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REMARKS ON MORREY'S QUASI-CONVEXITY

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Dedicated to Professor David Preiss on the occasion of his 75th birthday

Abstract. We revisit the question of whether the functions defined of the real $m \times n$ matrices that are convex along rank-one directions are also quasi-convex in the sense of Morrey. Using the linearity of the map $f \to \int_{\mathbf{T}^n} f(\nabla u(x))\, dx$, we propose to study the question as a problem in convex optimization. This might be useful when trying to resolve the open cases, such as the case $m = 2$, or various cases with symmetries.

1. Introduction

We consider variational integrals of the form

$$\int_\Omega f(\nabla u(x))\, dx \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ is an open set, $u\colon \Omega \to \mathbf{R}^m$ is a Lispchitz function, ∇u denotes the $m \times n$ matrix of the partial derivatives of u, and f is a continuous function on the set $M^{m\times n}$ of all real $m \times n$ matrices. We recall that f is *Morrey quasi-convex* if

$$\int_{\mathbf{R}^n} \left(f(A + \nabla\varphi(x)) - f(A)\right) dx \geq 0 \tag{1.2}$$

for each compactly supported $\varphi\colon \mathbf{R}^n \to \mathbf{R}^m$ and each $A \in M^{m\times n}$. Condition (1.2) was introduced by Ch. B. Morrey in [12], see also [13]. This will be the only notion of quasi-convexity discussed in this note, and therefore we can use the term *quasi-convex* instead of Morrey quasi-convex in what follows. Denoting by $\mathbf{T}^n$ the $n-$dimensional torus $\mathbf{R}^n/\mathbf{Z}^n$, it is well-known (and not hard to see) that an equivalent definition of quasi-convexity is

$$\int_{\mathbf{T}^n} \left(f(A + \nabla\varphi) - f(A)\right) dx \geq 0 \tag{1.3}$$

for each $\varphi\colon \mathbf{T}^n \to \mathbf{R}^m$ and $A \in M^{m\times n}$. A necessary condition for quasi-convexity is the convexity of f along any line in $M^{m\times n}$ the direction vector of which is a rank-one matrix. This is called *rank-one convexity.* For a C^2-function f this amounts to

$$\frac{\partial^2 f(X)}{\partial X^i_\alpha \partial X^j_\beta}\xi_\alpha\xi_\beta\eta^i\eta^j \geq 0\,, \qquad X \in M^{m\times n}\,,\ \xi \in \mathbf{R}^n\,,\ \eta \in \mathbf{R}^m\,. \tag{1.4}$$

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It is known that in general, this condition is not sufficient for quasi-convexity when $n \geq 2$ an $m \geq 3$, see [5, 16]. The case $m = 2$ seems to be open, although we would like to draw the attention of the reader to the recent preprint [6] where a numerical method for constructing a potential counterexample is discussed. The question also seems to be open for certain cases with $m \geq 3$ when we restrict the class of the admissible functions f by symmetries. In addition to [6], previous numerical exploration of the relations between rank-one convexity and quasi-convexity can be found for example in [8]. Important theoretical results relevant to Morrey quasi-convexity can be found for example in [1, 7, 10, 11, 14].

In this note, we wish to explore the fact that, for a fixed deformation φ, determining whether the basic quasi-convexity inequality (1.3) holds for all rank-one convex functions f essentially amounts to convex optimization. At a conceptual level, this is a straightforward consequence of the linearity of the expression in (1.3) with respect to f, and the fact that the set of rank-one convex functions forms a convex cone. In practice, convex optimization is most effective when optimizing over compact convex sets. For numerical experiments, these sets should be finite-dimensional as well.

A key notion in this context is that of *a convex cone with a compact base.* We say that a closed cone $\mathcal{Y}$ in a locally convex (Hausdorff) topological linear space $\mathcal{E}$ (over the real numbers) has a compact base if there exists a continuous linear functional $\ell \colon \mathcal{E} \to R$ that is strictly positive on $\mathcal{Y}$ such that the set $\mathcal{X} = \mathcal{Y} \cap \{\ell = 1\}$ is compact. (There are various definitions of the notion of the base of a cone in the literature, see, for example, [2, 4]. The definition above seems to be suitable for our purposes here.) When μ is another continuous linear functional on $\mathcal{E}$ and $\mathcal{Y}$ is a closed convex cone with a compact base $\mathcal{X}$, one can reduce the study of the question of whether μ is positive on $\mathcal{Y}$ to the minimization of μ over the compact set $\mathcal{X}$. This is a classical problem of convex optimization and we have various tools at our disposal in that situation. In particular, we can try to use finite-dimensional approximations, approximations by polytopes, and linear programming.

Rank-one convexity is often considered to be easier than quasi-convexity because its definition is local. While this is no doubt a valid viewpoint, there also seems to be some truth in the statement that the challenge of deciding the validity of (1.3) is often related to our incomplete understanding of the cone of the rank-one convex functions. Our effort will be directed toward finding a suitable setup in which the cone of rank-one convex functions, after suitably factoring out the rank-one affine functions[1], becomes as close to a cone with a compact base as possible. While we do not quite achieve this goal (at least not without some form of "completion"), we do obtain a satisfactory setup for polynomial approximations. The setup should be applicable to various other approximation methods as well. Our main results are Theorem 2.6 and Corollary 2.7. We also mention Lemma 4.3 that concerns approximations by polytopes. Various forms of the lemma can be probably found in the literature, but we include it here for completeness.

The main idea is that, after factoring out the rank-one affine functions, the suitable linear functional ℓ on rank-one convex functions on a unit ball B in $M^{m\times n}$

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[1] also known as *null Lagrangians*

that *almost* results in a compact base (although not quite) is

$$\frac{1}{|S|}\int_S f(X)\,dX\,, \qquad S=\partial B\,. \tag{1.5}$$

It turns out that rank-one convex functions are sub-harmonic, and one can use this fact in combination with simple representation formulae for harmonic functions and elementary estimates to obtain the desired results, utilizing the convexity properties of the cone of rank-one convex functions.

2. Compactness

Clearly the expression (1.3) is linear in f and the class of rank-one convex functions forms a cone. However, for optimization it is good to have compactness, in addition to convexity. Identifying a suitable class of the functions f where one can obtain compactness will be our next task.

Let us recall some terminology. Let $\mathcal{Y}$ be a closed convex cone in a locally convex (Hausdorff) topological vector space $\mathcal{E}$. As discussed in the introduction, we will say that $\mathcal{Y}$ has a compact base if there exists a continuous linear functional $l\colon \mathcal{E}\to\mathbf{R}$ that is strictly positive on $\mathcal{Y}\setminus\{0\}$ such that the set $\mathcal{X}=\mathcal{Y}\cap\{l=1\}$ is compact. This is a favorable situation in which one can essentially work with $\mathcal{X}$ rather than $\mathcal{Y}$, using the compactness of $\mathcal{X}$. As we mentioned in the introduction, our definition here is closely related to that in [4], but other definitions are used in the literature, see [2], for example.

2.1. A 1d Toy Model with convex functions.

As a very simple example, consider the cone $\mathcal{Y}_{\rm cf}$ in $C[-1,1]$ of the convex functions on the interval $[-1,1]$. One obstacle to the existence of a suitable base for this cone is that $\mathcal{Y}_{\rm cf}$ is not *proper* (using the terminology in [4]), in the sense that it contains a non-trivial linear subspace. We can replace $\mathcal{Y}_{\rm cf}$ by the cone $\mathcal{Y}$ of convex functions on $[-1,1]$ that satisfy $f(0)=0$ and $f\geq 0$. The cone $\mathcal{Y}$ is proper. Defining

$$l(f)=\frac{1}{2}(f(-1)+f(1))\,, \tag{2.1}$$

the set $\mathcal{X}=\mathcal{Y}\cap\{l=1\}$ almost works, the only minor issue is that to obtain compactness, we have to extend our set of functions by relaxing the continuity requirement: instead of requiring our functions to be continuous on $[-1,1]$, we require them to be continuous only on $(-1,1)$, while still imposing the convexity and the conditions $f(0)=0$ and $f\geq 0$ on $[-1,1]$. We denote by $\bar{\mathcal{Y}}$ the cone of such convex functions f and let $\bar{\mathcal{X}}=\bar{\mathcal{Y}}\cap\{l=1\}$. The set $\bar{\mathcal{X}}$ then can be identified with the set of probability measures on $[-1,1]$ via the representation

$$f(x)=\int F(x,t)\,d\nu(t)\,, \tag{2.2}$$

where F is defined for $t\in[0,1)$ by

$$F(x,t)=\begin{cases}0 & \text{when } x<t\\ 2(x-t)/(1-t) & \text{when } x\in[t,1]\end{cases}\,, \tag{2.3}$$

$$F(x,1) = \begin{cases} 2 & \text{when } x = 1 \\ 0 & \text{when } x \in [-1,1)\,, \end{cases} \tag{2.4}$$

and analogously for $t \in [-1,0)$. The measures ν which charge the set $\{-1,1\}$ have to be added to the original continuous functions setup to obtain compactness.

When we restrict our attention to suitable finite-dimensional subspaces, no completion is necessary. For example, letting $\mathcal{Y}_N$ to be the set of polynomials f of degree $\leq N$ such that $f(0)=0$, $f \geq 0$ in $[-1,1]$ and $f'' \geq 0$ in $[-1,1]$, the set

$$\mathcal{X}_N = \mathcal{Y}_N \cap \{\ell = 1\} \tag{2.5}$$

will clearly be a closed compact subset of a finite-dimensional linear space over the real numbers.

If our goal is to optimize some linear functionals over $\mathcal{X}$, we can try to avoid the issues related to the infinite-dimensionality by working in $\mathcal{X}_N$. (For the simple model above one can probably deal directly with $\bar{\mathcal{X}}$ in many problems.) In case one wishes to deal with finite-dimensional approximations, the simplicity of the toy model gives us many reasonable possibilities for finite-dimensional approximations. For example, instead of polynomials, one could use continuous piece-wise affine functions.[2] This aspect becomes more subtle when we work in the more complicated setting of rank-one convex functions in (subsets of) $M^{m\times n}$.

Another way to factor out the affine functions from $\mathcal{Y}_{\rm cf}$ and obtain a proper cone is to choose two points $-1 < \alpha < \beta < 1$ and restrict our attention to the cone $\mathcal{Y}_{\rm z} = \mathcal{Y}_{\rm cf} \cap \{f\,,\, f(\alpha)=0\,,\, f(\beta)=0\}$. Starting with $\mathcal{Y}_{\rm z}$, one can then proceed similarly as with $\mathcal{Y}$. This choice may be in fact more suitable for the polynomial approximations. We leave the details for interested readers. Below we will employ a similar choice in the context of the rank-one convex functions.

2.2. **Convex functions in dimensions $d \geq 2$.** One can adapt the model above to $\mathbf{R}^n$ in a natural way as follows. We replace the unit interval $[-1,1]$ by the unit ball $B^n = \{x \in \mathbf{R}^n\,,\, |x| \leq 1\}$ and the functional l above by

$$\ell(f) = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(x)\,dx\,, \quad S^{n-1} = \partial B^n\,. \tag{2.6}$$

One can now consider the cone $\mathcal{Y}$ of non-negative convex Borel-measurable functions on B^n that are finite in the interior of B^n, vanish at $x=0$, but are not necessarily finite or continuous at the boundary of B^n. We can try to take $\mathcal{X} = \mathcal{Y} \cap \{\ell = 1\}$ as the base of the cone. However, to obtain compactness, we would need to compactify $\mathcal{X}$ by allowing the restrictions of the functions on B^n to S^{n-1} to be probability measures. One important difference with the case $d=1$ is that it is not clear whether one has a nice representation similar to (2.2). The difficulty is that the set $\mathcal{X}$ or its suitable completion is presumably no longer a simplex in the sense of the Choquet theory.

[2] In fact, this choice would make the description of $\mathcal{X}_N$ easier than it would be when we work with polynomials. This is because, for piece-wise linear approximations on a fixed grid, one can easily identify the extremal points in the cone of the piece-wise affine convex functions. They are essentially the functions $F(x,t)$ above for suitable values of t. The situation for polynomials is more difficult because in terms of the terminology used in [4], for the polynomial approximation, the set $\mathcal{X}_N$ is no longer a simplex.

We will not go into the technicalities, as our main interest will again be in the case when $\mathcal{Y}$ is replaced by the intersection of $\mathcal{Y}$ with some finite-dimensional space of functions, such as polynomials of degree $\leq N$. The situation is then quite similar to the case $d = 1$, except for the fact that we are not dealing with simplices.

We remark that we could also factor out the affine functions by restricting our attention to convex functions that vanish at some given $n + 1$ points in the interior of B that are "affinely independent". We leave the details to the interested reader.

2.3. **Rank-One Convex Functions on $M^{m\times n}$.** We now turn our attention to the rank-one convex functions. Our first task is to show that the problem with the existence/non-existence of a suitable base of the cone of the rank-one convex functions is somewhat similar to the simple situation with the convex functions above.

Although ultimately we will be dealing with finite-dimensional submanifolds of the rank-one convex functions in which each function is finite and well-defined everywhere, for more general considerations it makes sense to allow functions with values in the extended real line $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$. A function $f : [a, b] \to \overline{\mathbf{R}}$ is convex if the usual convexity condition $f((1 - t)x_1 + tx_2) \leq (1 - t)f(x_1) + tf(x_2)$ is satisfied for each $x_1, x_2 \in [a, b]$ and each $t \in [0, 1]$, with the usual conventions that we have $\infty + x = \infty$, $0\infty = 0$, $t\infty = \infty$ and $x < \infty$ for any $x \in \mathbf{R}, t \in [0, 1]$.

Definition 2.1.

(a) Let $\mathcal{O} \subset M^{m\times n}$ be a convex set with a non-empty interior. A function $f : \mathcal{O} \to \overline{\mathbf{R}}$ is rank-one convex in $\mathcal{O}$ if it is finite in the interior of $\mathcal{O}$ and convex on the intersection of $\mathcal{O}$ with any line in a rank-one direction.

(b) A function f defined on an open subset of $M^{m\times n}$ is locally rank-one convex if it is rank-one convex on each ball contained in the set.

The following observation will be useful:

Lemma 2.2. *Let $\mathcal{O} \subset M^{m\times n}$ be an open set and let $f : \mathcal{O} \to \mathbf{R}$ be locally rank-one convex. Then f is subharmonic in the sense that $\Delta f \geq 0$ in distributions, where Δ is the standard Laplace operator in $M^{m\times n} \sim \mathbf{R}^{mn}$.*

Proof. Writing a matrix X in coordinates X_{ij} we can write the Laplacian as

$$\Delta = \sum_{i,j} \frac{\partial^2}{\partial X_{ij}^2} \,. \tag{2.7}$$

Assume first that f is twice differentiable. Each direction $\partial/\partial X_{ij}$ is a rank-one direction, and hence f is convex along it, by the local rank-one convexity. The general case of possibly non-smooth functions f can be handled for example by approximation: We replace f by $f_\varepsilon = f * \phi_\varepsilon$ for suitable mollifiers ϕ_ε. (The function f_ε will be defined on a set $\mathcal{O}_\varepsilon$ that is slightly smaller than $\mathcal{O}$, but the sets will $\mathcal{O}_\varepsilon$ will approach $\mathcal{O}$ as $\varepsilon \searrow 0$.) The mollification will preserve the rank-one convexity and hence f_ε will safisfy $\Delta f_\varepsilon \geq 0$, and this will be preserved in the limit (in the distributional sense). □

Lemma 2.3. *Let $B \subset M^{m\times n}$ be a closed ball and let $f : B \to \overline{\mathbf{R}}$ be rank-one convex. Then f is bounded from below.*

Proof. We can assume that B is centered at $X = 0$. Since any locally rank-one convex function is continuous in the interior of B, the function f must be bounded on any compact subset of the interior of B. In particular, for each $R < 1$ we have

$$\sup\{|f(X)|\,,\,|X| < R\} \leq c(R, f) < \infty. \tag{2.8}$$

Let $X \in B$ and let us write $X = \sum_{j=1}^{m} e_j \otimes a^j$ where a^j are $n-$vectors and e_j is the $j-$th vector of the canonical basis of $\mathbf{R}^m$. Choosing j_0 so that that $|a^{j_0}| \geq |X|/\sqrt{m}$, we see that the cosine of the angle between the rank-one matrix $e_{j_0} \otimes a^{j_0}$ and X is $\geq 1/\sqrt{m}$. Now the convexity of f on the line $X + te_{j_0} \otimes a^{j_0}$ together with the bound (2.8) give the desired result. $\square$

Lemma 2.4. *Let $B \subset M^{m\times n}$ be a closed ball with center $\overline{X}$ and let $f\colon B \to \overline{\mathbf{R}}$ be a Borel measurable rank-one convex function. Then*

$$\frac{1}{|\partial B|}\int_{\partial B} f(X)\,dX \geq f(\overline{X})\,. \tag{2.9}$$

Moreover, the equality holds if and only if f is rank-one affine (in the sense that both f and $-f$ are rank-one convex).

Proof. We can assume that $\overline{X} = 0$ and the radius of B is one. The function f is bounded below on ∂B by Lemma 2.3, so there is no issue with the definition of the integral. When f is continuous up to the boundary, the inequality (2.9) is a consequence of f being bounded below, and subharmonic. In the general case one can use the rank-one convexity of f "up to the boundary" to show that

$$\limsup_{R\nearrow 1}\int_{\partial B_R} f(X)\,dX \leq \int_{\partial B} f(X)\,dX\,. \tag{2.10}$$

Let us sketch an argument by which one can obtain (2.10). First, we note that a convex $\overline{\mathbf{R}}$-valued function g on a closed interval $[a, b]$ satisfies

$$g(b) \geq \lim\sup_{x\to b_-} g(x)\,. \tag{2.11}$$

If we take a small piece Σ of the boundary of B and move it inside along a rank-one line, we can establish a local version of (2.10) for Σ. (Note that we can use the rank-one line from the proof of Lemma 2.3 for this purpose.) Putting together the local pieces, we obtain (2.10).

In the case of equality, the function f has to be harmonic (as we know that it is subharmonic) and therefore it is smooth in the interior of B. Assume that f is not affine along some rank-one line with direction $a \otimes b \neq 0$. Let Q_1, Q_2 be matrices in $\mathrm{SO}(m)$ and $\mathrm{SO}(n)$, respectively, such that $Q_1 a = (1, 0, \ldots, 0) \in \mathbf{R}^m$ and $Q_2 b = (1, \ldots, 0) \in \mathbf{R}^n$, respectively. Let $g\colon B \to \mathbf{R}$ be defined by $g(X) = f(Q_1^{-1} X Q_2)$. Then $\partial^2 g/\partial {X_{11}}^2 > 0$ on some open set. Since the map $X \to Q_1 X Q_2^{-1}$ is an isometry of $M^{m\times n}$ and preserves the cone of rank-one matrices, we see that $\Delta g > 0$ on an open subset of B. Hence also $\Delta f > 0$ on an open subset of B and the inequality in (2.9) has to be strict. We see that we can only have equality when f is rank-one affine, as claimed. $\square$

We will use the following well-known statement.

Lemma 2.5. *Let $\mathcal{O} \subset M^{m\times n}$ be open and connected and let $f\colon \mathcal{O} \to R$ be locally affine along rank-one lines in $\mathcal{O}$. Then f is a linear combination of minors, in the sense that $f(X)$ is of the form $a_1M_1(X) + a_2M_2(X) + \dots a_rM_r(X) + b$, where $X \to M_j(X)$ is either a suitable subdeterminant of X or a constant.*

Proof. We refer the reader for example to [5]. □

We will denote the set of all rank-one affine functions (also known as *null Lagrangians*) by $\mathcal{L}$. Let $r = \dim \mathcal{L}$.

Let us now choose a finite set of matrices $0 = A^{(1)}, A^{(2)}, \dots, A^{(r)} \in M^{m\times n}$ with $|A^{(k)}| \le \frac{1}{2}$, $\quad k = 1, 2, \dots r$ so that the map

$$f \to \Big(f(A^{(1)}), f(A^{(2)}), \dots, f(A^{(r)}\Big)$$

is a linear space isomorphism of $\mathcal{L}$ and $\mathbb{R}^r$.

In what follows we will fix the notation

$$B = \{X \in M^{m\times n}\,,\, |X| \le 1\}\,, \qquad S = \{X \in M^{m\times n}\,,\, |X| = 1\}\,, \tag{2.12}$$

$$\mathcal{RC}(B) = \{f\colon B \to \overline{\mathbf{R}}\,,\ f \text{ is Borel measurable and rank-one convex in } B\} \tag{2.13}$$

and

$$\mathcal{RC}_0(B) = \{f \in \mathcal{RC}(B)\,,\, f(A^{(j)}) = 0\,,\, j = 1, 2, \dots, r\}\,. \tag{2.14}$$

We also define

$$\mathcal{X} = \{f \in \mathcal{RC}_0(B)\,,\, \frac{1}{|S|}\int_S f(X)\, dX = 1\,\}\,. \tag{2.15}$$

The set $\mathcal{X}$ seems to be the right analog of the set $\mathcal{X}$ from subsection 2.1.

Theorem 2.6. *For $f \in \mathcal{X}$ we have*

$$\sup_{B_R} |\nabla f| \le c(R) < \infty\,, \qquad 0 < R < 1\,, \tag{2.16}$$

where $B_R = \{X \in M^{m\times n}\,,\, |X| \le R\}\,.$

Proof. Let us first show that the functions in $\mathcal{X}$ have a common uniform bound from below:

$$\inf\{f(X)\,,\, f \in \mathcal{X}\,,\, X \in B\} > -\infty\,. \tag{2.17}$$

Arguing by contradiction, let us assume that this is not the case and that from some sequence $f_n \in \mathcal{X}$ there are $X_n \in B$ such that $f_n(X_n) \searrow -\infty$. Let us set $g_n = f_n/(-\inf_B f_n)$. Then $\inf_B g_n = -1$, and

$$\frac{1}{|S|}\int_S g_n(X)\, dX = \varepsilon_n \searrow 0\,. \tag{2.18}$$

Clearly

$$\frac{1}{|S|}\int_S |g_n| \le 2 + \varepsilon_n\,. \tag{2.19}$$

The functions g_n are subharmonic by Lemma 2.2. For sub-harmonic functions h on B we have

$$h(X) \le \int_S h(Y)P(X,Y)\,dY\,, \tag{2.20}$$

where

$$P(X,Y) = \frac{1}{|S|}\frac{1-|X|^2}{|X-Y|^{mn}} \tag{2.21}$$

is the Poisson kernel for the unit ball B. This gives
(2.22)
$$g_n(X) \le \int_S g_n(Y)P(X,Y)\,dY \le (2+\varepsilon_n)\sup_{Y\in S}\frac{1-|X|^2}{|X-Y|^{mn}} = (2+\varepsilon_n)\frac{1-|X|^2}{(1-|X|)^{mn}}\,.$$

This upper bound together with the convexity along the rank-one lines and $\inf_B g_n = -1$ easily give an upper bound

$$\sup_{B_R}|\nabla g_n| \le C(R)\,, \quad 0 \le R < 1\,, \tag{2.23}$$

where $C(R)$ is a finite increasing function on $[0,1)$. Recalling that f_n and hence also g_n belong to $\mathcal{R}C_0$, we have $g_n(0)=0$. The functions g_n are sub-harmonic and

$$\frac{1}{|S|}\int_S g_n(X)\,dX = \varepsilon_n \searrow 0\,. \tag{2.24}$$

In view of (2.23) and the Arzela-Ascoli Theorem, we can assume that g_n converge locally uniformly in the interior of B to some continuous function g. The function g clearly belongs to $\mathcal{R}C_0$, and is rank-one convex and hence subharmonic, due to Lemma 2.2. Denoting by $G_R(X,Y)$ the Green's function of the Laplacian in the ball B_R, we can write for $0<R<1$

$$0 = g_n(0) = \frac{1}{|\partial B_R|}\int_{\partial B_R} g_n(Y)\,dY + \int_{B_R} G_R(0,Y)\Delta g_n(Y)\,dY \tag{2.25}$$

and therefore

$$\int_{B_R} G_R(0,Y)\Delta g_n(Y)\,dY = -\frac{1}{|\partial B_R|}\int_{\partial B_R} g_n(Y)\,dY \ge -\varepsilon_n \nearrow 0\,. \tag{2.26}$$

Since $\Delta g_n \ge 0$ in B and $G_R(0,\cdot) \le 0$ in B, we see that $\Delta g = 0$, and by Lemma 2.4 this means that g is rank-one affine. Being also a member of $\mathcal{R}C_0$ we see that $g \equiv 0$. The functions g_n, therefore, converge locally uniformly to $g \equiv 0$ in the interior of the ball B. Now the condition $\inf_B g_n = -1$ easily leads to a contradiction of the convexity of g_n along rank-one lines. □

Theorem 2.6 shows that one can in some sense take $\mathcal{X}$ as a reasonable base of the rank-one convex functions, at least if we consider them modulo rank-one affine functions. One could try to compactify $\mathcal{X}$ by allowing the function $f|_{\partial B}$ to be in a suitable class of measures. However, when we work with suitable finite-dimensional subspaces, we get compactness without any additional technicalities. For example, let us define

$$\mathcal{X}_N = \mathcal{X} \cap \{f\colon M^{m\times n} \to \mathbf{R}\,,\ f \text{ is a polynomial of degree } \le N\}\,. \tag{2.27}$$

Corollary 2.7. *The set $\mathcal{X}_N$ is convex and compact for each positive integer N. Moreover, it can be considered as a base of the convex cone of the rank-one convex polynomials of degree $\le N$, taken modulo rank-one affine functions.*

Proof. This follows from the more general considerations above, but can also be proved more directly using the finite-dimensional nature of $\mathcal{X}_N$. Indeed, let us consider $\mathcal{X}_N$ as a subset of the finite-dimensional space $\mathcal{P}_N$ of polynomials of degree $\le N$ on $M^{m\times n}$. Consider a norm $\llbracket \cdot \rrbracket$ on $\mathcal{P}_N$. We can take for example $\llbracket f \rrbracket = \sup_B |f|$, although the exact form of the norm is unimportant for our purposes here (and we know that for a fixed N all norms of $\mathcal{P}_N$ are equivalent). We claim that $\mathcal{X}_N$ is bounded. If not, assume that $f_n \in \mathcal{X}_N$ is an unbounded sequence of polynomials. Setting $g_n = f_n/\llbracket f_n \rrbracket$, we can assume $g_n \to g \in \mathcal{X}_N$. Since $\int_S g_n(X)\,dx \to 0_+$ we have $\int_S g(X)\,dX = 0$ and since $g(0) = 0$ and g is subharmonic, we see — as before — that g must be rank-one affine and being in $\mathcal{RC}_0$ it must vanish, a contradiction with $\llbracket g \rrbracket = 1$. □

3. A convex optimization problem

3.1. **Gradient Young measures.** It is convenient to introduce the following notation. For $\varphi\colon \mathbf{T}^n \to \mathbf{R}^m$ and $A \in M^{m\times n}$ we will define the measure $\mu = \mu_{A,\varphi}$ on $M^{m\times n}$ by

$$\langle \mu, f\rangle = \frac{1}{|\mathbf{T}^n|}\int_{\mathbf{T}^n} f(A + \nabla\varphi)\,dX\,. \tag{3.1}$$

We will also use the notation $\bar\mu$ for the center of mass of μ. In the situation above we clearly have

$$\bar\mu_{A,\varphi} = A. \tag{3.2}$$

Any measure μ arising in this way from a Lipschitz function φ will be called a *gradient Young measure.* One can generate the gradient Young measure numerically simply by generating the smooth mappings $\varphi\colon \mathbf{T}^n \to \mathbf{R}^m$, for example by using trigonometric polynomials, piece-wise polynomial approximations (finite-elements), and other methods of representing functions.

Clearly, a continuous function $f\colon M^{m\times n} \to \mathbf{R}$ is quasi-convex if and only if

$$\langle \mu, f\rangle \ge f(\bar\mu) \tag{3.3}$$

for each gradient Young measure μ on $M^{m\times n}$.

It can be shown by a Hahn-Banach type argument, see [9], that every probability measure satisfying (3.3) for all continuous quasi-convex functions $f\colon M^{m\times n} \to \mathbf{R}$ is a gradient Young measure, but we will not need this statement in what follows.

3.2. **Optimization.** With the terminology introduced above, one can now formulate the question of whether rank-one convexity implies quasi-convexity on $M^{m\times n}$ as the question of whether (3.3) holds for each gradient Young measure μ and each rank-one convex function f. Therefore it is natural to consider the following question:

$(*)$ *Given a gradient Young measure μ on $M^{m\times n}$, do we have $\langle \mu, f\rangle \ge f(\bar\mu)$ for each rank-one convex function?*

By applying suitable shifts and rescalings to the functions f we can restrict our attention to the case $\bar{\mu} = 0$, $\operatorname{supp}(\mu) \subset B$ and $f(0) = 0$.

4. Finite-dimensional Approximations

Let us consider a gradient Young measure μ with $\operatorname{supp}(\mu) \subset B$ (where, as above, B is the closed unit ball in $M^{m\times n}$) and $\bar{\mu} = 0$. Let $\mathcal{X}_N$ be the finite-dimensional approximation of the set $\mathcal{X}$ defined in (2.27) by polynomials on $M^{m\times n}$ of degree $\leq N$. Recall that $B = \{X \in M^{m\times n}, |X| \leq 1\}$.

Lemma 4.1. *Let μ be a probabilistic measure in $B \subset M^{m\times n}$ and let $\mathcal{O} \subset M^{m\times n}$ be an open ball such that $B \subset \mathcal{O}$. If $\langle f, \mu\rangle < f(\bar{\mu})$ for some rank-one convex function $f\colon \mathcal{O} \to \mathbf{R}$, then there also exists an integer $N \geq 0$ and $g \in \mathcal{X}_N$ such that*

$$\langle \mu, g\rangle < g(\bar{\mu})\,. \tag{4.1}$$

Proof. Replacing f by its suitable mollification (defined on a set slightly smaller that $\mathcal{O}$ but still containing B), if necessary, we can assume that f is smooth. Given $\eta > 0$, we can find a polynomial $h\colon B \to R$ such that $\sup_B \big(|f-h| + |D^2 f - D^2 h|\big) < \eta$. We can now take

$$g(X) = h(X) + \varepsilon |X|^2 + L(X)\,. \tag{4.2}$$

for a suitable $\varepsilon = \varepsilon(f, h, \eta)$ and a suitable null-Lagrangian L, assuming η is sufficiently small (relatively to the quantity $f(\bar{\mu}) - \langle \mu, f\rangle$. The null Lagrangian can be used to make sure that the condition $g(A^{(k)}) = 0$ is satisfied for $k = 1, 2, \ldots, r$. We leave the standard technical details to the reader. □

Remark 4.2.

(i) The above proof can no doubt lead to $g \in \mathcal{X}_N$ with a large N. On the other hand, for possible numerical experiments, it is desirable to take N as low as possible. For general m, n and a fixed N one can very likely have, a situation with $\langle g, \mu\rangle \geq \langle g, \bar{\mu}\rangle$ for any $g \in \mathcal{X}_N$ but $\langle f, \mu\rangle < \langle f, \bar{\mu}\rangle$ for some rank-one convex $f\colon M^{m\times n} \to R$.

(ii) In general, a polynomial P that is rank-one convex in B may of course not be rank-one convex in $M^{m\times n}$. However, it is not hard to find a rank-one convex function $f\colon M^{m\times n} \to \mathbf{R}$ that is as close to P on B as we wish, and this is sufficient. A suitable extension method can be found for example in [15]. (This is one of the situations in which the locality of the rank-one convexity is important.)

4.1. **Possible numerical experiments.**

The simplest approach might be the following.

1. Generate a "random" gradient Young measure μ supported in B with $\bar{\mu} = 0$.
2. Minimize the linear functional $f \to \langle f, \mu\rangle$ over the compact convex set $\mathcal{X}_N$.

If the minimum found in step 2 drops below zero, we of course have a counterexample. In principle, one could consider the minimal value obtained in step 2 as function $\mathcal{F}(\mu)$ of μ and perform some steepest descend algorithm on $\mathcal{F}$ over some chosen finite-dimensional set of admissible gradient Young measures μ.

The dimension of the space of polynomials of degree $\leq N$ in $M^{m\times n}$ is

$$d = d(N,m,n) = \binom{N+mn}{N}.$$

The first interesting case seems to be $N = 4, m = n = 2$ and

$$d(4,2,2) = \binom{8}{4} = 70.$$

The null Lagragians in that case are of the form

$$L(x) = a_{11}X_{11} + a_{12}X_{12} + a_{21}X_{21} + a_{22}X_{22} + b\det(X) + c\,,$$

hence of dimension 6. Therefore the set $\mathcal{X}_4$ can be considered as a compact convex set in $\mathbf{R}^{64}$.

The corresponding numbers for $N = 4, n = 2, m = 3$ are

$$d(4,3,2) = \binom{10}{4} = 210$$

and the dimension of the null Lagrangians to be $6+3+1 = 10$ hence for $n = 2, m = 3$ the set $\mathcal{X}_4$ can be considered as a compact convex subset of $\mathbf{R}^{200}$. The evaluation of the linear functional

$$\ell(X) = \frac{1}{|S|}\int_S f(X)\,dX$$

should not present a problem.

The interesting question is how one should impose the requirement of the rank-one convexity of the polynomials numerically. For a C^2-function $f\colon M^{m\times n} \to \mathbf{R}$ this amounts to the condition

$$f''(X)\,(a\otimes b, a\otimes b) \geq 0 \tag{4.3}$$

for each $X \in M^{m\times n}$ and each rank-one matrix $a\otimes b$. When f is a quartic polynomial, the expression $X \to f''(X)\,(a\otimes b, a\otimes b)$ is a quadratic form and one could use some effective algorithms for verifying the positive definiteness of quadratic forms. However, in order to be able to use off-the-shelf linear programming[3] software, it may be natural to try the following method. Let us choose at random matrices $X_1, X_2, \ldots, X_p \in B$ and unit vectors $a_1, a_2, \ldots a_q, b_1, \ldots, b_{q'}$ and use the condition

$$f''(X_i)(a_j\otimes b_k, a_j\otimes b_k) \geq 0\,, \qquad 1\leq i\leq p,\ 1\leq j\leq q\,,\ 1\leq k\leq q' \tag{4.4}$$

as an approximation for rank-one convexity. For example, for $N = 4$, $n = 2$, $m = 2$, $p = 1000$, $q = 40, q' = 40$ one would end up with a linear programming problem in 64 variables and $1.6\cdot 10^6$ constraints. This might still be feasible with some efficient algorithms.

It would be interesting to see how the linear programming algorithms will perform. There are also some theoretical questions related to these approximations, see subsection 4.2 below.

Remark on invariant functions

As we already mentioned, for $n \geq 2, m \geq 3$ one has rank-one convex functions that are not quasi-convex. However, the question whether rank-one convexity is sufficient

[3] We refer the reader for example to [3] for details concerning linear programming.

for quasi-convexity remains open for various classes of functions with symmetries. For example, in non-linear elasticity one often deals with stored energy functions $W\colon M^{3\times 3} \to R$ that are *frame-indifferent* and *isotropic*. These two requirements mean that $W(Q_1XQ_2) = W(X)$ for all $Q_1, Q_2 \in \mathrm{SO}\,(3)$ and all X. By replacing the set $\mathcal{X}_N$ above with their natural symmetric variants, such as

$$\mathcal{X}_N^{\mathrm{sym}} = \{f \in \mathcal{X}_N\,,\ f \text{ has a desired symmetry}\} \tag{4.5}$$

one could test whether the rank-one convexity might imply quasi-convexity in some of the natural symmetry classes.

4.2. **Topics related to approximations.** For each X_i, a_j, b_k as in (4.4) let us consider the linear functional

$$\ell_{i,j,k}\colon f \to f''(X_i)(a_j \otimes b_k, a_j \otimes b_k) \tag{4.6}$$

In particular, the functional $\ell_{i,j,k}$ is well-defined on the (compact) set $\mathcal{X}_N$.

We will use the notation

$$\ell_0(f) = \frac{1}{|S|}\int_S f(X)\,dX\,. \tag{4.7}$$

Let us also denote by $\mathtt{B}$ the set of all functionals of the form

$$\ell\colon f \to f''(X)(a \otimes b, a \otimes b) \tag{4.8}$$

with $X \in B$ and unit vectors $a \in \mathbf{R}^m$, $b \in \mathbf{R}^n$. Here and in what follows we will use ℓ for the generic functional from $\mathtt{B}$, a slight change of notation in comparison with the meaning of ℓ used in the previous section, where it was used for what is now denoted ℓ_0. As above, we will use $\mathcal{P}_N$ to denote the set of all polynomials of degree $\le N$ on the set $M^{m\times n}$. (Strictly speaking, we should write $\mathcal{P}_{N,m,n}$ but the values of m, n will be clear from the context.)

Recalling the definition of the matrices A_j, $j = 1, 2, \ldots, r$ after Lemma 2.5, we have, by definition,

$$\mathcal{X}_N = \{\ f \in \mathcal{P}_N\,,\ l_0(f) = 1, f(A_j) = 0\,, j = 1, 2, \ldots, r\,,\ l(f) \ge 0 \text{ for each } \ell \in \mathtt{B}\} \tag{4.9}$$

The following lemma suggests that the approximation scheme leading to linear programming discussed above should be reasonable.

Lemma 4.3. *Let E be a finite-dimensional normed space and let E^* be its dual space. Let ℓ_0 be a non-zero element of E^* and let K be a compact subset of E^* such that the set*

$$\mathcal{K} = \{\ x \in E\,,\ l_0(x) = 1 \ \textit{and}\ l(x) \ge 0 \ \textit{for each}\ \ell \in K\ \}$$

is non-empty and compact. Then there exist finite sets $K_1 \subset K_2 \subset K_3 \subset \cdots \subset K$ such that the sets

$$\mathcal{K}_j = \{\ x \in E\,,\ l_0(x) = 1 \ \textit{and}\ l(x) \ge 0 \ \textit{for each}\ \ell \in K_j\ \}$$

are compact and their intersection is $\mathcal{K}$.

Proof. It is enough to show that there exists a finite set K_1 as above, the rest then follows by letting $K_j = K_1 \cup \{\ell_1, \dots, \ell_{j-1}\}$ for j=2, 3,..., where $\ell_1, \ell_2, \dots$ is a dense subset of K. Arguing by contradiction, let us assume that $\mathcal{K}_1$ is non-compact for each finite set $K_1 \subset K$. This means that the set $\mathcal{Z}_1 = \{y \in E\,;\, ||y|| = 1\,,\, \ell_0(y) = 0\,,\, \ell(y) \geq 0 \text{ for each } \ell \in K_1\}$ is non-empty for each finite $K_1 \subset K$. However, by standard compactness arguments, this means that it is also non-empty for $K_1 = K$. Let $y \in E$ with $||y|| = 1\,,\, \ell_0(y) = 0$, and $l(y) \geq 0$ for each $\ell \in K$. Then for any $x \in \mathcal{K}$ the ray $x + ty\,,\, t \in [0, \infty)$ is a subset of $\mathcal{K}$, a contradiction with $\mathcal{K}$ being bounded. □

The number of elements in K_1 has to be equal at least to the dimension of the affine set $\{x \in E\,,\, \ell_0(x) = 1\}$.

Assuming the set $\mathcal{K} = \mathcal{X}_N$ has a non-empty interior in the affine space $\mathcal{A} = \{f \in \mathcal{P}_N\,,\, f(A_j) = 0\,,\, j = 1, 2, \dots, r\,,\, \ell_0(f) = 1\}$ the discrepancy between $\mathcal{K}_j$ and $\mathcal{K}$ could in principle be measured by $|\mathcal{K}_j - \mathcal{K}|/|\mathcal{K}|$, where we denote by $|\mathcal{K}|$ the Lebesgue measure in the affine space $\mathcal{A}$.

Ideally, we would like to perform optimization on the set $\mathcal{K} = \mathcal{X}_N$. However, if we wish to use existing linear programming software, we probably will have to use approximations of the form $\mathcal{K}_j$. In the situation we consider here, it can be expected that the set $\mathcal{K} = \mathcal{X}_N$ cannot be described by finitely many linear functionals and the minimizers we get when we replace $\mathcal{K}$ with $\mathcal{K}_j$ will probably not be in $\mathcal{K}$. We can hope that if we achieve $\langle f, \mu \rangle < f(\bar{\mu})$ for some $f \in \mathcal{K}_j$ (with $\mathcal{K} = \mathcal{X}_N$) with j sufficiently large, the gap will be large enough to move f into $\mathcal{X}_N$ while still preserving some gap.

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