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SOLUTION OF A SINGULAR MINIMUM ENERGY CONTROL PROBLEM FOR TIME DELAY SYSTEM: REGULARIZATION APPROACH

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Dedicated to Professor Roger J.-B. Wets on the occasion of his 85th birthday

Abstract. In this paper, a minimum energy control problem for a linear time-dependent differential system with state delays (point-wise and distributed) is considered. The feature of this problem is that a weight matrix of the control in the problem's functional is singular (but non-zero). Due to this feature, the problem itself is singular. Using the regularization method and the asymptotic analysis of the regularized minimum energy control problem, the solution to the considered singular minimum energy control problem is derived. Illustrative example is presented. Along with this example, an example on non-uniqueness of solution to the singular minimum energy control problem is presented.

1. Introduction

An optimal control problem is called singular if it cannot be solved by application of the first-order solvability conditions, i.e., by application of the Pontriagin's Maximum Principle [30] and by application of the Hamilton-Jacobi-Bellman equation approach (Dynamic Programming approach) [3]. This occurs, because the problem of maximization of the problem's Variational Hamiltonian with respect to the control either has no solution or has infinitely many solutions. Singular optimal control problems with undelayed dynamics have been studied extensively in the literature (see e.g. [2, 6, 8, 14, 16–18, 23, 27–29, 32, 33] and references therein). Singular optimal control problems with delayed dynamics also were studied in the literature although less (see e.g., [1, 9–11, 31] and references therein).

Minimum energy control problem is an optimal control problem consisting in a minimization of the control's energy expenditure for the transfer of the state of a dynamic system from a given initial position to a given terminal position in a given time. Minimum energy control problem has been studied extensively in the literature in both, undelayed and delayed, dynamics versions (see e.g. [5, 12, 13, 22, 24–26] and references therein). In these works, the case of the regular minimum energy control problem is treated, i.e., the case where the first-order solvability conditions are applicable for the solution of the problem. In the present paper, we study a singular minimum energy control problem. Namely, we study the problem where

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a weight matrix of the control in the problem's functional, which represents the control's energy expenditure, is singular (but non-zero). Moreover, the differential system, describing the dynamics of the problem, has delays (point-wise and distributed) in the state variable. To the best of our knowledge, such a minimum energy control problem has not yet been studied in the literature. To solve the considered problem, the regularization method and the asymptotic analysis of the regularized problem are applied. This approach yields an open-loop optimal control of the considered singular problem.

The paper is organized as follows. In Section 2, the singular minimum energy control problem with delayed dynamics is rigorously formulated. In Section 3, a proper transformation of this problem is carried out. Due to this transformation, the initially formulated problem is converted to an equivalent singular problem with undelayed dynamics. Section 4 is devoted to the regularization of the singular minimum energy control problem with undelayed dynamics. The open-loop solution (optimal control) of the regularized problem also is presented in this section. In Section 5, the asymptotic analysis of the solution to the regularized problem is carried out. Solution of the singular minimum energy control problem is derived in Section 6. Illustrative example is given in Section 7. In Section 8 one more example, illustrating a non-uniqueness of solution to the singular minimum energy control problem in an extended set of admissible controls, is given. Conclusions are presented in Section 9.

The following main notations and notions are used in the paper:

(1) For an $n \times m$-matrix A, $(n \geq 1,\ m \geq 1)$, its norm is defined as: $\|A\| \triangleq \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|$, where a_{ij}, $(i = 1, ...n;\ j = 1, ..., m)$ are the entries of A.
(2) The upper index $''T''$ denotes the transposition either of a vector x (x^T) or of a matrix A (A^T).
(3) I_n denotes the identity matrix of dimension n.
(4) $L^2[t_1, t_2; \mathbb{R}^n]$ denotes the linear space of all functions $x(\cdot) : [t_1, t_2] \to \mathbb{R}^n$ square integrable in the interval $[t_1, t_2]$.
(5) $\mathcal{M}[t_1, t_2; \mathbb{R}^n]$ denotes the linear space of all pairs $\big(x_0, \varphi_0(t)\big)$, $t \in [t_1, t_2]$, where $x_0 \in \mathbb{R}^n$ and $\varphi_0(\cdot) \in L^2[t_1, t_2; \mathbb{R}^n]$.
(6) $\mathrm{col}(x, y)$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, denotes the column block-vector of the dimension $n + m$ with the upper block x and the lower block y, i.e., $\mathrm{col}(x, y) = (x^T, y^T)^T$.
(7) $\mathrm{diag}(A, B)$, where A and B are matrices of the dimensions $n \times n$ and $m \times m$, is a block-diagonal matrix with the upper left-hand block A and the lower right-hand block B.
(8) $O_{n_1 \times n_2}$ is used for the zero matrix of the dimension $n_1 \times n_2$, excepting the cases where the dimension of the zero matrix is obvious. In such cases, the notation 0 is used for the zero matrix.

2. Problem formulation

The system under the consideration is

$$\frac{dz(t)}{dt} = \sum_{i=0}^{N} A_i(t)z(t-h_i) + \int_{-h}^{0} G(t,\tau)z(t+\tau)d\tau \\ + B(t)u(t) + f(t), \qquad t \in [0, t_f], \tag{2.1}$$

where for any $t \in [-h, t_f]$, $z(t) \in \mathbb{R}^n$; for any $t \in [0, t_f]$, $u(t) \in \mathbb{R}^r$, $(r \le n)$, (u is a control); $0 = h_0 < h_1 < ... < h_N = h$ are given constant time delays in the state; $t_f > 0$ is a given final time-instant; for any $t \in [0, t_f]$ and $\tau \in [-h, 0]$, $A_i(t)$, $(i = 0, 1, ..., N)$, $G(t, \tau)$, $B(t)$ are given matrices of corresponding dimensions, and $f(t) \in \mathbb{R}^n$ is a given vector; the matrix-valued functions $A_i(t)$, $(i = 0, 1, ..., N)$, $B(t)$ and the vector-valued function $f(t)$ are continuous in the interval $[0, t_f]$; the matrix-valued function $G(t, \tau)$ is piece-wise continuous in $\tau \in [-h, 0]$ for any $t \in [0, t_f]$, and this function is continuous in $t \in [0, t_f]$ uniformly with respect to $\tau \in [-h, 0]$.

Let $g_0(\tau) = \big(z_0, \varphi(\tau)\big) \in \mathcal{M}[-h, 0; \mathbb{R}^n]$ and $z_f \in \mathbb{R}^n$ be arbitrary given. Using these points, we consider the following initial and terminal conditions for the system (2.1):

$$z(\tau) = \varphi(\tau), \qquad \tau \in [-h, 0); \qquad z(0) = z_0, \tag{2.2}$$

$$z(t_f) = z_f. \tag{2.3}$$

Definition 2.1. Consider the set U_z of all controls $u(t) \in L^2[0, t_f; E^r]$ such that the system (2.1) subject to the initial conditions (2.2) and the terminal condition (2.3) has a solution. The set U_z is called the set of admissible controls for the system (2.1).

Along with the system (2.1), we consider the functional

$$J\big(u(\cdot)\big) = \int_0^{t_f} u^T(t)R(t)u(t)dt, \tag{2.4}$$

where for any $t \in [0, t_f]$, $R(t)$ is a given symmetric matrix of the dimension $r \times r$; the matrix-valued function $R(t)$ is continuous in the interval $[0, t_f]$.

The minimum energy control problem for the system (2.1) is to find a control $u^0(\cdot) \in U_z$ such that

$$J^0 \triangleq J\big(u^0(\cdot)\big) \le J\big(u(\cdot)\big) \qquad \forall u(\cdot) \in U_z. \tag{2.5}$$

Remark 2.2. If, for any given $g_0(\tau) \in \mathcal{M}[-h, 0; \mathbb{R}^n]$ and $z_f \in \mathbb{R}^n$, the set U_z is not empty and the matrix $R(t)$ is positive definite for all $t \in [0, t_f]$, then the minimum energy control problem (2.1),(2.4),(2.5) has the solution (see e.g. [12], Theorem 6.1). Subject to the assumption on a positive definiteness of the matrix $R(t)$, $t \in [0, t_f]$, the minimum energy control problem (2.1),(2.4),(2.5) is called a regular problem. The necessary condition for the validity of the aforementioned result of [12] is the invertibility of the matrix $R(t)$ for any $t \in [0, t_f]$. Thus, this result is not applicable to solution of the minimum energy control problem (2.1),(2.4),(2.5) in the case where the matrix $R(t)$ is not invertible at least at one point of the interval $[0, t_f]$. In such a case, the minimum energy control problem (2.1),(2.4),(2.5) is called a singular problem. In the present paper, we solve this problem in the case where the matrix $R(t)$ is positive semi-definite and $\det R(t) = 0$ for all $t \in [0, t_f]$. More precisely, we assume that the matrix $R(t)$ has the block-diagonal form

$$R(t) = \mathrm{diag}\big(R_1(t)\ ,\ O_{(r-q)\times(r-q)}\big), \tag{2.6}$$

where $0 < q < r$; the $q \times q$-matrix $R_1(t)$ is positive definite for all $t \in [0, t_f]$.

3. Transformation of the minimum energy control problem (2.1),(2.4),(2.5),(2.6)

Consider the following terminal-value problem for $n \times n$-matrix-valued function $\Psi(t)$:

$$\begin{aligned} \frac{d\Psi(t)}{dt} &= -\sum_{i=0}^{N} \Psi(t + h_i)A_i(t + h_i) \\ &- \int_{-h}^{0} \Psi(t - \tau)G(t - \tau, \tau)d\tau, \quad t \in [0, t_f], \\ &\Psi(t_f) = I_n, \qquad \Psi(t) = 0, \quad t > t_f. \end{aligned} \tag{3.1}$$

By virtue of the results of [20] (Section 4.3), the problem (3.1) has the unique solution $\Psi(t)$, $t \geq 0$.

Using the Halanay Transformation for a linear system with state delays (see [21]) and the transformation for a linear nonhomogeneous system (see [15,19]), we make the following change of the state variable in the system (2.1):

$$\begin{aligned} x(t) &= \Psi(t)z(t) + \sum_{i=1}^{N} \int_{t}^{t+h_i} \Psi(s)A_i(s)z(s - h_i)ds \\ &+ \int_{t}^{t+h} \left(\int_{t}^{s} \Psi(\sigma)G\Big(\sigma, s - \sigma - h\Big)d\sigma \right) z(s - h)ds \\ &+ \int_{t}^{t_f} \Psi(\xi)f(\xi)d\xi, \qquad t \in [0, t_f], \end{aligned} \tag{3.2}$$

where $x(t)$ is a new state variable.

Let us denote

$$\mathcal{B}(t) \stackrel{\triangle}{=} \Psi(t)B(t), \qquad t \in [0, t_f], \tag{3.3}$$

$$\begin{aligned} x_0 &\stackrel{\triangle}{=} \Psi(0)z_0 + \sum_{i=1}^{N} \int_{-h_i}^{0} \Psi(\tau + h_i)A_i(\tau + h_i)\varphi(\tau)d\tau \\ &+ \int_{-h}^{0} \left(\int_{0}^{\tau+h} \Psi(\sigma)G\Big(\sigma, \tau - \sigma\Big)d\sigma \right) \varphi(\tau)d\tau + \int_{0}^{t_f} \Psi(\xi)f(\xi)d\xi. \end{aligned} \tag{3.4}$$

Direct differentiation of $x(t)$ in (3.2) with respect to t, and use of the problem (3.1) and the expressions (3.3),(3.4) yield the following assertion.

Proposition 3.1. *Let for a given $u(t) \in U_z$, the absolutely continuous function $z(t)$, $t \in [0, t_f]$ be the solution of the boundary-value problem (2.1),(2.2)-(2.3). Then, the function $x(t)$, given by (3.2), is the absolutely continuous solution of the boundary-value problem, consisting of the differential system*

$$\frac{dx(t)}{dt} = \mathcal{B}(t)u(t), \quad t \in [0, t_f], \tag{3.5}$$

the initial condition

$$x(0) = x_0, \tag{3.6}$$

and the terminal condition

$$x(t_f) = z_f. \tag{3.7}$$

Definition 3.2. Consider the set U_x of all controls $u(t) \in L^2[0, t_f; E^r]$ such that the system (3.5) subject to the initial condition (3.6) and the terminal condition (3.7) has a solution. The set U_x is called the set of admissible controls for the system (3.5).

The minimum energy control problem for the system (3.5) is to find a control $u^*(\cdot) \in U_x$ such that

$$J^* \stackrel{\triangle}{=} J\big(u^*(\cdot)\big) \leq J\big(u(\cdot)\big) \quad \forall u(\cdot) \in U_x. \tag{3.8}$$

By virtue of the results of [12], we directly have the following assertion.

Proposition 3.3. *If $u^0(t)$, $t \in [0, t_f]$ is a solution of the minimum energy control problem (2.1),(2.4),(2.5),(2.6), then it is a solution of the minimum energy control problem (3.5),(2.4),(3.8),(2.6). Vice versa: if $u^*(t)$, $t \in [0, t_f]$ is a solution of the minimum energy control problem (3.5),(2.4),(3.8),(2.6), then it is a solution of the minimum energy control problem (2.1),(2.4),(2.5), (2.6). Moreover,*

$$J^0 = J^*. \tag{3.9}$$

Remark 3.4. Due to Proposition 3.3, the initially formulated minimum energy control problem (2.1),(2.4),(2.5),(2.6) and the new minimum energy control problem (3.5),(2.4),(3.8),(2.6) are equivalent to each other. In what follows of the paper, we deal with the simpler problem (3.5),(2.4),(3.8),(2.6) and we call this problem the Original Minimum Energy Control Problem (OMECP). In the subsequent sections, we will derive the solution of the OMECP valid for all $x_0 \in \mathbb{R}^n$ and $z_f \in \mathbb{R}^n$.

4. Regularization of the OMECP

4.1. **Partial cheap control problem.** In order to derive a solution of the OMECP, we replace it by a parameter dependent regular minimum energy control problem, which is close in some sense to the OMECP. This new minimum energy control problem has the same dynamics (3.5) and the same set of admissible controls U_x as the OMECP has. However, the functional in the new problem differs from the one in the OMECP. Namely, this functional has the form

$$J_\varepsilon\big(u(\cdot)\big) \stackrel{\triangle}{=} \int_0^{t_f} u^T(t)\big(R(t) + \mathcal{E}\big)u(t)dt, \tag{4.1}$$

where $\mathcal{E} = \mathrm{diag}\big(\underbrace{0, ..., 0}_{q}, \underbrace{\varepsilon, ..., \varepsilon}_{r-q}\big)$, $\varepsilon > 0$ is a small parameter.

The minimum energy control problem for the system (3.5) with the functional (4.1) is to find a control $u_\varepsilon^*(\cdot) \in U_x$ such that

$$J_\varepsilon^* \stackrel{\triangle}{=} J_\varepsilon\big(u_\varepsilon^*(\cdot)\big) \le J_\varepsilon\big(u(\cdot)\big) \quad \forall u(\cdot) \in U_x. \tag{4.2}$$

Using (2.6), we have immediately

$$R(t) + \mathcal{E} = \mathrm{diag}\big(R_1(t), \varepsilon I_{r-q}\big). \tag{4.3}$$

Remark 4.1. Due to (4.3) and the assumption that $R_1(t)$ is a positive definite matrix for all $t \in [0, t_f]$, the matrix $R(t) + \mathcal{E}$ is positive definite for all $t \in [0, t_f]$ and $\varepsilon > 0$. Thus, the minimum energy control problem (3.5),(4.1),(4.2) is regular for all $\varepsilon > 0$. Moreover, due to (4.3) and the smallness of ε, this problem is a partial cheap control problem, i.e., the optimal control problem where the cost only of some (but not all) control coordinates in the functional is small. In what follows, we call the problem (3.5),(4.1),(4.2) the Partial Cheap Control Minimum Energy Problem (PCCMEP).

4.2. **Solution of the PCCMEP.** Consider the matrices

$$\mathcal{W}_{\mathcal{B}} \stackrel{\triangle}{=} \int_0^{t_f} \mathcal{B}(t)\mathcal{B}^T(t)dt. \tag{4.4}$$

$$W_{\mathcal{B}}(\varepsilon) \stackrel{\triangle}{=} \int_0^{t_f} \mathcal{B}(t)\big(R(t) + \mathcal{E}\big)^{-1}\mathcal{B}^T(t)dt. \tag{4.5}$$

By virtue of the results of [12], we have the following assertion.

Proposition 4.2. *Let the matrix $\mathcal{W}_{\mathcal{B}}$ be positive definite. Then, for any $x_0 \in \mathbb{R}^n$ and $z_f \in \mathbb{R}^n$, the set U_x is not empty. For any $\varepsilon > 0$, the matrix $W_{\mathcal{B}}(\varepsilon)$ also is positive definite, and the solution of the PCCMEP exists and has the form:*

$$u_\varepsilon^*(t) = \big(R(t) + \mathcal{E}\big)^{-1}\mathcal{B}^T(t)W_{\mathcal{B}}^{-1}(\varepsilon)(z_f - x_0), \qquad t \in [0, t_f]. \tag{4.6}$$

Moreover,

$$J_\varepsilon^* = (z_f - x_0)^T W_{\mathcal{B}}^{-1}(\varepsilon)(z_f - x_0). \tag{4.7}$$

5. Asymptotic analysis of the solution to the PCCMEP

5.1. **Asymptotic analysis of the matrix $W_{\mathcal{B}}^{-1}(\varepsilon)$.** To analyze the asymptotic behaviour of $W_{\mathcal{B}}^{-1}(\varepsilon)$ for $\varepsilon \to +0$, first we are going to transform equivalently this matrix. For this purpose, we partition the matrix $\mathcal{B}(t)$, $t \in [0, t_f]$ into blocks as:

$$\mathcal{B}(t) = \Big(\mathcal{B}_1(t)\ ,\ \mathcal{B}_2(t)\Big), \qquad t \in [0, t_f], \tag{5.1}$$

where the matrices $\mathcal{B}_1(t)$ and $\mathcal{B}_2(t)$ have the dimensions $n \times q$ and $n \times (r - q)$, respectively.

Using the equations (4.3),(5.1) and taking into account the invertibility of the matrix $R_1(t)$, we obtain for all $t \in [0, t_f]$ and $\varepsilon > 0$

$$\mathcal{B}(t)\big(R(t) + \mathcal{E}\big)^{-1}\mathcal{B}^T(t) = \mathcal{B}_1(t)R_1^{-1}(t)\mathcal{B}_1^T(t) + \frac{1}{\varepsilon}\mathcal{B}_2(t)\mathcal{B}_2^T(t). \tag{5.2}$$

Let us treat the matrix

$$K \triangleq \int_0^{t_f} \mathcal{B}_2(t)\mathcal{B}_2^T(t)dt. \tag{5.3}$$

This matrix is symmetric and positive semi-definite.

In what follows, we assume

A1. The matrix K has zero eigenvalue of the algebraic multiplicity k, $(n - r + q \leq k < n)$.

Due to this assumption and the results of [4], there exists an orthogonal $n \times n$-matrix L, $(L^T = L^{-1})$, such that the following equality is valid:

$$D \triangleq LKL^T = \begin{pmatrix} O_{k\times k} & O_{k\times(n-k)} \\ O_{(n-k)\times k} & \Theta \end{pmatrix}, \tag{5.4}$$

where the block Θ is of dimension $(n-k)\times(n-k)$, and it is a nonsingular (positive definite) matrix.

Since the integrand in (5.3) is a positive semi-definite matrix for all $t \in [0, t_f]$, the equations (5.3) and (5.4) yield

$$L\mathcal{B}_2(t) = \begin{pmatrix} O_{k\times(r-q)} \\ \Lambda(t) \end{pmatrix}, \quad t \in [0, t_f], \tag{5.5}$$

where the block $\Lambda(t)$ is of dimension $(n-k) \times (r-q)$.

Due to (5.3)-(5.5),

$$\int_0^{t_f} \Lambda(t)\Lambda^T(t)dt = \Theta. \tag{5.6}$$

Using the equations (4.5),(5.1)-(5.4) and the orthogonality of the matrix L, we can represent the matrix $W_{\mathcal{B}}^{-1}(\varepsilon)$ as:

$$W_{\mathcal{B}}^{-1}(\varepsilon) = L^T\left(\frac{1}{\varepsilon}D + L\int_0^{t_f} \mathcal{B}_1(t)R_1^{-1}(t)\mathcal{B}_1^T(t)dtL^T\right)^{-1} L. \tag{5.7}$$

Let us partition the matrix $L\int_0^{t_f} \mathcal{B}_1(t)R_1^{-1}(t)\mathcal{B}_1^T(t)dtL^T$ into blocks as:

$$L\int_0^{t_f} \mathcal{B}_1(t)R_1^{-1}(t)\mathcal{B}_1^T(t)dtL^T = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^T & \Omega_{13} \end{pmatrix}, \tag{5.8}$$

where the block Ω_{11} is of dimension $k\times k$; the block Ω_{12} is of dimension $k\times(n-k)$; the block $\Omega_{u,13}$ is of dimension $(n-k)\times(n-k)$; the matrices Ω_{11} and Ω_{13} are symmetric and positive semi-definite.

Due to the equations (5.4),(5.7),(5.8), the matrix $W_{\mathcal{B}}^{-1}(\varepsilon)$ can be rewritten in the form

$$W_{\mathcal{B}}^{-1}(\varepsilon) = L^T\Gamma^{-1}(\varepsilon)L, \tag{5.9}$$

where

$$\Gamma(\varepsilon) = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_2^T & (1/\varepsilon)\Gamma_3(\varepsilon) \end{pmatrix},$$

$$\Gamma_1 = \Omega_{11}, \quad \Gamma_2 = \Omega_{12}, \quad \Gamma_3(\varepsilon) = \Theta + \varepsilon\Omega_{13}.$$

(5.10)

Being positive definite, the matrix Θ is invertible. Therefore, there exists a positive number ε_1 such that, for all $\varepsilon \in (0, \varepsilon_1]$, the matrix $\Gamma_3(\varepsilon)$ is invertible and the following inequality is valid:

$$\left\|\Gamma_3^{-1}(\varepsilon) - \Theta^{-1}\right\| \leq a_1\varepsilon, \tag{5.11}$$

where $a_1 > 0$ is some constant independent of ε.

In what follows, we assume

A2. The matrix $\Gamma_1 = \Omega_{11}$ is invertible.

Consider the matrix

$$\Delta(\varepsilon) \stackrel{\triangle}{=} \Gamma_1 - \varepsilon\Gamma_2\Gamma_3^{-1}(\varepsilon)\Gamma_2^T = \Omega_{11} - \varepsilon\Omega_{12}\Gamma_3^{-1}(\varepsilon)\Omega_{12}^T. \tag{5.12}$$

By virtue of the inequality (5.11), the matrix $\Gamma_3^{-1}(\varepsilon)$ is bounded for all $\varepsilon \in (0, \varepsilon_1]$. Hence, due to the assumption A2, there exists a positive number $\varepsilon_2 \leq \varepsilon_1$ such that, for all $\varepsilon \in (0, \varepsilon_2]$, the matrix $\Delta(\varepsilon)$ is invertible and its inverse matrix satisfies the inequality

$$\left\|\Delta^{-1}(\varepsilon) - \Omega_{11}^{-1}\right\| \leq a_2\varepsilon, \tag{5.13}$$

where $a_2 > 0$ is some constant independent of ε.

Applying the Frobenius formula (see e.g. [7]) to the calculation of $\Gamma^{-1}(\varepsilon)$ and taking into account the equation (5.10) and the inequality (5.13), we obtain that for all $\varepsilon \in (0, \varepsilon_2]$

$$\Gamma^{-1}(\varepsilon) = \Phi(\varepsilon) = \begin{pmatrix} \Phi_1(\varepsilon) & \Phi_2(\varepsilon) \\ \Phi_2^T(\varepsilon) & \Phi_3(\varepsilon) \end{pmatrix}, \tag{5.14}$$

where

$$\begin{aligned} \Phi_1(\varepsilon) &= \Delta^{-1}(\varepsilon), \quad \Phi_2(\varepsilon) = -\varepsilon\Delta^{-1}(\varepsilon)\Omega_{12}\Gamma_3^{-1}(\varepsilon), \\ \Phi_3(\varepsilon) &= \varepsilon\Gamma_3^{-1}(\varepsilon) + \varepsilon^2\Gamma_3^{-1}(\varepsilon)\Omega_{12}^T\Delta^{-1}(\varepsilon)\Omega_{12}\Gamma_3^{-1}(\varepsilon). \end{aligned} \tag{5.15}$$

Let us estimate the matrices $\Phi_2(\varepsilon)$ and $\Phi_3(\varepsilon)$. Using the expressions of these matrices and the inequalities (5.11),(5.13), we obtain the existence of a positive number $\varepsilon_3 \leq \varepsilon_2$ such that, for all $\varepsilon \in (0, \varepsilon_3]$, the following inequalities are valid:

$$\left\|\Phi_2(\varepsilon)\right\| \leq a_3\varepsilon, \quad \left\|\Phi_3(\varepsilon)\right\| \leq a_3\varepsilon, \tag{5.16}$$

$$\left\|\frac{1}{\varepsilon}\Phi_2(\varepsilon) + \Omega_{11}^{-1}\Omega_{12}\Theta^{-1}\right\| \leq a_3\varepsilon, \quad \left\|\frac{1}{\varepsilon}\Phi_3(\varepsilon) - \Theta^{-1}\right\| \leq a_3\varepsilon. \tag{5.17}$$

Now, using the estimates (5.13),(5.16),(5.17), we can construct and justify the asymptotic expansions of the solution of the PCCMEP $u_\varepsilon^*(t)$ (see the equation (4.6)) and the corresponding value of the functional in the PCCMEP J_ε^* (see the equation (4.7)).

5.2. **Asymptotic expansion of $u^*_\varepsilon(t)$.** Substituting (4.3),(5.1) and (5.9) into (4.6), we obtain

$$u^*_\varepsilon(t) = \begin{pmatrix} R_1^{-1}(t) & 0 \\ 0 & \varepsilon^{-1} I_{r-q} \end{pmatrix} \begin{pmatrix} \mathcal{B}_1^T(t) \\ \mathcal{B}_2^T(t) \end{pmatrix} L^T \Gamma^{-1}(\varepsilon) L(z_f - x_0)$$
$$= \begin{pmatrix} u^*_{\varepsilon,1}(t) \\ u^*_{\varepsilon,2}(t) \end{pmatrix}, \quad t \in [0, t_f], \tag{5.18}$$

where

$$u^*_{\varepsilon,1}(t) = R_1^{-1}(t)\mathcal{B}_1^T(t) L^T \Gamma^{-1}(\varepsilon) L(z_f - x_0), \quad t \in [0, t_f],$$
$$u^*_{\varepsilon,2}(t) = \varepsilon^{-1}\mathcal{B}_2^T(t) L^T \Gamma^{-1}(\varepsilon) L(z_f - x_0), \quad t \in [0, t_f]. \tag{5.19}$$

Thus, we have represented the solution $u^*_\varepsilon(t)$ of the PCCMEP in the block-vector form with the upper block $u^*_{\varepsilon,1}(t)$ and the lower block $u^*_{\varepsilon,2}(t)$. We start to construct the asymptotic expansion of $u^*_\varepsilon(t)$ with its upper block.

Let us introduce into the consideration the following block-matrix of the dimension $n \times n$:

$$\Phi_0 \triangleq \begin{pmatrix} \Omega_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.20}$$

Using this matrix, we construct the vector-valued function

$$u^*_{0,1}(t) \triangleq R_1^{-1}(t)\mathcal{B}_1^T(t) L^T \Phi_0 L(z_f - x_0), \quad t \in [0, t_f]. \tag{5.21}$$

Now, using the equations (5.14),(5.15),(5.19),(5.20) and the inequalities (5.13), (5.16), we have the inequality

$$\left\| u^*_{\varepsilon,1}(t) - u^*_{0,1}(t) \right\| \le c_1 \varepsilon \quad \forall\, t \in [0, t_f],\ \varepsilon \in (0, \varepsilon_3], \tag{5.22}$$

where $c_1 > 0$ is some constant independent of ε.

Proceed to the asymptotic analysis of $u^*_{\varepsilon,2}(t)$. From (5.5), we directly have

$$\mathcal{B}_2^T(t) L^T = \big(O_{(r-q)\times k}\ ,\ \Lambda^T(t) \big), \quad t \in [0, t_f]. \tag{5.23}$$

Substitution of (5.14) and (5.23) into the expression for $u^*_{\varepsilon,2}(t)$ (see the equation (5.19)) yields after a routine matrix algebra the following expression for all $t \in [0, t_f]$:

$$u^*_2(t, \varepsilon) = \Big(\varepsilon^{-1}\Lambda^T(t)\Phi_2^T(\varepsilon)\ ,\ \varepsilon^{-1}\Lambda^T(t)\Phi_3(\varepsilon) \Big) L(z_f - x_0). \tag{5.24}$$

Consider the vector-valued function

$$u^*_{0,2}(t) \triangleq \Big(-\Lambda^T(t)\Theta^{-1}\Omega_{12}^T\Omega_{11}^{-1}\ ,\ \Lambda^T(t)\Theta^{-1} \Big) L(z_f - x_0), \quad t \in [0, t_f]. \tag{5.25}$$

Now, using the estimates (5.17), we obtain the inequality

$$\left\| u^*_{\varepsilon,2}(t) - u^*_{0,2}(t) \right\| \le c_2 \varepsilon \quad \forall\, t \in [0, t_f],\ \varepsilon \in (0, \varepsilon_3], \tag{5.26}$$

where $c_2 > 0$ is some constant independent of ε.

Let us introduce into the consideration the following vector-valued function:

$$u_0^*(t) \triangleq \mathrm{col}\Big(u_{0,1}^*(t)\ ,\ u_{0,2}^*(t)\Big), \qquad t \in [0, t_f]. \tag{5.27}$$

Using the equation (5.18) and the inequalities (5.22) and (5.26), we directly have the following inequality:

$$\big\|u_\varepsilon^*(t) - u_0^*(t)\big\| \le c\varepsilon \qquad \forall\, t \in [0, t_f],\ \varepsilon \in (0, \varepsilon_3], \tag{5.28}$$

where $c > 0$ is some constant independent of ε.

The inequality (5.28) means that $u_0^*(t)$ is the zero-order asymptotic expansion with respect to $\varepsilon > 0$ of $u_\varepsilon^*(t)$, and this expansion is uniform in $t \in [0, t_f]$.

5.3. **Asymptotic expansion of J_ε^*.** Consider the value

$$J_0^* \triangleq (z_f - x_0)^T L^T \Phi_0 L (z_f - x_0). \tag{5.29}$$

Using this value, as well as the equations (4.7),(5.9),(5.14),(5.20) and the inequalities (5.13), (5.16), we obtain the inequality

$$\big|J_\varepsilon^* - J_0^*\big| \le \alpha\varepsilon \qquad \forall\, \varepsilon \in (0, \varepsilon_3], \tag{5.30}$$

where $\alpha > 0$ is some constant independent of ε.

6. Solution of the original minimum energy control problem

In this section, we are going to prove that the control $u_0^*(t)$, $t \in [0, t_f]$, given by the equations (5.21),(5.25),(5.27), solves the OMECP. This proof consists of two stages. At the first stage, we will show that this control is admissible for the system (3.5), i.e., $u_0^*(\cdot) \in U_x$. At the second stage, we will show the validity of the inequality in (3.8) with $u^*(t) = u_0^*(t)$, $t \in [0, t_f]$.

6.1. **Admissibility of the control $u_0^*(t)$ for the system (3.5).** Substituting $u(t) = u_0^*(t)$, $t \in [0, t_f]$ into the equation (3.5) and using the equations (5.1) and (5.27), we obtain the following differential equation:

$$\frac{dx(t)}{dt} = \mathcal{B}_1(t) u_{0,1}^*(t) + \mathcal{B}_2(t) u_{0,2}^*(t), \qquad t \in [0, t_f]. \tag{6.1}$$

Due to Definition 3.2, to show the inclusion $u_0^*(\cdot) \in U_x$, it is necessary and sufficient to show the existence of solution to the equation (6.1) satisfying the initial condition (3.6) and the terminal condition (3.7).

Solving the equation (6.1) subject to the initial condition (3.6), we directly have

$$x(t) = x_0 + \int_0^t \mathcal{B}_1(\sigma) u_{0,1}^*(\sigma) d\sigma + \int_0^t \mathcal{B}_2(\sigma) u_{0,2}^*(\sigma) d\sigma, \qquad t \in [0, t_f]. \tag{6.2}$$

Substitution of $t = t_f$ into (6.2) yields

$$x(t_f) = x_0 + \int_0^{t_f} \mathcal{B}_1(\sigma) u_{0,1}^*(\sigma) d\sigma + \int_0^{t_f} \mathcal{B}_2(\sigma) u_{0,2}^*(\sigma) d\sigma. \tag{6.3}$$

Therefore, the inclusion $u_0^*(\cdot) \in U_x$ is fulfilled if and only if the expression in the right-hand side of (6.3) equals z_f. To show this equality, we will treat separately the second and the third addends of the expression in the right-hand side of (6.3). Let us start with the second addend $\int_0^{t_f}\mathcal{B}_1(\sigma)u^*_{0,1}(\sigma)d\sigma$. Substituting the expression for $u^*_{0,1}(t)$ (see the equation (5.21)) into this addend, we obtain

$$\int_0^{t_f}\mathcal{B}_1(\sigma)u^*_{0,1}(\sigma)d\sigma=\int_0^{t_f}\mathcal{B}_1(\sigma)R_1^{-1}(\sigma)\mathcal{B}_1^T(\sigma)L^T\Phi_0L(z_f-x_0)d\sigma. \tag{6.4}$$

Multiplying both sides of this equality from the left by $L^TL=I_n$, we have after some rearrangement

$$\int_0^{t_f}\mathcal{B}_1(\sigma)u^*_{0,1}(\sigma)d\sigma=L^T\Big(L\int_0^{t_f}\mathcal{B}_1(\sigma)R_1^{-1}(\sigma)\mathcal{B}_1^T(\sigma)d\sigma L^T\Big)\Phi_0L(z_f-x_0). \tag{6.5}$$

Using the equations (5.8) and (5.20), we can rewrite the equation (6.5) as:

$$\begin{aligned}
&\int_0^{t_f}\mathcal{B}_1(\sigma)u^*_{0,1}(\sigma)d\sigma\\
&=L^T\begin{pmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{12}^T & \Omega_{13}\end{pmatrix}\times\begin{pmatrix}\Omega_{11}^{-1} & O_{k\times(n-k)}\\ O_{(n-k)\times k} & O_{(n-k)\times(n-k)}\end{pmatrix}L(z_f-x_0)\\
&=L^T\begin{pmatrix}I_k & O_{k\times(n-k)}\\ \Omega_{12}^T\Omega_{11}^{-1} & O_{(n-k)\times(n-k)}\end{pmatrix}L(z_f-x_0).
\end{aligned} \tag{6.6}$$

Proceed to the third addend of the expression in the right-hand side of (6.3), which is $\int_0^{t_f}\mathcal{B}_2(\sigma)u^*_{0,2}(\sigma)d\sigma$. Substituting the expression for $u^*_{0,2}(t)$ (see the equation (5.25)) into this addend, we obtain after some rearrangement

$$\int_0^{t_f}\mathcal{B}_2(\sigma)u^*_{0,2}(\sigma)d\sigma=\int_0^{t_f}\mathcal{B}_2(\sigma)\Lambda^T(\sigma)\Theta^{-1}\Big(-\Omega_{12}^T\Omega_{11}^{-1}\ ,\ I_{n-k}\Big)L(z_f-x_0)d\sigma. \tag{6.7}$$

Multiplying both sides of this equality from the left by $L^TL=I_n$, we have after some rearrangement

$$\begin{aligned}
&\int_0^{t_f}\mathcal{B}_2(\sigma)u^*_{0,2}(\sigma)d\sigma\\
&\qquad=L^T\int_0^{t_f}\big(L\mathcal{B}_2(\sigma)\big)\Lambda^T(\sigma)d\sigma\Theta^{-1}\Big(-\Omega_{12}^T\Omega_{11}^{-1}\ ,\ I_{n-k}\Big)L(z_f-x_0).
\end{aligned} \tag{6.8}$$

Using the equations (5.5) and (5.6), we can rewrite the equation (6.8) as:

$$\begin{aligned}
&\int_0^{t_f}\mathcal{B}_2(\sigma)u^*_{0,2}(\sigma)d\sigma\\
=\ &L^T\int_0^{t_f}\begin{pmatrix}O_{k\times(r-k)}\\ \Lambda(\sigma)\end{pmatrix}\Lambda^T(\sigma)d\sigma\Theta^{-1}\Big(-\Omega_{12}^T\Omega_{11}^{-1}\ ,\ I_{n-k}\Big)L(z_f-x_0)\\
=\ &L^T\int_0^{t_f}\begin{pmatrix}O_{k\times(n-k)}\\ \Lambda(\sigma)\Lambda^T(\sigma)\end{pmatrix}d\sigma\Theta^{-1}\Big(-\Omega_{12}^T\Omega_{11}^{-1}\ ,\ I_{n-k}\Big)L(z_f-x_0)\\
=\ &L^T\begin{pmatrix}O_{k\times(n-k)}\\ \int_0^{t_f}\Lambda(\sigma)\Lambda^T(\sigma)d\sigma\end{pmatrix}\Theta^{-1}\Big(-\Omega_{12}^T\Omega_{11}^{-1}\ ,\ I_{n-k}\Big)L(z_f-x_0)\\
=\ &L^T\begin{pmatrix}O_{k\times(n-k)}\\ \Theta\end{pmatrix}\Theta^{-1}\Big(-\Omega_{12}^T\Omega_{11}^{-1}\ ,\ I_{n-k}\Big)L(z_f-x_0)
\end{aligned}$$

$$= L^T \begin{pmatrix} O_{k\times(n-k)} \\ I_{n-k} \end{pmatrix} \Big(-\Omega_{12}^T\Omega_{11}^{-1}\ ,\ I_{n-k} \Big) L(z_f - x_0)$$

$$= L^T \begin{pmatrix} O_{k\times k} & O_{k\times(n-k)} \\ -\Omega_{12}^T\Omega_{11}^{-1} & I_{n-k} \end{pmatrix} L(z_f - x_0). \tag{6.9}$$

Now, using the equations (6.3),(6.6),(6.9) and the orthogonality of the matrix L, yields

$$\begin{aligned} x(t_f) = x_0 + L^T & \begin{pmatrix} I_k & O_{k\times(n-k)} \\ \Omega_{12}^T\Omega_{11}^{-1} & O_{(n-k)\times(n-k)} \end{pmatrix} L(z_f - x_0) \\ & + L^T \begin{pmatrix} O_{k\times k} & O_{k\times(n-k)} \\ -\Omega_{12}^T\Omega_{11}^{-1} & I_{n-k} \end{pmatrix} L(z_f - x_0) \\ & = x_0 + L^T I_n L(z_f - x_0) = z_f, \end{aligned} \tag{6.10}$$

meaning that the solution (6.2) of the equation (6.1) satisfies the initial condition (3.6) and the terminal condition (3.7). Thus,

$$u_0^*(\cdot) \in U_x. \tag{6.11}$$

6.2. **Validity of the inequality in (3.8) for $u^*(t) = u_0^*(t)$, $t \in [0, t_f]$.** First of all, let us calculate the value $J\big(u_0^*(\cdot)\big)$. Using the equations (2.4),(2.6) and (5.27), we obtain

$$J\big(u_0^*(\cdot)\big) = \int_0^{t_f} \big(u_0^*(t)\big)^T R(t) u_0^*(t)dt = \int_0^{t_f} \big(u_{0,1}^*(t)\big)^T R_1(t) u_{0,1}^*(t)dt. \tag{6.12}$$

Substitution of the expression for $u_{0,1}^*(t)$ (see the equation (5.21)) into the right-hand side of the equation (6.12) yields after some rearrangement

$$J\big(u_0^*(\cdot)\big) = (z_f - x_0)^T L^T \Phi_0 \Big(L \int_0^{t_f} \mathcal{B}_1(t) R_1^{-1}(t) \mathcal{B}_1^T(t) dt L^T \Big) \Phi_0 L(z_f - x_0). \tag{6.13}$$

Further, using the equations (5.8) and (5.20), we can rewrite the equation (6.13) as:

$$\begin{aligned} J\big(u_0^*(\cdot)\big) &= (z_f - x_0)^T L^T \begin{pmatrix} \Omega_{11}^{-1} & O_{k\times(n-k)} \\ O_{(n-k)\times k} & O_{(n-k)\times(n-k)} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^T & \Omega_{13} \end{pmatrix} \begin{pmatrix} \Omega_{11}^{-1} & O_{k\times(n-k)} \\ O_{(n-k)\times k} & O_{(n-k)\times(n-k)} \end{pmatrix} L(z_f - x_0) \\ &= (z_f - x_0)^T L^T \begin{pmatrix} \Omega_{11}^{-1} & O_{k\times(n-k)} \\ O_{(n-k)\times k} & O_{(n-k)\times(n-k)} \end{pmatrix} L(z_f - x_0) \\ &= (z_f - x_0)^T L^T \Phi_0 L(z_f - x_0). \end{aligned} \tag{6.14}$$

Thus, comparing (5.29) and (6.14), we have

$$J\big(u_0^*(\cdot)\big) = J_0^*. \tag{6.15}$$

Using the equation (6.15), we are going to show the validity of the inequality in (3.8) for $u^*(t) = u_0^*(t)$, $t \in [0, t_f]$.

Based on (2.4),(2.6),(4.1),(4.2) and (5.30), we have the following chain of inequalities and equality:

$$0 \le \inf_{u(t)\in U_x} J\big(u(\cdot)\big) \le J\big(u_\varepsilon^*(\cdot)\big) \le J_\varepsilon\big(u_\varepsilon^*(\cdot)\big) = J_\varepsilon^* \le J_0^* + \alpha\varepsilon \quad \forall\, \varepsilon \in (0,\varepsilon_3]. \tag{6.16}$$

Since the values $\inf_{u(t)\in U_x} J\big(u(\cdot)\big)$ and J_0^* are independent of ε, then (6.16) yields the inequality

$$\inf_{u(t)\in U_x} J\big(u(\cdot)\big) \le J_0^*. \tag{6.17}$$

Let us show the validity of the equality

$$\inf_{u(t)\in U_x} J\big(u(\cdot)\big) = J_0^*. \tag{6.18}$$

For this purpose, we assume the opposite which, by virtue of (6.17), is

$$\inf_{u(t)\in U_x} J\big(u(\cdot)\big) < J_0^*. \tag{6.19}$$

Due to this strong inequality, there exists $\bar{u}(\cdot) \in U_x$ such that

$$\inf_{u(t)\in U_x} J\big(u(\cdot)\big) < J\big(\bar{u}(\cdot)\big) < J_0^*. \tag{6.20}$$

Using the inequalities (4.2) and (5.30), we obtain

$$J_0^* - \alpha\varepsilon \le J_\varepsilon^* = J_\varepsilon\big(u_\varepsilon^*(\cdot)\big) \le J_\varepsilon\big(\bar{u}(t)\big) = J\big(\bar{u}(\cdot)\big) + b\varepsilon \quad \forall\, \varepsilon \in (0,\varepsilon_3], \tag{6.21}$$

where

$$b = \int_0^{t_f} \bar{u}_{\mathrm{low}}^T(t)\bar{u}_{\mathrm{low}}(t)dt$$

and $\bar{u}_{\mathrm{low}}(t)$ is the lower block of the vector $\bar{u}(t)$ of the dimension $r-q$.

From the chain of the equalities and the inequalities (6.21), we have the validity of the inequality $J_0^* \le J\big(\bar{u}(\cdot)\big)+(b+\alpha)\varepsilon$ for all $\varepsilon \in (0,\varepsilon_3]$, which yields the inequality $J_0^* \le J\big(\bar{u}(\cdot)\big)$. The latter contradicts the right-hand side inequality in (6.20). This contradiction means that the inequality (6.19) is wrong, which implies the validity of the equality (6.18). The equalities (6.15) and (6.18) directly imply the validity of the inequality in (3.8) for $u^*(t) = u_0^*(t)$, $t \in [0,t_f]$. The latter, along with the inclusion (6.11), means that the control $u_0^*(t)$, $t \in [0,t_f]$ is the solution of the OMECP.

7. Illustrative example

Consider the following particular case of the system (2.1):

$$\frac{dz_1(t)}{dt} = z_1(t-1) + u_1(t) - u_2(t), \tag{7.1}$$

$$\frac{dz_2(t)}{dt} = z_2(t-1) + 2u_1(t) - u_2(t), \tag{7.2}$$

where $t \in [0,2]$; $z_1(t)$, $z_2(t)$, $u_1(t)$, $u_2(t)$ are scalar variables.

Comparing the system (7.1)-(7.2) with the system (2.1), we can conclude that in (7.1)-(7.2) $n = 2$, $r = 2$,

$$N = 1, \quad h_1 = 1, \quad t_f = 2; \quad f(t) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \in [0, 2], \tag{7.3}$$

and the matrices of the coefficients have the form

$$A_0(t) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_1(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad t \in [0, 2], \tag{7.4}$$

$$G(t, \tau) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad (t, \tau) \in [0, 2] \times [-1, 0], \tag{7.5}$$

$$B(t) \equiv \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \qquad t \in [0, 2]. \tag{7.6}$$

The system (7.1)-(7.2) is subject to the initial conditions

$$z_1(\tau) = 0, \quad \tau \in [-1, 0), \quad z_1(0) = 2, \tag{7.7}$$

$$z_2(\tau) = 0, \quad \tau \in [-1, 0), \quad z_2(0) = 1, \tag{7.8}$$

and the terminal conditions

$$z_1(2) = 5, \tag{7.9}$$

$$z_2(2) = 6. \tag{7.10}$$

Comparing the initial conditions (7.7)-(7.8) with the initial conditions (2.2), as well as the terminal conditions (7.9)-(7.10) with the terminal condition (2.3), we can see that in (7.7)-(7.8)

$$\varphi(\tau) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \tau \in [-1, 0); \qquad z_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{7.11}$$

and

$$z_f = \begin{pmatrix} 5 \\ 6 \end{pmatrix}. \tag{7.12}$$

In this example, the functional is chosen as:

$$J\big(u(\cdot)\big) = \int_0^2 u_1^2(t)dt, \tag{7.13}$$

where $u(t) = \mathrm{col}\big(u_1(t)\ ,\ u_2(t)\big)$, $t \in [0, 2]$.

Comparing the functional (7.13) with the functional (2.4),(2.6), we have that $q = 1$ and

$$R(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad R_1(t) \equiv 1, \qquad t \in [0, 2]. \tag{7.14}$$

In this example, the set U_z of admissible controls for the system (7.1)-(7.2) is the set of all controls $u(\cdot) \in L^2[0, 2; \mathbb{R}^2]$ such that this system has a solution satisfying the initial conditions (7.7)-(7.8) and the terminal conditions (7.9)-(7.10).

We look for the control $u^0(\cdot) \in U_z$, which satisfies the inequality

$$J\big(u^0(\cdot)\big) \leq J\big(u(\cdot)\big) \quad \forall u(\cdot) \in U_z. \tag{7.15}$$

Thus, the minimum energy control problem, consisting of the system (7.1)-(7.2), the initial conditions (7.7)-(7.8), the terminal conditions (7.9)-(7.10), the functional (7.13) and the condition of the control's optimality (7.15), is a particular case of the minimum energy control problem (2.1),(2.4),(2.5),(2.6). Following the results of Section 3, we are going to transform the minimum energy control problem (7.1)-(7.2),(7.7)-(7.8),(7.9)-(7.10),(7.13),(7.15) to the equivalent simpler form problem.

We start with the obtaining the matrix-valued function $\Psi(t)$, defined by the terminal-value problem (3.1). Using the data (7.3)-(7.5), we directly obtain $\Psi(t)$ corresponding to this data

$$\Psi(t) = \psi(t) I_2, \quad t \in [0,2], \tag{7.16}$$

where the scalar function $\psi(t)$ has the form

$$\psi(t) = \begin{cases} 2-t, & t \in [0,1], \\ 1, & t \in (1,2]. \end{cases} \tag{7.17}$$

Substitution of (7.6) and (7.16)-(7.17) into (3.3) yields

$$\mathcal{B}(t) = \begin{cases} (2-t)\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, & t \in [0,1], \\ \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, & t \in (1,2]. \end{cases} \tag{7.18}$$

Thus, the system (3.5) becomes

$$\frac{dx(t)}{dt} = \left(\begin{cases} (2-t)\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, & t \in [0,1], \\ \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, & t \in (1,2] \end{cases}\right) u(t), \quad t \in [0,2], \tag{7.19}$$

where $x(t) = \mathrm{col}\big(x_1(t)\ ,\ x_2(t)\big)$, ($x_1(t)$ and $x_2(t)$ are scalar variables), $u(t) = \mathrm{col}\big(u_1(t)\ ,\ u_2(t)\big)$.

Furthermore, using the data (7.3)-(7.5),(7.11) and the equations (7.16)-(7.17), we directly have the vector x_0, defined by (3.4),

$$x_0 = \mathrm{col}(4\ ,\ 2). \tag{7.20}$$

Thus, due to (3.6),

$$x(0) = \mathrm{col}(4\ ,\ 2). \tag{7.21}$$

Also, due to (3.7) and (7.12),

$$x(2) = \mathrm{col}(5\ ,\ 6). \tag{7.22}$$

In this example, the set U_x of admissible controls for the system (7.19) is the set of all controls $u(\cdot) \in L^2[0,2;\mathbb{R}^2]$ such that this system has a solution satisfying the initial condition (7.21) and the terminal condition (7.22).

The minimum energy control problem, equivalent to the above formulated problem (7.1)-(7.2),(7.7)-(7.8),(7.9)-(7.10),(7.13),(7.15), is to find the control $u_0^*(\cdot) \in U_x$, which satisfies the inequality

$$J\big(u_0^*(\cdot)\big) \le J\big(u(\cdot)\big) \quad \forall u(\cdot) \in U_x. \tag{7.23}$$

This problem is a particular case of the OMECP formulated in Section 3 (see Remark 3.4). In what follows of this example, we are going to derive the control $u_0^*(t)$, $t \in [0,2]$ using the results of Sections 5 and 6. To proceed with the application of the results of these sections to the derivation of the control $u_0^*(t)$, $t \in [0,2]$, first, we should check the fulfilment of the assumption of Proposition 4.2, i.e., the positive definiteness of the matrix $\mathcal{W}_{\mathcal{B}}$ (see the equation (4.4)). Using the equations (4.4),(7.3) and (7.18) yields

$$\mathcal{W}_{\mathcal{B}} = \frac{10}{3}\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \tag{7.24}$$

meaning the positive definiteness of this matrix.

Due to the equation (5.1), we partition the matrix $\mathcal{B}(t)$, given by (7.18), into two blocks $\mathcal{B}_1(t)$ and $\mathcal{B}_2(t)$ which are

$$\mathcal{B}_1(t) = \begin{cases} (2-t)\begin{pmatrix} 1 \\ 2 \end{pmatrix}, & t \in [0,1], \\ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, & t \in (1,2], \end{cases} \tag{7.25}$$

$$\mathcal{B}_2(t) = \begin{cases} (t-2)\begin{pmatrix} 1 \\ 1 \end{pmatrix}, & t \in [0,1], \\ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, & t \in (1,2], \end{cases} \tag{7.26}$$

Calculating the matrix K (see the equation (5.3)), we obtain

$$K = \begin{pmatrix} \frac{10}{3} & \frac{10}{3} \\ \frac{10}{3} & \frac{10}{3} \end{pmatrix}. \tag{7.27}$$

This matrix has a simple zero eigenvalue, meaning the fulfilment of the assumption A1 with $k = 1$. The orthogonal matrix L, appearing in the equation (5.4), can be chosen as

$$L = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{7.28}$$

Substituting the matrices K and L, given by the equations (7.27) and (7.28), into the equation (5.4), we obtain after a routine algebra the matrix D

$$D = \begin{pmatrix} 0 & 0 \\ 0 & \frac{20}{3} \end{pmatrix}. \tag{7.29}$$

The latter, along with (5.4), means that

$$\Theta = \frac{20}{3}. \tag{7.30}$$

Substitution of (7.26) and (7.28) into (5.5) yields after a direct calculation

$$L\mathcal{B}_2(t) = \begin{cases} (t-2)\begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}, & t \in [0,1], \\ \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix}, & t \in (1,2]. \end{cases} \tag{7.31}$$

Hence,

$$\Lambda(t) = \begin{cases} \sqrt{2}(t-2), & t \in [0,1], \\ -\sqrt{2}, & t \in (1,2]. \end{cases} \tag{7.32}$$

Using the equations (7.14),(7.25),(7.28) and calculating the expression in the left-hand side of the equation (5.8), we obtain

$$L\int_0^{t_f} \mathcal{B}_1(t)R_1^{-1}(t)\mathcal{B}_1^T(t)dtL^T = \begin{pmatrix} \frac{5}{3} & -5 \\ -5 & 15 \end{pmatrix}. \tag{7.33}$$

Comparing the matrices in the right-hand sides of the equations (5.8) and (7.33), we directly have

$$\Omega_{11} = \frac{5}{3}, \quad \Omega_{12} = -5, \quad \Omega_{13} = 15, \tag{7.34}$$

which, due to (5.20), yields

$$\Phi_0 \triangleq \begin{pmatrix} 0.6 & 0 \\ 0 & 0 \end{pmatrix}. \tag{7.35}$$

Using (7.14),(7.12),(7.21),(7.25),(7.28),(7.35), we can calculate the function $u_{0,1}^*(t)$, given by (5.21). Thus, we obtain

$$u_{0,1}^*(t) = \begin{cases} 0.9(2-t), & t \in [0,1], \\ 0.9, & t \in (1,2]. \end{cases} \tag{7.36}$$

Similarly, using (5.25),(7.12),(7.21),(7.28),(7.30),(7.34), we calculate the function $u_{0,2}^*(t)$ as:

$$u_{0,2}^*(t) = \begin{cases} 0.6(2-t), & t \in [0,1], \\ 0.6, & t \in (1,2]. \end{cases} \tag{7.37}$$

Thus, by virtue of the equation (5.27) and the results of Section 6, the solution of the OMECP in this example is

$$u_0^*(t) = \begin{cases} (2-t)\begin{pmatrix} 0.9 \\ 0.6 \end{pmatrix}, & t \in [0,1], \\ \begin{pmatrix} 0.9 \\ 0.6 \end{pmatrix}, & t \in (1,2]. \end{cases} \tag{7.38}$$

Due to Proposition 3.3, this control also solves the initially formulated problem of this example (7.1)-(7.2),(7.7)-(7.8),(7.9)-(7.10),(7.13),(7.15).

Finally, using (5.29),(7.12),(7.21),(7.28),(7.35), we calculate the value of the functional (7.13) corresponding to $u_0^*(t)$, $t \in [0,2]$ in both minimum energy control problems

$$J_0^* = 2.7 \ . \tag{7.39}$$

8. Example on non-uniqueness of solution to the OMECP

Consider the following particular case of the boundary-value problem (3.5)-(3.7):

$$\begin{aligned} \frac{dx_1(t)}{dt} &= (t-2)u_1(t) + (2t-4)u_2(t), \qquad t \in [0,2], \\ \frac{dx_2(t)}{dt} &= (4-2t)u_1(t) + (2-t)u_2(t), \qquad t \in [0,2], \\ x_1(0) &= x_{0,1}, \qquad x_2(0) = x_{0,2}, \\ x_1(2) &= z_{f,1}, \qquad x_2(2) = z_{f,2}, \end{aligned} \tag{8.1}$$

where $x_1(t)$, $x_2(t)$, $u_1(t)$, $u_2(t)$ are scalar variables; $x_{0,1}$, $x_{0,2}$, $z_{f,1}$ and $z_{f,2}$ are some given scalar values.

In this example, first, we consider the set U_x of admissible controls for the differential system of (8.1) as the set of all controls $u(\cdot) = \mathrm{col}\big(u_1(\cdot), u_2(\cdot)\big) \in L^2[0,2;\mathbb{R}^2]$ such that this system has a solution satisfying the corresponding boundary conditions (see the equation (8.1)).

Furthermore, in this example, the functional is chosen as:

$$J\big(u(\cdot)\big) = \int_0^2 u_1^2(t)dt. \tag{8.2}$$

The minimum energy control problem, consisting of the boundary-value problem (8.1), the set of admissible controls U_x and the functional (8.2) is a particular case of the OMECP formulated in Section 3 (see Remark 3.4), where

$$n = 2, \qquad r = 2, \qquad q = 1, \qquad t_f = 2, \tag{8.3}$$

$$\begin{aligned} \mathcal{B}(t) = \begin{pmatrix} t-2 & 2t-4 \\ 4-2t & 2-t \end{pmatrix}, \qquad R_1(t) \equiv 1, \qquad t \in [0,2], \\ x_0 = \mathrm{col}(x_{0,1}, x_{0,2}), \qquad z_f = \mathrm{col}(z_{f,1}, z_{f,2}). \end{aligned} \tag{8.4}$$

Based on the results of Sections 5 and 6, let us derive the control $u_0^*(t)$, $t \in [0,2]$ solving the aforementioned minimum energy control problem. In order to apply the results of these sections to the derivation of the control $u_0^*(t)$, $t \in [0,2]$, first, we should check the fulfilment of the assumption of Proposition 4.2, i.e., the positive definiteness of the matrix $\mathcal{W}_{\mathcal{B}}$ (see the equation (4.4)). Using the equations (4.4),(8.3) and (8.4) yields

$$\mathcal{W}_{\mathcal{B}} = \frac{8}{3}\begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}, \tag{8.5}$$

meaning the positive definiteness of this matrix.

Due to the equation (5.1), we partition the matrix $\mathcal{B}(t)$, given in (8.4), into two blocks $\mathcal{B}_1(t)$ and $\mathcal{B}_2(t)$ which are

$$\mathcal{B}_1(t) = \begin{pmatrix} t-2 \\ 4-2t \end{pmatrix}, \quad \mathcal{B}_2(t) = \begin{pmatrix} 2t-4 \\ 2-t \end{pmatrix}, \quad t \in [0,2]. \tag{8.6}$$

Calculation of the matrix K, defined by the equation (5.3), yields

$$K = \begin{pmatrix} \frac{32}{3} & -\frac{16}{3} \\ -\frac{16}{3} & \frac{8}{3} \end{pmatrix}, \tag{8.7}$$

meaning that K has zero eigenvalue of the algebraic multiplicity $k = 1$. Hence, the assumption A1 is fulfilled.

The orthogonal matrix L, appearing in the equation (5.4), can be chosen as

$$L = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}. \tag{8.8}$$

Substituting the matrices K and L, given by the equations (8.7) and (8.8), into the equation (5.4), we obtain by a routine algebra the matrix D

$$D = \begin{pmatrix} 0 & 0 \\ 0 & \frac{40}{3} \end{pmatrix}. \tag{8.9}$$

The latter, along with (5.4), means that

$$\Theta = \frac{40}{3}. \tag{8.10}$$

Calculating $\Lambda(t)$, defined by the equation (5.5), and using $\mathcal{B}_2(t)$ and L (see the equations (8.6) and (8.8)), we obtain

$$\Lambda(t) = \sqrt{5}(2-t), \quad t \in [0,2]. \tag{8.11}$$

Now, calculating the block-form matrix, defined by (5.8), and using (8.6) and (8.8), we obtain by a routine algebra

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^T & \Omega_{13} \end{pmatrix} = \begin{pmatrix} \frac{72}{15} & \frac{96}{15} \\ \frac{96}{15} & \frac{128}{15} \end{pmatrix}. \tag{8.12}$$

This block-form matrix and the equation (5.20) yield

$$\Phi_0 \triangleq \begin{pmatrix} \frac{5}{24} & 0 \\ 0 & 0 \end{pmatrix}. \tag{8.13}$$

Using (8.4),(8.6),(8.8),(8.13), we can calculate the component $u_{0,1}^*(t)$ of the control $u_0^*(t)$, given by (5.21). Thus, we obtain

$$u_{0,1}^*(t) = -\frac{1}{8}(t-2)[(z_{f,1} - x_{0,1}) + 2(z_{f,2} - x_{0,2})], \quad t \in [0,2]. \tag{8.14}$$

Similarly, using (5.25),(8.4),(8.8),(8.10),(8.11),(8.12), we calculate the component $u^*_{0,2}(t)$ of the control $u^*_0(t)$ as:

$$u^*_{0,2}(t) = \frac{1}{8}(t-2)[2(z_{f,1} - x_{0,1}) + (z_{f,2} - x_{0,2})], \quad t \in [0,2]. \tag{8.15}$$

Furthermore, using (5.29),(8.4),(8.8),(8.13), we calculate the value of the functional (8.2) corresponding to $u^*_0(t) = \mathrm{col}\big(u^*_{0,1}(t), u^*_{0,2}(t)\big)$, $t \in [0,2]$ in the minimum energy control problem (8.1),(8.2)

$$J^*_0 = \frac{1}{24}[(z_{f,1} - x_{0,1}) + 2(z_{f,2} - x_{0,2})]^2. \tag{8.16}$$

Proceed to the derivation of another solution to the minimum energy control problem (8.1),(8.2). For this purpose, we extend the set U_x of admissible controls for this problem. Namely, let us consider the following set of controls:

$$\mathcal{U} \stackrel{\triangle}{=} \left\{ u(t) = \mathrm{col}\big(u_1(t), u_2(t)\big) : u_1(t) \in L^2[0,2],\ u_2(t) \in \left(\mathcal{U}_\delta \bigcup L^2[0,2] \right) \right\}, \\ \mathcal{U}_\delta \stackrel{\triangle}{=} \{\beta\delta(t-\bar{t})\}, \tag{8.17}$$

where β is any real number; $\bar{t} \in [0,2]$ is any time instant; $\delta(t-\bar{t})$, $t \in [0,2]$ is the δ-function of Dirac with the impulse at $t = \bar{t}$.

Consider the set $\mathcal{U}_x$ of all controls $u(t) \in \mathcal{U}$ such that the boundary-value problem (8.1) has a solution with any $u(t) \in \mathcal{U}_x$. Thus, $\mathcal{U}_x \subset \mathcal{U}$, while $U_x \subset \mathcal{U}_x$. The set $\mathcal{U}_x$ is chosen as the extended set of admissible controls in the minimum energy control problem (8.1),(8.2).

Let us find out whether there exist β and $\bar{t} \in [0,2]$ such that the control $\bar{u}(t) = \mathrm{col}\big(\bar{u}_1(t), \bar{u}_2(t)\big) = \mathrm{col}\big(0, \beta\delta(t-\bar{t})\big) \in \mathcal{U}_x$.

Substituting $u(t) = \bar{u}(t)$ into the boundary-value problem (8.1), we have

$$\begin{aligned} \frac{dx_1(t)}{dt} &= (2t-4)\beta\delta(t-\bar{t}), \quad t \in [0,2], \\ \frac{dx_2(t)}{dt} &= (2-t)\beta\delta(t-\bar{t}), \quad t \in [0,2], \\ x_1(0) &= x_{0,1}, \quad x_2(0) = x_{0,2}, \\ x_1(2) &= z_{f,1}, \quad x_2(2) = z_{f,2}, \end{aligned} \tag{8.18}$$

Integrating the differential system in (8.18) from $t = 0$ to $t = 2$, and using the corresponding initial and terminal conditions yield

$$\begin{aligned} z_{f,1} &= x_{0,1} + (2\bar{t} - 4)\beta, \\ z_{f,2} &= x_{0,2} + (2 - \bar{t})\beta. \end{aligned} \tag{8.19}$$

For any $\bar{t} \neq 2$, this set of equations with respect to the unknown β has a solution if and only if

$$z_{f,1} - x_{0,1} = -2(z_{f,2} - x_{0,2}), \tag{8.20}$$

and this solution is $\beta = \beta(\bar{t}, x_{0,2}, z_{f,2}) \stackrel{\triangle}{=} (z_{f,2} - x_{02})/(2 - \bar{t})$ for any $\bar{t} \in [0, 2)$.

Hence, subject to the condition (8.20), the control

$$\bar{u}(t) = \mathrm{col}\big(\bar{u}_1(t), \bar{u}_2(t)\big) = \mathrm{col}\big(0, \beta(\bar{t}, x_{0,2}, z_{f,2})\delta(t - \bar{t})\big) \in \mathcal{U}_x \quad \forall\, \bar{t} \in [0, 2).$$

Since $\bar{u}_1(t) \equiv 0$, $t \in [0, 2]$, then $J\big(\bar{u}(t)\big) = 0$ meaning that $\bar{u}(t)$, $t \in [0, 2]$ is a solution of the minimum energy control problem (8.1),(8.2) for any $\bar{t} \in [0, 2)$.

It should be noted that, subject to the condition (8.20), the component $u^*_{0,1}(t)$ of the control $u^*_0(t)$ becomes zero ($u^*_{0,1}(t) \equiv 0$, $t \in [0, 2]$), the component $u^*_{0,2}(t)$ of the control $u^*_0(t)$ becomes as $u^*_{0,2}(t) = \frac{3}{8}(z_{f,2} - x_{0,2})(2 - t)$, $t \in [0, 2]$, the value of the functional (8.2) corresponding to the control $u^*_0(t)$ with these components in the minimum energy control problem (8.1),(8.2) (see the equation (8.16)) becomes zero. Moreover, subject to the condition (8.20), any control $\widetilde{u}(t) = \mathrm{col}\big(\widetilde{u}_1(t), \widetilde{u}_2(t)\big) = \mathrm{col}\big(0, \widetilde{u}_2(t)\big) \in U_x$, where $\widetilde{u}_2(t) \in L^2[0, 2]$ satisfies the Fredholm integral equation of the first kind

$$\int_0^2 (2 - t)\widetilde{u}_2(t)dt = z_{f,2} - x_{0,2},$$

is a solution of the minimum energy control problem (8.1),(8.2).

Thus, subject to the condition (8.20), we have derived infinitely many solutions to the minimum energy control problem (8.1),(8.2). Among these solutions are $u^*_0(t) = \mathrm{col}\big(0, u^*_{0,2}(t)\big) \in U_x \subset \mathcal{U}_x$ and $\widetilde{u}(t) = \mathrm{col}\big(0, \widetilde{u}_2(t)\big) \in U_x \subset \mathcal{U}_x$, while the other solutions are $\bar{u}(t) \in \mathcal{U}_x$ valid for any $\bar{t} \in [0, 2)$. The values of the functional in the problem, corresponding to these solutions are the same, namely, zero. It should be noted that, due to the presence of the Dirac δ-function in $\bar{u}(t)$, the solution $\bar{u}(t)$ can be useful rather for a theoretical analysis of the minimum energy control problem, while the solutions $u^*_0(t)$ and $\widetilde{u}(t)$ can be useful for both purposes, theoretical analysis of the minimum energy control problem and practical implementation of this solution.

9. Conclusions

In this paper, the minimum energy control problem was considered. The dynamics of the problem has multiple point-wise and distributed delays in the state variable. The weight matrix of the control cost is block-diagonal with two blocks on the main diagonal. One of these blocks is a positive definite matrix, while the other is zero matrix. Thus, the aforementioned weight matrix of the control cost is singular, meaning that the considered minimum energy control problem is singular. The control coordinates, which are present in the functional are regular, while the other control coordinates are singular. A set of admissible controls in the problem is chosen as the set of all square integrable control functions, transferring the problem's dynamic system from a given initial position to a given terminal position.

By the proper linear change of the state variable, the initially formulated control problem was transformed equivalently to a much simpler one. This new problem also is a singular minimum energy control problem, while its equation of dynamics does not have delays any more. In the sequel of the paper, this new undelayed problem was considered as an original minimum energy control problem. To solve this problem, the regularization method was applied. Namely, the original problem was replaced by a regular minimum energy control problem, which depends on a small positive parameter ε. This new minimum energy control problem has the same dynamics and the same set of admissible controls as the original problem has. However, the functional in the new problem differs from the one in the original problem. The new functional is the sum of the original functional and the finite-horizon integral of the squares of the singular control coordinates with the small positive weight ε. Thus obtained the parameter dependent minimum energy control problem is a partial cheap control problem, and it becomes the original problem for $\varepsilon = 0$. Asymptotic analysis with respect to ε of the solution to this partial cheap control problem was carried out. Based on this analysis, the solution (the open-loop optimal control) of the original singular minimum energy control problem was derived. The corresponding value of the functional also was obtained.

It was shown by example that, subject to some additional condition, the singular minimum energy problem can have infinitely many solutions in the original set and in some extended set of admissible controls.

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