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A REVIEW ON SEVERAL QUESTIONS RELATED TO MEASURE OF NONCOMPACTNESS

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Dedicated to Professor David Preiss on the occasion of his 75th birthday.

Abstract. This is a review paper, which gives an overview on the recent research progress of several basic questions related to measure of noncompactness (MNC) in Banach spaces, including representation and construction problems of MNC, existence problem of inequivalent regular MNCs, countable determination problem of the Kuratowski MNC, independence of the generalized Cantor intersection property assumption in definition of non-full MNC, some basic integral inequalities related to solvability of a Cauchy problem in Banach spaces and the problem that whether every pre-MNC controls an MNC.

1. Introduction

The study of measure of noncompactness (MNC) has continued for over 90 years. It has been shown that the theory of MNC was used in a wide variety of topics in nonlinear analysis (See, for instance , Akhmerov et al. [5], Appell [6], Ayerbe Toledano et al. [7], Banaś and Goebel [8, 10], Djebali et al. [28] and Meskhi [45]). Roughly speaking, an MNC μ is a nonnegative function defined on the family $\mathscr{B}(M)$ consisting of all nonempty bounded subsets of a (complete) metric space M and satisfies some specific properties such as non-decreasing monotonicity in the order of the set inclusion, noncompactness which states that $\mu(B) = 0$ implies that B is relatively compact in M, and the generalized Cantor intersection property (see Definition 1.2(11)).

The first MNC α was introduced and studied by K. Kuratowski [39] in 1930, which is now called the Kuratowski MNC:

$$\alpha(B) = \inf\{\ d > 0 : B \subset \cup_{j\in F} E_j,\ F^\sharp < \infty,\ d(E_j) \le d\ \},\ B \in \mathscr{B}(M), \tag{1.1}$$

where $F^\sharp$ is the cardinality of $F \subset \mathbb{N}$, and $d(E_j)$ denotes the diameter of $E_j \subset M$. This MNC has been widely used.

The earliest successful application of the Kuratowski MNC was applied in fixed point theory. It is well-known that in fixed point theory, one of the most important theorems is the Brouwer theorem, which states that every continuous self-mapping

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defined a nonempty bounded closed convex set of $\mathbb{R}^n$ has a fixed point. A remarkable generalization of the Brouwer theorem is called the Schauder theorem, which says that the Brouwer theorem holds again if we substitute a compact convex set of a Banach space for the bounded closed convex set of $\mathbb{R}^n$. A most general setting of the Brouwer theorem is Tychonoff's theorem [56]: A more general class of topological linear spaces in which the Schauder theorem holds is the class of locally convex spaces. In 1955, V. Klee [37] showed that if a nonempty bounded closed convex set C of a Banach space X satisfies that every continuous self-mapping has a fixed point, then C is compact. (In 1985, Lin Pee-Kee and Y. Sternfeld [41] further proved that we can substitute the class of Lipschitz mappings for that of continuous mappings in the Klee theorem.) Therefore, compactness plays an essential role in the Schauder theorem. In 1955, using the Kuratowski MNC, G. Darbo [25] extended the Schauder fixed point theorem to noncompact operators named set-contractive operators (see, [53] for a remarkable generalization). Since then, the study of MNCs and of their applications has become an active research area, and various MNCs have appeared. Among many other MNCs, the Hausdorff MNC β is another commonly used MNC (introduced by I. Gohberg, L.S. Gol'denśshteín and A.S.Markus [33] in 1957). It is defined for $B \in \mathscr{B}(M)$ by

$$\beta(B) = \inf\{r > 0 : B \subset \cup_{x\in F} B(x, r) \text{ for some finite } F \subset M\}, \tag{1.2}$$

where $B(x, r)$ denotes the closed ball centered at x with radius r.

It is easy to observe that if μ is either the Kuratowski MNC α or the Hausdorff MNC β, then it satisfies the following three conditions.

(1) $B \in \mathscr{B}(M)$, $\mu(B) = 0 \iff B$ is relatively compact;
(2) $A, B \in \mathscr{B}(M)$ with $A \supset B \implies \mu(A) \geq \mu(B)$;
(3) $A, B \in \mathscr{B}(M) \implies \mu(A \cup B) = \mu(A) \vee \mu(B)$.

If, in addition, M is a Banach space, then

(4) $B \in \mathscr{B}(M) \implies \mu(\mathrm{co}(B)) = \mu(B)$;
(5) $B \in \mathscr{B}(M) \implies \mu(kB) = |k|\mu(B)$, $\forall$ scalar k;
(6) $A, B \in \mathscr{B}(M) \implies \mu(A + B) \leq \mu(A) + \mu(B)$.

Since every metric space is isometric to a subset of a Banach space (see, for instance, [14, Lemma 1.1]), without loss of generality, we can assume that the metric space M in question is a Banach space in the sequel.

Definition 1.1 (MNC with positive homogeneity)**.** Let X be a Banach space, and $\mu : \mathscr{B}(X) \to \mathbb{R}^+$ be a nonnegative-valued function.

i) [11] μ is said to be a regular MNC on the space X provided it satisfies the six properties (1)-(6).

ii) [44] μ is called a homogeneous MNC on X provided it satisfies the five properties (1), (2) and (4)-(6).

iii) [2] We say that μ is a sublinear MNC on X provided it satisfies the five properties (1), (2), (4),(6) and

(7) $B \in \mathscr{B}(X) \implies \mu(kB) = k\mu(B)$, $\forall\, k \geq 0$.

Thus, a regular MNC is a homogeneous MNC, and a homogeneous MNC is a sublinear one.

It is easy to check that both the Kuratowski measure α and the Hausdorff measure β are regular measures.

A more general definition of MNC is the following one, which we call "convex MNC".

Definition 1.2 (Convex MNC)**.** Let X be a Banach space, and $\mu : \mathscr{B}(X) \to \mathbb{R}^+$ be a nonnegative-valued function.

μ is said to be a convex MNC on X provided it satisfies the four properties (1), (2), (4) and

(8) $A, B \in \mathscr{B}(X) \Longrightarrow \mu(\lambda A + (1-\lambda)B) \leq \lambda\mu(A) + (1-\lambda)\mu(B),\ \forall\, \lambda \in [0,1]$.

Every convex MNC μ satisfies the three properties (9-11) below (see, for instance, [21]).

(9) $\mu(B \cup \{x_0\}) = \mu(B)$, $x_0 \in X$, $B \in \mathscr{B}(X)$.

(10) $\mu(\overline{B}) = \mu(B)$, $B \in \mathscr{B}(X)$.

(11) (Generalized Cantor intersection) For every sequence $(B_n) \subset \mathscr{B}(X)$ of nonempty bounded sets of X with $\lim_n \mu(B_n) = 0$, we have

$$B_{n+1} \subset B_n, \forall\, n \in \mathbb{N},\ \lim_n \mu(B_n) = 0 \Longrightarrow \bigcap_n \overline{B}_n \neq \emptyset. \tag{1.3}$$

Remark 1.3. From the definitions of various kinds of MNCs above, we have already seen that for every MNC μ defined on a Banach space X, the following conditions are always satisfied.

$$B \in \mathscr{B}(X) \Longrightarrow \mu(\mathrm{co}(B)) = \mu(B),$$

and

$$\mu(\overline{B}) = \mu(B).$$

Thus, we often blur the distinction between $\mathscr{B}(X)$ and $\mathscr{C}(X)$, the cone of all nonempty bounded closed convex sets of X; and we also use $\mathscr{K}(X)$ to denote both the cone of all nonempty relatively compact sets and the cone of all nonempty compact convex sets of X.

1.1. **Order preserving embedding from $\mathscr{C}(X)$ to $C(K)$.** For a Banach space X, let $\mathscr{C}(X)$ be the cone of all nonempty bounded closed convex sets of X endowed with the following set addition and scalar multiplication:

$$A \oplus B = \overline{A + B} \equiv \overline{\{a + b : a \in A, b \in B\}},\ kA = \{ka : a \in A\} \tag{1.4}$$

(where $A, B \in \mathscr{C}(X)$ and k is a scalar), and endowed with the Hausdorff metric d_H defined for $A, B \in \mathscr{C}(X)$ by

$$d_H(A, B) = \inf\{r > 0 : A \subset B + rB_X,\ B \subset A + rB_X\}, \tag{1.5}$$

where B_X denotes the closed unit ball of X.

In Section 2, we will introduce a triple order preserving theorem from $\mathscr{C}(X)$ to $C(K)$ established in [23], where $C(K)$ is a Banach function space for some compact Hausdorff space K, which will play an important role in the sequel.

1.2. **On representation of MNCs.** There are a large number of publications related to constructions of various types of MNCs, and to representation of the Hausdorff MNC β on classical Banach spaces such as the space of continuous functions $C(K)$, the sequence spaces ℓ_p $(1 \leq p \leq \infty)$ and c_0, the integrable function spaces L_p $(1 \leq p \leq \infty)$. See, for example, [5–8, 10, 28] and [45]. However, general construction and representation of an arbitrary MNC on an abstract Banach space had never been considered before 2018. The main difficulty of such representation is how to turn a function μ defined on the super space $\mathscr{B}(X)$ into a function defined on a usual metric space in order preserving. We will introduce some representation theorems and construction theorems of convex MNCs in Section 3 and of sublinear MNCs in Section 4 which are presented in [21] and [23] respectively.

1.3. **On inequivalent MNCs.**

Definition 1.4. Let X be a Banach space, and $\mu,\ \nu : \mathscr{B}(X) \to \mathbb{R}^+$ be two MNCs on X. They are called equivalent provided there exist $a, b > 0$ such that

$$a\mu(B) \le \nu(B) \le b\mu(B),\ \forall\ B \in \mathscr{B}(X). \tag{1.6}$$

The following natural question was first asked by K. Goebel in 1978 by noticing all known examples before then are equivalent (see, [11]).

Problem 1.5. Do there exist inequivalent regular measures of noncompactness in every infinite dimensional Banach space?

In 1992, J. Banaś and A. Martinón [11] proved that for every infinite dimensional Banach space X, there are always inequivalent regular measures of noncompactness on $\ell_p(X)$ for $1 \le p \le \infty$, although their result was not widely known. However, Problem 1.5 remained open. (See, for example, [13, Remark 3.6].)

In 2011 ([43, 44]), J. Mallet-Paret and R. Nussbaum showed that for many of the classical Banach spaces which arise in analysis there always exist inequivalent measures of noncompactness. They further proposed the following "fundamental question":

Problem 1.6. For what infinite dimensional Banach spaces X do there exist inequivalent homogenous measures of noncompactness β_1 and β_2 on X?

In Section 5, we will see that the answer to this question is affirmative [1]: Every infinite dimensional Banach space admits a regular MNC not equivalent to the Hausdorff measure. This gives Problem 1.5 (hence, Problem 1.6) an affirmative answer.

1.4. **On countable determination of the Kuratowski MNC.** In the theory of MNC, the following "countable determination question" is a long-standing problem (see, for example, [5, §1.4.3, p.19]).

Problem 1.7. Given an MNC μ on a complete metric space M, is the following statement true?

$$\forall B \in \mathscr{B}(M),\ \exists \text{ countable subset } B_0 \subset B \text{ so that } \mu(B_0) = \mu(B). \tag{1.7}$$

This question is doubtlessly fundamental and important. For example, if the metric space M in question is a Banach space X consisting of measurable functions (say, an L_p-space), and if $B = \{f_n\} \in \mathscr{B}(X)$ is a countable subset of M, then all the functions $\sup B \equiv \sup_n f_n$, $\inf B \equiv \sup_n f_n$, $\limsup_n f_n$ and $\liminf_n f_n$ are in X. Besides, it is usually easier to deal with a sequence than to handle an uncountable subset. See, for example, [3,16]. B.N. Sadovskii ([51,53]) first studied the countable determination question, and gave a negative answer to it for the Hausdorff measure β. Indeed, he defined a sequential measure $\tilde{\beta}$ with respect to β by

$$\tilde{\beta}(B) = \sup\{\beta(C) : C \text{ is a countable subset of } B\},\ B \in \mathscr{B}(M), \tag{1.8}$$

and showed the following (sharp) inequalities.

$$\frac{1}{2}\beta(B) \leq \tilde{\beta}(B) \leq \beta(B),\ B \in \mathscr{B}(M). \tag{1.9}$$

But there is a bounded subset B in a Banach space X such that $\tilde{\beta}(B) = \frac{1}{2} < 1 = \beta(B)$ (see, also, [5, §.1.4]). However, the following question remained open.

Problem 1.8. For the Kuratowski MNC α on a metric space M, is the following assertion true?

$$\forall B \in \mathscr{B}(M),\ \exists \text{ countable subset } B_0 \subset B \text{ so that } \alpha(B_0) = \alpha(B). \tag{1.10}$$

An affirmative answer to this question is presented in [20]. In Section 6, we will give a sketch proof of the result.

1.5. **On an integral inequality related to a Cauchy problem.** Many mathematicians have made great efforts to expect the following type of integral inequalities with various assumptions:

$$\mu\Big\{\int_0^a x_n(\omega)d\omega : n \in \mathbb{N}\Big\} \leq \int_0^a \mu\Big\{x_n(\omega) : n \in \mathbb{N}\Big\}d\omega, \tag{1.11}$$

where $\{x_n\}$ is a sequence of continuous X-valued functions in $C([0,a],X)$ endowed with the sup-norm $\|\cdot\|_{C([0,a],X)}$ defined for $x \in C([0,a],X)$ by $\|x\| = \|x\|_{C([0,a],X)} = \sup_{t\in[0,a]} \|x(t)\|$, and μ is the Hausdorff MNC, or, the Kuratowski MNC. Such type of integral inequalities arise from the following initial value problem in Banach spaces.

$$\begin{cases} x'(t) = f(t,x), & a \geq t > 0; \\ x(0) = x_0 \end{cases} \tag{1.12}$$

Clearly, the problem 1.12 has a solution $x \in C([0,a],X)$ if and only if x is a fixed point of the following Picard-Lindelöf operator A defined for $x \in C([0,a],X)$ by

$$Ax(t) = x_0 + \int_0^t f(s,x(s))ds. \tag{1.13}$$

The classical Peano's theorem states if $f : \mathbb{R}^+ \oplus \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, then the problem 1.12 has a local solution. For an infinite dimensional Banach space X, under the condition due to Picard and Lindeloff that f is Lipschitz, i.e., $\|f(t,x) - f(t,y)\| \leq L\|x - y\|$ holds for all $x, y \in X$ (where $L > 0$ is a constant), then the problem 1.12 has a unique local solution, and it is extendable to a global solution. But there are a series of counterexamples showing that in an infinite dimensional Banach space X, the Picard-Lindeloff condition (i.e., the Lipschitz assumption on f) can not be dropped. The first one is in Dieudonné [27], where he showed that for $X = c_0$ (the Banach space of null sequences) there is a continuous function $f : \mathbb{R} \oplus c_0 \to c_0$ so that the problem 1.12 admits no local solution. Since then, many other counterexamples in various infinite dimensional Banach spaces has been constructed. See, for example, [40], [17], [57], [29]. Cellina [18] showed that such a counterexample exists in every nonreflexive Banach space. Especially, Godunov [31] gave a complete negative answer to the problem 1.12 for every infinite dimensional Banach space X.

Theorem 1.9 (Godunov)**.** *Let X be an infinite dimensional Banach space. Then there exists a continuous function $f : \mathbb{R} \oplus X \to X$ and an initial value $x(t_0) = x_0$ such that the problem 1.12 has no local solution.*

Godunov [30] further constructed a continuous function $f : \mathbb{R} \oplus H \to H$ in a Hilbert space H, such that the problem 1.12 has no solution on any open interval of $\mathbb{R}$. For more information, we refer the reader to Hájek and Vivi [34].

Among many extra conditions which are sufficient for the solvability of the problem 1.12, one of the most important types is under hypotheses in terms of MNCs. The procedure can be roughly described as follows. Assume that the function $f : \mathbb{R}^+ \times X \to X$ in the problem 1.12 is continuous and condensing with respect some MNC μ defined on X, that is, for each nonempty bounded subset $B \subset C([0, a], X)$, we have for all $t \in [0, a]$, $\mu\big(f(t, B(t)\big) < \mu\big(B(t)\big)$ whenever $\mu\big(B(t)\big) > 0$, where $B(t) = \{x(t) : x \in B\} \subset X$. Then we claim that Picard-Lindelöf operator A defined in (1.13) is also condensing. Once this claim is met, there is not much left to do to prove the existence of solutions for (1.12). And it boils down to proving the inequality (1.11).

Ambrosetti [4, (1967)] is the first one to use the Kuratowski MNC α to the solvability. He proved the existence theorem under the assumptions that f is uniformly continuous and Lipschitz with respect to the Kuratowski MNC α. Some further contributions have been made by a number of mathematicians. See, for example, Sadovskii [52, (1968)], Goebel and Rzymowdski [32, (1971)], Rzymowski [49, 50, (1971)], Szufla [54, (1968)] and [55, (1971)], Cellina [19, (1972)], Banaś and Goebel [10, (1980)], Mönch [46, (1980)], Mönch and G.-F. von Harten [47, (1982)], Heinz [35, (1983)], Kunze and Schlüchtermann [38, (1998)]. Nevertheless, it still comes as a surprise that the estimate (1.11) has been proved only under some very restrictive assumptions in the existing literatures. While a number of counterexamples constructed by Heinz [35] show that it is quite complicated and poses significant difficulties to present appropriate hypotheses to guarantee (1.11). In 1970, Goebel and Rzymowdski [32] first showed that (1.11) holds for the Hausdorff MNC β, with the assumptions that $\{x_n\} \subset C([0, a], X)$ is bounded and equi-continuous. In 1980, Banaś and Goebel [10] further proved (1.11) holds again under the same assumptions but one can substitute a sublinear MNC μ for β. Mönch [46, 1980], Mönch and G.-F. von Harten [47, 1982] proved (1.11) with the assumptions a) $\mu = \beta$, b) X is a weakly compactly generated space (in particular, a separable Banach space) and c) $\{x_n\} \subset C([0, a]; X)$ and there is $\psi \in L_1([0, a])$ such that $\sup_n \|x_n(t)\| \le \psi(t)$ a.e. In 1998, Kunze and Schlüchtermann [38] showed that (1.11) holds for a Grothendieck measure μ assuming that X is strongly generated by a Grothendieck class. But for an arbitrary Banach space, one has to insert the factor 2 in the right-hand side of (1.14), i. e.,

$$\alpha\Big\{ \int_0^t x_n(s)ds : n \ge 1\Big\} \le 2 \int_0^t \alpha\Big\{x_n(s) : n \ge 1\Big\}ds.$$

See, for instance, [16].

It is shown in [21] that for every nonempty subset $G \subset L_1([0,a],X)$ of integrable X-valued functions with $\psi(t) \equiv \sup_{g\in G} \|g(t)\|$ integrable on $[0,a]$ such that the mapping $\mathbb{J}_G : [0,a] \to C_b(\Omega)$ defined for $t \in [0,a]$ by

$$\mathbb{J}_G(t)(\omega) = \sup_{g\in G}\langle \omega, g(t)\rangle \equiv \sigma_{G(t)}(\omega),\ \omega \in \Omega \equiv B_{X^*}, \tag{1.14}$$

has separable range in $C_b(\Omega)$, then for every convex MNC (or convex measure of non-weak compactness, convex measure of non-superweak compactness, convex measure of non-Radon-Nikodým property etc.) μ defined on X, we have

$$\mu\Big\{\int_0^\tau G(s)ds\Big\} \le \frac{1}{\tau}\int_0^\tau \mu\Big\{\tau G(s)\Big\}ds,\ \forall 0 < \tau \le a. \tag{1.15}$$

In particular, if μ is a sublinear MNC, or, $\tau \le 1$, then

$$\mu\Big\{\int_0^\tau G(s)ds\Big\} \le \int_0^\tau \mu\Big\{G(s)\Big\}ds. \tag{1.16}$$

It is also shown in [21] that the mapping $\mathbb{J}_G(\cdot) : [0,a] \to C_b(\Omega)$ is always weakly measurable, and if G satisfies one of the following conditions

i) $G \subseteq C(I,X)$ is a nonempty equi-continuous subset;

ii) $G \subseteq R(I,X)$ is a nonempty separable equi-regulated subset;

iii) $G \subseteq L_1(I,X)$ is a nonempty uniformly measurable set,

then it is strongly measurable. Consequently, (1.16) holds.

In Section 7, we will introduce a sketch proof of the result mentioned above.

1.6. **On fullness of MNCs.** The notion of MNC has been generalized in various ways. Assume that μ is a nonnegative real-valued function defined on $\mathscr{B}(X)$. Roughly speaking, the notion of MNC has been generalized in two directions. One is to claim that $\ker(\mu) \supset \mathscr{K}(X)$. For example, if we choose $\ker(\mu) = \mathscr{W}(X)$, the family of all nonempty weakly relatively compact subsets of X, then the function μ is called a measure of non-weak compactness [15]. The other way is to claim $\emptyset \neq \ker(\mu) \subset \mathscr{K}(X)$ [10]. For example, let $\mu(B) = \mathrm{diam}(B)$, the diameter of B. Then $\ker(\mu) =$ all singletons of X. The notion of convex non-full MNC was introduced by Banaś and Goebel in 1980 [10] but they call it again MNC. A nonnegative-valued function $\mu : \mathscr{B}(X) \to \mathbb{R}^+$ is said to be a convex non-full MNC provided it satisfies that (1') $\emptyset \neq \ker\mu \subset \mathscr{K}(X)$, and the properties (2), (4), (8) in Definition 1.2, and, in addition, the generalized Cantor intersection property (11).

We should mention that the generalized Cantor intersection property is one of the most important properties in applications of MNC to fixed point theory. We have already known that every convex MNC admits the property naturally (See, for instance, [21, Theorem 3.4]). But we do not know whether it is independent of other conditions in the definition of non-full MNC. If the answer to this question is affirmative, then the following question is arising naturally: What conditions can guarantee a convex non-full MNC to dominate a convex MNC?

In Section 8, we will introduce some results including a positive answer to the former question in a recent paper [9].

2. Order preserving embedding of $\mathscr{C}(X)$

For a Banach space X, let $\mathscr{C}(X)$ (resp. $\mathscr{B}(X)$, $\mathscr{K}(X)$) be the collection of all non-empty bounded closed convex (resp. nonempty bounded, nonempty convex compact) sets of X endowed with the Hausdorff metric d_H, which is defined for $A, B \in 2^X$ (the power set of X) by

$$\begin{aligned} d_H(A,B) &= \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\} \\ &= \inf\{r>0 : A\subset B+rB_X,\ B\subset A+rB_X\}. \end{aligned} \tag{2.1}$$

If there is no confusion, we simply denote them by $\mathscr{C}$, $\mathscr{B}$ and $\mathscr{K}$, respectively.

In this section, we introduce an order preserving embedding of $\mathscr{C}(X)$ into $C(K)$ for some compact Hausdorff space K presented in [23].

We use Ω to denote the closed unit ball B_{X^*} of the dual X^*, and $C_b(\Omega)$, the Banach space of all real-valued bounded norm-continuous functions on Ω endowed with the sup-norm. Let $J : \mathscr{C}(X) \to C_b(\Omega)$ be defined for $C \in \mathscr{C}(X)$ by

$$J(C)(\omega) = \sigma_C(\omega) \equiv \sup_{c\in C}\langle \omega, c\rangle, \quad \omega\in\Omega. \tag{2.2}$$

The Banach lattices X and Y are said to be order isometric if there exists a linear isometry T from X onto Y which is also an order isomorphism.

With the symbols as above, we summarise the triple order preserving embedding procedure (presented in [23]) as follows.

Theorem 2.1 (Triple order preserving embedding theorem)**.** (1) *[23, Theorem 2.3 i)] Given a Banach space X, the collection $\mathscr{C}$ consisting of all nonempty closed bounded convex sets of X endowed with the set addition $A \oplus B = \overline{A+B}$, the usual scalar multiplication of sets $\lambda C = \{\lambda c : c \in C\}$, and with the norm $|||\cdot|||$ defined by $|||C||| = \sup_{c\in C}\|c\|$ is a complete normed convex cone.*

(2) *[23, Theorem 2.3 ii)] If we endow with the Hausdorff metric d_H on $\mathscr{C}$, order $\mathscr{C}$ by set inclusion, and order $C_b(\Omega)$ by the usual order of real-valued functions, then the mapping $J : \mathscr{C} \to C_b(\Omega)$ defined for $C \in \mathscr{C}$ and $\omega \in \Omega$ by*

$$J(C)(\omega) = \sigma_C(\omega) = \sup_{c\in C}\langle\omega, c\rangle,$$

is a positively linear order isometry, where $C_b(\Omega)$ denotes the Banach space of all bounded norm continuous functions on Ω endowed with the sup-norm.

(3) *[23, Theorem 3.2 i)] Both $E_{\mathscr{C}} = \overline{J\mathscr{C} - J\mathscr{C}}$ and $E_{\mathscr{K}} = \overline{J\mathscr{K} - J\mathscr{K}}$ are sublattices of $C_b(\Omega)$ and $E_{\mathscr{K}}$ is a lattice ideal of $E_{\mathscr{C}}$, where $\mathscr{K} \subset \mathscr{C}$ is the subcone of $\mathscr{C}$ consisting of all nonempty compact convex subsets of X;*

(4) *[23, Theorem 3.2 ii)] The quotient space $QE_{\mathscr{C}} = E_{\mathfrak{C}}/E_{\mathscr{K}}$ is an abstract M space, hence, order isometric to a sublattice of $C(K)$ for some compact Haudorff space K;*

(5) *[23, Theorem 3.2 iii)] The subcone $TQJ\mathscr{C}$ is contained in the positive cone of $C(K)$, where $T : QE_{\mathscr{C}} \to C(K)$ is an order isometry.*

3. Representation of convex MNCs

In this section, we will introduce a representation theorem of convex MNCs due to [21].

Recall that $\mathscr{B}(X)$ is the cone of all nonempty bounded subsets of a Banach space X, and $\mathscr{C}(X)$ (resp., $\mathscr{K}(X)$) denotes the cone of all nonempty bounded closed (resp., compact) convex subsets of X endowed the operations addition $\oplus$ and scalar multiplication of sets, and endowed with the Hausdorff metric d_H. The Banach spaces $C_b(\Omega)$, $E_{\mathscr{C}}$, $E_{\mathscr{K}}$ and $C(K)$, the mappings J, Q, T and $\mathbb{T} = TQJ$ are the same as those of Theorem 2.1. Let $V = \mathbb{T}\mathscr{C}(X)$. Then V is a closed subcone of the positive cone $C(K)^+$ of $C(K)$.

Now, we state the representation theorem as follows.

Theorem 3.1. *Suppose that X is a Banach space. Then there is a Banach function space $C(K)$ endowed with the sup-norm for some compact Hausdorff space K such that for every convex MNC μ on X, there is a function $\digamma$ on the cone $V \equiv \mathbb{T}(\mathscr{B}(X)) \subset C(K)^+$ satisfying*

i) *$\mu(B) = \digamma(\mathbb{T}B)$, for all $B \in \mathscr{B}(X)$;*
ii) *$\digamma$ is nonnegative-valued convex and monotone on V;*
iii) *$\digamma$ is bounded by $b_r = \digamma(r\mathbb{T}B_X)$ on $V \bigcap (rB_{C(K)})$, for all $r \geq 0$;*
iv) *$\digamma$ is c_r-Lipschitian on $V \bigcap (rB_{C(K)})$, for all $r \geq 0$, where $c_r = \digamma\big((1 + r)\mathbb{T}B_X\big) = \mu\big((1+r)B_X\big)$;*
v) *In particular, if μ is a sublinear MNC, then we can take $c_r = \mu(B_X)$ in iv).*

Before starting to give a sketch proof of Theorem 3.1, we require a sequence of lemmas.

Lemma 3.2. *Let X be a Banach space, and $f, g \in V \equiv \mathbb{T}\mathscr{C}(X)$. Then $f \leq g$ if and only if there exist $A, B \in \mathscr{C}(X)$ with $f = \mathbb{T}A$, $g = \mathbb{T}B$ such that for all $\varepsilon > 0$ there is $K_\varepsilon \in \mathscr{K}(X)$ satisfying*

$$A \subset B + K_\varepsilon + \varepsilon B_X. \tag{3.1}$$

Corollary 3.3. *Let X be a Banach space, and $f, g \in V \equiv \mathbb{T}\mathscr{C}(X)$. Then $f = g$ if and only if there exist $A, B \in \mathscr{C}(X)$ with $f = \mathbb{T}A$, $g = \mathbb{T}B$ such that for all $\varepsilon > 0$ there is $K_1, K_2 \in \mathscr{K}(X)$ satisfying*

$$B \subset A + K_1 + \varepsilon B_X, \text{ and } A \subset B + K_2 + \varepsilon B_X. \tag{3.2}$$

Lemma 3.4. *Let $f, g \in V$ with $\|f - g\| = r$. Then $|f - g| \leq r\mathbb{T}(B_X)$.*

Recall (Definition 1.2) that for a Banach space X, $\mu : \mathscr{B}(X) \to \mathbb{R}^+$ is said to be a convex MNC provided it satisfies the following four properties.

(P1) [**Noncompactness**] $B \in \mathscr{B}(X)$, $\mu(B) = 0 \Longleftrightarrow B$ is relatively compact;
(P2) [**Monotonicity**] $A, B \in \mathscr{B}(X)$ with $A \supset B \implies \mu(A) \geq \mu(B)$;
(P3) [**Convexification invariance**] $B \in \mathscr{B}(X) \Longrightarrow \mu(\mathrm{co}(B)) = \mu(B)$;
(P4) [**Convexity**] $\mu(\lambda A + (1-\lambda)B) \leq \lambda\mu(A) + (1-\lambda)\mu(B)$, $\forall A, B \in \mathscr{B}(X)$ and $0 \leq \lambda \leq 1$.

Every convex MNC admits the following basic properties.

Lemma 3.5. *Let X be a Banach space, and $\mu : \mathscr{B}(X) \to \mathbb{R}^+$ be a convex MNC. Then*

i) [**Density determination**] $\mu(\overline{B}) = \mu(B)$, $\forall B \in \mathscr{B}(X)$;
ii) [**Translation invariance**] $\mu(B + C) = \mu(B)$, $\forall B \in \mathscr{B}(X), C \in \mathscr{K}(X)$;
iii) [**Negligibility**] $\mu(B \cup C) = \mu(B)$, $\forall\, B \in \mathscr{B}(X), C \in \mathscr{K}(X)$;
iv) [**Generalized Cantor intersection**] $\emptyset \neq B_{n+1} \subset B_n \in \mathscr{B}(X)$, $n \in \mathbb{N}$; $\mu(B_n) \to 0 \Longrightarrow$

$$\bigcap_n \overline{B_n} \neq \emptyset.$$

For $B \in \mathscr{B}(X)$, we will simply denote $\mathbb{T}B$ by f_B in the sequel.

Lemma 3.6. *Let μ be a convex MNC on a Banach space X. Then*

i) $\digamma(f_B) = \mu(B)$, $B \in \mathscr{B}(X)$ *defines a monotone convex function on V;*
ii) *For each $r > 0$, $\digamma$ is bounded by $b_r \equiv \mu(rB_X)$ on $V \cap (rB_{C(K)})$;*
iii) *For each $f \in V$,*

$$p(g) = \lim_{t \to 0^+} \frac{\digamma(f + tg) - \digamma(f)}{t}, \; g \in V \tag{3.3}$$

defines a non-negative sublinear functional p on V satisfying

$$p(g) \leq \digamma(f + g) - \digamma(f), \;\; \forall g \in V. \tag{3.4}$$

Lemma 3.7. *Let μ be a convex MNC on a Banach space X, and $\digamma$ be defined as Lemma 3.6. Then $\digamma$ is continuous on V.*

For a subcone C of a Banach space Z, we say a linear functional $z^* \in Z^*$ is a positive functional on C if it is non-negative valued on C, or, equivalently, $z^*|_{X_C}$ is a positive functional of the subspace $X_C \equiv C - C$ with respect to the "positive" cone C in the usual sense.

Lemma 3.8. *Let $V_0 \subset V$ be a subcone of V with $f_{B_X} \in V_0$, and let $E_{V_0} = V_0 - V_0$. Then for every functional $x^* \in E_{V_0}^*$ which is positive on V_0, we have*

$$\|x^*\|_{E_{V_0}} = \langle x^*, f_{B_X} \rangle. \tag{3.5}$$

Lemma 3.9. *Suppose that $V_0 \subset V$ is a subcone of V, and $u \in V_0$ is in the relative interior* $\mathrm{int}(V_0)$ *of V_0. Let the function $\digamma$ be defined on V as Lemma 3.6. If u is a Gâteaux differentiability point of $\digamma|_{V_0}$ (the restriction of $\digamma$ to V_0). Then its relative Gâteaux derivative $x^* = d_G\digamma|_{V_0}(u)$ is a positive functional on V_0.*

Lemma 3.10. *Suppose that g is a continuous convex function on a closed convex subset D of a Banach space X with* $\mathrm{int}(D) \neq \emptyset$. *Then the following mean-value theorem holds.*

i) $\forall x, y \in \mathrm{int}(D)$, *there exist $\xi \in [x, y] \equiv \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$ and $x_\xi^* \in \partial g(\xi)$ such that*

$$g(y) - g(x) = \langle x_\xi^*, y - x \rangle. \tag{3.6}$$

ii) $\forall x^* \in \partial g(x)$, $y^* \in \partial g(y)$, *we have*

$$\langle y^*, y - x \rangle \geq g(y) - g(x) \geq \langle x^*, y - x \rangle. \tag{3.7}$$

Lemma 3.11. *Suppose that X is a Banach space, μ is a convex MNC on X, and that $V_0 \subset V$ is a closed subcone of V with $f_{B_X} \in V_0$ such that $E_{V_0} = V_0 - V_0$ is a finite dimensional subspace. Let the function $\digamma$ on V be defined as Lemma 3.6. Then*

i) *for each $r > 0$, $\digamma$ is c_r-Lipschitzian on $V_{0,r} \equiv V_0 \cap rB_{C(K)}$, where $c_r = \digamma((1+r)f_{B_X}) = \mu\big((1+r)B_X\big)$;*
ii) *if, in addition, μ is a homogeneous MNC, then $c_r = \digamma(f_{B_X}) = \mu(B_X)$.*

Now, we are ready to prove Theorem 3.1 as follows.

Proof of Theorem 3.1. Given a convex MNC μ on X, let

$$\digamma(\mathbb{T}B) = \mu(B), \ \ \forall B \in \mathscr{B}(X).$$

Then by Lemma 3.6, i), ii) and iii) follow. Next, we will show iv). Given $r \geq 0$, $f, g \in V$ with $\|f\|$, $\|g\| \leq r$, let $V_0 \subset V$ be the subcone generated by $\{f, g, \mathbb{T}(B_X)\}$. By Lemma 3.11 i), $\digamma$ is c_r-Lipschitian on $V_0 \bigcap rB_{C(K)}$. Therefore,

$$\big|\digamma(f) - \digamma(g)\big| \leq c_r\|f - g\|.$$

Consequently, iv) follows.
v) follows from Lemma 3.11 ii). □

4. Construction of sublinear MNCs

In this section, we will introduce some construction and representation theorem related to sublinear MNCs (See, Definition 1.1). All symbols will be the same as in the previous sections.

Theorem 4.1 ([23, Theorem 5.5])**.** *Let X be a Banach space, and $C(K)$ be the function space with respect to X defined in Theorem 2.1. Then for every bounded subset $F \subset C(K)^*$ of positive functionals satisfying that for each $0 \neq u \in TQJ\mathscr{C}$ there exists $\varphi \in F$ so that $\langle\varphi, u\rangle > 0$, the following formula defines a homogenous MNC μ on X:*

$$\mu(B) = \|T[QJ\overline{\text{co}}(B)]\|_F \ \text{ for all } B \in \mathscr{B}, \tag{4.1}$$

where $\|u\|_F = \sup_{\varphi\in F\cup -F}\langle\varphi, u\rangle$ for all $u \in C(K)$.

In particular, the Hausdorff MNC β can be reformulated as follows.

$$\beta(B) = \|TQJ[\overline{\text{co}}B]\|_{C(K)}, \ \text{ for all } B \in \mathscr{B}. \tag{4.2}$$

Theorem 4.2 ([23, Theorem 5.4])**.** *Let X be a Banach space. For every bounded subset $F \subset C(K)^*$ of*

$$\cup_{k\in K}\{\mathbb{R}^+\delta_k\} = \{r\delta_k : r \geq 0, \ k \in K\}$$

satisfying that for each $0 \neq u \in TQJ\mathscr{C}$ there exists $\varphi \in F$ so that $\langle\varphi, u\rangle > 0$, the following formula defines a regular MNC μ on X.

$$\mu(B) = \|T[QJ\overline{\text{co}}(B)]\|_F \ \text{ for all } B \in \mathscr{B}, \tag{4.3}$$

where δ_k is the evaluation functional at $k \in K$, $\|u\|_F = \sup_{\varphi\in F\cup -F}\langle\varphi, u\rangle$ for all $u \in C(K)$.

For a convex function f defined on a convex subset C of X, ∂f is the subdifferential mapping of f defined for $x \in C$ by

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle, \text{ for all } y \in C\}.$$

We again use σ_A to denote the support function of $A \subset X$, i.e.

$$\sigma_A(x^*) = \sup_{x \in A} \langle x^*, x\rangle, \ x^* \in X^*.$$

Next, we introduce a representation theorem of sublinear (hence, regular and homogeneous) MNC.

Theorem 4.3 ([2, Theorem 2.1]). *Suppose that μ is a sublinear MNC defined on a Banach space X. Then there is a bounded set $S \subset C(K)^*$ of positive functionals such that*

$$\mu(B) = \sigma_{S \cup -S}(f_B), \ \text{ for all } B \in \mathscr{B}(X), \tag{4.4}$$

where $f_B = TQJ(\overline{\mathrm{co}}(B)) \in C(K), \ B \in \mathscr{B}(X)$.

5. On inequivalent regular MNCs

In this section we will show that every infinite dimensional Banach space admits a regular MNC not equivalent to the Hausdorff MNC β defined as (1.2). Since the Hausdorff MNC β is a regular MNC, it gives Problem 1.5 a positive answer.

Lemma 5.1 ([1, Lemma 3.5]). *For a Banach space X, with the symbols as the same in Theorem 2.1, assume that the positive cone $V \equiv TQJ(\mathscr{C})$ of the space $C(K)$ contains a basic sequence. Then X has a homogenous measure of non-compactness not equivalent to the Hausdorff measure, where $E_{\mathscr{C}} = \overline{J\mathscr{C} - J\mathscr{C}}$, $E_{\mathscr{K}} = \overline{J\mathscr{K} - J\mathscr{K}}$ and $J : \mathscr{C} \to C_b(\Omega)$ are also defined as Theorem 2.1, and $Q : E_{\mathscr{C}} \to E_{\mathscr{C}}/E_{\mathscr{K}}$ denotes the quotient mapping.*

Lemma 5.2 ([1, Lemma 3.6]). *Suppose that Z is a Banach space admitting an infinite (respectively, unconditional) decomposition. Then there is a (respectively, an unconditional) normalized basic sequence of $E_{\mathscr{C}}$ contained in the cone $J\mathscr{C}$.*

Lemma 5.3 ([1, Corollary 3.7]). *Every Banach space containing a closed subspace admitting an infinite decomposition (in particular, any space containing an unconditional basic sequence, or, any infinite product of infinite dimensional Banach spaces) has inequivalent homogenous measures of noncompactness.*

Theorem 5.4. *Every infinite dimensional Banach space admits a regular MNC not equivalent to the Hausdorff MNC.*

Proof. Let X be an infinite dimensional Banach space. All symbols will be the same as mentioned previously.

Suppose that X is a Schur space, that is, the norm sequential convergence coincides with the weakly sequential convergence on X. Then it contains ℓ_1. Since the standard unit vector basis (e_n) of ℓ_1 is unconditional, we finish the proof by Lemma 5.3.

Suppose that X is not a Schur space. Then it contains a normalized weakly null sequence (y_n). Thus, there is a basic subsequence (x_n) $(\subset (y_n))$ of X. Without loss of generality, we can assume that $\Delta \equiv (x_n)$ is a monotone basic sequence. (Otherwise, we can renorm X so that Δ is a monotone basic sequence.) Let $X_0 = \overline{\text{span}}(\Delta)$. We denote the sequence of the coefficient functionals corresponding to Δ by $\Delta^* \equiv (\varphi_n) \subset X_0^*$. Then $\|\varphi_n\| \leq 2$ and $\varphi_n \to 0$ in the w^*-topology of X_0^*. $(\Delta_n)_{n=1}^\infty$ is an infintie σ-partition of Δ, i.e. it satisfies the following conditions: (1) $\cup_{n=1}^\infty \Delta_n = \Delta$; (2) $\Delta_n \cap \Delta_m = \emptyset$ whenever $n \neq m$; and (3) each Δ_n is an infinite subset of Δ. Let (Δ_n^*) be the partition of Δ^* corresponding to (Δ_n). For each $n \in \mathbb{N}$, put $C_n = \overline{\text{co}}(\Delta_n)$. Then it is easy to observe that

a) for each $j \in \mathbb{N}$, C_j is weakly compact, $C_m \cap C_n = \{0\}$ whenever $m \neq n$, and

b) for each selection $z_n \in C_n$ $(n = 1, 2, \dots)$ we have $z_n \to 0$ in the weak topology of X.

We denote by $C(K)^+$ (resp. $C(K)^{*+}$) the positive cone of $C(K)$ (resp. $C(K)^*$). Let

$$f_n = TQJ(C_n)(\in C(K)), \tag{5.1}$$

$$D = \overline{\text{co}(B_{C(K)} \cap C(K)^+ \cup \{nf_n : n \in \mathbb{N}\})}, \tag{5.2}$$

and let

$$D^\circ = \{\varphi \in C(K)^{*+} : \langle \varphi, d\rangle \leq 1, \text{ for all } d \in D\}. \tag{5.3}$$

We claim that D and D^0 have the following properties:

(i) D contains no nontrivial subcones of $V \equiv TQJ(\mathscr{C})$;

(ii) D^0 positively separates points of V, i.e. for every $0 \neq x \in V$, there is $\varphi \in D^0$ such that $\langle \varphi, x\rangle > 0$.

Proof of (i) Suppose that $g \in V$ with $g = TQJ(B)$ for some closed bounded nonempty convex set B in X satisfies $ng \in D$ for all $n \in \mathbb{N}$. Then by definitions of f_m, D and D^0, there exists a sequence of $(g_n) \subset V$ with

$$g_n = b_n + \sum_{j=1}^{m_n} \alpha_{n,j} j f_j \in \text{co}[B_{C(K)} \cap C(K)^+ \cup \{mf_m : m \in \mathbb{N}\}]$$

such that $\|ng - g_n\| \to 0$; where $b_n \in B_{C(K)} \cap C(K)^+$, $m_n \in \mathbb{N}$, and $\alpha_{n,j} \in \mathbb{R}^+$ with $\sum_{j=1}^{m_n} \alpha_{n,j} \leq 1$. Thus

$$V \ni (1/n)g_n = \frac{b_n}{n} + (1/n)\sum_{j=1}^{m_n} \alpha_{n,j} j f_j \to g$$

(as $n \to \infty$). By Theorem 2.1 (2)-(5), this is equivalent to that for every $\varepsilon > 0$, there exist a finite set $F \subset X$ and $N \in \mathbb{N}$ such that for all $n \geq N$

$$B \subset (1/n)\sum_{j=1}^{m_n} \alpha_{n,j} j C_j + \varepsilon B_X + F \text{ and } (1/n)\sum_{j=1}^{m_n} \alpha_{n,j} j C_j \subset B + \varepsilon B_X + F. \tag{5.4}$$

Since $(1/n)\alpha_{n,j} j \to 0$ for $j = 1, 2, \dots$ as $n \to \infty$, by b) for every selection $w_n \in (1/n)\sum_{j=1}^{m_n} \alpha_{n,j} j C_j$, $n = 1, 2, \dots$ we have $w_n \to 0$ in the weak topology of X. Consequently, $B \subset \varepsilon B_X + F$. Since ε is arbitrary, B is compact. Consequently, $g = 0$.

Proof of (ii) Since D does not contain nontrivial cone of V, then we have $D_1 \equiv \overline{\mathrm{co}(D \cup -C(K)^+)}$ does not contain nontrivial subcone of V. Given $f \neq 0$ in V. $nf \notin D$ for some $n \in \mathbb{N}$. By the separation theorem of convex sets, there is a functional $\varphi \in C(K)^*$ such that

$$\infty > M \equiv \langle \varphi, nf \rangle > \sup\{\langle \varphi, d \rangle : \ d \in D_1\} \geq 0$$

This entails that $\varphi \in C(K)^{*+}$ and bounded above by M on D. Let $\psi = \frac{1}{M}\varphi$. Then $\psi \in D^\circ$ with $\langle \psi, f \rangle > 0$. Therefore, by Theorem 4.1 (below),

$$\mu(B) = \sup_{e \in D^\circ \cup -D^\circ} \langle e, f_B \rangle, \quad f_B = TQJ(\overline{\mathrm{co}(B)}), B \in \mathscr{B}(X)$$

defines a homogenous measure of noncompactness on X. Since $\mu(C_n) \leq \frac{1}{n} \to 0$, and since $\beta(C_n) \geq 1/2$ for all $n \in \mathbb{N}$, μ is not equivalent to the Hausdorff measure β. Consequently, there is a regular measure of noncompactness ν such that it is equivalent to μ [44]. Thus, the regular measure ν is not equivalent to the Hausdorff measure β. □

Remark 5.5. The proof above is taken from [1, 62 (10) (2019), 2053-2056].

6. Countable determination of α

In this section, we will give a sketch proof of " For every nonempty bounded subset B of a metric space, there is a countable subset B_0 of B so that the Kuratowski MNC α satisfies $\alpha(B_0) = \alpha(B)$". The results of this section are due to [20].

We will assume that X is a Banach space, and X^* its dual. B_X denotes the closed unit ball of X. For a subset $A \subset X$, $\mathrm{co}(A)$ (resp. $\overline{A}$, $\mathrm{aff}(A)$) stands for the convex hull (resp. the closure, the affine hull) of A.

6.1. **Finite representability of subsets in Banach spaces.** Recall that a Banach space X is said to be finitely representable in another Banach space Y provided that for all $\varepsilon > 0$ and for every finite dimensional subspace $F \subset X$ there exist a (finite dimensional) subspace $G \subset Y$ and a linear isomorphism $T : F \to G$ so that $\|T\| \cdot \|T^{-1}\| < 1 + \varepsilon$. The following notion is a generalization of the classical finite representability of Banach spaces to general subsets, which is introduced by Cheng et al. [24] and [22]. It is done by substituting "simplexes" for the "finite dimensional subspaces". (An n-simplex in a linear space X is a convex set S satisfying that there exist $n+1$ affinely independent vectors $x_j \in X$, $j = 0, 1, \ldots, n$, i.e. $x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0$ are linearly independent, such that $S = \mathrm{co}(x_0, x_1, \ldots, x_n)$, the convex hull of $(x_j)_{j=0}^n$.)

Definition 6.1. Suppose that X and Y are Banach spaces, and $A \subset X$, $B \subset Y$ are two subsets.

i) Given $\varepsilon > 0$, the set A is said to be ε-finitely representable in the set B, if for all $n \in \mathbb{N}$ and for every n-simplex $S_n = \mathrm{co}(x_0, x_1, \ldots, x_n)$ with the vertices $(x_j)_{j=0}^n \subset A$, there exist an n-simplex $S_n' = \mathrm{co}(y_0, y_1, \ldots, y_n)$ with vertices $(y_j)_{j=0}^n \subset B$ and an affine isomorphism $T : \mathrm{aff}(x_j) \to \mathrm{aff}(y_j)$ satisfying that $T(x_j)_{j=0}^n = (y_j)_{j=0}^n$ (or, equivalently, $TS_n = S_n'$) and that

$$(1 - \varepsilon)\|x - y\| \leq \|Tx - Ty\| \leq (1 + \varepsilon)\|x - y\|, \ \forall \, x, y \in S_n. \tag{6.1}$$

ii) If, in addition, for all $\varepsilon > 0$, A is ε-finitely representable in B, then we say that A is finitely representable in the set B.

It is easy to observe that X is finitely representable in Y if and only if the unit ball B_X of X is finitely representable in the unit ball B_Y of Y. Before introducing the concept of strong finite representability, we also require some more notions below.

Definition 6.2. Let H be a convex set in a Banach space X. Then

i) H is said to be a convex polyhedron provided there exist $z_1, z_2, \ldots, z_k \in X$ for some $k \in \mathbb{N}$ such that $H = \mathrm{co}(z_1, z_2, \ldots, z_k)$; In this case, the set $\mathrm{extr}(H)(\subset (z_1, z_2, \ldots, z_m))$ of all extreme points of H is called the vertex set of H, and denoted by $\mathrm{vert}(H)(= \mathrm{extr}(H))$.
ii) A convex polyhedron H is called an n-dimensional convex polyhedron if its affine hull $\mathrm{aff}(H)$ is an n-dimensional affine subspace of X, i.e. $\dim[\mathrm{aff}(H)] = n$.
iii) We say that an n-dimensional convex polyhedron H is an (n, m)-polyhedron for some non-negative integer m if

$$\mathrm{vert}(H) = (x_0, x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}).$$

Since the vertex set $\mathrm{vert}(H)$ of the (n, m)-polyhedron H always contains a maximal affinely independent subset of $n + 1$ elements, we write an (n, m)-polyhedron H as

$$H = \mathrm{co}(x_0, x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}), \tag{6.2}$$

where $(x_0, x_1, \ldots, x_n)$ is a maximal affinely independent subset of $\mathrm{vert}(H)$. In this case, we have

$$\mathrm{aff}(H) = \mathrm{aff}(x_0, x_1, \ldots, x_n). \tag{6.3}$$

Remark 6.3. Unless stated otherwise, by an (n, m)-polyhedron

$$H = H(x_0, x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}),$$

we always mean that

$$\mathrm{vert}(H) = (x_0, x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$$

with $H = \mathrm{co}(x_0, x_1, \ldots, x_{n+m})$, that $S \equiv \mathrm{co}(x_0, x_1, \ldots, x_n)$ is an n-simplex, and that $\mathrm{aff}(S) = \mathrm{aff}(H)$.

Lemma 6.4. *Suppose that* $H_1 = H(x_0, x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ *and* $H_2 = H(y_0, y_1, \ldots, y_n, y_{n+1}, \ldots, y_{n+m})$ *are two* (n, m)*-polyhedrons in a Banach space* X. *Let*

$$\delta_1 = \min\{\|x_i - x_j\| : 0 \leq i \neq j \leq n + m\} > 0,$$

$$\delta_2 = \min\{\|y_i - y_j\| : 0 \leq i \neq j \leq n + m\} > 0,$$

and let

$$0 < 2\varepsilon < \delta = \min\{\delta_1, \delta_2\}. \tag{6.4}$$

Then

$$d_{\mathfrak{H}}(\mathrm{vert}(H_1), \mathrm{vert}(H_2)) < \varepsilon \tag{6.5}$$

if and only if there is a permutation $\pi : \{0, 1, \dots, n+m\} \to \{0, 1, \dots, n+m\}$ *such that*

$$\|x_j - y_{\pi(j)}\| < \varepsilon,\ j = 0, 1, \dots, n+m, \tag{6.6}$$

where $d_{\mathfrak{H}}$ *is the Hausdorff metric on* $\mathscr{B}(X)$.

Definition 6.5. Suppose that X and Y are Banach spaces, and $A \subset X$, $B \subset Y$ are two subsets.

i) Given $\varepsilon > 0$, the set A is said to be ε-strongly finitely representable in the set B, if for every pair n, m of non-negative integers, and for every (n, m)-polyhedron $H \subset X$ with $\mathrm{vert}(H) \subset A$, there exist an (n, m)-polyhedron $H' \subset Y$ with $\mathrm{vert}(H') \subset B$ and an affine isomorphism $T : \mathrm{aff}(H) \to \mathrm{aff}(H')$ such that

$$(1-\varepsilon)\|x-y\| \le \|Tx - Ty\| \le (1+\varepsilon)\|x-y\|,\ \ \forall\, x, y \in \mathrm{aff}(H), \tag{6.7}$$

and

$$d_{\mathfrak{H}}\Big(\mathrm{vert}(T(H)), \mathrm{vert}(H')\Big) < \varepsilon, \tag{6.8}$$

where $d_{\mathfrak{H}}$ is the Hausdorff metric on $\mathscr{B}(Y)$, the set of all nonempty bounded subsets of Y.

ii) A is said to be strongly finitely representable in B, if A is ε-strongly finitely representable in B for all $\varepsilon > 0$.

iii) In particular, if A itself is finite, and if A is strongly finitely representable in B, then we simply say that A is strongly representable in B.

Lemma 6.6. *Assume that* $A \subset X$ *is strongly finitely representable in* $B \subset Y$. *Then for all* $\varepsilon > 0$ *and for every* (n, m)*-polyhedron*

$$H = H(x_0, x_1, \dots, x_{n+m}) \ \text{ with } \ \mathrm{vert}(H) \subset A,$$

there exist an (n, m)*-polyhedron*

$$H' = H'(y_0, y_1, \dots, y_{n+m}) \ \text{ with } \mathrm{vert}(H') \subset B$$

and an affine isomorphism $T : \mathrm{aff}(H) \to \mathrm{aff}(H')$ *such that*

$$(1-\varepsilon)\|x-y\| \le \|Tx - Ty\| \le (1+\varepsilon)\|x-y\|,\ \ \forall\, x, y \in \mathrm{aff}(H), \tag{6.9}$$

$$T(S) = S', \ \text{ and } \ d_{\mathfrak{H}}\Big(\mathrm{vert}(T(H)), \mathrm{vert}(H')\Big) < \varepsilon, \tag{6.10}$$

where $d_{\mathfrak{H}}$ *is the Hausdorff metric on* $\mathscr{B}(Y)$, *and* S *(resp.* S'*) is the* n*-simplex* $\mathrm{co}(x_0, x_1, \dots, x_n)$ *(resp.* $\mathrm{co}(y_0, y_1, \dots, y_n)$*).*

Corollary 6.7. *Suppose that* A, B *are two subsets of a Banach space* X. *If* A *is strongly finitely representable in* B, *then it is finitely representable in* B.

Proposition 6.8. *Suppose that* X *and* Y *are Banach spaces. Then* X *is strongly finitely representable in* Y *if (and only if)* X *is finitely representable in* Y.

6.2. **Ultraproducts of subsets in Banach spaces.** A filter $\mathcal{F}$ on a nonempty set I is a family of subsets of I satisfying that a) $\emptyset \notin \mathcal{F}$; b) for all $n \in \mathbb{N}$, $(F_j)_{j=1}^n \subset \mathcal{F} \Longrightarrow \cap_j F_j \in \mathcal{F}$; and c) $A \in \mathcal{F}$, and $A \subset B \subset I \Longrightarrow B \in \mathcal{F}$. A filter $\mathcal{F}$ is called a free filter if $\bigcap\{F \in \mathcal{F}\} = \emptyset$. We say that a filter $\mathcal{F}$ is an ultrafilter if it satisfies that either $S \in \mathcal{F}$ or $I \setminus S \in \mathcal{F}$ for any $S \subset I$. We will always use $\mathcal{U}$ to denote an ultrafilter on a set I.

Let X_i, $i \in I$ be Banach spaces, $A_i \subset X_i$, $i \in I$, and let $\prod_I A_i$ be the Cartesian product of these sets, i.e. the set of all families $(a_i)_{i\in I}$ with $a_i \in A_i$. Two families (a_i), (b_i) are said to be equivalent with respect to the ultrafilter $\mathcal{U}$, if for every $\varepsilon > 0$

$$\{i \in I : \|a_i - b_i\| < \varepsilon\} \in \mathcal{U}.$$

This defines an equivalence relation on $\prod_I A_i$.

Definition 6.9. The set of all equivalence classes of $\prod_I A_i$ with respect to the ultrafilter $\mathcal{U}$ is called the ultraproduct of the subsets $(A_i)_{i\in I}$ denoted by $(A_i)_{\mathcal{U}}$. In particular, if $A_i \equiv A$ for all $i \in I$, we simply denote $(A_i)_{\mathcal{U}}$ by $(A)_{\mathcal{U}}$, and call it the ultrapower of A.

Proposition 6.10. *Let E_i, $i \in I$ be a family of Banach spaces, and $A_i, B_i \subset E_i$, $i \in I$. Then for every ultrafilter $\mathcal{U}$ on I,*

$$(A_i \cup B_i)_{\mathcal{U}} = (A_i)_{\mathcal{U}} \cup (B_i)_{\mathcal{U}};$$
$$(A_i \cap B_i)_{\mathcal{U}} = (A_i)_{\mathcal{U}} \cap (B_i)_{\mathcal{U}};$$
$$(A_i \setminus B_i)_{\mathcal{U}} = (A_i)_{\mathcal{U}} \setminus (B_i)_{\mathcal{U}}.$$

6.3. **A lemma about strongly finite representability and ultraproducts.** It was shown in [22, Prop.2.4] that for two Banach spaces X and Y, and for two bounded subsets $A \subset X$ and $B \subset Y$, if A is finitely representable in B, then for every affinely independent subset A_0 of A, there exist a free ultrafilter $\mathcal{U}$ on some index set I and an affine isometry $T : \mathrm{aff}(A_0) \to (\mathrm{aff}(B))_{\mathcal{U}}$ with $T(A_0) \subset (B)_{\mathcal{U}}$. If A is strongly finitely representable in B, then we can further show the following result.

Lemma 6.11. *Suppose that A is a subset of a Banach space X, B is a convex subset of a Banach space Y. If A is strongly finitely representable in B, then there exist a free ultrafilter $\mathcal{U}$ on some index set I and an affine isometry $T : \mathrm{aff}(A) \to (\mathrm{aff}(B))_{\mathcal{U}}$ such that $T(A) \subset (B)_{\mathcal{U}}$.*

6.4. **Strongly finite representability of sets in their countable subsets.** In this subsection, we will show that every subset of a Banach space is strongly finitely representable in a countable subset of it. For the proof of this result, we will need a sequence of lemmas.

For m elements $a_j : j = 1, 2, \dots, m$ of a set A, $(a_1, a_2, \dots, a_m)$ is either to denote the subset $\{a_j : j = 1, 2, \dots, m\}$, or, the "vector" $(a_1, a_2, \dots, a_m)$ in the Cartesian product A^m. We often blur the distinction if it arises no confusion.

Given $1 \le p \le \infty$, let $\|\cdot\|_p$ be the ℓ_p-norm defined on $\mathbb{R}^n$, i.e.

$$\|x\|_p = (\sum_{j=1}^n |x(j)|^p)^{\frac{1}{p}}, \quad x = (x(1), x(2), \dots, x(n)) \in \mathbb{R}^n. \tag{6.11}$$

Let K be the closed unit ball of the space $\ell_\infty^n \equiv (\mathbb{R}^n, \|\cdot\|_\infty)$,

$$\Omega = \big\{ |||\cdot||| \text{ is a seminorm on } \mathbb{R}^n \text{ with } |||\cdot||| \le \|\cdot\|_1 \big\},$$

and $C(K)$ be the Banach space of all continuous functions on K with the sup-norm $\|\cdot\|_\infty$.

Lemma 6.12. *Assume that K and Ω are defined as above. Then Ω is a convex compact set of $C(K)$.*

Lemma 6.13. *For every n dimensional (real) normed space X, there exists a norm $|||\cdot|||$ on $\mathbb{R}^n$ satisfying*

$$\|\cdot\|_\infty \le |||\cdot||| \le \|\cdot\|_1, \tag{6.12}$$

and there is a linear isometry $T : X \to (\mathbb{R}^n, |||\cdot|||)$.

Lemma 6.14. *Let $p_1, p_2 \in C(K)$ be two norms on $\mathbb{R}^n$ with $p_1 \ge \|\cdot\|_\infty$, and let $\varepsilon > 0$. If $\|p_1 - p_2\|_{C(K)} \le \varepsilon$, i.e.*

$$\big|p_1(x) - p_2(x)\big| \le \varepsilon, \ \ \forall\, x \in K, \tag{6.13}$$

then

$$(1-\varepsilon)p_1(x) \le p_2(x) \le (1+\varepsilon)p_1(x), \ \forall\, x \in \mathbb{R}^n. \tag{6.14}$$

Lemma 6.15. *With notations as above, for every subset A of a Banach space X, we have*

$$\mathcal{H}_n(A) = \bigcup_{m=0}^{\infty} \mathcal{H}_{n,m}(A), \ \text{ and } \ \mathcal{H}(A) = \bigcup_{n=0}^{\infty} \mathcal{H}_n(A). \tag{6.15}$$

Given two nonnegative integers m, n and $r > 0$, let $\mathcal{H}_{n,m} \equiv \mathcal{H}_{n,m}(rB_{\ell_\infty^n})$, i.e. the set of all (n,m)-polyhedrons H contained in $\ell_\infty^n \equiv (\mathbb{R}^n, \|\cdot\|_\infty)$ of the form

$$H = \mathrm{co}(x_0, x_1, \ldots, x_n, u_1, \ldots, u_m)$$

with its vertexes

$$\mathrm{vert}(H) = (x_0, x_1, \ldots, x_n, u_1, \ldots, u_m) \subset rB_{\ell_\infty^n} \equiv \{rx : x \in B_{\ell_\infty^n}\}$$

such that $(x_0, x_1, \ldots, x_n)$ is a maximal affinely independent subset of $\mathrm{vert}(H)$. Since $d_{\mathfrak{H}}$-convergence of a sequence $(\mathrm{vert}(H_j))$ for $(H_j) \subset \mathcal{H}_{n,m}$ is equivalent to the convergence of the corresponding vertex vector sequence

$$\{(x_{0,j}, x_{1,j}, \ldots, x_{n,j}, u_{1,j}, \ldots, u_{m,j})\} \text{ in } \ell_\infty^{n(n+1+m)}$$

within some permutations (Lemma 6.4), the following result follows easily.

Lemma 6.16. *For each pair m, n of nonnegative integers, $(\mathrm{vert}(\mathcal{H}_{n,m}), d_{\mathfrak{H}})$ is relatively compact in $(\mathscr{B}(\ell_\infty^n), d_{\mathfrak{H}})$, where $\mathrm{vert}(\mathcal{H}_{n,m}) = \{\mathrm{vert}(H) : H \in \mathcal{H}_{n,m}\}$.*

Lemma 6.17. *Suppose that A is a bounded subset of a Banach space X. Let $r = \sup_{a\in A} \|a\|$. Then for any fixed $n, m \in \mathbb{N}$,*

$$\mathcal{T}_{n,m,0}(A) \equiv \{\mathrm{vert}(T_H(H)) \subset \ell_\infty^n : H \in \mathcal{H}_{n,m,0}(A)\}$$

$$(\text{resp. } \mathcal{T}_{n,m,1}(A) \equiv \{\mathrm{vert}(T_H(H)) \subset \ell_\infty^{n+1} : H \in \mathcal{H}_{n,m,1}(A)\})$$

is a bounded subset of $(\mathscr{B}(\ell_\infty^n), d_{\mathfrak{H}})$ (resp. $(\mathscr{B}(\ell_\infty^{n+1}), d_{\mathfrak{H}})$) and bounded by $2r$. Consequently, by Lemma 6.16, $\mathcal{T}_{n,m}(A) = \mathcal{T}_{n,m,0}(A) \cup \mathcal{T}_{n,m,1}(A)$ is relatively compact.

Lemma 6.18. *Suppose that A is a bounded subset of a Banach space X. Then the products $\mathcal{P}_{n,m,0}(A) \equiv \mathcal{T}_{n,m,0}(A) \times \Omega_n$ and $\mathcal{P}_{n,m,1}(A) \equiv \mathcal{T}_{n,m,1}(A) \times \Omega_{n+1}$ are again relatively compact.*

Lemma 6.19. *Assume that A be a bounded subset of a Banach space X. Given $U_0 = (\mathrm{vert}(T_0H_0), |\cdot|_0), U_j = (\mathrm{vert}(T_jH_j), |\cdot|_j) \in \mathcal{P}_{n,m,0}(A)$ (resp. $\mathcal{P}_{n,m,1}(A)$) $j = 1, 2, \ldots,$ if $U_j \to U_0$ in $\mathcal{P}_{n,m,0}(A)$ (resp. $\mathcal{P}_{n,m,1}(A)$), then $\mathrm{vert}(H_0)$ is strongly (finitely) representable in $\bigcup_{j=1}^{\infty} \mathrm{vert}(H_j)$.*

Now, we are ready to state the main result of this subsection.

Theorem 6.20. *Suppose that A is a nonempty subset of a Banach space X. Then there exists a countable subset A_0 such that A is strongly finitely representable in A_0.*

Corollary 6.21. *Suppose that X and Y are Banach spaces, and that $A \subset X$ and $B \subset Y$ are two subsets.*

i) *If A is finitely representable in B, then there is a countable subset B_0 of B such that A is finitely representable in B_0.*
ii) *If A is strongly finitely representable in B, then there is a countable subset B_0 of B such that A is strongly finitely representable in B_0.*

Corollary 6.22. *Suppose that X and Y are Banach spaces. If X is finitely representable in Y, then there is a separable subspace Y_0 such that X is finitely representable in Y_0. Consequently, every Banach space is finitely representable in a separable subspace of it.*

6.5. **Countable determination of the Kuratowski measure.** In this subsection, we will show the main result of the paper: the Kuratowski measure α defined on any metric space X satisfies that for every bounded set $B \subset X$, there is a countable subset B_0 such that $\alpha(B_0) = \alpha(B)$.

Lemma 6.23. *Let α_M be the Kuratowski measure defined on a metric space M. Then there exist a Banach space X and an isometric mapping $T : M \to X$ so that $\alpha|_{T(M)}$ (the restriction of the Kuratowski measure α on X to $T(M)$) coincides with α_M, i.e.*

$$\alpha(T(B)) = \alpha_M(B), \text{ for all } B \in \mathscr{B}(M).$$

Assume that M is a metric space and $\mathcal{U}$ is an ultrafilter. For distinction, we use α_M to denote the Kuratowski measure on M, and $\alpha_{\mathcal{U}}$ to denote the Kuratowski measure on the ultrapower $(M)_{\mathcal{U}}$ of M.

Lemma 6.24 ([36, Corollary. 2.3])**.** *For any subset B of a metric space M, and for any ultrafilter $\mathcal{U}$, we have*

$$\alpha_M(B) = \alpha_{\mathcal{U}}[(B)_{\mathcal{U}}]. \tag{6.16}$$

Now, we are ready to prove the main theorem of this section.

Theorem 6.25. *Suppose that X is a metric space. Then for every bounded subset $B \subset X$, there is a countable subset B_0 of B such that $\alpha(B_0) = \alpha(B)$.*

Proof. By Lemma 6.23, we can assume that X is a Banach space. Given $B \in \mathscr{B}(X)$, by Theorem 6.20, there exists a countable subset B_0 of B such that B is strongly finitely representable in B_0. Applying Lemma 6.11, we can obtain a free ultrafilter $\mathcal{U}$, and an affine isometry $T : \mathrm{aff}(B) \to [\mathrm{aff}(B_0)]_{\mathcal{U}}$ such that $T(B) \subset (\mathrm{co}B_0)_{\mathcal{U}}$. It follows from Lemma **??** that

$$\alpha(B) = \alpha_{\mathcal{U}}[T(B)] \leq \alpha_{\mathcal{U}}[(\mathrm{co}B_0)_{\mathcal{U}}] = \alpha(\mathrm{co}B_0) = \alpha(B_0). \tag{6.17}$$

On the other hand, non-deceasing monotonicity of the Kuratowski measure α in the order of set inclusion entails that $\alpha(B) \geq \alpha(B_0)$. Therefore, $\alpha(B_0) = \alpha(B)$. □

7. A basic integral inequality

In this section, we will introduce some results concerning the integral inequality (1.11) related to the Cauchy problem (1.12), which are due to [21]. The letter X again denotes a real Banach space, and I is the interval $[0, a] \subset \mathbb{R}^+$ with $a > 0$. We use (I, Σ, m) to denote the Lebesgue measure space. For a set A, χ_A stands for the characteristic function of A. Unless stated otherwise, all notions and symbols will be the same as previously defined. We use $L_p(I, X)$ $(1 \leq p \leq \infty)$ to denote the space of all X-valued measurable functions defined on I such that $|f|^p$ is Lebesgue-Bochner integrable endowed with the L_p-norm $\|f\| = (\int_I |f|^p d\mu)^{1/p}$. $L_0(I, X)$ stands for the space of all X-valued strongly measurable functions.

We are going to convert whether the integral inequality (1.11) is true to whether the mapping $\mathbb{J}_G : I \to C_b(\Omega)$ defined blow is (Lebesgue-Bochner) measurable, where $\Omega = B_{X^*}$ and $C_b(\Omega)$ denotes again the Banach space of all continuous bounded functions on Ω endowed with the sup-norm. For a Banach space X, the Banach spaces $E_{\mathscr{C}}$, $E_{\mathscr{K}}$ and $C(K)$, the mappings J, Q, T and $\mathbb{T} = TQJ$, the positive cone $V = \mathbb{T}\mathscr{C}$ are the same as in Section 2.

We first recall some definitions and known results.

Definition 7.1. Let $G \subset L_0(I, X)$ be a nonempty subset satisfying $\sup \|G(t)\| \equiv \sup_{g \in G} \|g(t)\| < \infty$ a.e. The mapping $\mathbb{J}_G : I \to C_b(\Omega)$ is defined for $t \in I$ by

$$\mathbb{J}_G(t)(x^*) = \sup_{g \in G} \langle x^*, g(t) \rangle, \ x^* \in \Omega. \tag{7.1}$$

Note that $\mathbb{J}_G(t) = J(G(t))$, where $G(t) = \{g(t) : g \in G\}$. If it arises no confusion, we simply write $\mathbb{J}$ for $\mathbb{J}_G$.

We recall some notions and basic properties related to X-valued functions defined on the interval I (from Definition 7.2 to Lemma 7.6), which can be found in J. Diestel [26].

Definition 7.2. Let $f : I \to X$ be a function.

i) f is said to be a simple function provided there exist a finite Σ-partition $\{E_j\}_{j=1}^n$ and n vectors $\{x_j\}_{j=1}^n \subset X$ such that $f = \sum_{j=1}^n x_j \chi_{E_j}$. The integral of f is defined by $\int_I f dm = \sum_j x_j m(E_j)$.

ii) f is called (strongly) measurable if there is a sequence $\{f_n\}$ of simple functions such that $f_n(s) \to f(s)$ for almost all $s \in I$. If, in addition, the limit $\lim_n \int_I f_n dm$ exists, then we say that f is (Lebesgue-Bochner) integrable and $\int_I f dm \equiv \lim_n \int_I f_n dm$ is called the integral of f on I.

iii) We say that f is weakly measurable if for each $x^* \in X^*$ the numerical function $\langle x^*, f\rangle$ is (Σ-) measurable.

Lemma 7.3. *A function $f : I \to X$ is (Lebesgue-Bochner) integrable if and only if f is strongly measureable and $\int_I \|f\|dm < \infty$.*

Lemma 7.4 (Pettis's measurability theorem)**.** *A function $f : I \to X$ is strongly measurable if and only if*

(1) *$f(I) \subset X$ is essentially separable, i.e., there exists a null set $I_0 \subset I$ such that $f(I\backslash I_0)$ is separable; and*
(2) *f is weakly measurable.*

Lemma 7.5. *Let F be a closed linear operator defined inside X and having values in a Banach space Y. If both $f : I \to X$ and Ff are Bochner integrable, then*

$$F\Big(\int_I f ds\Big) = \int_I F f ds.$$

Lemma 7.6 (Jensen's inequality)**.** *Assume $f : [0, 1] \to X$ is integrable, and $p : X \to \mathbb{R}$ is a continuous convex function. If $p \circ f$ is integrable, then*

$$p\,\Big(\int_0^1 f ds\Big) \leq \int_0^1 \Big(p \circ f\Big)ds.$$

Lemma 7.7. *Let X be a Banach space, $I = [0, a]$, $G \subseteq L_1(I, X)$ be a nonempty set, $\psi \in L_1(I, \mathbb{R}_+)$ such that $\sup_{d\in G} \|g(t)\| \leq \psi(t)$ a.e. $t \in I$. Assume that $\mathbb{J}_G : I \to C_b(\Omega)$ is strongly measurable. Then $TQ\mathbb{J}_G : I \to C(K)$ defined for $t \in I$ by $TQ\mathbb{J}_G(t) = TQJ\Big(G(t)\Big)$ is integrable on I and satisfies*

$$0 \leq TQJ\Big(\int_0^t G(s)ds\Big) \leq \int_0^t TQJ\Big(G(s)\Big)ds. \tag{7.2}$$

Theorem 7.8. *Let μ be a convex MNC on a Banach space X, $I = [0, a]$, $G \subseteq L_1(I, X)$ be a nonempty bounded subset, and $\psi \in L_1(I, \mathbb{R}^+)$ such that $\sup_{g\in G} \|g(t)\| \leq \psi(t)$ a.e. $t \in I$. Assume that $\mathbb{J}_G : I \to C_b(\Omega)$ is strongly measurable. Then $\mu\Big(G(t)\Big)$ is measurable and*

$$\mu\Big(\int_0^t G(s)ds\Big) \leq \frac{1}{t}\int_0^t \mu\Big(tG(s)\Big)ds, \ \ \forall 0 < t \leq a. \tag{7.3}$$

Corollary 7.9. *Let μ be a convex MNC on a Banach space X, $I = [0, a]$, $G \subseteq L_1(I, X)$ be a nonempty bounded subset, and $\psi \in L_1(I, \mathbb{R}^+)$ such that $\sup_{g\in G} \|g(t)\| \leq \psi(t)$ a.e. $t \in I$. Assume that $\mathbb{J}_G : I \to C_b(\Omega)$ is strongly measurable. Then the following inequality holds*

$$\mu\Big(\int_0^t G(s)ds\Big) \leq \int_0^t \mu\Big(G(s)\Big)ds, \ \ \forall 0 < t \leq a, \tag{7.4}$$

if one of the following conditions is satisfied.

i) $0 < t \leq \min\{1, a\}$;
ii) *μ is a sublinear MNC.*

In the following, we will discuss measurability and weak measurability of the mapping $\mathbb{J}_G : I \to C_b(\Omega)$ for a bounded set $G \subset L_1(I, X)$. As a result, we show that if G is separable in $L_1(I, X)$ and there is $u \in L_p(I, X)$ for some $0 < p$ such that $\sup_{g\in G} \|g(t)\| < u(t)$ a.e., then $\mathbb{J}_G$ is weakly measurable. Making use of this result, we further prove that $\mathbb{J}_G$ is strongly measurable if G is one of the following classical classes:

a) $G \subset C(I, X)$ is an equi-continuous subset;
b) G is a separable equi-regulated subset of $R(I, X)$ (the Banach space of bounded functions on I satisfying that $\lim_{t\to t_0^\pm} u(t)$ exist for all $u \in R(I, X)$ and $t_0 \in I$) endowed with the sup-norm; and
c) $G \subset L_1(I, X)$ is uniformly measurable.

Lemma 7.10. *Let $\mathbf{T}$ be a Hausdorff topological space, and $C_b(\mathbf{T})$ be the Banach space of all bounded continuous functions on $\mathbf{T}$ endowed with the sup-norm $\|f\| = \sup_{t\in\mathbf{T}} |f(t)|$ for $f \in C_b(\mathbf{T})$. Then the closed unit ball $B_{C_b(\mathbf{T})^*}$ of the dual $C_b(\mathbf{T})^*$ satisfies*

$$B_{C_b(\mathbf{T})^*} = w^*\text{-}\overline{\mathrm{co}}\{\pm\delta_t : t \in \mathbf{T}\},$$

where $\delta_t \in C_b(\mathbf{T})^*$ is the evaluation functional defined for $f \in C_b(\mathbf{T})$ by $\langle \delta_t, f\rangle = f(t)$, and $w^*\text{-}\overline{\mathrm{co}}(A)$ denotes the w^*-closed convex hull of A in $C_b(\mathbf{T})^*$.

Theorem 7.11. *Let X be a Banach space, $I = [0, a]$, $G \subseteq L_1(I, X)$ be a separable subset satisfying that there is $u \in L_p(I, \mathbb{R}^+)$ for some $p > 0$ such that $\sup_{g\in G} \|g(t)\| \le u(t)$ for almost all $t \in I$. Then $\mathbb{J}_G : [0, a] \to C_b(\Omega)$ is weakly measurable.*

Theorem 7.12. *Let $G \subset C(I, X)$ be a nonempty equi-continuous subset. Then $\mathbb{J}_G : I \to C_b(\Omega)$ is continuous, hence, strongly measurable.*

Corollary 7.13. *Let X be a Banach space, μ be a convex MNC on X, $I = [0, a]$, and $G \subseteq C(I, X)$ be a nonempty equi-continuous subset. Then the following inequality holds*

$$\mu\Big(\int_0^t G(s)ds\Big) \le \frac{1}{t}\int_0^t \mu\Big(tG(s)\Big)ds, \ \ \forall 0 < t \le a. \tag{7.5}$$

In particular,

$$\mu\Big(\int_0^t G(s)ds\Big) \le \int_0^t \mu\Big(G(s)\Big)ds, \ \ \forall 0 < t \le a. \tag{7.6}$$

if one of the following conditions is satisfied.

i) $0 < t \le \min\{1, a\}$;
ii) *μ is a sublinear MNC.*

Remark 7.14. Corollary 7.13 is a generalization of K. Goebel et al. In 1970, K. Goebel and W. Rzymowski [32] showed that the inequality (7.6) holds for equi-continuous subsets $G \subset C(I, X)$ and for the Hausdorff MNC β, i.e. when $\mu = \beta$. (See, also, [49].) In 1980, J. Banaś and K. Goebel [10] showed the inequality (7.6) holds for equi-continuous subsets $G \subset C(I, X)$ and for homogeneous MNC μ.

Before stating next result, we recall the notions of regulated functions and of equi-regulated sets of such functions (see, for example, [48]).

Definition 7.15. i) A function $f : [a, b] \to X$ is said to be regulated provided for every $t \in [a, b)$ the right-sided limit $\lim_{s \to t^+} f(s) \equiv f(t^+)$ exists and for every $t \in (a, b]$ the left-sided limit $\lim_{s \to t^-} f(s) \equiv f(t^-)$ exists.

We denote by $R([a, b], X)$ the Banach space of all regulated functions defined on the interval $[a, b]$ endowed with the sup-norm.

ii) A nonempty subset $G \subset R(I, X)$ is called equi-regulated if

$$\forall t \in (a, b], \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall g \in G,\ \forall t_1, t_2 \in (t - \delta, t) \cap [a, b], \|g(t_2) - g(t_1)\| \le \varepsilon,$$

$$\forall t \in [a, b), \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall g \in G,\ \forall t_1, t_2 \in (t, t + \delta) \cap [a, b], \|g(t_2) - g(t_1)\| \le \varepsilon.$$

Theorem 7.16. *Let X be a Banach space and $I = [0, a]$. Assume that $G \subset R(I, X)$ is a separable equi-regulated set. Then the mapping $\mathbb{J}_G : I \to C_b(\Omega)$ is strongly measurable.*

Corollary 7.17. *Let X be a Banach space, μ be a convex MNC on X, $I = [0, a]$, and $G \subseteq R(I, X)$ be a nonempty separable equi-regulated subset. Then the following inequality holds*

$$\mu\Big(\int_0^t G(s)ds\Big) \le \frac{1}{t}\int_0^t \mu\Big(tG(s)\Big)ds, \ \ \forall 0 < t \le a. \tag{7.7}$$

In particular,

$$\mu\Big(\int_0^t G(s)ds\Big) \le \int_0^t \mu\Big(G(s)\Big)ds, \ \ \forall 0 < t \le a, \tag{7.8}$$

if one of the following conditions is satisfied.

i) $0 < t \le \min\{1, a\}$;

ii) *μ is a sublinear MNC.*

Remark 7.18. Corollary 7.17 is a generalization of L. Olszowy and T. Zajac [48, Th. 3.1] in 2020, where they showed that the inequality (7.8) holds for the Hausdorff MNC $(\mu =)\beta$.

For a finite measure space (Γ, Σ, η) and a Banach space X, we denote by $L_1(\Gamma, \eta, X)$ the Banach space of all X-valued η-integrable functions f endowed with the L_1-norm $\|f\| = \int_\Gamma \|f(\gamma)\|d\eta$. In particular, if $\Gamma = I = [a, b] \subset \mathbb{R}$ and η is the Lebesgue measure on $\mathbb{R}$, then simply denote it by $L_1(I, X)$.

The following useful notion of uniform measurability of functions was introduced by M. Kunze and G. Schlüchtermann [38, Def. 3.16].

Definition 7.19. A bounded set $G \subset L_1(\Gamma, \eta, X)$ is said to be uniformly η-measurable provided for all $\varepsilon > 0$ and $A \in \Sigma$ there exist $A_\varepsilon \in \Sigma$ and mutually disjoint $A_1, \dots A_n \in \Sigma$ with $\cup_{j=1}^n A_j = A_\varepsilon$ and with $\eta(A \setminus A_\varepsilon) < \varepsilon$ such that for $j = 1, 2, \dots, n$ we can choose $\gamma_j \in A_j$ satisfying

$$\sup_{g \in G} \|g(\cdot)\chi_{A_\varepsilon} - \sum_{j=1}^n g(\gamma_j)\chi_{A_j}\|_{L_1} < \varepsilon. \tag{7.9}$$

Given a subset $G \subset L_0(\Gamma, \eta, X)$ of X-valued η-measurable functions, we denote by $G(\gamma) = \{g(\gamma) : g \in G\}$, $\gamma \in \Gamma$; and let $\mathbb{J}_G : \Gamma \to C_b(\Omega)$ be again defined for $\gamma \in \Gamma$ by

$$\mathbb{J}_G(\gamma)(x^*) = J\Big(G(\gamma)\Big)(x^*) = \sup_{g \in G}\langle x^*, g(\gamma)\rangle = \sigma_{G(\gamma)}(x^*),\ x^* \in \Omega \equiv B_{X^*}. \tag{7.10}$$

Theorem 7.20. *Let $G \subset L_0(\Gamma, \eta, X)$ be a subset with*

$$\sup_{g \in G}\|g(\gamma)\| < \infty, \text{ for almost all } \gamma \in \Gamma. \tag{7.11}$$

If G is uniformly η-measurable, then $\mathbb{J}_G : \Gamma \to C_b(\Omega)$ is strongly measurable.

Corollary 7.21. *Let X be a Banach space, μ be a convex MNC on X, $I = [0, a]$, and $G \subseteq L_1(I, X)$ be a nonempty separable and uniformly measurable subset. Then the following inequality holds*

$$\mu\Big(\int_0^t G(s)ds\Big) \leq \frac{1}{t}\int_0^t \mu\Big(tG(s)\Big)ds,\ \ \forall 0 < t \leq a. \tag{7.12}$$

In particular,

$$\mu\Big(\int_0^t G(s)ds\Big) \leq \int_0^t \mu\Big(G(s)\Big)ds,\ \ \forall 0 < t \leq a, \tag{7.13}$$

if one of the following conditions is satisfied,

i) $0 < t \leq \min\{1, a\}$;
ii) *μ is a sublinear MNC.*

Remark 7.22. Corollary 7.21 is a generalization of Kunze and G. Schlüchtermann [38, Corollary 3.19], where they showed that the inequality (7.13) holds for the Hausdorff MNC β, i.e. $\mu = \beta$.

8. On fullness of MNC

In this section, we will introduce some very recent results [9] related to fullness of non-full convex MNC and the generalized Cantor intersection property. To begin with this section, we recall the definition of non-full MNC which was introduced by Banaś and Goebel [10], but they still call it MNC.

Definition 8.1. Let X be a Banach space. A function $\mu : \mathscr{B} \to \mathbb{R}^+$ is said to be a non-full MNC provided

(P1) [**Noncompactness**] ker$\mu \neq \emptyset$ and $B \in \mathscr{B}(X)$, $\mu(B) = 0 \Longrightarrow B$ is relatively compact;
(P2) [**Monotonicity**] $A, B \in \mathscr{B}(X)$ with $A \supset B \implies \mu(A) \geq \mu(B)$;
(P3) [**Convexification invariance**] $B \in \mathscr{B}(X) \Longrightarrow \mu(\mathrm{co}(B)) = \mu(B)$;
(P4) [**Convexity**] $\mu(\lambda A + (1-\lambda)B) \leq \lambda\mu(A) + (1-\lambda)\mu(B)$, $\forall A, B \in \mathscr{B}(X)$ and $0 \leq \lambda \leq 1$;
(P5) [**Generalized Cantor intersection property**] If $\{B_n\} \subset \mathscr{B}(X)$ with $B_{n+1} \subset B_n$ for all $n \in \mathbb{N}$, then

$$\mu(B_n) \to 0 \Longrightarrow \bigcap \overline{B}_n \neq \emptyset.$$

Remark 8.2. μ is said to be a pre-MNC, provided it satisfies (P1)–(P4).

If μ satisfies (P2)–(P4) and $\ker(\mu) \supset \mathscr{K}(X)$, then we say it is a generalized MNC.

The generalized Cantor intersection property (P5) is one of the most important properties in applications of MNC to fixed point theory. We have already known that every convex MNC admits the property naturally (See, for instance, [21, Theorem 3.4]). But we do not know whether it is independent of other conditions in the definition of non-full convex MNC, that is, is a pre-MNC necessarily a non-full convex MNC. If the answer to this question is affirmative, then the following question is arising naturally: What conditions can guarantee that a non-full convex MNC to dominate a convex MNC?

The following theorem states that the generalized Cantor intersection property (P5) is independent of other conditions (P1)–(P4) in Definition 8.1.

Theorem 8.3. *For every infinite dimensional Banach space X, there is a translation invariant homogenous pre-MNC μ on it so that μ does not admit the generalized Cantor intersection property.*

Should we mention here that the representation theorem (Theorem 3.1) is valid again for every generalized MNC but not for a non-full convex MNC. Because Theorem 3.1 is based on $\ker(\mu) \supset \mathscr{K}(X)$. However, paralleling to Theorem 3.1, we have the following theorem.

Theorem 8.4. *Suppose that X is a Banach space. Then for every pre-MNC μ on X, there is a function Λ on the cone $\mathfrak{S}(\Omega) \equiv J\mathscr{C}(X) \subset C_b(\Omega)$ satisfying*

i) *$\mu(B) = \Lambda(JB)$, for all $B \in \mathscr{B}(X)$;*

ii) *Λ is nonnegative-valued convex and monotone increasing on $\mathfrak{S}(\Omega)$;*

iii) *Λ is bounded by $b_r = \Lambda(rJB_X)$ on $\mathfrak{S}(\Omega) \bigcap (rB_{C_b(\Omega)})$, for all $r \geq 0$;*

iv) *Λ is c_r-Lipschitzian on $\mathfrak{S}(\Omega) \bigcap (rB_{C_b(\Omega)})$, for all $r \geq 0$, where $c_r = \Lambda\big((1+r)JB_X\big) = \mu\big((1+r)B_X\big)$;*

v) *In particular, if μ is a sublinear pre-MNC, then we can take $c_r = \mu(B_X)$ in iv).*

For an extended real-valued function $f : D \to \mathbb{R} \cup \{+\infty\}$ defined on a closed convex set D of a Banach space X, its epigraph is defined by

$$\mathrm{epi}(f) = \{(x, r) \in D \times \mathbb{R}, f(x) \leq r\}.$$

The following properties are simple and well-known.

Proposition 8.5. *Let D be a nonempty closed convex set of a Banach space X, and $f : D \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. Then*

i) *f is convex on D if and only if $\mathrm{epi}(f)$ is convex in $X \times \mathbb{R}$;*

ii) *f is lower semi-continuous on D if and only if $\mathrm{epi}(f)$ is closed in $X \times \mathbb{R}$.*

Definition 8.6. Let D be a nonempty closed convex set of a Banach space X, and $f : D \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. $\bar{f}$ is said to be the (lower semi-continuous, resp.,) convexification of f provided it is convex and satisfies $\bar{f} \leq f$ on D and it is the maximum among all (lower semi-continuous, resp.,) convex functions g with $g \leq f$ on D.

For an extended real-valued function f, we denote by $f_{\rm conv}$ and successively, $\bar{f}_{\rm conv}$ its convexification and lower semicontinuous convexification. Note that every extended real-valued function uniquely determines its epigraph, and vice versa. By the definition above, the following properties can be easily verified.

Proposition 8.7. *Let D be a nonempty closed convex set of a Banach space X, and $f : D \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. Then*

i) $\mathrm{epi}(f_{\rm conv}) = \mathrm{co}[\mathrm{epi}(f)]$;
ii) $\mathrm{epi}(\bar{f}_{\rm conv}) = \overline{\mathrm{co}}[\mathrm{epi}(f)]$;
iii) $f_{\rm conv}(x) = \inf\Big\{ \sum_{j=1}^{n} \lambda_j f(x_j) : n \in \mathbb{N}, 1 \le j \le n, 0 \le \lambda_j, \sum_{j=1}^{n} \lambda_j = 1, x_j \in D, x = \sum_{j=1}^{n} \lambda_j x_j \Big\}$, $x \in D$;
iv) $\bar{f}_{\rm conv}(x) = \inf\Big\{\alpha : \exists\{x_n\} \subset D, x_n \to x \text{ such that } \liminf_n f_{\rm conv}(x_n) = \alpha\Big\}$, $x \in D$.

Proposition 8.8. *Suppose that μ is a generalized MNC defined on a Banach space X, and that Λ is the locally Lipschitz convex function associated with μ defined in Theorem 8.4. Then for every nonempty compact convex set $K \subset X$,*

$$\Lambda(C + K) = \Lambda(C),\ \forall\, C \in \mathscr{C}(X).$$

For a pre-MNC defined on a Banach space X, let Λ be the locally Lipschitz convex function associated with μ defined in Theorem 8.4. We define

$$\Lambda'(JC) = \begin{cases} 0, & C \in \mathscr{K}(X), \\ \Lambda(JC), & C \in \mathscr{B}(X) \setminus \mathscr{K}(X). \end{cases} \tag{8.1}$$

Lemma 8.9. *Let Λ' be defined by (8.1). Then its convexification $\Lambda'_{\rm conv}$ satisfies that for all $C \in \mathscr{C}(X)$,*

$$\Lambda'_{\rm conv}(JC) = \inf\{\Lambda(JD) : C = D + K, D \in \mathscr{C}(X), K \in \mathscr{K}(X)\}. \tag{8.2}$$

Theorem 8.10. *Suppose that μ is a pre-MNC defined on a Banach space X, and that Λ' is defined by (8.1). If $\Lambda'_{\rm conv}$ is monotone non-decreasing on $\mathfrak{S}(\Omega)$, then*

i) $\Lambda'_{\rm conv} = \bar{\Lambda}'_{\rm conv} \le \Lambda$.
ii) *$\Lambda'_{\rm conv}$ is locally Lipschitz on $\mathfrak{S}(\Omega)$.*
iii) *The function ν defined by*

$$\nu(B) = \Lambda'_{\rm conv}(JB),\ \ B \in \mathscr{B}(X)$$

is the largest generalized MNC controlled by μ.

Theorem 8.11. *Suppose that μ is a pre-MNC defined on a Banach space X, and that Λ is the locally Lipschitz convex function associated with μ defined in Theorem 8.4. Then the function ν defined by*

$$\nu(B) = \Lambda'_{\rm conv}(JB),\ \ B \in \mathscr{B}(X)$$

is a convex MNC if only if $\Lambda'_{\rm conv}$ is monotone increasing and there is not a nonempty closed bounded convex noncompact set C with the following decomposition

$$C = C_n + K_n,\ n = 1, 2, \dots, \tag{8.3}$$

where $C_n \in \mathscr{B}(X), K_n \in \mathscr{K}(X)$ *such that*

$$\lim_{n\to\infty} \Lambda(JC_n) = 0. \tag{8.4}$$

A pre-MNC is said to be translation invariant if $\mu(x_0 + B) = \mu(B)$ for all $B \in \mathscr{B}(X)$ and $x_0 \in X$. For example, $\mu(B) = \mathrm{diam}(B)$, $\forall B \in \mathscr{B}(X)$ is a translation invariant MNC. The following theorem gives a necessary condition for a translation invariant MNC to dominate a full MNC.

Theorem 8.12. *Suppose that μ is a translation invariant MNC. Then there does not exist a nonempty closed bounded convex noncompact set C with the following decomposition*

$$C = C_n + K_n,\ n = 1, 2, \dots, \tag{8.5}$$

where $C_n \in \mathscr{B}(X), K_n \in \mathscr{K}(X)$ *such that*

$$\lim_{n\to\infty} \Lambda(C_n) = 0. \tag{8.6}$$

Theorem 8.13. *Suppose that μ is a translation invariant non-full MNC. Then μ dominates a convex MNC if and only if the convex function Λ associated with μ defined in Theorem 8.4 dominates a convex monotone increasing function Λ_1 such that there does not exist a nonempty closed bounded convex noncompact set C with the following decomposition*

$$C = C_n + K_n,\ n = 1, 2, \dots, \tag{8.7}$$

where $C_n \in \mathscr{B}(X), K_n \in \mathscr{K}(X)$ *such that*

$$\lim_{n\to\infty} \Lambda_1(C_n) = 0. \tag{8.8}$$

Problem 8.14. We do not know whether the convexification Λ'_{conv} of the truncation Λ' defined by (8.1) is monotone increasing although Λ' is monotone increasing.

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