Pure and Applied Functional Analysis
Volume 8, Number 6, 2023, 1733–1746

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

VISUALIZATION OF ENUMERATION QUESTIONS WITH FACTOR SPECTRUM TILING

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Abstract. The Fibonacci word $\mathbb{F}$ is the fixed point beginning with a of morphism $\sigma(a) = ab$ and $\sigma(b) = a$. Since $\mathbb{F}$ is uniformly recurrent, each factor ω appears infinitely many times in the sequence which is arranged as ω_p (the pth occurrence of ω, $p \geq 1$). Here we distinguish $\omega_p \neq \omega_q$ if $p \neq q$. In this paper, we calculate the number of repeated palindromes in $\mathbb{F}[n_1, n_2]$.

1. Introduction

Combinatorics on words, is a field which grew simultaneously within disparate branches of mathematics such as group theory and probability. It has grown into an independent theory finding substantial applications in computer science, automata theory and linguistics [12, 13]. The numbers of special types of factors have been investigated in recent years, such as palindromes, squares, cubes, r-powers, etc. See [3–6, 14–17, 19].

The Fibonacci word $\mathbb{F}$, also called the Fibonacci sequence in some papers, is the fixed point beginning with a of morphism $\sigma(a) = ab$ and $\sigma(b) = a$. We define $F_{-2} = \varepsilon$ (the empty word), $F_{-1} = b$, and $F_m = \sigma^m(a)$ for $m \geq 0$. Then $F_0 = a$, $F_m = F_{m-1}F_{m-2}$ for $m \geq 1$. We call $|F_m| = f_m$ the mth Fibonacci number for $m \geq -2$. Here $|\omega|$ means the length of ω. Let $|\omega|_a$ denote the number of letter a occurring in ω.

Denote $\phi = \frac{\sqrt{5}-1}{2}$ and $\lfloor \alpha \rfloor$ is the largest integer not more than α. We abbreviate $[i, i+1, \dots, i+j]$ as $[i..i+j]$ or $[0..j] \oplus i$ for $i \in \mathbb{N}$ and $j \in \mathbb{N}^+$. Let $\Sigma[d_1, d_2, \dots, d_n] = \sum_{i=1}^{n} d_i$.

Let $\mathrm{One}(i) = [\underbrace{1, 1, \dots, 1}_{i}]$ and $\mathrm{Zero}(j) = [\underbrace{0, 0, \dots, 0}_{j}]$.

Since $\mathbb{F}$ is uniformly recurrent, each factor ω appears infinitely many times in the sequence which is arranged as ω_p (the pth occurrence of ω for $p \geq 1$). Here we distinguish $\omega_p \neq \omega_q$ if $p \neq q$. We let $\mathrm{Occ}(\omega, p)$ (resp. $\mathrm{Pos}(\omega, p)$) denote the position of the first (resp. last) letter of ω_p in $\mathbb{F}$.

A palindrome is a finite word that reads the same backwards as forwards, such as "noon". Let $\omega = x_1 \cdots x_n$. We define $\omega[i, j] = x_i x_{i+1} \cdots x_{j-1} x_j$ for $1 \leq i \leq j \leq n$.

2020 *Mathematics Subject Classification.* 68R15.
Key words and phrases. Combinatorics on words, palindrome, derived sequence, factor spectrum, factor spectrum tiling.

$\omega[i] = \omega[i,i] = x_i$ and $\omega[i,i-1] = \varepsilon$. In Huang-Wen [9], we gave an algorithm for counting the number of repeated palindromes in $\mathbb{F}[1,n]$ (the prefix of $\mathbb{F}$ of length n). In this paper, we calculate the number of repeated palindromes in $\mathbb{F}[n_1,n_2]$, denoted by $\mathrm{Num}(n_1,n_2)$. Without causing confusion, we omit "repeated". A natural idea is that we consider two functions: (1) the number of palindromes in $\mathbb{F}[1,n_2]$, (2) the number of palindromes whose first letter occurs in $\mathbb{F}[1,n_1]$. But unfortunately, $\mathrm{Num}(n_1,n_2)$ is not equal to the difference between the two functions.

The calculation method adopted in this paper is based on the exact structure of the derived sequence of the Fibonacci word $\mathbb{F}$, and the structure of palindromes in $\mathbb{F}$. The main tool of study the derived sequence is "kernel word" in Huang-Wen [7]. Denote by δ_m the last letter of F_m for $m \geq -1$, then $\delta_m = a$ if and only if m is even. The mth kernel word of $\mathbb{F}$ is defined as $K_m = \delta_{m+1}F_m\delta_m^{-1}$ for $m \geq -2$, which is also called singular word in Wen-Wen [17]. Let $\mathrm{Ker}(\omega)$ be the maximal kernel word occurring in factor ω. Let $L = |\mathbb{F}[n_1,n_2]| = n_2 - n_1 + 1$ and $m = \min\{m \mid f_m \leq L \leq f_{m+3} - 2\}$. Then there exist several palindromes with kernel K_h $(h = m, m+1, m+2)$ satisfying $\mathrm{Occ}(\omega,p) < n_1$ and $\mathrm{Pos}(\omega,p) > n_2$.

In order to calculate $\mathrm{Num}(n_1,n_2)$, we only need to consider sequences $\mathrm{Num}_{\leq m}^{pos}$, $\mathrm{Num}_{\leq m}^{occ}$ and Num_m. For $n \geq 1$ and $L \in \{f_m..f_{m+3}-2\}$

$$\begin{cases}\mathrm{Num}_{\leq m}^{pos}[n] & = \#\big\{(\omega,p) \mid \mathrm{Ker}(\omega) = K_h,\ -1 \leq h \leq m,\ \mathrm{Pos}(\omega,p) = n\big\};\\ \mathrm{Num}_{\leq m}^{occ}[n] & = \#\big\{(\omega,p) \mid \mathrm{Ker}(\omega) = K_h,\ -1 \leq h \leq m,\ \mathrm{Occ}(\omega,p) = n\big\};\\ \mathrm{Num}_m[L,n] & = \#\big\{(\omega,p) \mid \mathrm{Ker}(\omega) = K_m,\\ & \qquad\qquad \mathrm{Occ}(\omega,p) \geq n,\ \mathrm{Pos}(\omega,p) \leq n+L-1\big\}.\end{cases}$$

Thus the number of repeated palindromes in $\mathbb{F}[n_1,n_2]$ is

$$\begin{aligned}&\mathrm{Num}(n_1,n_2)\\ &= \sum_{h=-1} \#\big\{(\omega,p) \mid \mathrm{Ker}(\omega) = K_h, \mathrm{Occ}(\omega,p) \geq n_1, \mathrm{Pos}(\omega,p) \leq n_2\big\}\\ &= \sum_{h=-1}^{m-1} \#\big\{(\omega,p) \mid \mathrm{Ker}(\omega) = K_h, \mathrm{Pos}(\omega,p) \leq n_2\big\}\\ &\quad - \sum_{h=-1}^{m-1} \#\big\{(\omega,p) \mid \mathrm{Ker}(\omega) = K_h, \mathrm{Occ}(\omega,p) \leq n_1 - 1\big\}\\ &\quad + \sum_{h=m}^{m+2} \#\big\{(\omega,p) \mid \mathrm{Ker}(\omega) = K_h, \mathrm{Occ}(\omega,p) \geq n_1, \mathrm{Pos}(\omega,p) \leq n_2\big\}\\ &= \Sigma\mathrm{Num}_{\leq m-1}^{pos}[1,n_2] - \Sigma\mathrm{Num}_{\leq m-1}^{occ}[1,n_1-1] + \sum_{h=m}^{m+2} \mathrm{Num}_h[L,n_1].\end{aligned} \tag{1.1}$$

We will prove all of the three sequences are the Fibonacci words in Sections 3-5. So we can give the expressions of the enumeration questions using the Zeckendorf numeration system. Besides that, we give some basic notions in Section 2, and give the main result and an example in Section 6.

2. Basic notions

The calculation method adopted in this paper is based on the exact structure of the derived sequence of the Fibonacci word $\mathbb{F}$, the factor spectrum tiling, and the structure of palindromes in $\mathbb{F}$.

2.1. The Derived Sequence.

The definitions of both the return word and derived sequence are from Durand [2]. Recall that ω_p denotes the pth occurrence of ω, and $\mathrm{Occ}(\omega,p)$ denotes the position of the first letter of ω_p in $\mathbb{F}$. Let $R_p(\omega)=\mathbb{F}[\mathrm{Occ}(\omega,p),$
$\mathrm{Occ}(\omega,p+1)-1]$ be the pth return word of ω. Denote by $\mathcal{H}_\omega$ the set of return words over factor $\omega\prec\mathbb{F}$. Then the Fibonacci word $\mathbb{F}$ can be written in a unique way as a concatenation $\mathbb{F}=\mathbb{F}[1,h-1]R_1(\omega)R_2(\omega)\cdots$ where $R_p(\omega)\in\mathcal{H}_\omega$ and $\mathbb{F}[1,h-1]$ is the prefix of $\mathbb{F}$ occurring before the first occurrence of ω. Let us give to $\mathcal{H}_\omega$ the linear order defined by the rank of the first occurrence in $\mathbb{F}$. This defines a one to one and onto map $\Lambda_\omega:\mathcal{H}_\omega\to\{1,\ldots,\mathrm{Card}(\mathcal{H}_\omega)\}=\mathcal{N}_\omega$, and the sequence $\mathcal{D}_\omega:=\Lambda_\omega(R_1(\omega))\Lambda_\omega(R_2(\omega))\Lambda_\omega(R_3(\omega))\cdots$. This sequence of alphabet $\mathcal{N}_\omega$ is called a derived sequence of $\mathbb{F}$. Notice that we omit the prefix $\mathbb{F}[1,h-1]$. Wen-Wen [17] and Huang-Wen [7] characterized the exact structure of the derived sequence of the Fibonacci word $\mathbb{F}$. More precisely, Wen-Wen [17] proved that for any kernel word $K_m\in\mathbb{F}$ (a special type of factors) the derived sequence $\{R_p(K_m)\}_{p\geq1}$ is still a Fibonacci word; and Huang-Wen [7] proved that for all factors $\omega\in\mathbb{F}$ the derived sequence $\{R_p(\omega)\}_{p\geq1}$ is still a Fibonacci word.

2.2. The factor spectrum and the factor spectrum tiling.

Huang-Wen [7–11] introduced and researched "the factor spectrum". As a new tool, the definitions of the factor spectrum in these papers are different in details. Essentially, the factor spectrum consider two variables: factor variable and positional variable. In this paper, we visualize the number of repeated palindromes in $\mathbb{F}[n_1,n_2]$ with factor spectrum tiling. We believe that by analogous arguments we can visualize some more enumeration questions with factor spectrum tiling.

2.3. The Structure of Palindromes.

Let $\mathcal{P}_F$ be all palindromes occurring in $\mathbb{F}$. Huang-Wen [9] gave the structure of palindromes in $\mathbb{F}$. Any palindrome with kernel K_m can be expressed uniquely as

$$K_{m+1}[i+1,f_{m+1}]K_mK_{m+1}[1,f_{m+1}-i]=K_{m+3}[i+1,f_{m+3}-i], \tag{2.1}$$

where $1\leq i\leq f_{m+1}$ and $m\geq-1$.

Proposition 2.1 (Properties 3.1 and 3.3 in Huang-Wen [9]).
For $m\geq-1$ and $p\geq1$,

1. $\mathrm{Pos}(K_m,p)=pf_{m+1}+(\lfloor\phi p\rfloor+1)f_m-1$; *in particular,* $\mathrm{Pos}(a,p)=p+\lfloor\phi p\rfloor$ *and* $\mathrm{Pos}(b,p)=2p+\lfloor\phi p\rfloor$.
2. *Let ω be a palindrome with kernel K_m satisfying Expression (2.1), then* $\mathrm{Pos}(\omega,p)=\mathrm{Pos}(K_m,p)+f_{m+1}-i=(p+1)f_{m+1}+(\lfloor\phi p\rfloor+1)f_m-i-1$ *for* $1\leq i\leq f_{m+1}$.

2.4. **The Zeckendorf numeration system.**

In the Zeckendorf numeration system, natural numbers are represented as sums of Fibonacci numbers [18]. Recall $f_m = |F_m|$ be the Fibonacci numbers for $m \geq -2$. Ignoring leading zeros, any natural number N can be written uniquely as $N = \sum_{i=0}^{\infty} d_i f_i$, with digits $d_i = 0$ or 1, and where $d_i d_{i+1} = 11$ is not allowed. We write $Z(N) = d_M \cdots d_2 d_1 d_0$. For instance, $Z(6) = 1001$, since $f_3 = 5$, $f_0 = 1$, see Dekking [1]. Define inverse mapping of $Z(n) = d_M \cdots d_2 d_1 d_0$ that $(d_M \cdots d_2 d_1 d_0)_F = \sum_{j=0}^{M} d_j f_j$. For instance, $(1001)_F = 6$.

Now we list some classical conclusions based on the Zeckendorf numeration system. We will use the Zeckendorf numeration system to rewrite the algorithms of enumeration questions into expressions.

Proposition 2.2. *Let ρ be a Fibonacci word over alphabet $\{R_A, R_B\}$ with prefix R_0 where $|R_A| = f_{m+1}$ and $|R_B| = f_m$ for $m \geq -1$.*

When $n > |R_0|$, let $Z(n - |R_0|) = d_M d_{M-1} \ldots d_{m+1} d_m \ldots d_1 d_0$, then

(1) *there exist $N := (d_M d_{M-1} \ldots d_{m+1})_F$'s return words $\{R_A, R_B\}$ in $\rho[1, n]$, omit the last return word (maybe incomplete or equal to the empty word ε);*
(2) *there exist $N_A := \lfloor (N+1)\phi \rfloor$'s R_A in $\rho[1, n]$; in particular, $|\mathbb{F}[1, n]|_a = \lfloor (n+1)\phi \rfloor$; and there exist $N_B := \lfloor (N+1)\phi^2 \rfloor$'s R_B in $\rho[1, n]$;*
(3) *the length of the last return word is $n - |R_0| - N_A|R_A| - N_B|R_B| = (d_m d_{m-1} \ldots d_0)_F$; moreover, the last return word is $P[1, (d_m d_{m-1} \ldots d_0)_F]$ where $P = d_{m+1} R_B + (1 - d_{m+1}) R_A$.*

See Figure 1 as an example. (1) When $n = 27 = 21 + 5 + 1 = f_6 + f_3 + f_0$, thus $Z(n - |R_0|) = 1001001$. There exist $N = (1001)_F = 6$'s return words $\{R_A, R_B\}$ in $\rho[1, 27]$. In this case, the last return word is incomplete. When $n = 26 = 21 + 5 = f_6 + f_3$, thus $Z(n - |R_0|) = 1001000$. In this case, the last return word is the empty word ε. (2) There exist $N_A = \lfloor (N+1)\phi \rfloor = 4$'s R_A and $N_B = \lfloor (N+1)\phi^2 \rfloor = 2$'s R_B in $\rho[1, 27]$. (3) The length of the last return word is $n - |R_0| - N_A|R_A| - N_B|R_B| = 27 - 0 - 4 \times 5 - 2 \times 3 = 1$, which is equal to $(001)_F = 1$. Moreover, since $d_3 = 1$, the last return word in $\rho[1, 27]$ is $R_B[1, 1] = [a]$.

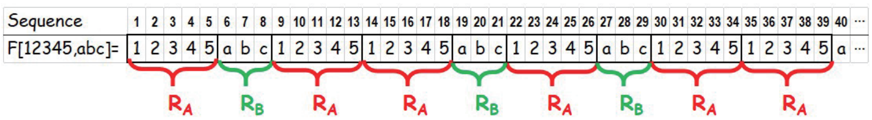

FIGURE 1. The Fibonacci word ρ over alphabet $\{R_A, R_B\} = \{12345, abc\}$ with prefix $R_0 = \varepsilon$. In this case, $m = 2$.

3. THE SEQUENCE $\mathrm{Num}_{\leq m}^{pos}$

We first consider the sequence Num_m^{pos} that

$$\mathrm{Num}_m^{pos}[n] = \#\big\{(\omega, p) \mid \mathrm{Ker}(\omega) = K_m,\ \mathrm{Pos}(\omega, p) = n\big\}. \tag{3.1}$$

Obviously, $\mathrm{Num}_{\leq m}^{pos}[n] = \sum_{h=-1}^{m} \mathrm{Num}_h^{pos}[n]$ for all $n \geq 1$.

3.1. The sequence Num_m^{pos}.

Lemma 3.1. *The number of factor b occurring before a_p in $\mathbb{F}$ is $\lfloor \phi p \rfloor$.*

Proof. Using the exact structure of the derived sequence, the derived sequence $\{R_p(a)\}_{p\geq 1}$ is a Fibonacci word over the alphabet $\{R_A, R_B\} = \{R_1(a), R_2(a)\} = \{ab, a\}$ with prefix $R_0(a) = \varepsilon$. Thus the number of factor b occurring before a_p in $\mathbb{F}$ is equal to the number of return word R_A occurring before the pth return word. Furthermore, it is equal to the number of factor a occurring in $\mathbb{F}[1, p-1]$. By Proposition 2.2(2), the number is $|\mathbb{F}[1, p-1]|_a = \lfloor \phi p \rfloor$. □

Proposition 3.2. *For $m \geq -1$, the sequence Num_m^{pos} is a Fibonacci word over the alphabet $\{R_A, R_B\}$ with prefix R_0, where $R_A = \mathrm{One}(f_{m+1})$, $R_B = \mathrm{Zero}(f_m)$ and $R_0 = \mathrm{Zero}(f_{m+2} - 2)$.*

Proof. On one hand, by Proposition 2.1(2), for $m \geq -1$ and $p \geq 1$

$$\begin{aligned} &\{\mathrm{Pos}(\omega, p) \mid \omega \in \mathcal{P}_F, \mathrm{Ker}(\omega) = K_m\} \\ &= [pf_{m+1} + (\lfloor \phi p \rfloor + 1)f_m - 1..pf_{m+1} + (\lfloor \phi p \rfloor + 1)f_m + f_{m+1} - 2] \\ &= [1..f_{m+1}] \oplus \{pf_{m+1} + (\lfloor \phi p \rfloor + 1)f_m - 2\}. \end{aligned} \tag{3.2}$$

These positions correspond to all digits 1's in sequence Num_m^{pos}.

On the other hand, let ρ be a Fibonacci word over the alphabet $\{R_A, R_B\} = \{\mathrm{One}(f_{m+1}), \mathrm{Zero}(f_m)\}$ with prefix $R_0 = \mathrm{Zero}(f_{m+2} - 2)$. We let $R_{A,p}$ denote the pth occurrence of R_A. Obviously, the number of R_A occurring before $R_{A,p}$ is $p - 1$. The number of R_B occurring before $R_{A,p}$ is equal to the number of b occurring before a_p. By Lemma 3.1, it is equal to $\lfloor \phi p \rfloor$.

So the position of the first letter of $R_{A,p}$ in ρ is

$$\begin{aligned} &|R_0| + (p-1)|R_A| + \lfloor \phi p \rfloor |R_B| + 1 \\ &= f_{m+2} - 2 + (p-1)f_{m+1} + \lfloor \phi p \rfloor f_m + 1 \\ &= pf_{m+1} + (\lfloor \phi p \rfloor + 1)f_m - 1. \end{aligned} \tag{3.3}$$

Since $|R_A| = f_{m+1}$, the positions (from the first letter to the last letter) of the pth occurrence of R_A are $\{\mathrm{Pos}(\omega, p) \mid \omega \in \mathcal{P}_F, \mathrm{Ker}(\omega) = K_m\}$ for $p \geq 1$. This completes the proof. □

As an example of Proposition 3.2, Figure 2(a-b) gives the first several occurrences of palindromes with kernel K_m for $m = 2$, and the first 40 digits of the sequence Num_m^{pos}.

3.2. The factor spectrum tiling.

Define matrices $\Gamma_{-1} = \begin{bmatrix}1\end{bmatrix}$, $\Gamma_0 = \begin{bmatrix}0 & 1 \\ 1 & 1\end{bmatrix}$, and for $m \geq 1$

$\Gamma_m = \left[\begin{array}{c|c} \Gamma_{m-2} & \Gamma_{m-1} \\ \hline 0 \cdots 0 & \\ \hline \multicolumn{2}{c}{1 \cdots \;\cdots\; \cdots 1} \end{array}\right]_{(m+2)\times f_{m+1}}$. For instance, $\Gamma_1 = \begin{bmatrix}1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1\end{bmatrix}$.

Lemma 3.3 (Lemma 3.4 in Huang-Wen [9])**.**

(1) $\lfloor \phi(p + \lfloor \phi p \rfloor + 1) \rfloor = p$,

(2) $\lfloor \phi(2p + \lfloor \phi p \rfloor + 1) \rfloor = p + \lfloor \phi p \rfloor$,

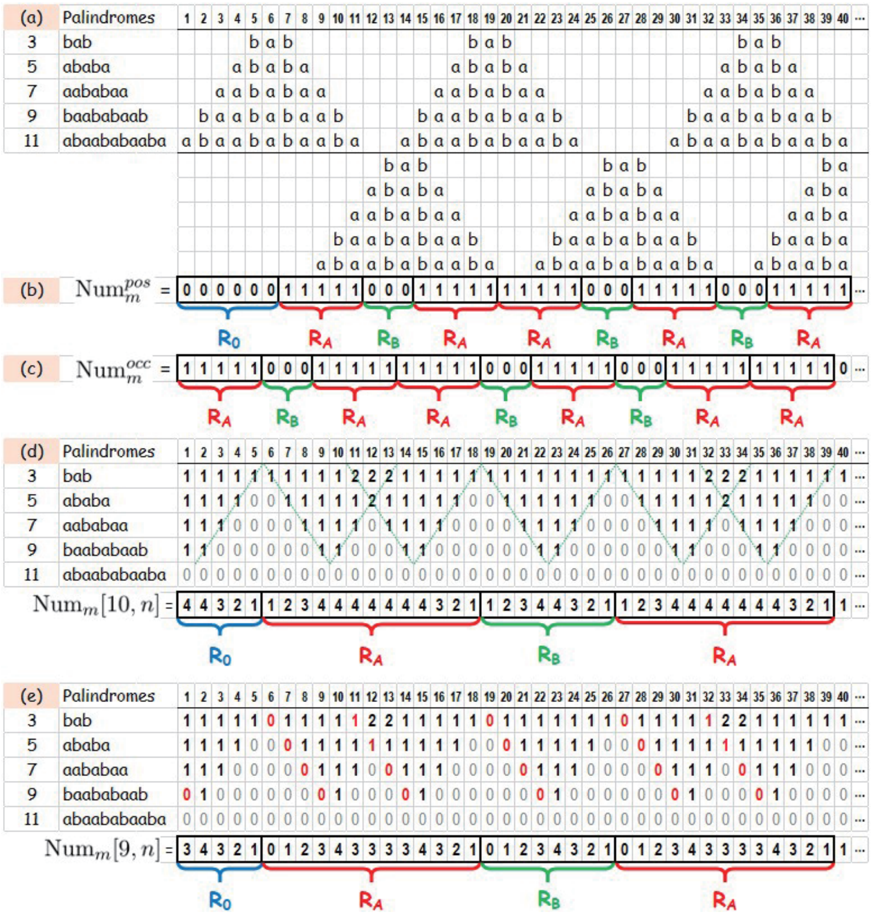

FIGURE 2. Examples of sequences $\mathrm{Num}_{\leq m}^{pos}$, $\mathrm{Num}_{\leq m}^{occ}$ and Num_m. All of them are the Fibonacci words. (a) The first several occurrences of palindromes with kernel K_m for $m = 2$. (b) The sequence Num_m^{pos} is the Fibonacci word over the alphabet $\{R_A, R_B\} = \{[1,1,1,1,1],[0,0,0]\}$ with prefix $R_0 = [0,0,0,0,0,0]$. (c) The sequence Num_m^{occ} is the Fibonacci word over the alphabet $\{R_A, R_B\} = \{[1,1,1,1,1],[0,0,0]\}$ with prefix $R_0 = \varepsilon$. (d-e) The sequence $\mathrm{Num}_m[L,n]$ for $L = 10$ and 9.

(3) $\lfloor \phi(p + \lfloor \phi p \rfloor) \rfloor = p - 1$,
(4) $\lfloor \phi(2p + \lfloor \phi p \rfloor) \rfloor = p + \lfloor \phi p \rfloor$ *for* $p \geq 1$.

Proposition 3.4. *For $m \geq 0$, the sequence* $\mathbb{M}_m^{pos} := \begin{bmatrix} \mathrm{Num}_{-1}^{pos} \\ \mathrm{Num}_{0}^{pos} \\ \vdots \\ \mathrm{Num}_{m}^{pos} \end{bmatrix}$ *is an $(m+2) \times \infty$ matrix. It is a Fibonacci word over $\{R_A, R_B\}$ with prefix R_0, where $R_A = \Gamma_m$, $R_B = \Gamma_{m-1}$ and*

$$R_0 = \left[\begin{array}{c|c|c|c} \Gamma_{-1} & \Gamma_{-2} & \cdots & \Gamma_{m-1} \\ \cline{1-1} & & & \\ \cline{2-2} & & & \\ \cline{3-3} & & & \\ \cline{4-4} 0 & & \cdots & 0 \end{array}\right]_{(m+2)\times(f_{m+2}-2)}.$$

Proof. By Proposition 3.2, the sequence Num_m^{pos} is a Fibonacci word. Thus we only need to prove two fact: (1) each occurrence of Γ_m in sequence $\mathbb{M}_m^{pos}$ is with structure

$$\Gamma_{m,p} = \left[\begin{array}{ccc|ccc} \Gamma_{m-2,\mathrm{Pos}(b,p)+1} & & & \Gamma_{m-1,\mathrm{Pos}(a,p)+1} & & \\ \hline 0 & \cdots & 0 & & & \\ \hline 1 & \cdots & \cdots & & \cdots & 1 \end{array}\right]_{(m+2)\times f_{m+1}},$$

where $\Gamma_{m,p}$ is the pth occurrence of Γ_m; (2) the prefix R_0 can be tiled with $\Gamma_{i,1}$ for $-1 \leq i \leq m-1$.

(1) By the structure of Γ_m, the last line of the pth occurrence of Γ_m in sequence $\mathbb{M}_m^{pos}$ is the pth occurrence of $R_A = \mathrm{One}(f_{m+1})$ in sequence Num_m^{pos}. Thus by Equation (3.2), the columns of $\Gamma_{m,p}$ is

$$\begin{aligned} &[pf_{m+1} + (\lfloor \phi p \rfloor + 1)f_m - 1..(p+1)f_{m+1} + (\lfloor \phi p \rfloor + 1)f_m - 2] \\ &= [1..f_{m+1}] \oplus \{pf_{m+1} + (\lfloor \phi p \rfloor + 1)f_m - 2\}. \end{aligned} \tag{3.4}$$

Furthermore, the columns of $\Gamma_{m-2,\mathrm{Pos}(b,p)+1} = \Gamma_{m-2,2p+\lfloor \phi p \rfloor}$ is

$$\begin{aligned} &[1..f_{m-1}] \oplus \{(2p + \lfloor \phi p \rfloor + 1)f_{m-1} + (\lfloor \phi(2p + \lfloor \phi p \rfloor)\rfloor + 1)f_{m-2} - 2\} \\ &= [1..f_{m-1}] \oplus \{(2p + \lfloor \phi p \rfloor + 1)f_{m-1} + (p + \lfloor \phi p \rfloor + 1)f_{m-2} - 2\} \\ &= [1..f_{m-1}] \oplus \{pf_{m+1} + (\lfloor \phi p \rfloor + 1)f_m - 2\}, \end{aligned} \tag{3.5}$$

the first equality holds by Lemma 3.3(4).

Similarly, the columns of $\Gamma_{m-1,\mathrm{Pos}(a,p)+1} = \Gamma_{m-1,p+\lfloor \phi p \rfloor+1}$ is

$$\begin{aligned} &[1..f_m] \oplus \{(p + \lfloor \phi p \rfloor + 1)f_m + (\lfloor \phi(p + \lfloor \phi p \rfloor + 1)\rfloor + 1)f_{m-1} - 2\} \\ &= [1..f_m] \oplus \{(p + \lfloor \phi p \rfloor + 1)f_m + (p+1)f_{m-1} - 2\} \\ &= [f_{m-1} + 1..f_{m+1}] \oplus \{pf_{m+1} + (\lfloor \phi p \rfloor + 1)f_m - 2\}, \end{aligned} \tag{3.6}$$

the first equality holds by Lemma 3.3(1).

Comparing Equations (3.4)-(3.6), $\Gamma_{m-2,\mathrm{Pos}(b,p)+1}$ and $\Gamma_{m-1,\mathrm{Pos}(a,p)+1}$ can tile to $\Gamma_{m,p}$.

(2) By Equation (3.4), the columns of $\Gamma_{i,1}$ is $[f_{i+2} - 1..f_{i+2} - 2]$. Thus the prefix R_0 can be tiled with $\Gamma_{i,1}$ for $-1 \leq i \leq m-1$.

These two points complete the proof. □

3.3. **The sequence** $\mathrm{Num}_{\leq m}^{pos}$.

Since $\mathrm{Num}_{\leq m}^{pos}[n] = \sum_{h=-1}^{m} \mathrm{Num}_h^{pos}[n]$ for all $n \geq 1$, the sequence $\mathrm{Num}_{\leq m}^{pos}$ is column sum of $\mathbb{M}_m^{pos}$. We let P_m denote the column sum of Γ_m. By the definition of

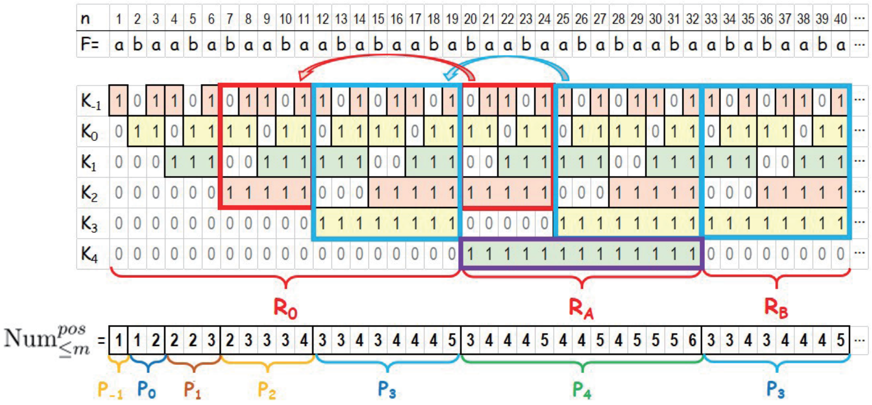

FIGURE 3. The factor spectrum tiling in sequence $\mathbb{M}_m^{pos}$, and the column sum of $\mathbb{M}_m^{pos}$ for $m = 4$.

Γ_m, $P_{-1} = [1]$, $P_0 = [1, 2]$, $P_m = [P_{m-2}, P_{m-1}] + \mathrm{One}(f_{m+1})$ for $m \geq 1$. Obviously, $|P_m| = f_{m+1}$, $\Sigma P_{-1} = 1$, $\Sigma P_0 = 3$, $\Sigma P_m = \Sigma P_{m-2} + \Sigma P_{m-1} + f_{m+1}$ for $m \geq 1$. Moreover, for $m \geq -1$

$$\begin{cases} \Sigma P_m = \frac{m+3}{5} f_{m+3} + \frac{m}{5} f_{m+1}, \\ \Sigma P_m - \Sigma P_{m-1} = \frac{2m+4}{5} f_{m+1} - \frac{m-2}{5} f_m, \\ \sum_{h=-1}^{m} \Sigma P_h = \frac{m}{5} f_{m+5} + \frac{m+2}{5} f_{m+3} + 2. \end{cases} \tag{3.7}$$

All of them can be verified by induction.

As a corollary of Proposition 3.4, $\mathrm{Num}_{\leq m}^{pos}$ is a Fibonacci word too.

Proposition 3.5. *For $m \geq 0$, the sequence $\mathrm{Num}_{\leq m}^{pos}$ is a Fibonacci word over the alphabet $\{R_A, R_B\}$ with prefix R_0, where $R_A = P_m$, $R_B = P_{m-1}$ and $R_0 = [P_{-1}, P_0, P_1, \ldots, P_{m-1}]$.*

3.4. The expression of $\Sigma\mathrm{Num}_{\leq m}^{pos}[1, n]$.

(A) When $n \leq |R_0| = \sum_{h=-1}^{m-1} |P_h| = \sum_{h=-1}^{m-1} f_{h+1} = f_{m+2} - 2$, the number of palindromes in $\mathbb{F}[1, n]$ with kernel K_i $(-1 \leq i \leq m)$ is $\Sigma\mathrm{Num}_{\leq m}^{pos}[1, n] = \Sigma R_0[1, n]$. Furthermore, since the first occurrence of palindromes with kernel K_i $(i > m)$ must larger than n, $\Sigma R_0[1, n]$ is the number of palindromes in $\mathbb{F}[1, n]$ too.

(B) When $n > |R_0|$, let $Z(n - |R_0|) = d_M d_{M-1} \ldots d_{m+1} d_m \ldots d_1 d_0$. Using the Zeckendorf numeration system and Proposition 2.2, there exist $N = (d_M d_{M-1} \ldots d_{m+1})_F$'s return words $\{R_A, R_B\}$ in $\rho[1, n]$; and there exist $N_A = \lfloor (N + 1)\phi \rfloor$'s R_A (resp. $N_B = \lfloor (N + 1)\phi^2 \rfloor$'s R_B) in $\rho[1, n]$. Thus the number of palindromes in $\mathbb{F}[1, n]$ with kernel K_i $(-1 \leq i \leq m)$ is

$$\begin{aligned} &\Sigma\mathrm{Num}_{\leq m}^{pos}[1, n] \\ &= \Sigma R_0 + N_A \Sigma R_A + N_B \Sigma R_B + \Sigma\Lambda[1, (d_m d_{m-1} \ldots d_0)_F], \end{aligned} \tag{3.8}$$

where $\Lambda = d_{m+1} R_B + (1 - d_{m+1}) R_A$.

Furthermore,

$$\begin{aligned} &\Sigma R_0 + N_A\Sigma R_A + N_B\Sigma R_B = \Sigma R_0 + N\Sigma R_B + N_A(\Sigma R_A - \Sigma R_B) \\ &= \left\{\tfrac{m-1}{5}f_{m+4} + \tfrac{m+1}{5}f_{m+2} + 2\right\} + N\left\{\tfrac{m+2}{5}f_{m+2} + \tfrac{m-1}{5}f_m\right\} \\ &\quad + \lfloor (N+1)\phi\rfloor \left\{\tfrac{2m+4}{5}f_{m+1} - \tfrac{m-2}{5}f_m\right\}. \end{aligned} \tag{3.9}$$

Theorem 3.6. *For $m \geq -1$, when $n \leq f_{m+2} - 2$, the number of palindromes in $\mathbb{F}[1,n]$ is $\Sigma R_0[1,n]$. When $n > f_{m+2}-2$, let $Z(n-f_{m+1}+2) = d_M d_{M-1}...d_{m+1}d_m...d_1d_0$. Let $N = (d_M d_{M-1}...d_{m+1})_F$. Then the number of palindromes in $\mathbb{F}[1,n]$ with kernel K_i $(-1 \leq i \leq m)$ is*

$$\begin{aligned} &\Sigma \mathrm{Num}^{pos}_{\leq m}[1,n] \\ &= \left\{\tfrac{m-1}{5}f_{m+4} + \tfrac{m+1}{5}f_{m+2} + 2\right\} + N\left\{\tfrac{m+2}{5}f_{m+2} + \tfrac{m-1}{5}f_m\right\} \\ &\quad + \lfloor (N+1)\phi\rfloor \left\{\tfrac{2m+4}{5}f_{m+1} - \tfrac{m-2}{5}f_m\right\} + \Sigma\Lambda[1,(d_m d_{m-1}...d_0)_F], \end{aligned} \tag{3.10}$$

where $\Lambda = d_{m+1}R_B + (1-d_{m+1})R_A$.

4. The sequence $\mathrm{Num}^{occ}_{\leq m}$

By an analogous argument, we can prove that the sequence $\mathrm{Num}^{occ}_{\leq m}$ is a Fibonacci word too, and obtain the expression of the number of palindromes whose first letter occurrence in $\mathbb{F}[1,n]$ with kernel K_i $(-1 \leq i \leq m)$. We just list some main results.

Proposition 4.1. *For $m \geq -1$, the sequence Num^{occ}_m is a Fibonacci word over the alphabet $\{R_A, R_B\}$, where $R_A = \mathrm{One}(f_{m+1})$ and $R_B = \mathrm{Zero}(f_m)$.*

Define matrices $\Gamma_{-1} = \begin{bmatrix}1\end{bmatrix}$, $\Gamma_0 = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$, and for $m \geq 1$,

$\Gamma_m = \left[\begin{array}{c|c} \Gamma_{m-1} & \begin{array}{c}\Gamma_{m-2}\\ \hline 0\cdots 0\end{array} \\ \hline \multicolumn{2}{c}{1\cdots\ \cdots\ \cdots 1} \end{array}\right]_{(m+2)\times f_{m+1}}$. For instance, $\Gamma_1 = \begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}$. Notice that, we use the same notations in Section 3. We can distinguish these notations according to the sections where they appear.

Proposition 4.2. *For $m \geq 0$, the sequence $\mathbb{M}^{occ}_m := \begin{bmatrix}\mathrm{Num}^{occ}_{-1}\\ \mathrm{Num}^{occ}_0\\ \vdots \\ \mathrm{Num}^{occ}_m\end{bmatrix}$ is an $(m+2)\times\infty$ matrix. It is a Fibonacci word over the alphabet $\{R_A, R_B\}$, where $R_A = \Gamma_m$, $R_B = \Gamma_{m-1}$.*

Since $\mathrm{Num}^{occ}_{\leq m}[n] = \sum_{h=-1}^{m} \mathrm{Num}^{occ}_h[n]$ for all $n \geq 1$, the sequence $\mathrm{Num}^{occ}_{\leq m}$ is column sum of $\mathbb{M}^{occ}_m$. We let P_m denote the column sum of Γ_m. By the definition of Γ_m, $P_{-1} = [1]$, $P_0 = [2,1]$, $P_m = [P_{m-1}, P_{m-2}] + \mathrm{One}(f_{m+1})$ for $m \geq 1$. Notice that, the values of $|P_m|$, ΣP_h $(-1 \leq h \leq m)$, $\sum_{h=-1}^{m}\Sigma P_h$ are equal to the corresponding values in Section 3.

Proposition 4.3. *For $m \geq 0$, the sequence $\mathrm{Num}^{occ}_{\leq m}$ is a Fibonacci word over the alphabet $\{R_A, R_B\}$, where $R_A = P_m$ and $R_B = P_{m-1}$.*

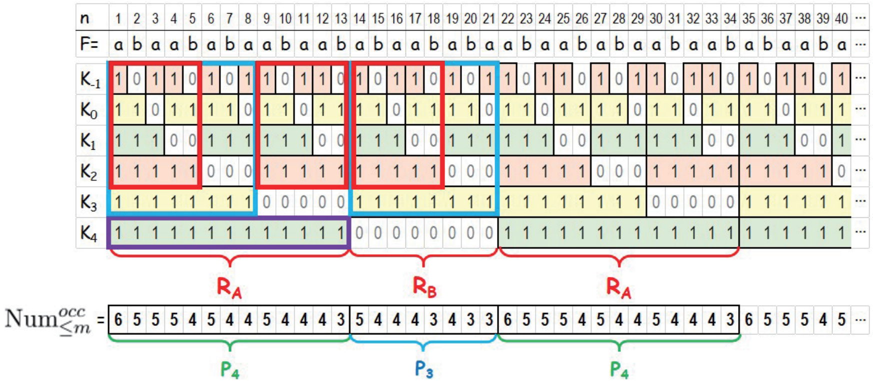

FIGURE 4. The factor spectrum tiling in sequence $\mathbb{M}_m^{occ}$, and the column sum of $\mathbb{M}_m^{occ}$ for $m = 4$.

Theorem 4.4. *For $m \geq -1$, let $Z(n) = d_M d_{M-1} \dots d_{m+1} d_m \dots d_1 d_0$. Let $N = (d_M d_{M-1} \dots d_{m+1})_F$. Then the number of palindromes whose first letter occurrence in $\mathbb{F}[1, n]$ with kernel K_i $(-1 \leq i \leq m)$ is*

$$
\begin{aligned}
&\Sigma \mathrm{Num}_{\leq m}^{occ}[1, n] \\
&= N\left\{\tfrac{m+2}{5} f_{m+2} + \tfrac{m-1}{5} f_m\right\} + \lfloor (N+1)\phi \rfloor \left\{\tfrac{2m+4}{5} f_{m+1} - \tfrac{m-2}{5} f_m\right\} \\
&\quad + \Sigma\Lambda[1, (d_m d_{m-1} \dots d_0)_F],
\end{aligned} \tag{4.1}
$$

where $\Lambda = d_{m+1} R_B + (1 - d_{m+1}) R_A$.

5. The sequence Num_m

For any $m \geq -1$, now we consider sequence Num_m that

$$
\begin{aligned}
\mathrm{Num}_m[L, n] = \#\{(\omega, p) \mid\ & \mathrm{Ker}(\omega) = K_m, \\
& \mathrm{Occ}(\omega, p) \geq n,\ \mathrm{Pos}(\omega, p) \leq n + L - 1\},
\end{aligned} \tag{5.1}
$$

where $n \geq 1$ and $L \in \{f_m .. f_{m+3} - 2\}$. By an analogous argument, the sequence Num_m is a Fibonacci word. The expressions of R_A, R_B and R_0 is given in Figure 5. Specifically, in order to determine the three expressions, we only need to given the triangles with digits 0's.

6. Main result and an example

6.1. Main Result.

Let $L = |\mathbb{F}[n_1, n_2]| = n_2 - n_1 + 1$ and $m = \min\{m \mid f_m \leq L \leq f_{m+3} - 2\}$. By Equation (1.1), the number of repeated palindromes in $\mathbb{F}[n_1, n_2]$ is

$$
\begin{aligned}
&\mathrm{Num}(n_1, n_2) \\
&= \Sigma \mathrm{Num}_{\leq m-1}^{pos}[1, n_2] - \Sigma \mathrm{Num}_{\leq m-1}^{occ}[1, n_1 - 1] + \sum_{h=m}^{m+2} \mathrm{Num}_h[L, n_1].
\end{aligned} \tag{6.1}
$$

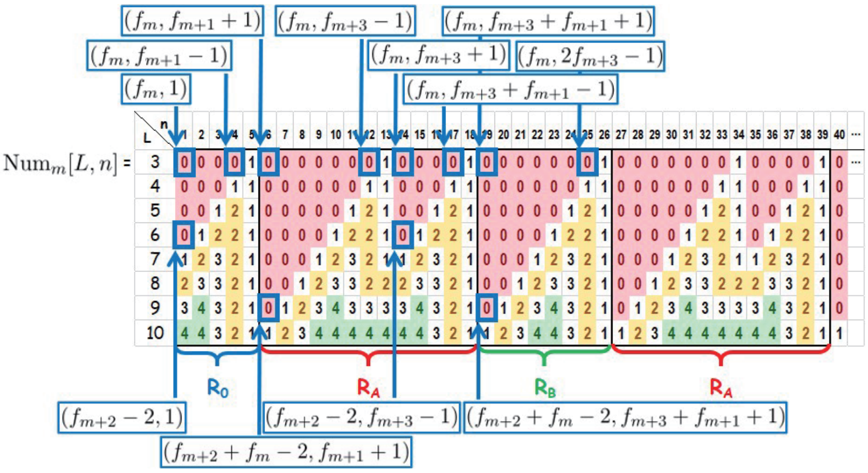

FIGURE 5. The sequence Num_m for $m = 2$, and the expressions of R_A, R_B and R_0 for $m \geq -1$. Here the tuple (a, b) means $L = a$ and $n = b$.

Let

$$\begin{cases} Z(n_1-1) = e_{M'}e_{M'-1}...e_1e_0, \;\; N_1 = (e_{M'}e_{M'-1}...e_m)_F; \\ P'_{-1} = [1], P'_0 = [2,1], P'_m = [P'_{m-1}, P'_{m-2}] + \mathrm{One}(f_{m+1}), m \geq 1; \\ \Lambda_1 = e_m P'_{m-2} + (1-e_m)P'_{m-1}; \\ Z(n_2 - f_{m+1} + 2) = d_M d_{M-1}...d_1 d_0, \;\; N_2 = (d_M d_{M-1}...d_m)_F; \\ P_{-1} = [1], P_0 = [1,2], P_m = [P_{m-2}, P_{m-1}] + \mathrm{One}(f_{m+1}), m \geq 1; \\ \Lambda_2 = d_m P_{m-2} + (1-d_m)P_{m-1}. \end{cases} \tag{6.2}$$

By Theorems 3.6 and 4.4, we get further simplification that

$$\begin{aligned} &\Sigma\mathrm{Num}^{pos}_{\leq m-1}[1, n_2] - \Sigma\mathrm{Num}^{occ}_{\leq m-1}[1, n_1 - 1] \\ &= \left\{\tfrac{m-2}{5} f_{m+3} + \tfrac{m}{5} f_{m+1} + 2\right\} + N_2 \left\{\tfrac{m+1}{5} f_{m+1} + \tfrac{m-2}{5} f_{m-1}\right\} \\ &\quad + \lfloor (N_2+1)\phi \rfloor \left\{\tfrac{2m+2}{5} f_m - \tfrac{m-3}{5} f_{m-1}\right\} + \Sigma\Lambda_2[1, (d_{m-1}d_{m-2}...d_0)_F] \\ &\quad - N_1 \left\{\tfrac{m+1}{5} f_{m+1} + \tfrac{m-2}{5} f_{m-1}\right\} - \lfloor (N_1+1)\phi \rfloor \left\{\tfrac{2m+2}{5} f_m - \tfrac{m-3}{5} f_{m-1}\right\} \\ &\quad - \Sigma\Lambda_1[1, (e_{m-1}e_{m-2}...e_0)_F] \\ &= \left\{\tfrac{m-2}{5} f_{m+3} + \tfrac{m}{5} f_{m+1} + 2\right\} + (N_2 - N_1) \left\{\tfrac{m+1}{5} f_{m+1} + \tfrac{m-2}{5} f_{m-1}\right\} \\ &\quad + (\lfloor (N_2+1)\phi \rfloor - \lfloor (N_1+1)\phi \rfloor) \left\{\tfrac{2m+2}{5} f_m - \tfrac{m-3}{5} f_{m-1}\right\} \\ &\quad + \Sigma\Lambda_2[1, (d_{m-1}d_{m-2}...d_0)_F] - \Sigma\Lambda_1[1, (e_{m-1}e_{m-2}...e_0)_F]. \end{aligned} \tag{6.3}$$

Notice that, there are two cases in Theorem 3.6. But by the definitions of L and m, when we consider $\mathrm{Num}(n_1, n_2)$, $n_2 \geq L$ must be larger than $f_{m+1} - 2$. That means we don't need to consider the first case in Theorem 3.6, when we calculate $\Sigma\mathrm{Num}^{pos}_{\leq m-1}[1, n_2]$.

Theorem 6.1. *Let $L = n_2 - n_1 + 1$ and $m = \min\{m \mid f_m \leq L \leq f_{m+3} - 2\}$, the number of repeated palindromes in $\mathbb{F}[n_1, n_2]$ is*

$$\begin{aligned} &\mathrm{Num}(n_1, n_2) \\ = &\left\{ \tfrac{m-2}{5} f_{m+3} + \tfrac{m}{5} f_{m+1} + 2 \right\} + (N_2 - N_1) \left\{ \tfrac{m+1}{5} f_{m+1} + \tfrac{m-2}{5} f_{m-1} \right\} \\ &+ (\lfloor (N_2+1)\phi \rfloor - \lfloor (N_1+1)\phi \rfloor) \left\{ \tfrac{2m+2}{5} f_m - \tfrac{m-3}{5} f_{m-1} \right\} \\ &+ \Sigma\Lambda_2[1, (d_{m-1}d_{m-2}...d_0)_F] - \Sigma\Lambda_1[1, (e_{m-1}e_{m-2}...e_0)_F] \\ &+ \sum_{h=m}^{m+2} \mathrm{Num}_h[L, n_1]. \end{aligned} \tag{6.4}$$

The expressions of N_1, N_2, Λ_1, Λ_2, $(d_{m-1}d_{m-2}...d_0)_F$ and $(e_{m-1}e_{m-2}...e_0)_F$ are given in Equation (6.2). The values of $\sum_{h=m}^{m+2} \mathrm{Num}_h[L, n_1]$ are given in Figure 5.

6.2. **An Example.**

According to the different kernel $\mathrm{Ker}(\omega)$, we can divide all repeated palindromes in $\mathbb{F}[17, 25] = ababaabaa$ into five parts as below, where $\omega(n)$ means there are n's palindrome ω occurring in $\mathbb{F}[17, 25]$.

$$\begin{cases} \{\omega \mid \omega \in \mathcal{P}_F, \mathrm{Ker}(\omega) = K_{-1} = a\} = \{a(6)\}; \\ \{\omega \mid \omega \in \mathcal{P}_F, \mathrm{Ker}(\omega) = K_0 = b\} = \{b(3), aba(3)\}; \\ \{\omega \mid \omega \in \mathcal{P}_F, \mathrm{Ker}(\omega) = K_1 = aa\} = \{aa(2), baab(1), abaaba(1)\}; \\ \{\omega \mid \omega \in \mathcal{P}_F, \mathrm{Ker}(\omega) = K_2 = bab\} = \{bab(1), ababa(1)\}; \\ \{\omega \mid \omega \in \mathcal{P}_F, \mathrm{Ker}(\omega) = K_3 = aabaa\} = \{aabaa(1)\}; \\ \{\omega \mid \omega \in \mathcal{P}_F, \mathrm{Ker}(\omega) = K_m, m \geq 4\} = \emptyset \text{ (empty set)}. \end{cases} \tag{6.5}$$

Thus the number of repeated palindromes in $\mathbb{F}[n_1, n_2]$ is 19.

By Theorem 6.1, $L = |\mathbb{F}[17, 25]| = 9$ and $m = 2$. By Equation (6.2)

$$\begin{cases} 17 - 1 = 16; \\ 16 = f_5 + f_2 \Rightarrow Z(17) = 100100, N_1 = (100)_F = 3; \\ P'_{-1} = [1], P'_0 = [2, 1], P'_1 = [P'_0, P'_{-1}] + [1, 1, 1] = [3, 2, 2]; \\ e_m = 1 \Rightarrow \Lambda_1 = P'_0 = [2, 1], \Lambda_1[1, (00)_F] = \varepsilon. \\ 25 - f_3 + 2 = 22; \\ 22 = f_6 + f_0 \Rightarrow Z(22) = 1000000, N_2 = (1000)_F = 5; \\ P_{-1} = [1], P_0 = [1, 2], P_1 = [P_{-1}, P_0] + [1, 1, 1] = [2, 2, 3]; \\ d_m = 0 \Rightarrow \Lambda_2 = P_1 = [2, 2, 3], \Lambda_2[1, (01)_F] = [2]. \end{cases} \tag{6.6}$$

By Figure 5, $\mathrm{Num}_2[9, 17] = 2$, $\mathrm{Num}_3[9, 17] = 1$, $\mathrm{Num}_4[9, 17] = 0$.

Thus by Theorem 6.1 the number of repeated palindromes in $\mathbb{F}[n_1, n_2]$ is

$$\begin{aligned}
&\mathrm{Num}(n_1, n_2)\\
=&\left\{\tfrac{m-2}{5}f_{m+3} + \tfrac{m}{5}f_{m+1} + 2\right\} + (N_2 - N_1)\left\{\tfrac{m+1}{5}f_{m+1} + \tfrac{m-2}{5}f_{m-1}\right\}\\
&+ (\lfloor (N_2+1)\phi\rfloor - \lfloor (N_1+1)\phi\rfloor)\left\{\tfrac{2m+2}{5}f_m - \tfrac{m-3}{5}f_{m-1}\right\}\\
&+ \Sigma\Lambda_2[1, (d_{m-1}d_{m-2}...d_0)_F] - \Sigma\Lambda_1[1, (e_{m-1}e_{m-2}...e_0)_F]\\
&+ \sum\nolimits_{h=m}^{m+2} \mathrm{Num}_h[L, n_1]\\
=&\left\{\tfrac{2}{5}f_3 + 2\right\} + (5-3)\left\{\tfrac{3}{5}f_3\right\}\\
&+ (\lfloor (5+1)\phi\rfloor - \lfloor (3+1)\phi\rfloor)\left\{\tfrac{6}{5}f_2 + \tfrac{1}{5}f_1\right\}\\
&+ \Sigma[2] - \Sigma\varepsilon + 2 + 1 + 0\\
=&19.
\end{aligned} \tag{6.7}$$

This value is equal to the conclusion in Equation (6.5).

Acknowledgements. Yuke Huang is supported by National Key R&D Program of China (Grant No. 2018YFC2001400).

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Manuscript received October 12 2022
revised November 2 2022

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