(g_{k,n}(z_0) - g_{k,n}(x) \right) \right| \\ &\geq |f_n(z_0) - f_n(x)| - \sum_{k=1}^{n-1} |a_k| \, |g_{k,n}(z_0) - g_{k,n}(x)| \qquad (3.7) \\ &\geq s - \sum_{k=1}^{n-1} |a_k| \, |g_{k,n}(z_0) - g_{k,n}(x)| \,, \qquad (3.8) \end{aligned}$$

where the last inequality follows from (3.6). We show

$$\sum_{k=1}^{n-1} |a_k| \, |g_{k,n}(z_0) - g_{k,n}(x)| \leq \frac{s}{2}. \tag{3.9}$$

Combining (3.9) with (3.8) implies (3.5) which proves F_1 is pointwise $\frac{1}{2}$-co-Lipschitz at z_0 as claimed.

To see (3.9) recall Notation 3.1, in particular, recall (b). As $R \geq D_k$, by Lemma 2.11, $g_{k,n}$ is $1/(2n|a_k|)$-Lipschitz on $\mathbb{C} \setminus B_R(0)$ for those $k \in \{1, \ldots, n-1\}$ where $a_k \neq 0$. Hence

$$\sum_{k=1}^{n-1} |a_k| \, |g_{k,n}(z_0) - g_{k,n}(x)| \leq \sum_{k=1}^{n-1} \frac{|z_0 - x|}{2n} = \sum_{k=1}^{n-1} \frac{s}{2n} \leq \frac{s}{2}.$$

$\diamondsuit$

Remark 3.9. Recall by Claim 3.6 that F_1 is pointwise c_0-co-Lipschitz at each $z \in U_2 \setminus W$ and by Claim 3.8 that F_1 is pointwise (1/2)-co-Lipschitz at each $z \in \mathbb{C} \setminus \overline{U_1}$. Therefore defining $c_1 := \min\left\{c_0, \frac{1}{2}\right\}$ we conclude $c_1 > 0$ and

$$F_1 \text{ is pointwise } c_1\text{-co-Lipschitz at each } z \in \mathbb{C} \setminus W. \tag{3.10}$$

We continue by defining the amended homeomorphism $h_2 : \mathbb{C} \to \mathbb{C}$, which coincides with h_1 on $\mathbb{C} \setminus V$, and prove the pointwise co- and Lipschitz properties of the amended function $F_2 = P \circ h_2$. Indeed, define $h_2 : \mathbb{C} \to \mathbb{C}$ via

$$h_2(z) = \begin{cases} h_1(z), & \text{if } z \notin V; \\ z_j + r^{1 - \frac{1}{m_j}} |z - z_j|^{1/m_j} e^{i \arg(z - z_j)}, & \text{if } |z - z_j| \leq r, \ z_j \in S(P'). \end{cases} \tag{3.11}$$

See Notation 2.1 for definition of m_j. To check that h_2 is a homeomorphism first note that $h_2\big|_{\mathbb{C}\setminus\mathrm{Int}(V)} = h_1\big|_{\mathbb{C}\setminus\mathrm{Int}(V)}$ and $h_2\big|_{\overline{B}_r(z_j)}$ is continuous for each $z_j \in S(P')$, thus h_2 is continuous. Further, as $h_2(\overline{B}_r(z_j)) = h_1(\overline{B}_r(z_j)) = \overline{B}_r(z_j)$, both $h_2\big|_{\overline{B}_r(z_j)}$ and $h_2\big|_{\mathbb{C}\setminus\mathrm{Int}(V)}$ are bijective, and $h_2(\mathbb{C}\setminus V)\cap h_2(V) = h_1(\mathbb{C}\setminus V)\cap h_1(V) = \emptyset$, we conclude that $h_2 : \mathbb{C}\to\mathbb{C}$ is bijective. Finally as $h_2^{-1}\big|_{\overline{B}_r(z_j)}$ is continuous for each $z_j \in S(P')$ and $h_2^{-1}\big|_{\mathbb{C}\setminus\mathrm{Int}(V)} = h_1^{-1}\big|_{\mathbb{C}\setminus\mathrm{Int}(V)}$, we conclude h_2 is a homeomorphism of the plane to itself.

Recall from (2.2) that $P(w) = (w-z_j)^{m_j}Q_j(w) + P(z_j)$ and so $F_2(z) = P(h_2(z))$ has the following form:
(3.12)

$$F_2(z) = \begin{cases} F_1(z), & \text{if } z \notin V;\\ P(z_j) + r^{m_j-1} f_{m_j}(z-z_j)Q_j(h_2(z)), & \text{if } |z-z_j| \le r,\ z_j \in S(P'),\end{cases}$$

where f_{m_j} is defined as in Lemma 2.9.

Clearly, $F_1(z) = F_2(z)$ for each $z \in \partial V$ as $h_1\big|_{\partial B_r(z_j)} = h_2\big|_{\partial B_r(z_j)}$ for all $z_j \in S(P')$. Moreover, since P is a complex polynomial, hence an open map, and as h_2 is a homeomorphism, we conclude that F_2 is an open map.

Remark 3.10. If there exists $z_j \in S(P')$ such that $m_j = n$, then $P(z) = P(z_j) + Q_j(z_j)(z-z_j)^n$ where $Q_j(z_j) \neq 0$. Therefore, $S(P') = \{z_j\}$ and so $F_2(z) = P(z_j) + Q_j(z_j)r^{n-1}f_n(z-z_j)$ for $z \in B_r(z_j)$. Hence, in such a case by Lemma 2.9, F_2 is pointwise $\left(|Q_j(z_j)|\, r^{n-1}\right)$-co-Lipschitz and pointwise $\left(|Q_j(z_j)|\, nr^{n-1}\right)$-Lipschitz at each $z \in B_r(z_j)$.

Claim 3.11. For each $z_j \in S(P')$ there exists $d_j > 0$ such that $F = F_2\big|_{\overline{B}_r(z_j)}$ is d_j-Lipschitz when considered as a function from $\overline{B}_r(z_j)$ to $F_2(\overline{B}_r(z_j))$.

Proof. Fix $z_j \in S(P')$. We shall show that F_2 is pointwise d_j-Lipschitz at each $x \in B_r(z_j)$ for some $d_j > 0$; the claim then follows by applying Lemma 2.6 followed by Lemma 2.5.

If $m_j = n$, then by Remark 3.10 it follows F_2 is pointwise $\left(|Q_j(z_j)|\, nr^{n-1}\right)$-Lipschitz at each $z \in B_r(z_j)$.

Suppose that $m_j < n$. If $x = z_j$, then for each $y \in B_r(z_j)$, as $F_2(x) = F_2(z_j) = P(z_j)$ and $|f_{m_j}(y-z_j)| = |y-z_j|$,

$$|F_2(x) - F_2(y)| = r^{m_j-1}|Q_j(h_2(y))|\cdot|y-z_j| = r^{m_j-1}|Q_j(h_2(y))|\cdot|x-y|.$$

Since $h_2\left(B_r(z_j)\right) = B_r(z_j)$, by Claim 3.5 (iv), F_2 is pointwise $2r^{m_j-1}|Q_j(z_j)|$-Lipschitz at $x = z_j$.

Suppose now that $x \in B_r(z_j)\setminus\{z_j\}$. Let $\rho_1 > 0$ be such that $B_{\rho_1}(x) \subseteq B_r(z_j)$. Further, for each $l \in \{1,\dots,n-m_j\}$, let $\rho_{2,l} > 0$ be given by Corollary 2.3, where $w = x - z_j \neq 0$, so that for each $z \in B_{\rho_{2,l}}(w)$, $\Phi_{l,m_j}(z,w)$ is well-defined and

$$\left|\Phi_{l,m_j}(z,w)\right| < 1 + \left|\Phi_{l,m_j}(w,w)\right|. \tag{3.13}$$

Define $\rho_2 := \min\{\rho_{2,l} : 1 \le l \le n-m_j\}$ and $\rho := \min(\rho_1,\rho_2)$. Note if $y \in B_\rho(x)$, then $z = y - z_j \in B_\rho(w)$. Considering (2.4), (3.11), (3.12) and Lemma 2.9 we deduce that if $y \in B_\rho(x)$, then

$$(3.14)\quad \begin{aligned} F_2(y) - F_2(x) &= F_2\left(z_j + |y - z_j| e^{i\arg(y-z_j)}\right) - F_2\left(z_j + |x - z_j| e^{i\arg(x-z_j)}\right) \\ &= r^{m_j-1}\left(f_{m_j}(z) - f_{m_j}(w)\right)\left(c_{0,j} + \sum_{l=1}^{n-m_j} r^{\frac{l(m_j-1)}{m_j}} c_{l,j}\cdot \Phi_{l,m_j}(z,w)\right), \end{aligned}$$

where $z = y - z_j$ and $w = x - z_j$. To see that F_2 is pointwise Lipschitz at x, as f_{m_j} is Lipschitz and $|z - w| = |y - x|$, it suffices to observe that $|\Phi_{l,m_j}(z,w)|$ are uniformly bounded over $z \in B_\rho(w)$ and $|w| = |x - z_j| < r < 1$. Indeed, by (3.13) as $l \in \{1, \ldots, n - m_j\}$, observe that

$$\left|\Phi_{l,m_j}(z,w)\right| < 1 + |w|^{l/m_j}\frac{l+m_j}{m_j} \le 1 + \frac{nr^{1/m_j}}{m_j} \le 1 + n.$$

Hence, we conclude that there exists $d_j > 0$ such that F_2 is pointwise d_j-Lipschitz at each $x \in B_r(z_j)$, which as explained above, implies the statement of Claim 3.11. ♢

Claim 3.12. There exists $L > 0$ such that F_2 is L-Lipschitz on $\mathbb{C}$.

Proof. Recall Remark 3.4. Since $F_1(z) = F_2(z)$ for $z \in (\mathbb{C} \setminus V) \cup \partial V$ we conclude F_2 is pointwise L_1-Lipschitz at each $z \in \mathbb{C} \setminus V$ and, moreover,

$$|F_2(z) - F_2(w)| \le L_1|z - w| \quad \text{for } z \in \partial V \text{ and } w \in \mathbb{C} \setminus V.$$

Therefore, by Claim 3.5 (i), Claim 3.11 and by defining L to be the maximum of L_1 and $\max_{j:z_j\in S(P')} d_j$, we conclude F_2 is pointwise L-Lipschitz at each $z \in \mathbb{C}$. Hence Lemma 2.6 implies that F_2 is L-Lipschitz on $\mathbb{C}$. ♢

We now turn our attention to the co-Lipschitzness of F_2.

Claim 3.13. For each $z_j \in S(P')$ and $z \in B_r(z_j)$, the mapping F_2 is pointwise α_j-co-Lipschitz at z, where α_j is defined in (3.15).

Proof. Fix $z_j \in S(P')$ and define

$$(3.15)\quad \alpha_j := \frac{r^{m_j-1}\,|Q_j(z_j)|}{2}.$$

If $m_j = n$, then by Remark 3.10 it follows that, as $\alpha_j < r^{n-1}|Q_j(z_j)|$, F_2 is pointwise α_j-co-Lipschitz at each $z \in B_r(z_j)$.

Suppose that $m_j < n$. By (2.3) we have that $\alpha_j > 0$. To show F_2 is pointwise α_j-co-Lipschitz at each $z \in B_r(z_j)$ we first show for each $z \in \overline{B}_r(z_j)$ that there exists $\rho = \rho(z) > 0$ such that

$$(3.16)\quad |F_2(z) - F_2(y)| \ge \alpha_j\,|z - y|$$

for each $y \in B_\rho(z) \cap \overline{B}_r(z_j)$. We emphasize that (3.16) holds not only for $z \in B_r(z_j)$ but also for $z \in \partial B_r(z_j)$, and this fact is used later in the proof of Claim 3.15.

Consider first when $z = z_j$. Let $\rho = r$ and $y \in B_\rho(z)$. From (3.12), we deduce that

$$|F_2(z) - F_2(y)| = r^{m_j-1}|y - z||Q_j(h_2(y))|.$$

Since $h_2(B_r(z_j)) = B_r(z_j)$, by Claim 3.5 (iv), we conclude that F_2 satisfies (3.16) when $z = z_j$.

Fix $z \in \overline{B}_r(z_j) \setminus \{z_j\}$. Let $\rho_1 = \rho_1(z) > 0$ be defined by

$$\rho_1(z) = \begin{cases} r, & \text{if } z \in \partial B_r(z_j); \\ r - |z - z_j|, & \text{if } z \in B_r(z_j) \setminus \{z_j\}. \end{cases} \tag{3.17}$$

By Corollary 2.26, since f_{m_j} is strongly 1-co-Lipschitz at $(z - z_j) \in \overline{B}_r(0)$ there exists $\rho_2 = \rho_2(z) > 0$ such that for any $x \in B_{\rho_2}(z - z_j)$ it follows that

$$\left| f_{m_j}(x) - f_{m_j}(z - z_j) \right| \geq |x - (z - z_j)|. \tag{3.18}$$

Further by Corollary 2.3, for each $l \in \{1, \dots, n - m_j\}$, let $\rho_{3,l} > 0$ be such that for each $y \in B_{\rho_{3,l}}(z)$, $\Phi_{l,m_j}(y - z_j, z - z_j)$ is well-defined and

$$\left| \Phi_{l,m_j}(y - z_j, z - z_j) \right| < r^{1/m_j} + |z - z_j|^{l/m_j} \frac{l + m_j}{m_j}. \tag{3.19}$$

Define $\rho_3 := \min \{\rho_{3,l} : 1 \leq l \leq n - m_j\}$ and let $\rho = \rho(z) > 0$ be given by $\rho = \min(\rho_1, \rho_2, \rho_3)$. We claim for $y \in B_\rho(z) \cap \overline{B}_r(z_j)$ that

$$|F_2(y) - F_2(z)| \geq \alpha_j \left| f_{m_j}(y - z_j) - f_{m_j}(z - z_j) \right|. \tag{3.20}$$

Fix $y \in B_\rho(z) \cap \overline{B}_r(z_j)$. By using $y \in \overline{B}_r(z_j)$ for $F_2(y)$, $z \neq z_j$ and $y \in B_\rho(z)$ for the well-definedness of $\Phi_{l,m_k}(y - z_j, z - z_j)$ and recalling (3.14), it follows that

$$\begin{aligned} &|F_2(y) - F_2(z)| \geq \\ &\quad r^{m_j - 1} \left(|c_{0,j}| - \max_{l \in \{1,\dots,n-m_j\}} \left| \Phi_{l,m_j}(y - z_j, z - z_j) \right| \cdot \sum_{k=1}^{n-m_j} r^{\frac{k(m_j - 1)}{m_j}} |c_{k,j}| \right) \\ &\quad \times \left| f_{m_j}(y - z_j) - f_{m_j}(z - z_j) \right|. \end{aligned}$$

Therefore, since $r < 1$, see Claim 3.5, to show (3.20) it suffices to prove, as $c_{0,j} = Q_j(z_j)$, that for all $l \in \{1, \dots, n - m_j\}$,

$$\left| \Phi_{l,m_j}(y - z_j, z - z_j) \right| \sum_{k=1}^{n-m_j} |c_{k,j}| \leq \frac{|Q_j(z_j)|}{2}. \tag{3.21}$$

This is trivial when $\sum_{k=1}^{n-m_j} |c_{k,j}| = 0$. Suppose $\sum_{k=1}^{n-m_j} |c_{k,j}| \neq 0$. By property (iii) of Claim 3.5, since $|y - z_j| < \rho \leq \rho_3$, $z \in \overline{B}_r(z_j)$, $m_j \geq 1$ and $l \leq n - m_j$, note that

$$\begin{aligned} \left| \Phi_{l,m_j}(y - z_j, z - z_j) \right| &< r^{1/m_j} + |z - z_j|^{l/m_j} \frac{l + m_j}{m_j} && \text{by (3.19)}, \\ &\leq (1 + n) r^{1/m_j} \\ &\leq \frac{|Q_j(z_j)|}{2 \sum_{k=1}^{n-m_j} |c_{k,j}|} && \text{by Claim 3.5 (iii)}. \end{aligned}$$

Thus (3.21) follows and so (3.20) is satisfied, as claimed.

Since $\rho \le \rho_2$ and $y \in B_\rho(z)$ it follows $(y - z_j) \in B_{\rho_2}(z - z_j)$. Therefore, by (3.18),

$$|f_{m_j}(y - z_j) - f_{m_j}(z - z_j)| \ge |(y - z_j) - (z - z_j)| = |y - z|.$$

Hence, combining this with (3.20) yields

$$|F_2(y) - F_2(z)| \ge \alpha_j |f_{m_j}(z - z_j) - f_{m_j}(y - z_j)| \ge \alpha_j |y - z|.$$

Thus we deduce that for each $z \in \overline{B}_r(z_j)$ there exists $\rho > 0$ such that (3.16) holds for all $y \in B_\rho(z) \cap \overline{B}_r(z_j)$.

If $z \in B_r(z_j)$, by (3.17) and since $\rho \le \rho_1$ we note $B_\rho(z) \subseteq B_r(z_j)$. Hence for each $y \in B_\rho(z)$, (3.16) is satisfied. Therefore, since $F_2 = P \circ h_2$ is an open map, by Corollary 2.15, Remark 2.16 and since $B_r(z_j)$ is open in $\mathbb{C}$, we conclude that F_2 is pointwise α_j-co-Lipschitz at any $z \in B_r(z_j)$. ♢

Remark 3.14. Taking $c_2 := \min_{z_j \in S(P')} \alpha_j > 0$ we deduce

$$F_2 \text{ is pointwise } c_2\text{-co-Lipschitz at each } z \in \mathrm{Int}(V). \tag{3.22}$$

Claim 3.15. There exists $c_3 > 0$ such that $F_2 : \mathbb{C} \to \mathbb{C}$ is pointwise c_3-co-Lipschitz at each $z \in \partial V$.

Proof. Let $c_3 := \min(c_0, c_2)$, where $c_0 > 0$ is given by Claim 3.6 and $c_2 > 0$ is given by Remark 3.14. Since F_2 is an open map, it suffices by Corollary 2.15 to show for each $z \in \partial V$ there exists $\rho = \rho(z) > 0$ such that if $x \in B_\rho(z)$, then

$$|F_2(z) - F_2(x)| \ge c_3 \, |z - x| \, . \tag{3.23}$$

Fix $z \in \partial V$ and let j be such that $z \in \partial B_r(z_j)$. Let $\rho_1 > 0$ be such that $B_{\rho_1}(z) \subseteq U_2$ and $B_{\rho_1}(z) \cap V \subseteq \overline{B}_r(z_j)$; note such $\rho_1 > 0$ exists by Claim 3.5 (i). Since $\partial V \subseteq U_2 \setminus \mathrm{Int}(V)$ and $F_1\big|_{U_2 \setminus \mathrm{Int}(V)} = F_2\big|_{U_2 \setminus \mathrm{Int}(V)}$, by (3.4) and $c_3 \le c_0$ there exists $\rho_2 \in (0, \rho_1)$ such that (3.23) is satisfied for each $x \in B_{\rho_2}(z) \cap (U_2 \setminus \mathrm{Int}(V)) = B_{\rho_2}(z) \setminus B_r(z_j)$.

Further, by (3.16) there exists $\rho \in (0, \rho_2)$ such that (3.23) is satisfied for each $x \in B_\rho(z) \cap \overline{B}_r(z_j)$ since $c_3 \le c_2 \le \alpha_j$; see Remark 3.14.

We then conclude that (3.23) is satisfied for each $x \in B_\rho(z)$. As F_2 is an open map, Corollary 2.15 implies the statement of Claim 3.15. ♢

Claim 3.16. There exists $c > 0$ such that F_2 is c-co-Lipschitz on $\mathbb{C}$.

Proof. Let $c := \min(c_1, c_2, c_3)$, where c_1 is given by Remark 3.9, c_2 is given by Remark 3.14 and c_3 is given by Claim 3.15. Recall by (3.10) of Remark 3.9 that F_1 is pointwise c_1-co-Lipschitz at each $z \in \mathbb{C} \setminus W$. As $F_1(z) = F_2(z)$ for $z \in \mathbb{C} \setminus V$ and $W \subseteq V$, we conclude

$$F_2 \text{ is pointwise } c\text{-co-Lipschitz at each } z \in \mathbb{C} \setminus V. \tag{3.24}$$

Also, Remark 3.14 implies that

$$F_2 \text{ is pointwise } c\text{-co-Lipschitz at each } z \in \mathrm{Int}(V). \tag{3.25}$$

From Claim 3.15, (3.24) and (3.25), we conclude that F_2 is pointwise c-co-Lipschitz at each $z \in \mathbb{C}$. Hence an application of Lemma 2.7 implies F_2 is c-co-Lipschitz on $\mathbb{C}$. ♢

Finally, Claims 3.12 and 3.16 together imply that $f := F_2 = P \circ h_2$ is an L-Lipschitz and c-co-Lipschitz mapping of the plane. □

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R. Hutchins
School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, U.K.

Current address: Mathematical Sciences, Loughborough University, Epinal Way, Loughborough, LE11 3TU, U.K.
E-mail address: R.Hutchins@lboro.ac.uk

O. Maleva
School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, U.K.
E-mail address: O.Maleva@bham.ac.uk