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APPROXIMATE SOLUTIONS OF A SET-VALUED INCLUSION IN A METRIC SPACE

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ABSTRACT. In our recent research we studied a feasibility problem with infinitely many sets in a metric space, presented a novel algorithm and analyzed its convergence taking into account computational errors. We showed that our algorithm generates a good approximate solution, if computational errors are bounded from above by a small positive constant. In the present paper we generalize these results a larger class of fixed point problems induced by a set-valued inclusion.

1. INTRODUCTION

During more than sixty years now, there has been a lot of activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [3, 5, 12, 14, 15, 18–20, 22–25, 28, 29] and the references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also covers the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility and common fixed point problems, which find important applications in engineering and medical sciences [9–11, 13, 16, 26–29].

The convex feasibility problem is to find a feasible point in the intersection of finitely many convex and closed sets. This problem, which was studied in [2, 4–7, 9], finds important applications in engineering, medical and the natural sciences [11, 13, 17, 21]. The convex feasibility problem is a special case of a more general common fixed point problem for which we need to find a common fixed point of a family of operators [26–29].

Assume that C_i, $i = 1, \dots, m$, where $m \geq 2$ is a natural number, are closed and convex sets in a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and a complete norm $\| \cdot \|$ which is induced by the inner product. We consider the problem

$$\text{Find } z \in \cap_{i=1}^m C_i$$

under the assumption that $\cap_{i=1}^m C_i$ is nonempty. It is well-known that for each $i \in \{1, \dots, m\}$ and each $x \in X$, there exists a unique element $P_{C_i}(x) \in C_i$ such that

$$\|x - P_{C_i}(x)\| = \inf\{\|x - y\| : \ y \in C_i\},$$

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$$\|P_{C_i}(x) - P_{C_i}(y)\| \le \|x - y\|,\ x, y \in X$$

and

$$\|z - x\|^2 \ge \|z - P_i(x)\|^2 + \|x - P_i(x)\|^2$$

for each $x \in X$ and each $z \in C_i$. In practice for solving convex feasibility problem the following iterative method is used.

Fix an integer $\bar{N} \ge m$ and denote by $\mathcal{R}$ the collection of all maps $r : \{1, 2, \dots, \} \to \{1, \dots, m\}$ such that for every positive integer s,

$$\{1, \dots, m\} \subset \{r(s), \dots, r(s + \bar{N} - 1)\}.$$

We associate with any map $r \in \mathcal{R}$ the following iterative algorithm:

Initialization: choose any starting point x_0 of the space X.

Iterative step: given a current iterate $x_k \in X$ calculate

$$x_{k+1} = P_{r(k+1)}(x_k).$$

It is known that iterates obtained by this method converge weakly to a solution of our feasibility problem. The same result is also guaranteed by the well-known Cimmino algorithm described below:

Initialization: choose any starting point x_0 of the space X.

Iterative step: given a current iterate $x_k \in X$ calculate

$$x_{k+1} = \sum_{i=1}^{m} m^{-1} P_i(x_k).$$

In [8] Y. Censor, T. Elfving, and G. T. Herman introduced dynamic string-averaging methods, which are in some sense a combination of the iterative algorithm and the Cimmino algorithm. In these dynamic string-averaging methods, which became very popular in the literature, a family of sets is divided into blocks and the algorithms operate in such a manner that all the blocks are processed in parallel.

In [30] we studied a feasibility problem with a collection of sets which is not necessarily finite. Clearly, the algorithms described above cannot be applied if the collection of sets is infinite. The main feature of these algorithms is that for iterative steps we need to calculate the values of all the operators belonging to our family of maps and even their sums with weighted coefficients. Of course, this is impossible if the family of maps is not finite. In [30] we introduced a new algorithm for solving feasibility problems with infinite families of sets and studied its convergence. It turns out that our results hold for feasibility problems in a general metric space.

In this paper we use the following notation.

For each $z \in R^1$ set

$$\lfloor z \rfloor = \sup\{i : \ i \text{ is an integer and } i \le z\}.$$

Let (X, ρ) be a metric space endowed with a metric ρ. For every element $x \in X$ and every positive number r put

$$B(x, r) = \{y \in X : \ \rho(x, y) \le r\}.$$

For every element $x \in X$ and every nonempty set $D \subset X$ define

$$\rho(x, D) = \inf\{\rho(x, y) : \ y \in D\}.$$

Fix $\theta \in X$. Denote by $\mathrm{Card}(E)$ the cardinality of a set E. We assume that the sum over an empty set is zero.

The following feasibility problem was considered in [30]. Assume that $\mathcal{A}$ is a nonempty set, for each $\alpha \in \mathcal{A}$, $C_\alpha \subset X$ is a nonempty, closed set and that there exists $P_\alpha : X \to C_\alpha$ such that

$$P_\alpha(x) = x, \; x \in C_\alpha.$$

We also assume that there exists $\bar{c} \in (0,1)$ such that for each $\alpha \in \mathcal{A}$, each $z \in C_\alpha$ and each $x \in X$,

$$\rho(z,x)^2 \geq \rho(z, P_\alpha(x))^2 + \bar{c}\rho(x, P_\alpha(x))^2$$

and that there exists

$$\widehat{z} \in \cap_{\alpha \in \mathcal{A}} C_\alpha.$$

In [30] we consider the problem

$$\text{Find } z \in \cap_{\alpha \in \mathcal{A}} C_\alpha$$

and use the following algorithm.

Let a sequence $\{\Delta_i\}_{i=1}^\infty \subset (0,\infty)$ satisfy

$$\lim_{i \to \infty} \Delta_i = 0.$$

Initialization: choose any element $x_0 \in X$.

Iterative step: given a current iterate x_k calculate $\alpha_k \in \mathcal{A}$ such that

$$\rho(x_k, P_{\alpha_k}(x_k)) \geq \sup\{\rho(x_k, P_\alpha(x_k)) : \; \alpha \in \mathcal{A}\} - \Delta_{k+1}$$

and calculate

$$x_{k+1} = P_{\alpha_k}(x_k).$$

It was shown in [30] that for each sequences $\{x_t\}_{t=0}^\infty \subset X$ generated by the algorithm we have

$$\lim_{t \to \infty} \rho(x_t, x_{t+1}) = 0$$

and

$$\lim_{t \to \infty} \sup\{\rho(x_t, P_\alpha(x_t)) : \; \alpha \in \mathcal{A}\} = 0.$$

In [31] we studied our algorithm introduced in [30] for common fixed problems taking into account computational errors which always present in practice. In this case the convergence to a solution does not take place. We showed that our algorithms generate a good approximate solution, if computational errors are bounded from above by a small positive constant. Clearly, in practice it is sufficient to find a good approximate solution instead of constructing a sequence which converges to a solution. On the other hand in practice computations induce numerical errors and if one uses methods in order to solve problems these methods usually provide only approximate solutions. Our main goal is, for a known computational error, to find out what an approximate solution can be obtained and how many iterates one needs for this.

Assume that (X, ρ) is a metric space. For each $S : X \to X$ set

$$\mathrm{Fix}(S) = \{z \in X : \; S(z) = z\}.$$

Fix

$$\theta \in X.$$

Assume that $\mathcal{A}$ is a nonempty set and that for each $\alpha \in \mathcal{A}$, $P_\alpha : X \to X$ satisfies

$$\mathrm{Fix}(P_\alpha) \neq \emptyset.$$

Assume that $\bar{c} \in (0, 1)$ and the following assumption holds:

for each $\alpha \in \mathcal{A}$, each $z \in \mathrm{Fix}(P_\alpha)$ and each $x \in X$,

$$\rho(z, x)^2 \geq \rho(z, P_\alpha(x))^2 + \bar{c}\rho(x, P_\alpha(x))^2.$$

Assume that $z_* \in X, \delta_C \in (0, 1]$ and that for each $\alpha \in \mathcal{A}$,

$$B(z_*, \delta_C) \cap \mathrm{Fix}(P_\alpha) \neq \emptyset.$$

The next theorem is the main result of [31].

Theorem 1.1. *Let*

$$M > \max\{1, \rho(\theta, z_*)\},$$
$$\delta_0, \delta_1, \in (0, 1),$$
$$\epsilon_0 = \max\{(8(4M + 8)(\delta_0 + \delta_C)\bar{c}^{-1})^{1/2}, \delta_1\}$$

and

$$n_0 = \lfloor 32M^2\bar{c}^{-1}\epsilon_0^{-2}\rfloor + 1.$$

Assume that $\{x_t\}_{t=0}^\infty \subset X$, $\{\alpha_t\}_{t=0}^\infty \subset \mathcal{A}$,

$$\rho(x_0, \theta) \leq M$$

and that for each integer $t \geq 0$,

$$\delta_1 + \rho(x_t, P_{\alpha_t}(x_t)) \geq \rho(x_t, P_\alpha(x_t)),\ \alpha \in \mathcal{A},$$
$$\rho(x_{t+1}, P_{\alpha_t}(x_t)) \leq \delta_0.$$

Then there exists an integer $q \in [0, n_0 - 1]$ *such that*

$$\rho(x_i, \theta) \leq 3M,\ i \in \{0, \dots, q\}$$

and

$$\rho(x_q, x_{q+1}) \leq \epsilon_0.$$

Moreover, if an integer $q \in [0, n_0 - 1]$ *and the inequality above holds, then*

$$\rho(x_q, P_\alpha(x_q)) \leq 3\epsilon_0 \text{ for each } \alpha \in \mathcal{A}.$$

In order to study the problem discussed above we can use another framework. For each $x \in X$, define

$$T(x) = \{P_\alpha(x) : \alpha \in \mathcal{A}\},\ x \in X.$$

Now we can study the behavior of the iterative process induced by the map T. Assume that $M \geq \rho(\theta, z_*)$. For each $\alpha \in \mathcal{A}$ there is

$$z_\alpha \in \mathrm{Fix}(P_\alpha)$$

such that

$$\rho(z_*, z_\alpha) \leq \delta_C.$$

Let $\alpha \in \mathcal{A}$. Clearly,

$$\rho(\theta, z_\alpha) \leq M + 1.$$

Assume that

$$x \in B(\theta, M).$$

We have

$$\rho(z_\alpha, x)^2 \geq \rho(z_\alpha, P_\alpha(x))^2 + \bar{c}\rho(x, P_\alpha(x))^2$$

and

$$\rho(z_\alpha, P_\alpha(x)) \leq \rho(z_\alpha, x).$$

By the equations above,

$$\begin{aligned}|\rho(z_\alpha, x)^2 - \rho(z_*, x)|^2 &\leq \rho(z_*, z_\alpha)(\rho(z_\alpha, x) + \rho(z_*, x)) \\ &\leq \delta_C(4M+1), |\rho(z_\alpha, P_\alpha(x))^2 - \rho(z_*, P_\alpha(x))|^2 \\ &\leq \rho(z_*, z_\alpha)(\rho(z_\alpha, P_\alpha(x)) + \rho(z_*, P_\alpha(x))) \\ &\leq \delta_C(4M+3)\end{aligned}$$

and

$$\begin{aligned}\rho(z_*, x)^2 \geq \rho(z_\alpha, x)^2 - \delta_C(4M+1) &\geq \bar{c}\rho(x, P_\alpha(x))^2 + \rho(z_\alpha, P_\alpha(x))^2 - \delta_C(4M+1) \\ &\geq \bar{c}\rho(x, P_\alpha(x))^2 + \rho(z_*, P_\alpha(x))^2 - \delta_C(8M+4).\end{aligned}$$

Thus

$$\rho(z_*, x)^2 \geq \bar{c}\rho(x, P_\alpha(x))^2 + \rho(z_*, P_\alpha(x))^2 - \delta_C(8M+4)$$

$x \in B(\theta, M)$.

In this paper we consider a set-mapping T satisfying the equation above and obtained a generalization of Theorem 1.1.

2. The main result

Let (X, ρ) be a metric space. We use the notation and definitions introduced in Section 1. Assume that

$$T : X \to 2^{\setminus}\{\emptyset\},$$

$z_* \in X$, $M > 1$, $\delta_M \in (0, 1]$, $\bar{c} \in (0, 1)$ and

$$\rho(z_*, \theta) < M. \tag{2.1}$$

and that for each $x \in B(\theta, 3M+2)$ and each $y \in T(x)$,

$$\rho(z_*, x)^2 \geq \rho(z_*, y)^2 + \bar{c}\rho(x, y)^2 - \delta_M. \tag{2.2}$$

We prove the following result.

Theorem 2.1. *Let*

$$\delta_0, \delta_1, \in (0, 1),$$

$$\epsilon_0 = \max\{2\delta_0,\ (8\delta_M \bar{c}^{-1})^{1/2},\ (16\bar{c}^{-1}\delta_0(4M+5))^{1/2}, \delta_1\} \tag{2.3}$$

and

$$n_0 = 1 + \lfloor 16(2M+2)^2 \bar{c}^{-1}\epsilon_0^{-2} \rfloor + 1. \tag{2.4}$$

Assume that $\{x_t\}_{t=0}^\infty \subset X$,

$$\rho(x_0, \theta) \leq M \tag{2.5}$$

and that for each integer $t \geq 0$,

$$B(x_{t+1}, \delta_0) \cap \{y \in T(x_t) :\ \rho(x_t, y) + \delta_1 \geq \rho(x_t, \xi),\ \xi \in T(x_t)\} \neq \emptyset. \tag{2.6}$$

Then there exists an integer $q \in [1, n_0]$ *such that*

$$\rho(x_t, \theta) \le 3M + 2, \ t \in \{0, \dots, q\}$$

and

$$\rho(x_q, x_{q+1}) \le \epsilon_0.$$

Moreover, if an integer $q \ge 0$ *and the inequality above holds, then*

$$T(x_q) \subset B(x_q, 3\epsilon_0).$$

Proof. By (2.5), for each integer $t \ge 0$ there exists

$$y_t \in T(x_t) \cap B(x_{t+1}, \delta_0) \tag{2.7}$$

such that

$$\rho(x_t, y_t) + \delta_1 \ge \rho(x_t, \xi), \ \xi \in T(x_t). \tag{2.8}$$

In view of (2.7),

$$y_0 \in T(x_0). \tag{2.9}$$

By (2.2), (2.5) and (2.9),

$$\rho(z_*, x_0) \ge \rho(z_*, y_0)^2 + \bar{c}\rho(x_0, y_0)^2 - \delta_M. \tag{2.10}$$

Equations (2.1), (2.5) and (2.10) imply that

$$\rho(z_*, y_0) \le \rho(z_*, x_0) + 1 \le 2M + 1. \tag{2.11}$$

It follows from (2.1), (2.7) and (2.11) that

$$\rho(z_*, x_1) \le 2M + 2, \ \rho(\theta, x_1) \le 3M + 3. \tag{2.12}$$

Assume that s is a natural number and that for each integer $k \in [1, s]$,

$$\rho(x_k, x_{k+1}) > \epsilon_0. \tag{2.13}$$

Assume that

$$k \in \{1, \dots, s\}$$

and

$$\rho(x_k, z_*) \le 2M + 2. \tag{2.14}$$

(In view of (2.12), our assumption holds for $k = 1$.) By (2.3), (2.7) and (2.14),

$$\begin{aligned} \rho(x_k, y_k) &\ge \rho(x_k, x_{k+1}) - \rho(x_{k+1}, y_k) \\ &> \epsilon_0 - \delta_0 \ge \epsilon_0/2. \end{aligned} \tag{2.15}$$

Equations (2.1) and (2.14) imply that

$$\rho(x_k, \theta) \le \rho(x_k, z_*) + \rho(z_*, \theta) \le 3M + 2. \tag{2.16}$$

It follows from (2.2), (2.3), (2.7), (2.15) and (2.16) that

$$\begin{aligned} \rho(z_*, x_k)^2 &\ge \rho(z_*, y_k)^2 + \bar{c}\rho(x_k, y_k)^2 - \delta_M \\ &\ge \rho(z_*, y_k)^2 + \bar{c}\epsilon_0/^2 4 - \delta_M \\ &\ge \rho(z_*, y_k)^2 + \bar{c}\epsilon_0/^2 8. \end{aligned} \tag{2.17}$$

In view of (2.14) and (2.17),

$$\rho(z_*, y_k) \le \rho(z_*, x_k) \le 2M + 2. \tag{2.18}$$

By (2.17) and (2.18),

$$\rho(z_*, x_{k+1}) \le \rho(z_*, y_k) + \rho(y_k, x_{k+1}) \le 2M + 3. \tag{2.19}$$

By (2.7), (2.18) and (2.19),

$$\begin{aligned} &|\rho(z_*, y_k)^2 - \rho(z_*, x_{k+1})^2| \\ &\quad \le |\rho(z_*, y_k) - \rho(z_*, x_{k+1})|(\rho(z_*, y_k) + \rho(z_*, x_{k+1})) \\ &\quad \le \rho(x_{k+1}, y_k)(4M + 5) \\ &\quad \le \delta_0(4M + 5). \end{aligned} \tag{2.20}$$

Equations (2.3), (2.17) and (2.20),

$$\begin{aligned} \rho(z_*, x_{k+1})^2 \le \rho(z_*, y_k)^2 + \delta_0(4M+5) &\le \rho(z_*, x_k)^2 - \bar{c}\epsilon_0^2/8 + \delta_0(4M+5) \\ &\le \rho(z_*, x_k)^2 - \bar{c}\epsilon_0^2/16. \end{aligned}$$

In view of (2.21),

$$\rho(z_*, x_{k+1}) \le \rho(z_*, x_k).$$

Thus by induction we showed that for each $k \in \{1, \dots, s+1\}$,

$$\rho(z_*, x_k) \le 2M + 2, \ \rho(\theta, x_k) \le 3M + 2$$

and that for each $k \in \{1, \dots . s\}$.

$$\rho(z_*, x_{k+1})^2 \le \rho(z_*, x_k)^2 - \bar{c}\epsilon_0^2/16. \tag{2.21}$$

It follows from (2.4), (2.12) and (2.21) that

$$\begin{aligned} (2M+2)^2 \ge \rho(z_*, x_1)^2 &\ge \rho(z_*, x_1)^2 - \rho(z_*, x_{s+1})^2 \\ &= \sum_{k=1}^{s} (\rho(z_*, x_k)^2 - \rho(z_*, x_{k+1})^2) \\ &\ge 16^{-1}\bar{c}\epsilon_0^2 s \end{aligned}$$

and

$$s \le 16(2M+2)^2 \bar{c}^{-1} \epsilon_0^{-2} \le n_0 - 1.$$

This implies that there exists an integer $q \in \{1, \dots, n_0\}$ such that

$$\rho(x_q, x_{q+1}) \le \epsilon_0 \tag{2.22}$$

and

$$\rho(\theta, x_k) \le 3M + 2, \ k \in \{0, \dots . q\}.$$

Assume that $q \in \{0, 1, \dots, \}$ and that (2.22) holds. It follows from (2.7) and (2.22) that

$$\rho(y_q, x_q) \le \epsilon_0 + \delta_0$$

and that for each $\xi \in T(x_q)$,

$$\rho(x_q, \xi) \le \rho(x_q, y_q) + \delta_1 \le \epsilon_0 + \delta_0 + \delta_1 \le 3\epsilon_0.$$

Theorem 2.1 is proved. □

3. An extension

We use all the notation introduced in Sections 1 and 2 and assume that all the assumptions made in Section 2 holds.

Theorem 3.1. *Let*

$$M > \bar{M} > \max\{1, \rho(\theta, z_*)\}, r_0 \in (0, 1], \delta_0, \delta_1 \in (0, 1), \tag{3.1}$$

$$\{z \in X : \ \rho(z, \xi) \le r_0, \ \xi \in T(z)\} \subset B(\theta, \bar{M}), \tag{3.2}$$

$$\epsilon_0 := \max\{2\delta_0, \ (8\delta_M \bar{c}^{-1})^{1/2}, \ (16\bar{c}^{-1}\delta_0(4M + 5))^{1/2}, \ \delta_1\} \le r_0/3 \tag{3.3}$$

and

$$n_0 = \lfloor 16(2M + 2)^2 c^{-1} \epsilon_0^{-2} \rfloor + 1. \tag{3.4}$$

Assume that $\{x_t\}_{t=0}^{\infty} \subset X$*,*

$$\rho(x_0, \theta) \le M \tag{3.5}$$

and that for each integer $t \ge 0$*,*

$$B(x_{t+1}, \delta_0) \cap \{y \in T(x_t) : \ \rho(x_t, y) + \delta_1 \ge \rho(x_t, \xi), \ \xi \in T(x_t)\} \ne \emptyset. \tag{3.6}$$

Then for each integer $t \ge 0$*,*

$$\rho(x_t, \theta) \le 3M + 2 \tag{3.7}$$

and there exists a strictly increasing sequence of natural numbers $\{q_p\}_{p=1}^{\infty}$ *such that*

$$1 \le q_p \le n_0 \tag{3.8}$$

and that for each integer $p \ge 1$*,*

$$1 \le q_{p+1} - q_p \le n_0, \tag{3.9}$$

$$\rho(x_{q_p}, x_{q_p+1}) \le \epsilon_0 \tag{3.10}$$

and

$$\rho(x_{q_p}, \xi) \le 3\epsilon_0, \ \xi \in T(x_q). \tag{3.11}$$

Proof. By Theorem 2.1, there exists an integer

$$q_1 \in [1, n_0]$$

such that

$$\rho(x_{q_1}, x_{q_1+1}) \le \epsilon_0, \ T(x_{q_1}) \subset B(x_{q_1}, 3\epsilon_0)$$

and

$$\rho(x_t, \theta) \le 3M + 2, \ t \in \{0, \dots, q_1\}.$$

Together with (3.2) this implies that

$$\rho(x_{q_1}, \theta) \le \bar{M} \le M.$$

Assume that $k \ge 1$ is an integer and we defined natural numbers $q_p, \ p = 1, \dots, k$ such that (3.8) holds, for each integer $p \in \{1, \dots, k\} \setminus \{k\}$, (3.9) holds and (3.10) and (3.11) hold for $p = 1, \dots, k$ and

$$\rho(x_i, \theta) \le 3M + 2, \ i = 0, \dots, q_k.$$

(Clearly our assumption holds for $k = 1$.) It follows from (3.2) and (3.11) that

$$\rho(x_{q_k}, \theta) \leq \bar{M} \leq M.$$

By Theorem 2.1 applied to $\{x_i\}_{i=q_k}^{\infty}$ there exits an integer

$$q_{k+1} \in [1 + q_k, n_0 + q_k]$$

such that

$$\rho(x_i, \theta) \leq 3M + 2, \ i \in \{q_k, \dots, q_{k+1}\},$$
$$\rho(x_{q_{k+1}}, x_{q_{k+1}+1}) \leq \epsilon_0$$

and

$$T(x_{q_{k+1}}) \subset B(x_{q_{k+1}}, 3\epsilon_0).$$

Thus by induction Theorem 3.1 is proved. □

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