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# CLARKE SUBDIFFERENTIAL FOR LIPSCHITZ FUNCTIONS ON ASPLUND SPACES

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ABSTRACT. Based on Preiss' Fréchet differentiability theorem, this note proves that the Clarke subdifferential $\partial f(x)$ of a locally Lipschitz real-valued function $f$ on an open set in an Asplund space is equal to the weak$^*$-closed convex hull of the set of all weak$^*$ limit points $w^*-\lim_{k\to\infty} \nabla f(x_k)$ with $\{x_k\}$ converging strongly to $x$ and $f$ being Fréchet differentiable at each $x_k$. This extends Clarke's classical result about the Clarke subdifferential from the finite dimensional case to the infinite dimensional one.

## 1. Introduction

Given an open set $G$ in a Banach space $X$ and a locally Lipschitz function $f : G \to \mathbb{R}$, recall that the Clarke subdifferential of $f$ at $x \in G$ is the following set

$$\partial f(x) := \{x^* \in X^* : \langle x^*, h\rangle \leq f^\circ(x, h) \quad \forall h \in X\},$$

where $f^\circ(x,h) = \limsup_{y\to x, t\to 0^+} \frac{f(y+th)-f(y)}{t}$ is the Clarke directional derivative of $f$ at $x$ in direction $h$. For convenience, let

$$\mathcal{G}_F(f) = \{u \in G : f \text{ is Frechet differentiable at } u\}.$$

It is well known, as the Radmader theorem, that if $X$ is finite dimensional then $G\backslash\mathcal{G}_F(f)$ is of Lebesgue measure 0. With the help of Radmader's theorem, Clarke [1] proved that if $X$ is finite dimensional then

$$\partial f(x) = \mathrm{co}\left\{\lim_{k\to\infty} \nabla f(x_k) : x_k \to x, x_k \in \mathcal{G}_F(f)\right\} \tag{1.1}$$

(cf. [1, Theorem 2.5.1]). Hiriart-Urruty and Thibault [3] extended formula (1.1) to separable Banach spaces (also see [1, page 285]). Recall that the Fréchet subdifferential and Mordukhovich limiting subdifferential $\hat\partial f(x)$ and $\bar\partial f(x)$ of $f$ at $x \in G$ are defined by

$$\hat\partial f(x) := \left\{x^* \in X^* : \liminf_{h\to 0} \frac{f(x+h) - f(x) - \langle x^*, h\rangle}{\|h\|} \geq 0\right\}$$

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and

$$\bar{\partial} f(x) := \{x^* :\ x_k \to x,\ x_k^* \overset{w^*}{\to} x^* \text{ and } x_k^* \in \hat{\partial} f(x_k)\}, \tag{1.2}$$

respectively. Note that there exists a globally Lipschitz function $\varphi$ on the classical Banach space $\ell^1$ such that $\hat{\partial}\varphi(x) = \bar{\partial}\varphi(x) = \emptyset$ for all $x \in \ell^1$. However, in the case when $X$ is an Asplund space, $\mathrm{dom}(\hat{\partial} f) = \{x \in G :\ \hat{\partial} f(x) \neq \emptyset\}$ is dense in the domain $G$ of $f$ and

$$\partial f(x) = \overline{\mathrm{co}}^{w^*}(\bar{\partial} f(x)) \quad \forall x \in G, \tag{1.3}$$

where $\overline{\mathrm{co}}^{w^*}$ denotes the weak*-closed convex hull (cf. [5, 6]). Recall that $X$ is an Asplund space if every continuous convex function $\varphi$ on an open convex set $G$ in $X$ is Fréchet differentiable on a dense $G_\delta$ subset of $G$. Many important and interesting results on Asplund spaces have been established (cf. [2, 7]). In particular, $X$ is an Asplund space if and only if every separable subspace of $X$ has a separable dual space (cf. [7]). Moreover, Preiss [8] proved that if $X$ is an Asplund space then $G_F(f)$ is dense in $G$ for any locally Lipschitz function $f$ on an open subset $G$ of $X$. Given a locally Lipschitz function $f$ on an open set $G$ in an Asplund space, motivated by (1.1)—(1.3) and Preiss' result, it is nature to ask whether or not

$$\partial f(x) = \overline{\mathrm{co}}^{w^*}\left(\left\{ w^*\!-\lim_{k\to\infty} \triangledown f(x_k) :\ x_k \to x,\ x_k \in \mathcal{G}_F(f)\right\}\right).$$

This note will give a positive answer to the above question. Moreover, this note considers extending Clarke's generalized Jacobian chain rule to infinite dimension spaces from the finite dimensional spaces.

## 2. Preliminaries

Throughout this section, we make the following assumptions:

**A1)** $X$ is an Asplund space.
**A2)** $G$ is an open nonempty set in $X$.
**A3)** $f : G \to \mathbb{R}$ is a locally Lipschitz function.

Recall that $\mathcal{G}_F(f)$ denotes the set of all Fréchet differentiable points of $f$ in $G$. To prove the main result in this note, we need the following very profound result established by Preiss (cf. [8, Theorem 2.5]).

**Lemma 2.1.** *Let $u, v \in X$ be such that the segment $[u,\ v] := \{tu + (1-t)v :\ 0 \le t \le 1\}$ is contained in $G$, and suppose that A1)—A3) are satisfied. Then*

$$\begin{aligned}\inf\{\langle \triangledown f(x), v-u\rangle :\ x \in G \cap \mathcal{G}_F(f)\} &\le f(v) - f(u)\\ &\le \sup\{\langle \triangledown f(x), v-u\rangle :\ x \in G \cap \mathcal{G}_F(f)\}.\end{aligned}$$

The following Lebourg mean-value theorem (cf. [1, Theorem 2.3.7]) is also useful for our analysis.

**Lemma 2.2.** *For any $x, y \in X$ with $[x,\ y] \subset G$, there exists $u \in (x,\ y) := \{tu + (1-t)v :\ 0 < t < 1\}$ such that*

$$f(y) - f(x) \in \{\langle u^*, y-x\rangle :\ u^* \in \partial f(u)\}.$$

For a locally Lipschitz vector-valued function $F : G \to \mathbb{R}^n$, we adopt the generalized Jacobian $\partial F(x)$ of $F$ at $x \in G$ as follows

$$\partial F(x) := \left\{ \lim_{k\to\infty} \nabla F(x_k) :\ x_k \to x,\ x_k \in \mathcal{G}_F(F) \right\} \tag{2.1}$$

(cf. [1,10]), where $\mathcal{G}_F(F)$ denotes the set of all points at which $F$ are differentiable. Even for $n = 2$, it is unknown whether or not $\partial F(x)$ is nonempty (c.f [2]). However, when the dimension $\dim(X)$ of $X$ is finite, $\partial F(x) \neq \emptyset$ for all $x \in G$ and the following result holds (cf. [1, Proposition 2.6.5]).

**Lemma 2.3.** *Suppose that $X$ is finite dimensional. Then, for any $x, y \in X$ with $[x,\ y] \subset G$,*

$$F(y) - F(x) \in \mathrm{co}(\partial F([x,\ y]))(y - x).$$

## 3. Main result

First we extend Clarke's result on the Clarke subdifferential formula for a locally Lipschitz real-valued function from the finite dimensional space to the more general Asplund space.

**Theorem 3.1.** *Let $G$ be an open set in an Asplund space $X$ and let $f : G \to \mathbb{R}$ be a locally Lipschitz function. Then, for any $x \in G$,*

$$f^\circ(x, h) = \limsup_{y \xrightarrow{\mathcal{G}_f} x} \langle \nabla f(y), h\rangle \quad \forall h \in X \tag{3.1}$$

*and*

$$\partial f(x) = \overline{\mathrm{co}}^{w^*}\left( \left\{ w^*\!-\lim_{k\to\infty} \nabla f(x_k) :\ x_k \to x,\ x_k \in \mathcal{G}_F(f) \right\} \right),$$

*where $y \xrightarrow{\mathcal{G}_F(f)} x$ means $\|y - x\| \to 0$ and $y \in \mathcal{G}_F(f)$.*

*Proof.* Take $L, r \in (0,\ +\infty)$ such that $B(x, r) \subset G$ and

$$|f(y) - f(z)| \le L\|y - z\| \quad \forall y, z \in B(x, r), \tag{3.2}$$

where $B(x, r)$ is the open ball with center $x$ and radius $r$. Let

$$A := \left\{ w^*\!-\lim_{k\to\infty} \nabla f(x_k) :\ x_k \to x,\ x_k \in \mathcal{G}_F(f) \right\}.$$

For any $h$ in $X$, take a sequence $\{y_n\}$ in $B(x, r)$ such that

$$\lim_{n\to\infty} \langle \nabla f(y_n), h\rangle = \limsup_{y \xrightarrow{\mathcal{G}_F(f)} x} \langle \nabla f(y), h\rangle.$$

Then $\|\nabla f(y_n)\| \le L$ for all $n \in \mathbb{N}$ (thanks to (3.2)). Since $X$ is an Asplund space, the unit ball $B_{X^*}$ of $X^*$ is sequentially compact with respect to the weak* topology (cf. [5, Proposition 1.123]). Hence there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\nabla f(y_{n_k}) \xrightarrow{w^*} y^* \in LB_{X^*}$. It follows that $y^* \in A$ and

$$\langle y^*, h\rangle = \lim_{k\to\infty} \langle \nabla f(y_k), h\rangle = \limsup_{y \xrightarrow{\mathcal{G}_F(f)} x} \langle \nabla f(y), h\rangle.$$

Hence $\limsup_{y \xrightarrow{\mathcal{G}_F(f)} x} \langle \nabla f(y), h\rangle \leq \sigma_A(h) = \sigma_{\overline{\mathrm{co}}^{w^*}(A)}(h)$, where $\sigma_A$ denotes the support functional of $A$ (i.e., $\sigma_A(h) = \sup_{y^* \in A} \langle y^*, h\rangle$). Thus, by (1.2) and (1.3), one has

$$\limsup_{y \xrightarrow{\mathcal{G}_F(f)} x} \langle \nabla f(y), h\rangle \leq \sigma_{\overline{\mathrm{co}}^{w^*}(A)}(h) \leq \sigma_{\partial f(x)}(h) = f^\circ(x, h) \tag{3.3}$$

(the last equality holds because of [1, Proposition 2.1.2]). On the other hand, take a sequence $\{(x_n, t_n)\}$ in $X \times (0, +\infty)$ such that $[x_n, x_n + t_n h] \subset B(x, r)$,

$$x_n \to x, \ t_n \to 0 \ \text{ and } \ \frac{f(x_n + t_n h) - f(x_n)}{t_n} \to f^\circ(x, h).$$

Then there exists a sequence $\{\varepsilon_n\}$ in $(0, +\infty)$ converging to 0 such that

$$U_n = \{u \in X : \ d(u, [x_n, \ x_n + t_n h]) < \varepsilon_n\} \subset B(x, r) \quad \forall n \in \mathbb{N}.$$

By Lemma 2.1, there exists $u_n \in U_n \cap \mathcal{G}_F(f)$ such that

$$f(x_n + t_n h) - f(x_n) < \langle \nabla f(u_n), t_n h\rangle + t_n^2.$$

Hence $\|u_n - x\| \leq \|u_n - x_n\| + \|x_n - x\| \leq \varepsilon_n + t_n\|h\| + \|x_n - x\| \to 0$ and

$$\begin{aligned} f^\circ(x, h) &= \lim_{n\to\infty} \frac{f(x_n + t_n h) - f(x_n)}{t_n} \\ &\leq \limsup_{n\to\infty} \langle \nabla f(u_n), h\rangle \\ &\leq \limsup_{y \xrightarrow{\mathcal{G}_F(f)} x} \langle \nabla f(y), h\rangle. \end{aligned}$$

This and (3.3) show that (3.1) holds and $f^\circ(x, h) = \sigma_{\overline{\mathrm{co}}^{w^*}(A)}(h)$. Noting that $\partial f(x)$ is a weak*-closed convex set and $f^\circ(x, \cdot) = \sigma_{\partial f(x)}$, it follows from the separation theorem that $\partial f(x) = \overline{\mathrm{co}}^{w^*}(A)$. The proof is complete. □

The following Jacobian chain rule is known (cf. [1, Theorem 2.6.6]).

**Theorem 3.2.** *Let $F : \mathbb{R}^m \to \mathbb{R}^n$ be Lipschitz near $x \in \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz near $F(x)$. Then*

$$\partial f(x) \subset \mathrm{co}\big(\partial g(F(x))\partial F(x)\big).$$

Naturally, one may try to extend Theorem 3.2 by replacing $\mathbb{R}^m$ with an infinite dimensional Asplund space $X$. However, even in the case that $X$ is an infinite dimensional Hilbert space, it is not known whether or not a Lipschitz function $G : X \to \mathbb{R}^3$ is Fréchet differentiable at some point in $X$, that is, it is unknown whether or not $\mathcal{G}_F(F)$ is nonempty. This motivates us to consider the case of piecewise linear vector-valued functions. Recall that $P \subset X$ is a convex polyhedron if there exist $(x_1^*, r_1), \cdots, (x_k^*, r_k) \in X^* \times \mathbb{R}$ such that

$$P = \{x \in X : \ \langle x_i^*, x\rangle \leq r_i, \ i = 1, \cdots, k\}.$$

We say that a vector-valued function $F : X \to \mathbb{R}^n$ is piecewise linear if its graph $\mathrm{gph}(F) := \{(x, F(x)) : \ x \in X\}$ is the union of finitely many convex polyhedra in $X \times \mathbb{R}^n$. The following lemma is known (cf. [11, Proposition 3.1 and Theorem 3.1]) and useful for us.

**Lemma 3.3.** *Let $F$ be a piecewise linear vector-valued function from a normed space $X$ to $\mathbb{R}^n$. Then there exist convex polyhedra $P_i$ in $X$, $T_i \in \mathcal{L}(X, \mathbb{R}^n)$ and $b_i \in \mathbb{R}^n$ ($i \in \overline{1m} := \{1, \cdots, m\}$) such that*

$$X = \bigcup_{i=1}^{m} P_i, \ \mathrm{int}(P_i) \neq \emptyset, \ \mathrm{int}(P_i) \cap P_j = \emptyset \ \ \forall i, j \in \overline{1m} \text{ with } i \neq j, \tag{3.4}$$

$$F(x) = T_i(x) + b_i \quad \forall i \in \overline{1m} \text{ and } \forall x \in P_i \tag{3.5}$$

*and*

$$T_1(x) = \cdots = T_m(x) \quad \forall x \in \mathrm{lin}\left(\bigcap_{i=1}^{m} P_i\right) \tag{3.6}$$

*where $\mathcal{L}(X, \mathbb{R}^n)$ denotes the space of all continuous linear operators from $X$ to $\mathbb{R}^n$ and $\mathrm{lin}\left(\bigcap_{i=1}^{m} P_i\right)$ denotes the largest subspace contained in $\bigcap_{i=1}^{m} P_i$.*

**Theorem 3.4.** *Let $X$ be a normed space, $F : X \to \mathbb{R}^n$ be a piecewise linear vector-valued function and let $g : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. Then, for any $x \in X$,*

$$\partial(g \circ F)(x) \subset \mathrm{co}\big(\partial g(F(x))\partial F(x)\big). \tag{3.7}$$

*Proof.* Since $F : X \to \mathbb{R}^n$ is piecewise linear, there exist convex polyhedra $P_i$ in $X$, $T_i \in \mathcal{L}(X, \mathbb{R}^n)$ and $b_i \in \mathbb{R}^n$ ($i = 1, \cdots, m$) such that (3.4)–(3.6) hold (thanks to Lemma 3.3). For each $i \in \overline{1m}$, take $x_{ij}^* \in X^*$ and $r_{ij} \in \mathbb{R}$ ($j = 1, \cdots, \nu_i$) such that

$$P_i = \{x \in X : \ \langle x_{ij}^*, x\rangle \leq r_{ij}, \ j = 1, \cdots, \nu_i\}.$$

It is easy to verify that

$$X_F := \mathrm{lin}\left(\bigcap_{i=1}^{m} P_i\right) = \bigcap_{i=1}^{m} \bigcap_{j=1}^{\nu_i} \ker(x_{ij}^*),$$

where $\ker(x_{ij}^*) := \{x \in X : \ \langle x_{ij}^*, x\rangle = 0\}$, and hence $X_F$ is a closed subspace of $X$ whose codimension is less than or equal to $\sum_{i=1}^{m} \nu_i$. Therefore, there exists a subspace $\widehat{X}$ of $X$ such that

$$\dim(\widehat{X}) < \infty, \ \widehat{X} \cap X_F = \{0\} \ \text{ and } \ X = \widehat{X} + X_F. \tag{3.8}$$

For each $i \in \overline{1m}$, let

$$\widehat{P}_i := \left\{u \in \widehat{X} : \ \langle x_{ij}^*, u\rangle \leq r_{ij}, \ j = 1, \cdots, \nu_i\right\}.$$

Then, by (3.4), $\widehat{P}_i$ is a convex polyhedron in $\widehat{X}$,

$$P_i = \widehat{P}_i + X_F \ \text{ and } \ \widehat{X} = \bigcup_{i=1}^{m} \widehat{P}_i.$$

Moreover, by (3.5) and (3.6), one has

$$F(u + v) = T_i(u) + T_0(v) + b_i \quad \forall (u, v) \in \widehat{P}_i \times X_F, \ i = 1, \cdots, m$$

where $T_0$ is a continuous linear operator from $X_F$ to $\mathbb{R}^n$. Let $\widehat{F} : \widehat{X} \to \mathbb{R}^n$ be such that $\widehat{F}(u) := F(u)$ for all $u \in \widehat{X}$. Then $\widehat{F}$ is a Lipschitz vector-valued function from $\widehat{X}$ to $\mathbb{R}^m$, and

$$F(u+v) = (\widehat{F} \oplus T_0)(u+v) := \widehat{F}(u) + T_0(v) \quad \forall (u,v) \in \widehat{X} + X_F. \tag{3.9}$$

Thus, for $(u,v) \in \widehat{X} \times X_F$, it is easy from (3.8) and (3.9) to verify that $F$ is Fréchet differentiable at $u+v$ if and only if $\widehat{F}$ is Fréchet differentiable at $u$ with

$$\nabla F(u+v)(u'+v') = \nabla \widehat{F}(u)(u') + T_0(v') \quad \forall (u',v') \in \widehat{X} \times X_F.$$

This implies that

$$\partial F(u+v) = \partial \widehat{F}(u) \oplus \widehat{T} := \left\{ \widehat{S} \oplus T_0 : \ \widehat{S} \in \partial \widehat{F}(u) \right\} \quad \forall (u,v) \in \widehat{X} \times X_F. \tag{3.10}$$

Take $(u_0, v_0) \in \widehat{X} \times X_F$ such that $x = u_0 + v_0$. Then, for any $(h_1, h_2) \in \widehat{X} \times X_F$, there exists a sequence $\{(u_k, v_k, t_k)\}$ in $\widehat{X} \times X_F \times (0, +\infty)$ such that

$$(u_k, v_k, t_k) \to (u_0, v_0, 0) \ \text{ and } \ (g \circ F)^\circ(x, h_1 + h_2) = \lim_{k \to \infty} \Delta_k, \tag{3.11}$$

where

$$\Delta_k := \frac{g(F(u_k + v_k + t_k(h_1+h_2)) - g(F(u_k+v_k))}{t_k}.$$

By (3.9) and Lemma 2.2, there exist $y_k \in [F(u_k+v_k), \ F(u_k+v_k+t_k(h_1+h_2))]$ and $y_k^* \in \partial g(y_k)$ such that

$$\begin{aligned} \Delta_k &= \left\langle y_k^*, \frac{F(u_k + v_k + t_k(h_1+h_2) - F(u_k+v_k)}{t_k} \right\rangle \\ &= \left\langle y_k^*, \frac{\widehat{F}(u_k + t_k h_1)) - F(u_k)}{t_k} \right\rangle + \langle y_k^*, T_0(h_2) \rangle. \end{aligned} \tag{3.12}$$

Noting that $F$ is locally Lipschitz, it follows from (3.11) that $y_k \to F(u_0+v_0) = F(x)$. Since $g$ is also locally Lipschitz, we can assume without loss of generality that

$$y_k^* \to y^* \in \partial g(F(x)) \tag{3.13}$$

(thanks to [1, Proposition 2.1.5]). On the other hand, by Lemma 2.3, one has

$$\frac{\widehat{F}(u_k + t_k h_1)) - F(u_k)}{t_k} \in \mathrm{co}\left(\partial \widehat{F}([u_k, \ u_k + t_k h_1])\right)(h_1),$$

and hence there exists $\widehat{S}_k \in \partial \widehat{F}([u_k, \ u_k + t_k h_1])$ such that

$$\left\langle y_k^*, \frac{\widehat{F}(u_k + t_k h_1)) - F(u_k)}{t_k} \right\rangle \le \langle y_k^*, \widehat{S}_k(h_1) \rangle.$$

Therefore, by (3.12), one has

$$\Delta_k \le \langle y_k^*, \widehat{S}_k(h_1) + T_0(h_2) \rangle = \langle y_k^*, (\widehat{S}_k \oplus T_0)(h_1+h_2) \rangle. \tag{3.14}$$

By the definition of $\partial \widehat{F}$, for any $k \in \mathbb{N}$ there exist $\hat{u}_k \in \widehat{X}$ such that

$$d\left(\hat{u}_k, [u_k, \ u_k + t_k h_1]\right) < \frac{1}{k} \ \text{ and } \ \left\| \nabla \widehat{F}(\hat{u}_k) - \widehat{S}_k \right\| < \frac{1}{k}.$$

It follows from (3.11) that $\lim_{k\to\infty} \hat{u}_k = u_0$. Since $\hat{F}$ is a locally Lipschitz function between finite dimensional spaces $\widehat{X}$ and $\mathbb{R}^n$, we can assume without loss of generality that $\nabla\widehat{F}(\hat{u}_k) \to \hat{S} \in \partial\widehat{F}(u_0)$ (taking a subsequence if necessary). Thus, by (3.10)—(3.14), one has

$$y^* \circ (\widehat{S} \oplus T_0) \in \partial g(F(x))\partial F(x) \ \text{ and } \ (g \circ F)^\circ(x, h_1 + h_2) \leq \langle y^*, (\widehat{S} \oplus T_0)(h_1 + h_2)\rangle.$$

Noting that the Clarke directional derivative $(g \circ F)^\circ(x, h_1 + h_2)$ is equal to the value of the support functional $\sigma_{\partial(g\circ F)(x)}$ at $h_1 + h_2$, it follows that

$$\sigma_{\partial(g\circ F)(x)}(h_1 + h_2) \leq \sigma_{\partial g(F(x))\partial F(x)}(h_1 + h_2).$$

This means that $\partial(g \circ F)(x) \subset \overline{\text{co}}^{w^*}\big(\partial g(F(x))\partial F(x)\big)$. Thus, to prove (3.7), we only need to show that $\text{co}\big(\partial g(F(x))\partial F(x)\big)$ is compact. Noting that $\widehat{F} : \widehat{X} \to \mathbb{R}^n$ is a locally Lipschitz function, it is easy to verify that $\partial\widehat{F}(u_0)$ is a bounded closed set in $\mathcal{L}(\widehat{X}, \mathbb{R}^n)$. Since $\dim\Big(\mathcal{L}(\widehat{X}, \mathbb{R}^n)\Big) = \dim(\widehat{X}) \times n < \infty$, $\partial\widehat{F}(u_0)$ is a compact set in $\mathcal{L}(\widehat{X}, \mathbb{R}^n)$. Since $\partial f(F(x))$ is a compact set in $\mathbb{R}^n$, this and (3.10) show that $\partial g(F(x))\partial F(x)$ is a bounded closed set in a finite dimensional subspace of $X^*$. It follows that $\text{co}\big(\partial g(F(x))\partial F(x)\big)$ is compact (cf. [9, Theorem 3.20]). The proof is complete. □

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