Pure and Applied Functional Analysis
Volume 8, Number 6, 2023, 1603–1620

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

CONVERGENCE OF REMOTE PROJECTIONS ONTO CONVEX SETS

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Abstract. Let $\{C_\alpha\}_{\alpha\in\Omega}$ be a family of closed and convex sets in a Hilbert space H, having a nonempty intersection C. We consider a sequence $\{x_n\}$ of remote projections onto them. This means, $x_0 \in H$, and x_{n+1} is the projection of x_n onto such a set $C_{\alpha(n)}$ that the ratio of the distances from x_n to this set and to any other set from the family is at least $t_n \in [0,1]$. We study properties of the weakness parameters t_n and of the sets C_α which ensure the norm or weak convergence of the sequence $\{x_n\}$ to a point in C. We show that condition (T) is necessary and sufficient for the norm convergence of x_n to a point in C for any starting element and any family of closed, convex, and symmetric sets C_α. This generalizes a result of Temlyakov who introduced (T) in the context of greedy approximation theory. We give examples explaining to what extent the symmetry condition on the sets C_α can be dropped. Condition (T) is stronger than $\sum t_n^2 = \infty$ and weaker than $\sum t_n/n = \infty$. The condition $\sum t_n^2 = \infty$ turns out to be necessary and sufficient for the sequence $\{x_n\}$ to have a partial weak limit in C for any family of closed and convex sets C_α and any starting element.

1. Introduction

In the entire paper H is a real Hilbert space; its scalar product we denote by $\langle\cdot,\cdot\rangle$ and the corresponding norm by $|\cdot|$.

Let $\{C_\alpha\}_{\alpha\in\Omega}$ be a family of closed and convex sets in H, $|\Omega|\ge 2$, so that $C = \bigcap_{\alpha\in\Omega} C_\alpha \ne \emptyset$. Let P_α denote the metric projection onto C_α. Sometimes we also denote the metric projection onto a closed convex set A by P_A.

A fixed sequence $\{\alpha(n)\}\subset\Omega$ and a starting element $x_0\in H$ generate the sequence $x_{n+1} = P_{\alpha(n)}x_n$, $n = 0,1,2,\dots$, of consecutive projections that can be examined for convergence.

In the case when C_α are closed linear subspaces of H and Ω is a finite set the convergence properties of the sequence $\{x_n\}$ are well understood. If the sequence of indices $\{\alpha(n)\}$ is cyclic, that is, $\Omega = \{0,1,\dots,K-1\}$ and $\alpha(n)\equiv n\,(\mathrm{mod}\,K)$, then $\{x_n\}$ converges in norm to a point of C [19], [11]. The rate of convergence depending on the position of the subspaces and of the initial point can be estimated (see [6], [20], and the bibliography there). Already for three closed linear subspaces divergence might occur if no extra information about the sequence of indices, or

2020 *Mathematics Subject Classification.* Primary: 46C05, 47J25 Secondary: 41A65.
Key words and phrases. Hilbert space, metric projection, convex set, convergence, symmetry.
*The first author was supported by the Russian Science Foundation (grant no. 23-71-30001) in Moscow State University.

about the geometry of the subspaces is known (see [16], [17], and [15]). The sequence $\{x_n\}$, however, always converges weakly to a point in C if Ω is finite, each $\alpha \in \Omega$ occurs infinitely many times in the sequence $\{\alpha(n)\}$ and the sets C_α are closed linear subspaces [1].

In the lack of linearity, when the sets C_α are just closed and convex, the situation is different. Already for $|\Omega| = 2$ the sequence $\{x_n\}$ might diverge in norm ([12], see also [14] and [18]). Weak convergence is known only under additional conditions [7]: when $|\Omega| \leq 3$ [9], or when $|\Omega| < \infty$ and the indices $\{\alpha(n)\}$ are cyclic [8], or when $|\Omega| < \infty$ and the sets are "somewhat symmetric" [9], or when Ω is arbitrary but each successive projection takes place on the farthest set [8]. In this article we develop the last plot.

For any starting element $x_0 \in H$, we consider the sequence of *remote* projections

$$x_{n+1} = P_{\alpha(n)} x_n, \qquad n = 0, 1, 2, \ldots, \tag{0.1}$$

where $\alpha(n) \in \Omega$ is chosen so that

$$\operatorname{dist}(x_n, C_{\alpha(n)}) \geq t_n \sup_\alpha \operatorname{dist}(x_n, C_\alpha),$$

and $t_n \in [0, 1]$ are prescribed *weakness parameters*. If there is at least one $n \in \mathbb{N}$ with $t_n = 1$, that is, when the nth projection is the *remotest*, we require the maximum $\max_\alpha \operatorname{dist}(x, C_\alpha)$ to be attained for each $x \in H$. Note that any sequence of consecutive projections onto the family $\{C_\alpha\}$ can be regarded as a sequence of remote projections with some, possibly very small, t_n's.

We prove a convergence result for the remote projections (0.1). If all the convex sets C_α are closed subspaces of codimension one, this convergence theorem is already known within the greedy approximation theory [24].

We recall the corresponding definitions. A subset D of the unit sphere $S(H) = \{s \in H : |s| = 1\}$ is called a *dictionary* if $\overline{\operatorname{span}} D = H$. For any dictionary $D \subset S(H)$, any sequence $\{t_n\}$ in $[0, 1]$ of weakness parameters, and any $x_0 \in H$, the *Weak Greedy Algorithm* (WGA) generates a sequence x_n defined inductively by

$$x_{n+1} = x_n - \langle x_n, g_n \rangle g_n, \qquad n = 0, 1, 2, \ldots, \tag{0.2}$$

where the element $g_n \in D$ is such that

$$|\langle x_n, g_n \rangle| \geq t_n \sup\{|\langle x_n, g \rangle| : g \in D\}.$$

The weakness parameters were introduced by Temlyakov in [22]. In the case when $t_n \equiv 1$ of the *Pure Greedy Algorithm*, or even if $t_n = 1$ for at least one n, we require the maximum $\max\{|\langle x_n, g \rangle| : g \in D\}$ to be attained for each $x \in H$. We say that the WGA converges if $|x_n| \to 0$.

Clearly, the WGA coincides with the process of remote projecting onto the family of hyperplanes $\{g^\perp : g \in D\}$ orthogonal to the dictionary elements. Since D is spanning, the origin is the only point in the intersection of these hyperplanes. The other way round it works as well.

Remark 1. *Let $\{C_\alpha\}$ be a family of closed linear subspaces with $\bigcap_\alpha C_\alpha = \{0\}$. Then $\bigcup_\alpha C_\alpha^\perp$ is spanning and the remote projections (0.1) correspond to the WGA with respect to the dictionary $\bigcup_\alpha C_\alpha^\perp \cap S(H)$ with the same sequence $\{t_n\}$ of weakness parameters.*

According to Temlyakov [23] the condition

$$\text{(T)} \qquad \forall \{a_\nu\} \in \ell_2 \text{ with } a_\nu \geq 0: \liminf_{m\to\infty} \frac{a_m}{t_m} \sum_{\nu=1}^{m} a_\nu = 0$$

on the sequence $\{t_m\}$ is necessary and sufficient for the convergence of all realizations of the WGA with the weakness sequence $\{t_m\}$ for each $x_0 \in H$ and all dictionaries $D \subset S(H)$.

The condition (T) is rather subtle. For instance,

$$\sum \frac{t_m}{m} = \infty \Rightarrow \text{(T)} \Rightarrow \sum t_m^2 = \infty,$$

but none of the implications can be reversed.

In Section 1 we generalize the convergence theorem of [23] to the case of remote projections onto a family of closed convex sets C_α that are uniformly quasi-symmetric with respect to their common point; see Theorem 1 below. Our proof partially leans on Temlyakov's paper [23] and on the proof of Jones' theorem on the convergence of the Pure Greedy Algorithm in Hilbert space [13], [24, Ch. 2]. The quasi-symmetry condition is essential in view of Hundal's example [12], [14]. In Corollary 1.1 we show that cyclic projections onto finitely many closed, convex and quasi-symmetric sets converge in norm. This generalizes a result from [2].

In Section 2 we discuss different versions of the quasi-symmetry condition, and show that the uniform quasi-symmetry is essential in Theorem 1.

Section 3 is devoted to the weak convergence of remote projections. In Theorem 2 we give a condition on $\{t_n\}$ sufficient for weak convergence to a point in C. As Corollary 2.1 we get a result of [9]: quasi-periodic projections onto finitely many closed and convex sets converge weakly to a point in their intersection. In Theorem 3 we give another condition on $\{t_n\}$ that is necessary and sufficient for all sequences of remote projections to have a partial weak limit in C. We construct an example showing certain sharpness of these theorems.

1. Norm convergence

In this section we show when remote projections onto a family of closed and convex sets converge. A symmetry assumption on the sets is needed; we define this weakened symmetry below. In Section 2 we will show to what extent this symmetry condition is necessary. We will also compare it to another weakened symmetry condition.

In what follows $B(a, r)$ denotes the closed ball with center a and radius $r > 0$.

Definition 1. Let C and C_α, $\alpha \in \Omega$, be closed convex sets in a Banach space X, all containing the origin.

(i) We call the set C *quasi-symmetric*, if
$\forall r > 0\ \exists\, \theta = \theta(r) \in (0, 1]: \ x \in C \cap B(0, r) \ \Rightarrow -\theta x \in C.$

(ii) We say that the family of sets $\{C_\alpha\}_{\alpha\in\Omega}$ is uniformly quasi-symmetric if
$\forall r > 0\ \exists\, \theta = \theta(r) \in (0, 1]\, \forall \alpha: \ x \in C_\alpha \cap B(0, r) \Rightarrow -\theta x \in C_\alpha.$

Moreover, we say that C is *quasi-symmetric* with respect to a point $a \in C$ if the set $(C - a)$ is quasi-symmetric. Similarly, the family of sets $\{C_\alpha\}_{\alpha\in\Omega}$ is uniformly quasi-symmetric with respect to a point $a \in \bigcap_{\alpha\in\Omega} C_\alpha$ if the family $\{C_\alpha - a\}_{\alpha\in\Omega}$ is uniformly quasi-symmetric.

In the above definition we can equivalently write "$\exists\, r > 0$" instead of "$\forall\, r > 0$". Indeed, $\theta(r) = \theta_0 \min\{1, r_0/r\}$ works for a given $r > 0$, if θ_0 works for some $r_0 > 0$.

Theorem 1. *For a sequence $\{t_n\}_{n=0}^\infty \subset [0,1]$, the following two statements are equivalent:*

(i) *The sequence $\{t_n\}$ satisfies the condition (T).*

(ii) *For any family $\{C_\alpha\}_{\alpha\in\Omega}$ of closed and convex sets in a Hilbert space H which is uniformly quasi-symmetric with respect to a point $a \in C = \bigcap_{\alpha\in\Omega} C_\alpha$ and for any starting element $x_0 \in H$ the sequence $\{x_n\}$ of remote projections (0.1) converges in norm to a point in C.*

Proof. We will prove here that (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (i) follows from [23] where Temlyakov shows that the condition (T) is necessary already when all C_α's are hyperplanes.

1. We can assume that $a = 0$. Let $x_n - x_{n+1} = y_n$, $\angle 0 x_{n+1} x_n = \pi/2 + \varepsilon_n$. We have $\varepsilon_n \in [0, \pi/2]$; otherwise x_{n+1} is not the nearest point for x_n in the segment $[0, x_{n+1}]$ and hence also in $C_{\alpha(n)}$. Consequently, $|x_n|^2 \geq |x_{n+1}|^2 + |y_n|^2$, so that the norms $|x_n|$ decrease to some $R \geq 0$. We suppose $R > 0$, otherwise $x_n \to 0$. Moreover,

$$\sum_{n=0}^{\infty} |y_n|^2 \leq \sum_{n=0}^{\infty} (|x_n|^2 - |x_{n+1}|^2) < \infty. \tag{1.1}$$

Consequently, since $\{t_m\}$ satisfies (T), we can choose a subsequence $\Lambda \subset \mathbb{N}$ with the property

$$b_m := \frac{|y_m|}{t_m} \sum_{\nu=0}^{m} |y_\nu| \to 0, \qquad m \to \infty,\ m \in \Lambda. \tag{1.2}$$

2. Now we prove that

$$\sum_{n=0}^{\infty} |y_n| \sin\varepsilon_n < \infty. \tag{1.3}$$

By the law of cosines,

$$\begin{aligned} |x_n|^2 &= |x_{n+1}|^2 + |y_n|^2 - 2|x_{n+1}||y_n| \cos\left(\frac{\pi}{2} + \varepsilon_n\right) \\ &= |x_{n+1}|^2 + |y_n|^2 + 2|x_{n+1}||y_n| \sin\varepsilon_n, \end{aligned}$$

so that

$$|y_n| \sin\varepsilon_n = \frac{|x_n|^2 - |x_{n+1}|^2 - |y_n|^2}{2|x_{n+1}|} \leq \frac{|x_n|^2 - |x_{n+1}|^2}{2R},$$

and (1.3) follows.

3. The vector y_ν is externally normal to a hyperplane supporting the set $C_{\alpha(\nu)}$ at the point $x_{\nu+1}$. Hence,

$$\langle y_\nu, z - x_{\nu+1} \rangle \leq 0 \tag{1.4}$$

for any $z \in C_{\alpha(\nu)}$, so that

$$\langle y_\nu, z\rangle \le \langle y_\nu, x_{\nu+1}\rangle = |y_\nu||x_{\nu+1}| \cos\left(\frac{\pi}{2} - \varepsilon_\nu\right)$$
$$= |y_\nu||x_{\nu+1}| \sin\varepsilon_\nu \le |y_\nu||x_0| \sin\varepsilon_\nu.$$

Since $C_{\alpha(\nu)}$ is quasi-symmetric with respect to 0, we get

$$|\langle y_\nu, z\rangle| \le \theta^{-1}|y_\nu||x_0| \sin\varepsilon_\nu, \qquad z \in C_{\alpha(\nu)} \cap B(0, 2|x_0|), \tag{1.5}$$

where $\theta = \theta(2|x_0|)$ is from the definition of uniform quasi-symmetry.

4. Next, for any m and ν,

$$\begin{aligned}
|y_m| &= \operatorname{dist}(x_m, C_{\alpha(m)}) \ge t_m \sup_\alpha \operatorname{dist}(x_m, C_\alpha) \ge t_m \operatorname{dist}(x_m, C_{\alpha(\nu)}) \\
&= t_m \min_{z\in C_{\alpha(\nu)}} |x_m - z| = t_m \min_{z\in C_{\alpha(\nu)}\cap B(0,2|x_0|)} |x_m - z| \\
&\ge t_m \min_{z\in C_{\alpha(\nu)}\cap B(0,2|x_0|)} |\langle x_m - z, y_\nu/|y_\nu|\rangle| \\
&\ge t_m \left(|\langle x_m, y_\nu/|y_\nu|\rangle| - \max_{z\in C_{\alpha(\nu)}\cap B(0,2|x_0|)} |\langle z, y_\nu/|y_\nu|\rangle|\right) \\
&\ge t_m \left(|\langle x_m, y_\nu/|y_\nu|\rangle| - \theta^{-1}|x_0| \sin\varepsilon_\nu\right);
\end{aligned}$$

we have used (1.5) in the last inequality. The above estimate implies that

$$|\langle x_m, y_\nu\rangle| \le \frac{|y_m||y_\nu|}{t_m} + \theta^{-1}|x_0||y_\nu| \sin\varepsilon_\nu \tag{1.6}$$

for any $m, \nu = 0, 1, 2, \ldots$

5. Now we prove that $\{x_n\}$ is a Cauchy sequence and hence it converges to some $w \in H$. Given any $n, k \in \mathbb{N}$ assume that $m \in \Lambda$ and $m > \max\{n, k\}$. Since $|x_n - x_k| \le |x_n - x_m| + |x_k - x_m|$, it is enough to show that $|x_n - x_m| \to 0$ as $n, m \to \infty$, $n < m$ and $m \in \Lambda$. We use the identity

$$|x_n - x_m|^2 = |x_n|^2 - |x_m|^2 - 2\langle x_n - x_m, x_m\rangle.$$

Since $|x_n| \to R$, we have $|x_n|^2 - |x_m|^2 \to 0$ as $n, m \to \infty$. The last term we estimate using (1.6) as follows:

$$\begin{aligned}
|\langle x_n - x_m, x_m\rangle| &= \left|\sum_{\nu=n}^{m-1} \langle y_\nu, x_m\rangle\right| \le \sum_{\nu=n}^{m-1} |\langle y_\nu, x_m\rangle| \\
&\le \frac{|y_m|}{t_m} \sum_{\nu=n}^{m-1} |y_\nu| + \theta^{-1}|x_0| \sum_{\nu=n}^{m-1} |y_\nu| \sin\varepsilon_\nu.
\end{aligned}$$

The first sum does not exceed b_m by (1.2), so it tends to 0 as $m \to \infty$, $m \in \Lambda$. The second sum tends to 0 as $n, m \to \infty$ in view of (1.3).

6. Finally we show that $w = \lim x_n$ is contained in all C_α's. If $w \notin C_\beta$ for some β, then

$$\operatorname{dist}(x_n, C_\beta) > \delta > 0$$

for all $n \geq n_0$. This implies that

$$\begin{aligned}|x_{n+1}|^2 &\leq |x_n|^2 - |y_n|^2 = |x_n|^2 - \operatorname{dist}(x_n, C_{\alpha(n)})^2 \\ &\leq |x_n|^2 - t_n^2 \operatorname{dist}(x_n, C_\beta)^2 \leq |x_n|^2 - t_n^2 \delta^2 \\ &\leq \cdots \leq |x_{n_0}|^2 - \delta^2 \sum_{\nu=n_0}^{n} t_\nu^2.\end{aligned}$$

Since (T) implies that $\sum t_\nu^2 = \infty$, this contradicts $|x_n| \downarrow R > 0$. □

Assume $C_1, \ldots, C_K$ are closed linear subspaces of H. Assume $\{\alpha(n)\}$ is a quasi-periodic sequence of the indices $1, \ldots, K$. This means that there is a constant $M \in \mathbb{N}$ so that for every interval I of length M the set $\{\alpha(n) : n \in I\}$ contains all of the indices:

$$\{\alpha(n) : n \in I\} = \{1, \ldots, K\}.$$

Then the sequence $x_{n+1} = P_{\alpha(n)} x_n$ of projections converges in norm [19], [11], [21]. Already for two closed and convex sets this is not true, as the example of Hundal exhibits [12], [14], [18]. Theorem 1 implies easily, that as soon as the closed and convex sets $C_1, \ldots, C_K$ are also quasi-symmetric, convergence occurs. For *symmetric* in place of *quasi-symmetric* this was established in [2].

Corollary 1.1. *Assume $C_1, \ldots, C_K$ are finitely many closed, convex and quasi-symmetric subsets of H with a nonempty intersection $C = \bigcap_1^K C_j$. Assume $\{\alpha(n)\}$ is a quasi-periodic sequence of the indices $1, \ldots, K$. Then the sequence $x_{n+1} = P_{\alpha(n)} x_n$ of nearest point projections converges in norm to a point in C for any starting point $x_0 \in H$.*

Proof. The given sets are quasi-symmetric and there are only finitely many of them, so the family is uniformly quasi-symmetric. We will show there are weakness parameters $t_n \in [0, 1]$ satifying $\sum t_n/n = \infty$ so that the sequence $\{x_n\}$ corresponds to a sequence of remote projections with these parameters. Hence according to Theorem 1 the sequence $\{x_n\}$ converges in norm.

We choose $\beta(n) \in \{1, \ldots, K\}$ and define $b_n > 0$ and $t_n \in [0, 1]$ as follows:

$$\operatorname{dist}(x_n, C_{\beta(n)}) = \max_k \operatorname{dist}(x_n, C_k) = b_n,$$

$$t_n = |x_{n+1} - x_n|/b_n.$$

We will prove that for each interval I of length M there is an $n \in I$ so that $t_n \geq 1/(6M)$ and hence $\sum t_n/n = \infty$; here M is the constant of the quasi-periodicity of $\{\alpha_n\}$.

Assume for a contradiction that there is $m \in \mathbb{N}$ so that $t_{m+j} < 1/(6M)$ for all $j \in \{0, \ldots, M\}$. We will show that then $\beta(m) \notin \{\alpha(m + j) : j = 0, \ldots, M\}$ contradicting the sequence $\{\alpha(n)\}$ of indices being quasi-periodic with constant M. Indeed, by the triangle inequality,

$$\begin{aligned}b_{n+1} &= |x_{n+1} - P_{\beta(n+1)} x_{n+1}| \\ &\leq |x_{n+1} - x_n| + |x_n - P_{\beta(n+1)} x_n| + |P_{\beta(n+1)}(x_n - x_{n+1})| \\ &\leq \frac{b_n}{6M} + b_n + \frac{b_n}{6M} = \left(1 + \frac{1}{3M}\right) b_n,\end{aligned}$$

for $m \le n \le m + M - 1$. By induction, for any $1 \le k \le M$,

$$b_{m+k} \le \left(1 + \frac{1}{3M}\right)^M b_m \le 2b_m.$$

Again by triangle inequalities

$$|x_m - x_{m+k}| \le \frac{k}{6M} 2b_m < b_m,$$

hence $x_{m+k} \notin C_{\beta(m)}$. □

Assume $\{C_\alpha\}_{\alpha\in\Omega}$ is a family of closed subspaces in H and that $|\Omega| \ge 2$ is at most countable. Assume that in a sequence $\{\alpha(n)\} \subset \Omega$ each element of Ω appears infinitely many times. The sequence of consecutive projections $x_{n+1} = P_{\alpha(n)}x_n$, $n = 0, 1, 2, \ldots$, $x_0 \in H$, generated by α does not have to converge in general. However, if the norm limit (or even just the weak limit) of the sequence exists, then it is equal to $P_C x_0$, where $C = \bigcap_\alpha C_\alpha$.

Already for three closed subspaces C_1, C_2, C_3 and the sequence of remote projections (0.1) we can choose some of the weakness parameters $t_n \in [0, 1]$ so small that the subspace C_3 can be completely avoided. This causes (0.1) to converge to $P_{C_1\cap C_2}x_0$ which can be arranged to differ from $P_C x_0$.

Already for two closed convex sets things can go awry even for the remotest projections, that is, if in (0.1) we set $t_n = 1$ for all $n \in \mathbb{N}$.

Example 1. *In the Euclidean plane $H = \mathbb{R}^2$ there are two closed, convex and symmetric sets C_1 and C_2, and a starting point x_0 so that the limit point of the remotest projections is not equal to $P_{C_1\cap C_2}x_0$.*

Proof. In the coordinate representation (s, t) of vectors in $\mathbb{R}^2$, we set

$$C_1 = \{s = 0\}, \qquad C_2 = \{s - 2 \le t \le s + 2\}.$$

The line C_1 and the stripe C_2 are both symmetric with respect to 0, and their intersection is the segment $C = \{(0, t) : t \in [-2, 2]\}$. For the starting point $x_0 = (-4, 4)$, we have

$$\operatorname{dist}(x_0, C_2) = 3\sqrt{2} > 4 = \operatorname{dist}(x_0, C_1).$$

Hence $x_1 = P_2x_0 = (-1, 1)$ and $x_2 = P_1x_1 = (0, 1) \in C$, whereas $P_C x_0 = (0, 2)$. □

Note that for finitely many closed convex sets there are special projection algorithms converging to the projection of the starting point onto their intersection [3, Ch. 30].

2. Symmetry conditions

Dye and Reich [9] introduced the following property of weakened symmetry.

Definition 2. Let C be a closed convex set in a Banach space X. The origin is a *weak internal point* (shortly WIP) of C if

$$\forall x \in C\ \exists\, \delta = \delta(x) > 0 : \ -\delta x \in C. \tag{2.1}$$

Moreover, we say that $a \in X$ is a WIP-point of C if the origin is a WIP-point of the set $(C - a)$.

Clearly, the origin is a WIP-point of C if and only if it is a WIP-point of $C_1 = C \cap B(0,1)$. It is also easy to see that the origin is a WIP-point of a quasi-symmetric set: the condition (i) of Definition 1 seems to be stronger than (2.1). Surprisingly, the converse is also true: $\delta(x)$ in (2.1) can be chosen independently of x lying in the unit ball, say. Closed convex sets in Banach spaces cannot be too asymmetric.

Remark 2. *Let C be a closed and convex set in a Banach space X. A point $a \in X$ is a weak internal point of C if and only if C is quasi-symmetric with respect to a.*

Proof. We show only the less obvious implication. We assume that $a = 0$ and that $C = C \cap B(0,1)$. We take the maximal possible δ which works in (2.1): for every $0 \neq x \in C$ there exists $\delta(x) > 0$ so that

(i) $-\delta(x)x \in C$;
(ii) if $\eta > \delta(x)$, then $-\eta x \notin C$.

We claim that $\inf_{x\in C} \delta(x) > 0$ and give an elementary proof of this fact first. If not, then there are non-zero elements $e_n \in C$ having $\delta(e_n) < 1/3^n$, $n \in \mathbb{N}$. Then

$$e = \sum_{n=1}^{\infty} \frac{e_n}{2^n} \in C$$

and we may assume $e \neq 0$; otherwise we take $(1-\varepsilon)e_1$ instead of e_1 for sufficiently small $\varepsilon > 0$. Then $-\delta e \in C$ for some $\delta > 0$. For a fixed $k \in \mathbb{N}$ we observe that

$$\frac{1}{1+\delta(1-1/2^k)} + \sum_{\mathbb{N}\ni n\neq k} \frac{1}{1+\delta(1-1/2^k)} \frac{\delta}{2^n} = 1;$$

all the summands on the left-hand side are positive. Consequently,

$$\frac{1}{1+\delta(1-1/2^k)}(-\delta e) + \sum_{\mathbb{N}\ni n\neq k} \frac{1}{1+\delta(1-1/2^k)} \frac{\delta}{2^n} e_n \in C,$$

that is,

$$\frac{1}{1+\delta(1-1/2^k)} \left(-\delta \sum_{n=1}^{\infty} \frac{e_n}{2^n} + \sum_{\mathbb{N}\ni n\neq k} \frac{\delta}{2^n} e_n \right) = \frac{-\delta/2^k}{1+\delta(1-1/2^k)} e_k \in C.$$

Hence,

$$\frac{\delta/2^k}{1+\delta(1-1/2^k)} \leq \delta(e_k) < \frac{1}{3^k}.$$

The last inequality implies that

$$\frac{\delta}{2^k} < \frac{1+\delta}{3^k} - \frac{\delta}{6^k},$$

which is impossible for large k's; how large exactly depends on δ.

Here is a "Baire category" proof of the fact that $\inf_{x\in C} \delta(x) > 0$ due to V.I. Bogachev. According to [4, Proposition 2.5.1] both sets $C \cap (-C)$ and $\operatorname{conv}(C \cup (-C))$ generate norms on $\operatorname{span} C$ in which $\operatorname{span} C$ is a Banach space. The open mapping theorem implies that the two norms are equivalent, hence the above infimum is positive.

□

Next we exhibit that the *uniform* quasi-symmetry assumption on the sets C_α in Theorem 1 is essential. In [12] and [14], an example of a closed convex cone C with the vertex at the origin was constructed so that iterating the nearest point projection between C and a hyperplane D converges weakly but not in norm for a starting point $x_0 \in D$. In the example the hyperplane $D = e^\perp$ for an $0 \neq e \in H = \ell_2$ and the set C is the epigraph in $\ell_2 = D + \operatorname{span}\{e\}$ of a suitably chosen nonnegative convex sublinear function defined on D. Those familiar with the example readily "see", that the family of closed convex sets consisting of D and $C - c_n e$ for some suitable $c_n \searrow 0$ consists of quasi-symmetric sets for which the remote projections algorithm starting at x_0 closely traces the iterates of nearest points projections of x_0 between C and D. Consequently it converges weakly but not in norm. Rather than writing this up rigorously we give here a construction which is easier to present.

Example 2. *In any infinite dimensional Hilbert space H, there exists a countable family of closed, convex and quasi-symmetric sets so that the sequence of remotest projections on this family does not converge in norm for a certain starting point.*

Proof. We assume that H is separable as if it is not, then we build the example in a closed separable infinite dimensional subspace of H. Also, we construct a family of sets and a point in their intersection so that each set in the family is quasi-symmetric with respect to this point. To center at the origin, we translate, if need be.

We use as a building stone an example constructed in [5]; we first recall its relevant properties.

Let $\{e, e_k : k \in \mathbb{N}\}$ be an orthonormal basis of H. For each $k \in \mathbb{N}$, we choose vectors $v_1^k, \dots, v_{n_k}^k \in \operatorname{span}\{e_k, e_{k+1}\}$ as in [5]. Their number n_k increases in a particular way, the norms $|v_n^k|$ decrease in a particular way. Their only property relevant here are their directions:

$$\arg v_n^k = -\frac{\pi}{2} + \frac{\pi n}{n_k}, \qquad k \in \mathbb{N}, n = 1, \dots, n_k; \tag{2.2}$$

here the polar angle arg in the plane $\operatorname{span}\{e_k, e_{k+1}\}$ is measured from the positive direction of e_k.

The diverging greedy algorithm with respect to the dictionary containing $\pm e$ and all vectors $(e + v_n^k)/|e + v_n^k|$ which is constructed in [5] can be interpreted as the process of remotest projections onto the family of closed convex sets consisting of the hyperplane $D = e^\perp$ and the half-spaces

$$C_{n,k} = \{y \in H : \langle y, e + v_n^k \rangle \le 0\}.$$

Starting with $x_0 = e_1$, the remotest projections algorithm generates $x_{m+1} = P_{C_{n,k}} x_m$ for even m (k and n depending on m) and $x_{m+1} = P_D x_m$ for odd m. For all m and k, the inequalities $\langle x_m, e_k \rangle \ge 0$ and $\langle x_m, e \rangle \le 0$ hold. The sequence $\{x_m\}$ converges to 0 weakly but not in norm; for more details see [5].

The hyperplane D and all the half-spaces $C_{n,k}$ are quasi-symmetric with respect to any point

$$a \in D \cap \left(\cap_{n,k} C_{n,k}^\circ \right),$$

where $C_{n,k}^\circ$ denotes the interior of the half-space $C_{n,k}$. We define the coordinates of such a point

$$a = (0, a^1, a^2, \dots)$$

with respect to the basis $\{e, e_k : k \in \mathbb{N}\}$, recursively:

$$a^1 = -1, a^{k+1} = a^k \tan \frac{\pi}{4n_k}.$$

Clearly, $a \in \ell_2$, $a \in D$, and (2.2) implies that

$$\langle a, e + v_n^k \rangle = \langle a, v_n^k \rangle = \langle a^k e_k + a^{k+1} e_{k+1}, v_n^k \rangle < 0,$$

since

$$\arg(a^k e_k + a^{k+1} e_{k+1}) = -\pi + \frac{\pi}{4n_k}$$

in the plane span $\{e_k, e_{k+1}\}$. Hence, $a \in C^\circ_{n,k}$ for each k and n. □

If the interior of the intersection of a family of closed convex sets is non-empty, then, clearly, the family is uniformly quasi-symmetric. In such a case, any sequence of projections onto these sets converges. For remote projections we even give an estimate of the rate.

Remark 3. *Let each closed convex set C_α contain the ball $B(a, r)$, $a \in H$, $r > 0$.*

(a) *The sequence (0.1) of remote projections converges in norm for each starting element $x_0 \in H$ and for any sequence $\{t_n\}$. In particular, random projections converge.*

(b) *If, moreover, $\sum t_n^2 = \infty$, then the limit point w belongs to $\bigcap_{\alpha \in \Omega} C_\alpha$, and the rate of convergence is estimated by*

$$|x_n - w| \le 2|x_0 - a| \prod_{k=0}^{n-1} \left(1 - \frac{t_k^2 r^2}{|x_0 - a|^2}\right)^{1/2}. \tag{2.3}$$

The statement (b) clarifies a result from [10]. There the convergence to a point in the intersection was shown under the condition $\sup_\alpha \operatorname{dist}(x_n, C_\alpha) \to 0$ as $n \to \infty$. Also, an exponential rate of convergence was established for remotest projections ($t_n \equiv 1$) with an estimate similar to ours.

Proof. (a) We assume $a = 0$ and use the notations $y_n = x_n - x_{n+1}$, $\varepsilon_n = \pi/2 - \angle 0 x_{n+1} x_n$, and also several inequalities from the proof of Theorem 1.

In view of (1.4), we have $\langle y_n, z - x_{n+1} \rangle \le 0$ for any $z \in B(0, r)$. Consequently,

$$|y_n||x_{n+1}| \sin \varepsilon_n = \langle y_n, x_{n+1} \rangle \ge \sup_{z \in B(0,r)} \langle y_n, z \rangle = r|y_n|,$$

so that

$$\sin \varepsilon_n \ge \frac{r}{|x_{n+1}|} \ge \frac{r}{|x_0|}.$$

This estimate together with (1.3) yields $\sum |y_n| < \infty$, meaning that x_n converge in norm.

(b) To prove that the limit point w belongs to $C = \bigcap_{\alpha \in \Omega} C_\alpha$ in case $\sum t_n^2 = \infty$, one can use the same arguments as in part 6 of the proof of Theorem 1.

Now we proceed to prove (2.3). Note that for any $n \in \mathbb{N}$ we have

$$|x_n - w| \le 2 \operatorname{dist}(x_n, C), \tag{2.4}$$

otherwise

$$|x_m - y| < \frac{|x_m - w|}{2}$$

for some $y \in C$ and $m \in \mathbb{N}$, so that

$$|x_n - y| \to |w - y| \geq |x_m - w| - |x_m - y| > |x_m - y|,$$

which contradicts the fact that the sequence $\{|x_n - y|\}$ is decreasing.

Let $n \in \mathbb{N}$ and

$$d_n = \sup_\alpha \operatorname{dist}(x_n, C_\alpha).$$

The ball $B(x_n, d_n)$ contains a point $p_\alpha \in C_\alpha$ for each α. Since the point

$$u_n = \frac{d_n}{d_n + r} a + \frac{r}{d_n + r} x_n$$

belongs to conv $\{p, B(a,r)\}$ for each $p \in B(x_n, d_n)$, we get $u_n \in C$, so that

$$\operatorname{dist}(x_n, C) \leq |x_n - u_n| = |x_n - a| \cdot \frac{d_n}{d_n + r}.$$

Consequently,

$$d_n \geq \frac{r \operatorname{dist}(x_n, C)}{|x_n - a| - \operatorname{dist}(x_n, C)} \geq \frac{r \operatorname{dist}(x_n, C)}{|x_0 - a|}.$$

Let $P_C x_n = b$. Since $b \in C_{\alpha(n)}$, the angle $\angle x_n x_{n+1} b$ is not less than $\pi/2$, so that

$$\operatorname{dist}(x_{n+1}, C)^2 \leq |x_{n+1} - b|^2 \leq |x_n - b|^2 - |x_n - x_{n+1}|^2$$
$$\leq \operatorname{dist}(x_n, C)^2 - t_n^2 d_n^2 \leq \operatorname{dist}(x_n, C)^2 \left(1 - \frac{t_n^2 r^2}{|x_0 - a|^2}\right).$$

Hence,

$$\operatorname{dist}(x_{n+1}, C) \leq \operatorname{dist}(x_0, C) \prod_{k=0}^{n} \left(1 - \frac{t_k^2 r^2}{|x_0 - a|^2}\right)^{1/2},$$

which together with (2.4) gives (2.3).

□

Assume that unlike the assumption in Remark 3 we deal with a family of slim sets: all C_α are hyperplanes $g_\alpha^\perp$. Then remote projections implement the Weak Greedy Algorithm with respect to the dictionary $D = \{\pm g_\alpha : \alpha \in \Omega\}$ and there are estimates of the rate of convergence for starting elements from $\overline{\operatorname{conv}} D$ [24, Ch. 2]. We wonder if any such estimates can be shown for a class of starting elements in the general setting of Theorem 1.

3. Weak convergence

Bregman [8] proved that for any family of general (non-symmetric) closed convex sets with nonempty intersection the remotest projections (0.1) with $t_n \equiv 1$ always converge weakly. He assumed that $\max_\alpha \operatorname{dist}(x, C_\alpha)$ is attained for each $x \in H$. It is quite natural to generalize this result to remote projections by slightly changing his arguments.

Theorem 2. *Assume $\{C_\alpha\}$ is a family of closed and convex sets in a Hilbert space H with a nonempty intersection $C = \bigcap_{\alpha \in \Omega} C_\alpha$. Let the sequence $\{t_n\}$ in $[0, 1]$ satisfy the following condition: there are $\delta > 0$ and $K \in \mathbb{N}$ so that for any $n \in \mathbb{N}$ at least one of the values $t_n, \ldots, t_{n+K}$ is greater than δ. Then the sequence (0.1) of remote projections converges weakly to some point of C for any starting element $x_0 \in H$.*

Proof. Take any $n \in \mathbb{N}$ and $k \in \{n, \dots, n+K\}$ so that $t_k > \delta$. We use the notation $y_\nu = x_\nu - x_{\nu+1}$. For any $\alpha \in \Omega$, we have

$$\operatorname{dist}(x_n, C_\alpha) \le |x_n - x_k| + \operatorname{dist}(x_k, C_\alpha) \le \sum_{i=n}^{k-1} |y_i| + \frac{\operatorname{dist}(x_k, C_{\alpha(k)})}{t_k}$$

$$\le \sqrt{K} \left(\sum_{i=n}^{n+K-1} |y_i|^2 \right)^{1/2} + \frac{|y_k|}{\delta} \to 0 \qquad (n \to \infty),$$

since $\sum |y_\nu|^2 < \infty$ in view of (1.1). Consequently, each partial weak limit w of the x_n's belongs to C. In its turn, this implies that the whole sequence $|x_n - w|$ is decreasing, since each P_α is a 1-Lipschitz retraction onto C_α.

A partial weak limit does exist, so we have to prove its uniqueness. Let v and w be two partial weak limits, so that x_{n_i} converge weakly to v and x_{m_j} converge weakly to w. The numbers

$$d_n := |x_n - v|^2 - |x_n - w|^2 = 2\langle v - x_n, v - w\rangle - |v - w|^2 = 2\langle x_n - w, w - v\rangle + |w - v|^2$$

tend to a single limit, as we have just mentioned. On the other hand, $d_{n_i} \to -|v-w|^2$ and $d_{m_j} \to |w - v|^2$. Hence, $v = w$. ☐

The following result was established by Dye and Reich in [9].

Corollary 2.1. *Assume $C_1, \dots, C_K$ are finitely many closed and convex subsets of H with a nonempty intersection $C = \bigcap_1^K C_j$. Assume $\{\alpha(n)\}$ is a quasi-periodic sequence of the indices $1, \dots, K$. Then the sequence $x_{n+1} = P_{\alpha(n)} x_n$ of nearest point projections converges weakly to a point in C for any starting point $x_0 \in H$.*

Proof. The sequence $\{\alpha(n)\}$ is quasi-periodic, which means that there is a constant $M \in \mathbb{N}$ so that for every interval I of length M the set $\{\alpha(n) : n \in I\}$ contains all of the indices $1, \dots, K$. As in the proof of Corollary 1.1 we choose $\beta(n) \in \{1, \dots, K\}$ and define $b_n > 0$ and weakness parameters $t_n \in [0, 1]$ as follows:

$$\operatorname{dist}(x_n, C_{\beta(n)}) = \max_k \operatorname{dist}(x_n, C_k) = b_n,$$

$$t_n = |x_{n+1} - x_n|/b_n.$$

Then for each interval I of length M there is an $n \in I$ so that $t_n \ge 1/(6M)$, as shown in the proof of Corollary 1.1. According to Theorem 2 the sequence $\{x_n\}$ converges weakly to a point of C. ☐

We do not know if the condition on the sequence $\{t_n\}$ in Theorem 2 is necessary for the weak convergence of remote projections. It is much stronger than the condition (T) implying the norm convergence of remote projections in the uniformly quasi-symmetric case. We do not know of an equivalent condition for the weak convergence in the uniformly quasi-symmetric case either. We give, however, criteria for remote projections to have a partial weak limit in the intersection of the sets considered.

Theorem 3. *For a sequence $\{t_n\}_{n=0}^\infty \subset [0, 1]$, the following statements are equivalent:*

(i) $\sum t_n^2 = \infty$;

(ii) *the sequence* $\{x_n\}$ *of remote projections (0.1) with parameters* t_n *has a partial weak limit in* $\bigcap_{\alpha\in\Omega} C_\alpha$ *for any starting element* $x_0 \in H$ *and any family* $\{C_\alpha\}_{\alpha\in\Omega}$ *of closed and convex sets in* H *with nonempty intersection;*

(iii) *the residuals* $\{x_n\}$ *in the Weak Greedy Algorithm (0.2) with parameters* t_n *have a partial weak limit 0 for any starting element* $x_0 \in H$ *and any dictionary* $D \subset S(H)$.

Proof. (i) ⇒ (ii). Let $\sum t_n^2 = \infty$, $0 \in C = \bigcap_{\alpha\in\Omega} C_\alpha$ and $\{x_n\}$ be the sequence (0.1). We denote $y_n = x_n - x_{n+1}$ again. Since $\sum |y_n|^2 < \infty$ in view of (1.1), we can choose a subsequence $\Lambda \subset \mathbb{N}$ with the property $|y_n|/t_n \to 0$, $n \in \Lambda$, $n \to \infty$. Taking any $\alpha \in \Omega$ and $n \in \Lambda$, we get

$$\operatorname{dist}(x_n, C_\alpha) \le \frac{\operatorname{dist}(x_n, C_{\alpha(n)})}{t_n} = \frac{|y_n|}{t_n} \to 0, \qquad n \to \infty.$$

Consequently, any partial weak limit of $\{x_n\}_{n\in\Lambda}$ belongs to C. In fact, this partial weak limit in C is unique, as we have seen in the proof of Theorem 2. However, there may be other partial weak limits outside of C, as Example 3 below shows.

(ii) ⇒ (iii). This is obvious, since the WGA is a particular case of remote projections onto a family of hyperplanes having unique common point 0.

(iii) ⇒ (i). Let $\sum t_n^2 < \infty$. In what follows we construct a countable family $\{C_n\}$ of one-dimensional subspaces and a sequence (0.1) of remote projections onto this family with parameters t_n, which does not converge weakly and does not have 0 as a partial weak limit. Remark 1 then supplies an example of the WGA with parameters t_n whose residuals do not have 0 as a partial weak limit.

We choose a sequence $\{\tau_n\}$ with the properties $\tau_n \ge t_n$ for all n, $\sum \tau_n^2 < \infty$, $\sum \tau_n = \infty$, and fix m so that

$$\sum_{n=m}^{\infty} \tau_n^2 < \frac{1}{4}. \tag{3.1}$$

We fix a point s on the unit sphere of H and take the spherical cap

$$V(s) = \left\{ v \in S(H) : \langle v, s\rangle \ge \frac{\sqrt{3}}{2} \right\}.$$

We choose two *opposite* points a and b on the boundary of the cap: $\langle a, s\rangle = \langle b, s\rangle = \sqrt{3}/2$. We also choose a sequence $\{s_n\}_{n=m}^{\infty} \subset V(s)$ so that $s_m = a$ and $\langle s_n, s_{n+1}\rangle = \sqrt{1-\tau_n^2}$ for all $n \ge m$. This sequence can be constructed inductively: the choice of each next $s_{n+1} \in V(s)$ is possible, since $\sqrt{1-\tau_n^2} \ge \sqrt{3}/2$ by (3.1). Moreover, we have

$$|s_n - s_{n+1}| = \sqrt{2 - 2\sqrt{1-\tau_n^2}} \ge \tau_n,$$

so that $\{s_n\}_{n=m}^{\infty}$ may be made dense in $V(s)$, since $\sum \tau_n = \infty$.

Denoting by $L(v) := \operatorname{span}\{v\}$ the line spanned by a vector $v \in S(H)$, we consider the family of lines

$$L(a), L(b), L(s_n), \qquad n = m, m+1, \dots,$$

and the following sequence of remote projections onto this family of lines with starting element $x_0 = s$. The projections $x_1, \dots, x_m$ alternately lie on the lines

$L(a)$ and $L(b)$ so that they are remotest and satisfy the inequalities needed for any given parameters $t_0, \dots, t_{m-1}$. We choose the first projection x_1 lying either on $L(a)$ or $L(b)$ so that $x_m \in L(a)$, depending on the parity of m. As for $n \geq m$, we set x_{n+1} to be the projection of x_n onto $L(s_{n+1})$:

$$\begin{aligned}\operatorname{dist}(x_n, L(s_{n+1})) =& |x_n| \sin \angle(s_n 0 s_{n+1}) \\ =& |x_n|\tau_n \geq |x_n| t_n \geq t_n \sup_{v \in V(s)} \operatorname{dist}(x_n, L(v))\end{aligned}$$

Clearly, the sequence $\{x_n\}$ is contained in the cone $\{\lambda v : v \in V(s),\ \lambda > 0\}$. For any $n > m$, we have

$$|x_n|^2 = |x_m|^2 - \sum_{k=m}^{n-1} |x_k|^2 \tau_k^2 \geq |x_m|^2 \left(1 - \sum_{k=m}^{\infty} \tau_k^2\right) \geq \frac{3}{4}|x_m|^2.$$

This means that $|x_n| \to r > 0$, and the set of all partial weak limits of the sequence $\{x_n\}$ is the closed convex hull of the cap $rV(s)$.

□

The following Example shows that the conditions on the sequence $\{t_n\}$ in Theorem 2 cannot be replaced by $\liminf_{n\to\infty} t_n > 0$ and that in Theorem 3 one cannot claim the uniqueness of the weak limit.

Example 3. *Let H be an infinite dimensional Hilbert space. Then there exists a countable family of closed convex sets in H with non-empty intersection and a sequence (0.1) of remote projections on this family which does not converge weakly and its weakness parameters satisfy* $\liminf_{n\to\infty} t_n > 0$.

Proof. 1. We use the following local construction.

Lemma A. *[14, Section 2.2] Let $a, b, c \in H$ be such that $|a| = |b| \neq 0$ and $0 \neq c \in \{a, b\}^{\perp}$. For every $\varepsilon > 0$ there exists a convex closed cone $C = C(a, b, c, \varepsilon) \subset$* span $\{a, b, c\}$ *with vertex 0 so that alternating projections between C and the plane $D =$* span $\{a, b\}$ *move the point a close to the point b:*

$$|(P_D P_C)^m a - b| < \varepsilon \tag{3.2}$$

for some $m = m(a, b, c, \varepsilon)$.

The cone from Lemma A also satisfies

$$\operatorname{dist}(a, C) \leq \sqrt{2|a|\varepsilon}, \tag{3.3}$$

since otherwise

$$|(P_D P_C)^m a| \leq |P_C a| = \sqrt{|a|^2 - \operatorname{dist}(a, C)^2} < |a| - \varepsilon = |b| - \varepsilon,$$

which contradicts (3.2). Similarly,

$$\left|P_C (P_D P_C)^{m-1} a - (P_D P_C)^m a\right| = \operatorname{dist}(P_C (P_D P_C)^{m-1} a, D) \leq \sqrt{2|a|\varepsilon},$$

and hence

$$\begin{aligned}\operatorname{dist}(b, C) \leq& \left|b - P_C (P_D P_C)^{m-1} a\right| \leq |b - (P_D P_C)^m a| \\ &+ \left|P_C (P_D P_C)^{m-1} a - (P_D P_C)^m a\right| \leq \varepsilon + \sqrt{2|a|\varepsilon}.\end{aligned} \tag{3.4}$$

2. We may assume that H is separable and fix an orthonormal basis $\{u, v, e_k : k \in \mathbb{N}\}$ of H. We choose a decreasing sequence $\varepsilon_n \searrow 0$ so that

$$\sum_{k=1}^{\infty} \sqrt{\varepsilon_k} < \frac{1}{10}.$$

We set

$$\begin{aligned} D &= v^{\perp} = \overline{\operatorname{span}}\{u, e_k : k \in \mathbb{N}\}, \\ C_1 &= C(e_1, u, v, \varepsilon_1) + \overline{\operatorname{span}}\{e_n : n \in \mathbb{N}, n \neq 1\}, \\ m_1 &= m(e_1, u, v, \varepsilon_1), \\ C_2 &= C(u, e_2, v, \varepsilon_2) + \overline{\operatorname{span}}\{e_n : n \in \mathbb{N}, n \neq 2\}, \\ m_2 &= m(u, e_2, v, \varepsilon_2), \\ &\dots \\ C_{2k-1} &= C(e_k, u, v, \varepsilon_{2k-1}) + \overline{\operatorname{span}}\{e_n : n \in \mathbb{N}, n \neq k\}, \\ m_{2k-1} &= m(e_k, u, v, \varepsilon_{2k-1}), \\ C_{2k} &= C(u, e_{k+1}, v, \varepsilon_{2k}) + \overline{\operatorname{span}}\{e_n : n \in \mathbb{N}, n \neq k+1\}, \\ m_{2k} &= m(u, e_{k+1}, v, \varepsilon_{2k}), \\ &\dots . \end{aligned}$$

Clearly, (3.2) works for the extended cones C_{2k-1} and C_{2k} as well:

$$\left|(P_D P_{C_{2k-1}})^{m_{2k-1}} e_k - u\right| < \varepsilon_{2k-1}, \tag{3.5}$$

$$|(P_D P_{C_{2k}})^{m_{2k}} u - e_{k+1}| < \varepsilon_{2k}. \tag{3.6}$$

3. We have $e_k \in D$ for all k and $e_k \in C_n$ for $n \neq 2k-2, 2k-1$ by construction, hence also

$$\begin{aligned} \operatorname{dist}(e_k, C_{2k-1}) &< \sqrt{2\varepsilon_{2k-1}}, \\ \operatorname{dist}(e_k, C_{2k-2}) &< \sqrt{2\varepsilon_{2k-2}} + \varepsilon_{2k-2} \end{aligned}$$

by (3.3) and (3.4). This implies for P being a projection onto C_n or D that

$$|e_k - Pe_k| < 3\sqrt{\varepsilon_{2k-2}}, \qquad k = 2, 3, \dots . \tag{3.7}$$

4. Now we define the required sequence of remote projections on the family $\{D, C_n : n \in \mathbb{N}\}$.

We start with $x_0 = e_1$ and make m_1 alternating projections on C_1 and D:

$$y_1 = (P_D P_{C_1})^{m_1} e_1.$$

Then we make m_2 alternating projections on C_2 and D:

$$y_2 = (P_D P_{C_2})^{m_2} y_1.$$

Then we make the projection P_2 on one of the sets C_n so that

$$|y_2 - P_2 y_2| \geq \frac{1}{2} \sup_n \operatorname{dist}(y_2, C_n),$$

and we set

$$z_2 = P_2 y_2.$$

We proceed by induction: having defined $y_1, y_2, \ldots, y_{2k-2}$ and $z_2, z_4, \ldots, z_{2k-2}$, we make m_{2k-1} alternating projections on C_{2k-1} and D:

$$y_{2k-1} = (P_D P_{C_{2k-1}})^{m_{2k-1}} z_{2k-2},$$

then m_{2k} alternating projections on C_{2k} and D:

$$y_{2k} = (P_D P_{C_{2k}})^{m_{2k}} y_{2k-1},$$

and then one projection P_{2k} on one of the sets C_n so that

$$|y_{2k} - P_{2k}y_{2k}| \geq \frac{1}{2} \sup_n \operatorname{dist}(y_{2k}, C_n), \tag{3.8}$$

and we set

$$z_{2k} = P_{2k}y_{2k}.$$

For this sequence of projections, containing subsequences $\{y_k\}$ and $\{z_{2k}\}$, we have

$$\liminf_{n\to\infty} t_n \geq \frac{1}{2},$$

since projections via P_{2k} have $t_n \geq 1/2$ by (3.8).

5. At last we have to prove that the sequence $\{y_k\}$ does not converge weakly, and hence the whole sequence of projections has no weak limit.

We have $|y_1 - u| < \varepsilon_1$ by (3.5) for $k = 1$. Using (3.6) for $k = 1$, we get

$$|y_2 - e_2| \leq |(P_D P_{C_2})^{m_2}(y_1 - u)| + |(P_D P_{C_2})^{m_2} u - e_2| < \varepsilon_1 + \varepsilon_2,$$

which together with (3.7) implies that

$$|z_2 - e_2| = |P_2 y_2 - e_2| \leq |P_2(y_2 - e_2)| + |P_2 e_2 - e_2| < \varepsilon_1 + \varepsilon_2 + 3\sqrt{\varepsilon_2}.$$

In the same way, by induction on k, we get

$$|y_{2k-1} - u| < \sum_{\nu=1}^{2k-1} \varepsilon_\nu + 3\sum_{\nu=1}^{k-1} \sqrt{\varepsilon_{2\nu}},$$

$$|y_{2k} - e_{k+1}| < \sum_{\nu=1}^{2k} \varepsilon_\nu + 3\sum_{\nu=1}^{k-1} \sqrt{\varepsilon_{2\nu}},$$

$$|z_{2k} - e_{k+1}| < \sum_{\nu=1}^{2k} \varepsilon_\nu + 3\sum_{\nu=1}^{k} \sqrt{\varepsilon_{2\nu}}.$$

Consequently,

$$|y_{2k-1} - u| < 0.4 \Rightarrow \langle y_{2k-1}, u\rangle > 0.6$$
$$|y_{2k} - e_{k+1}| < 0.4 \Rightarrow \langle y_{2k}, u\rangle < 0.4$$

and the sequence $\{y_n\}$ does not converge weakly. □

The following statement is a parallel to Remark 3 (a). We consider here weak convergence instead of norm convergence. The intersection of the sets is not contained in any affine hyperplane in place of having non-empty interior.

Remark 4. *Let $\{C_\alpha\}$ be a family of closed convex subsets of a Hilbert space H. Assume that the affine hull of the intersection $C = \bigcap C_\alpha$ is dense in H. Then the sequence (0.1) of remote projections converges weakly for each starting element $x_0 \in H$ and for any sequence $\{t_n\}$ of weakness parameters. In particular, random projections converge weakly in this case.*

Proof. Fix a point $a \in C$. Then $\overline{\text{span}}\{v - a : v \in C\} = H$. For each $v \in C$ and any $n \in \mathbb{N}$, we have $v \in C_{\alpha(n)}$, hence $\angle v x_{n+1} x_n \geq \pi/2$, so that $|x_n - v| \geq |x_{n+1} - v|$, and the decreasing sequence

$$|x_n - v|^2 = |x_n - a|^2 - 2\langle x_n - a, v - a\rangle + |v - a|^2 \tag{3.9}$$

has a limit. In particular, the sequence $\{|x_n - a|^2\}$ has a limit, which together with (3.9) implies that the sequence of scalar products

$$\langle x_n - a, v - a\rangle$$

has a limit as well. The sequence $\{x_n - a\}$ is bounded and the set span $\{v - a : v \in C\}$ is dense in H, hence the sequence $\{x_n - a\}$ converges weakly, and so does the sequence $\{x_n\}$. ☐

Dye and Reich [9] proved weak convergence of random projections on a finite family of closed convex sets that are all WIP sets with respect to their common point, see also [7]. Such sets are uniformly quasi-symmetric with respect to this point by Remark 2. We wonder if Theorem 2 and Theorem 3 can be clarified under the additional condition of uniform quasi-symmetry of the sets C_α. We also note that the problem of weak convergence of random projections, that is, remote projections with arbitrary t_n's, onto a finite family of closed convex sets having nonempty intersection is still open [7].

Acknowledgements

We thank S. Reich, V.N. Temlyakov, and V.I. Bogachev for fruitful discussions.

References

[1] I. Amemiya and T. Ando, *Convergence of random products of contractions in Hilbert space*, Acta Sci. Math. (Szeged) **26** (1965), 239–244.

[2] J. B. Baillon, R. E. Bruck and S. Reich, *On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces*, Houston J. Math. **4** (1978), 1–9.

[3] H. H. Bauschke and P. L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*, Springer, 2017.

[4] V. I. Bogachev and O. G. Smolyanov, *Topological vector spaces and their applications*, Springer, 2017.

[5] P. A. Borodin, *Example of divergence of a greedy algorithm with respect to an asymmetric dictionary*, Math. Notes **109** (2021), 379–385.

[6] P. Borodin and E. Kopecká, *Alternating projections, remotest projections, and greedy approximation*, J. Approx. Th. **260** (2020), 105486.

[7] P. A. Borodin and E. Kopecká, *Weak limits of consecutive projections and of greedy steps*, Proc. Steklov Inst. Math. **319** (2022), 56–63.

[8] L. M. Bregman, *The method of successive projection for finding a common point of convex sets*, Soviet Mathematics Doklady **6** (1965), 688–692.

[9] J. M. Dye and S. Reich, *Unrestricted iterations of nonexpansive mappings in Hilbert space*, Nonlinear Analysis, **18**:2 (1992), 199–207.

[10] L. G. Gubin, B. T. Polyak and E. V. Raik, *The method of projections for finding the common point of convex sets*, USSR Computational Mathematics and Mathematical Physics, **7**:6 (1967), 1–24.
[11] I. Halperin, *The product of projection operators*, Acta Sci. Math. (Szeged) **23** (1962), 96–99.
[12] H. S. Hundal, *An alternating projection that does not converge in norm*, Nonlinear Anal. **57** (2004), 35–61.
[13] L. Jones, On a conjecture of Huber concerning the convergence of projection pursuit regression, Ann. Stat., **15**:2 (1987), 880–882.
[14] E. Kopecká, *Spokes, mirrors and alternating projections*, Nonlinear Analysis, **68** (2008), 1759–1764.
[15] E. Kopecká, *When products of projections diverge*, J. London Math. Soc. (2) **102** (2020), 345–367.
[16] E. Kopecká and V. Müller, *A product of three projections*, Studia Math. **223** (2014), 175–186.
[17] E. Kopecká and A. Paszkiewicz, *Strange products of projections*, Israel J. Math. **219** (2017), 271–286.
[18] E. Matoušková and S. Reich, *The Hundal example revisited*, J. Nonlinear Convex Anal. **4** (2003), 422–427.
[19] J. von Neumann, *On rings of operators. Reduction theory*, Ann. of Math. **50** (1949), 401–485.
[20] S. Reich and R. Zalas, *Polynomial estimates for the method of cyclic projections in Hilbert spaces*, Numer. Algor. **94** (2023), 1217–1242.
[21] M. Sakai, *Strong convergence of infinite products of orthogonal projections in Hilbert space*, Appl. Anal. **59** (1995), 109–120.
[22] V. N. Temlyakov, *Weak greedy algorithms*, Advances in Comp. Math., **12** (2000), 213–227.
[23] V. Temlyakov, *A criterion for convergence of Weak Greedy Algorithms*, Adv. Comp. Math., **17** (2002), 269–280.
[24] V. Temlyakov, *Greedy approximation*, Cambridge, 2011.

Manuscript received November 30 2023
revised December 10 2023

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