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DIRECTIONAL DERIVATIVES FOR SET-VALUED MAPS BASED ON SET CONVERGENCES

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Dedicated to Professor Roger J.-B. Wets on the occasion of his 85th birthday

Abstract. We explore the possibility to define and to meaningfully apply some new concepts of directional derivative which incorporate in their construction set convergences to set-optimization problems. We connect these new constructions with other directional derivatives for set-valued maps and we emphasize the flexibility and the potential applicability of this new approach. In this vein, we indicate a possible axiomatic perspective that allows one to significantly increase the number and (maybe) the efficiency of these derivatives when applied to concrete problems.

1. Introduction

In this work we continue the ideas that led in [2] and [3] to the development of some classes of directional derivatives for set-valued maps that seems to be of interest for getting a new perspective on optimality conditions for set-optimization problems. More specifically, in this paper we add a new layer of generality and flexibility to the aforementioned constructions by considering set convergences. This is a natural step to take for generalized differentiation objects that are intended to be applied to set-optimization problems because, by their very nature, these classes of problems deal primarily with sets and this feature is their main conceptual and technical difference with respect to vector optimization problems. In using the idea of constructing directional derivatives using set convergences, we have to face a problem of choice, since there are many such convergences, everyone with its specific features. In order to start with we choose the lower Kuratowski set convergence for two reasons: firstly, this is a rather weak set convergence and secondly, it is recognized as a prototype set convergence. However, even if the main study is done for this set convergence, we show then how the corresponding directional derivative behaves in comparison to directional derivatives built on different, stronger set-convergences. This comparison, as well as other considerations lead us to indicate a possible axiomatic approach in the construction of these directional derivatives.

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We want to emphasize that we do not propose ourselves to present here in detail many of the results that can be obtained, but to rather describe some principles that can generate consequences for the questions under study. In this sense, we mainly show the links of the new concepts with some of other already studied directional derivatives and then we exemplify the novelties that can be achieved by their employment in the study of optimality conditions for set-optimization problems.

The paper is organized as follows. In the second section we briefly present the setting of our study and we introduce the set-optimization problem on which we intend to test the directional derivatives we are going to introduce. A recently studied directional derivative for set-valued maps is recalled at the end of this section. The third section deals with a new directional derivative for set-valued maps built on the basis of lower Kuratowski set convergence. We study the relationship of this new concept with that mentioned in the previous section and we pay attention to the special case of single-valued maps. The fourth section proposes optimality conditions for two concepts of minimality for the set-optimization problem under consideration. In both cases, we underline the links, but also the differences with respect to recent results in literature. The fifth section pursuits the idea of constructing directional derivatives for set-valued maps based on different set convergences. We consider the lower Hausdorff-Pompeiu convergence as an extreme (strong) case and the lower Attouch-Wets convergence as an intermediate one. Finally, these multiple considerations lead us to the indication of a possible axiomatic approach that can encompass the particular situations discussed here.

2. Preliminaries

We work on real normed vector spaces. If X is such a space we denote by $B(x, \varepsilon)$ and $D(x, \varepsilon)$ the open and (respectively) the closed ball centered at $x \in X$ and with radius $\varepsilon > 0$. For notation and general facts about the set-convergences we work with we use, mainly, [10] and [11].

Let X, Y be normed spaces over the real field $\mathbb{R}$. Consider $K \subset Y$ a closed convex pointed proper cone. The set approach in vector optimization is based on some order relations on sets defined by Kuroiwa: see [5] for details. We work here with one such relation. We collect some known concepts and results, mainly from [6] and [2].

Let $A, B \subset Y$ be nonempty sets. Define $\preceq_K^l$ by

$$A \preceq_K^l B \iff B \subset A + K.$$

If K is solid, that is $\operatorname{int} K \neq \emptyset$, then one defines as well the strict relation $\prec_K^l$ by

$$A \prec_K^l B \iff B \subset A + \operatorname{int} K.$$

Let $F : X \rightrightarrows Y$ be a set-valued map with nonempty values and $M \subset X$ be a nonempty closed set. Consider the problem

$$(P) \text{ minimize } F(x) \text{ subject to } x \in M.$$

Definition 2.1. An element $\overline{x} \in M$ is said to be $l-$minimum for F on M or for the problem (P) if

$$x \in M, F(x) \preceq_K^l F(\overline{x}) \Longrightarrow F(\overline{x}) \preceq_K^l F(x).$$

The local counterpart is obvious.

The above concept means that for all $x \in M$ the inclusion $F(\overline{x}) \subset F(x) + K$ implies that $F(x) \subset F(\overline{x}) + K$, or, in other words, since K is convex, for any $x \in M$ one can have

$$F(\overline{x}) \not\subset F(x) + K \text{ or } F(\overline{x}) \subset F(x) + K \subset F(\overline{x}) + K.$$

Definition 2.2. An element $\overline{x} \in M$ is said to be $l-$weak minimum for F on M or for the problem (P) if

$$x \in M, \; F(x) \prec_K^l F(\overline{x}) \Longrightarrow F(\overline{x}) \prec_K^l F(x).$$

Remark 2.3. An element $\overline{x} \in M$ is a $l-$minimum for the problem (P) if and only if $\overline{x}$ is a $l-$minimum for $\operatorname{Epi} F$ on M, where $\operatorname{Epi} F : X \rightrightarrows Y$ is the epigraphical set-valued map defined by

$$\operatorname{Epi} F(x) = F(x) + K.$$

The same statement holds for $l-$weak minimality.

For a set $\emptyset \neq A \subset Y$, the set of weakly minimal points is

$$\operatorname{WMin}(A, K) := \{a \in A \mid (A - a) \cap - \operatorname{int} K = \emptyset\}.$$

Remark 2.4. Obviously, if $A \subset Y$ is a nonempty set such that $\operatorname{WMin}(A, K) \neq \emptyset$ then

$$A \not\subset A + \operatorname{int} K.$$

Therefore, for $\overline{x} \in M$, if $\operatorname{WMin}(F(\overline{x}), K) \neq \emptyset$ then $\overline{x}$ is $l-$weak minimum for the problem (P) if and only if $F(x) \not\prec_K^l F(\overline{x})$ for all $x \in M$ (see also [4] and the references therein). A similar assertion holds for local $l-$weak minimality.

We end this section by recalling the following generalized directional derivatives from [3] (see also [2]).

Definition 2.5. Let $F : X \rightrightarrows Y$ be a set-valued map and $\overline{x}, u \in X$.

(i) One calls the lower directional derivative of F at $\overline{x}$ in direction u the set, denoted $D^- F(\overline{x})(u)$, of elements $v \in Y$ such that for all $\varepsilon > 0$ there exist $(t_n) \downarrow 0$, $(u_n) \to u$ and $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$F(\overline{x}) + t_n v \subset F(\overline{x} + t_n u_n) + t_n B(0, \varepsilon).$$

(ii) One calls the upper directional derivative of F at $\overline{x}$ in direction u the set, denoted $D^+ F(\overline{x})(u)$, of elements $v \in Y$ such that for all $\varepsilon > 0$ and all $(t_n) \downarrow 0$, $(u_n) \to u$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$F(\overline{x}) + t_n v \subset F(\overline{x} + t_n u_n) + t_n B(0, \varepsilon).$$

3. A prototype for directional derivative concepts

In this section we define and study a directional derivative for set-valued maps based on a well-known set convergence and for this we turn back to the original definition of a set-valued map $F : X \rightrightarrows Y$ as a function from X into 2^Y.

As mentioned by many authors (see, for instance, [1], [9]) the Wijsman convergence is a prototype for topological set convergences and we consider here the "lower half" of it, which is as well the "lower halves" of Vietoris and Kuratowski convergences (see [10]). This convergence is not a topological one and the limit is not unique. We denote this set convergence by (K_-) and the fact that a sequence of sets $(V_n) \subset 2^Y \setminus \{\emptyset\}$ is convergent to a set $V \in 2^Y \setminus \{\emptyset\}$ in the sense of (K_-) by $V \in (K_-) - \lim V_n$ or $(V_n) \overset{(K_-)}{\to} V$. More precisely, the definition is: $V \in (K_-) - \lim V_n$ if and only if

$$V \subset \liminf V_n := \{y \in Y \mid \forall n \in \mathbb{N},\ \exists y_n \in V_n : \lim y_n = y\}.$$

The set convergence (K_-) has the following immediate properties:

- if $(y_n) \to y$, then $\{y_n\} \overset{(K_-)}{\to} \{y\}$;
- if $V \in (K_-) - \lim V_n$ then $V + K \in (K_-) - \lim (V_n + K)$;
- if $V \in (K_-) - \lim V_n$ then for every $v \in V$ there is a sequence (v_n) such that $v_n \in V_n$ for all n and $(v_n) \to v$;
- if $V \in (K_-) - \lim V_n$ then for every $U \subset V$, $U \in (K_-) - \lim V_n$.

We introduce now the new concept which we are going to study. Let F take values in $2^Y \setminus \{\emptyset\}$. For $\overline{x} \in X$ and $u \in X$ we define the Bouligand-type directional derivative of F at $\overline{x}$ in direction u with respect to (K_-) as

$$D^{(K_-)}F(\overline{x})(u) = \left\{ \begin{array}{c} V \in 2^Y \setminus \{\emptyset\} \mid \exists (t_n) \downarrow 0,\ \exists (u_n) \to u,\ \exists (V_n) \overset{(K_-)}{\to} V, \\ \forall n : F(\overline{x}) + t_n V_n \subset F(\overline{x} + t_n u_n) \end{array} \right\}.$$

Remark 3.1. Observe that if $A \in D^{(K_-)}F(\overline{x})(u)$ and $B \subset A$, then $B \in D^{(K_-)}F(\overline{x})(u)$. However, in general, if $A, B \in D^{(K_-)}F(\overline{x})(u)$ it is not true that $A \cup B \in D^{(K_-)}F(\overline{x})(u)$ (see the example in Remark 3.6).

Let us compare this new concept with that already studied and recalled in Definition 2.5.

Proposition 3.2. *In the above notation, if* $V \in D^{(K_-)}F(\overline{x})(u)$*, then* $V \subset D^{-}F(\overline{x})(u)$.

Proof. According to the above definition, there are some sequences $(t_n) \downarrow 0$, $(u_n) \to u$, $(V_n) \overset{(K_-)}{\to} V$, such that for all n

$$F(\overline{x}) + t_n V_n \subset F(\overline{x} + t_n u_n).$$

Take $v \in V$. Since $(V_n) \overset{(K_-)}{\to} V$, there is a sequence (v_n) such that $v_n \in V_n$ for all n and $(v_n) \to v$, so for all $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that for $n \geq n_\varepsilon$,

$$v \in \{v_n\} + B(0, \varepsilon).$$

Therefore, for n large enough,

$$\begin{aligned} F(\overline{x}) + t_n \{v\} &\subset F(\overline{x}) + t_n (\{v_n\} + B(0, \varepsilon)) \\ &\subset F(\overline{x}) + t_n V_n + t_n B(0, \varepsilon) \subset F(\overline{x} + t_n u_n) + t_n B(0, \varepsilon). \end{aligned}$$

This implies that $\{v\} \in D^{-}F(\overline{x})(u)$, whence the conclusion. □

Remark 3.3. Similarly, if $V \in D^{(K_-)} \operatorname{Epi} F(\overline{x})(u)$ for some $u \in X$, then $V \subset D^{-} \operatorname{Epi} F(\overline{x})(u)$.

Remark 3.4. Observe as well that if $V \in D^{(K_-)}F(\overline{x})(u)$ then $V + K \in D^{(K_-)} \operatorname{Epi} F(\overline{x})(u)$. The converse of the latter assertion is not true, in general: for instance in $\mathbb{R}$, $\{n^{-1}\} \overset{(K_-)}{\not\to} \{0,1\}$, but $\{n^{-1}\} + [0,\infty) \overset{(K_-)}{\to} \{0,1\} + [0,\infty)$.

Proposition 3.5. *If* $v \in D^- F(\overline{x})(u)$, *then* $\{v\} \in D^{(K_-)}F(\overline{x})(u)$.

Proof. Indeed, the inclusion $v \in D^- \operatorname{Epi} F(\overline{x})(u)$ means that for every $\varepsilon > 0$ there exist $(t_n) \downarrow 0$, $(u_n) \to u$ and $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$F(\overline{x}) + t_n v \subset F(\overline{x} + t_n u_n) + t_n B(0,\varepsilon).$$

For all $k \in \mathbb{N} \setminus \{0\}$, take $\varepsilon = k^{-1}$. For the sequences $(t_n) \downarrow 0$, $(u_n) \to u$ given by the above statement, there is a strictly increasing subsequence (n_k) with $t_{n_k} < k^{-1}$, $\|u_{n_k} - u\| < k^{-1}$ for all k. Therefore, for large k,

$$F(\overline{x}) + t_{n_k} v \subset F(\overline{x} + t_{n_k} u_{n_k}) + t_{n_k} B\left(0, k^{-1}\right).$$

This implies that for some $\rho_k \in B\left(0, k^{-1}\right)$,

$$F(\overline{x}) + t_{n_k}(v + \rho_k) \subset F(\overline{x} + t_{n_k} u_{n_k}).$$

Since $\{v + \rho_k\} \overset{(K_-)}{\to} \{v\}$, we get

$$\{v\} \in D^{(K_-)}F(\overline{x})(u),$$

and this is the conclusion. ☐

Remark 3.6. In general, one cannot conclude that a subset of $D^- F(\overline{x})(u)$ which consists of more than one point belongs to $D^{(K_-)}F(\overline{x})(u)$. For instance, let us consider $F : \mathbb{R} \rightrightarrows \mathbb{R}$, given by

$$F(x) = \begin{cases} \{x\}, & \text{if } x = \frac{1}{n} \text{ with } n \in \mathbb{N} \setminus \{0\}, \\ \{2x\}, & \text{if } x = \frac{1}{n\sqrt{n}} \text{ with } n \in \mathbb{N} \setminus \{0\}, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Then, clearly, $\{1,2\} \subset D^- F(0)(1)$, but $\{1,2\} \notin D^{(K_-)}F(0)(u)$.

Let us provide a direct calculation of this new directional derivative in some particular situations. Before that, we recall the following well known concepts. Let $f : X \to \mathbb{R} \cup \{+\infty\}$; the upper Hadamard directional derivative of f at $\overline{x} \in \operatorname{dom} f$ in the direction $u \in X$ is

$$d_+ f(\overline{x}, u) = \limsup_{t \downarrow 0, u' \to u} \frac{f(\overline{x} + tu') - f(\overline{x})}{t},$$

while the lower Hadamard directional derivative of f at $\overline{x}$ in the direction u is

$$d_- f(\overline{x}, u) = \liminf_{t \downarrow 0, u' \to u} \frac{f(\overline{x} + tu') - f(\overline{x})}{t}.$$

We identify, as usual, f with the naturally associated set-valued map and for $\mathbb{R}$ one takes $\mathbb{R}_+$ as the ordering cone.

Proposition 3.7. *In the above notation, the following assertions hold:*

(i) *if* $d_{-}f(\overline{x},u) \in \mathbb{R}$ *then* $\{d_{-}f(\overline{x},u)\} \in D^{(K_{-})}f(\overline{x})(u)$ *and for all* $\alpha < d_{-}f(\overline{x},u)$, $\{\alpha\} \notin D^{(K_{-})}f(\overline{x})(u)$;

(ii) *if* $d_{+}f(\overline{x},u) \in \mathbb{R}$ *then* $\{d_{+}f(\overline{x},u)\} \in D^{(K_{-})}f(\overline{x})(u)$ *and for all* $\beta > d_{+}f(\overline{x},u)$, $\{\beta\} \notin D^{(K_{-})}f(\overline{x})(u)$.

(iii) *if* $d_{-}f(\overline{x},u) \in \mathbb{R}$ *then* $[d_{-}f(\overline{x},u),\infty)$ *is the biggest set (in the sense of inclusion) which belongs to* $D^{(K_{-})}\operatorname{Epi} f(\overline{x})(u)$, *and if* $d_{-}f(\overline{x},u) = -\infty$, *then*

$$\mathbb{R} \in D^{(K_{-})}\operatorname{Epi} f(\overline{x})(u).$$

Proof. (i) and (ii) These inclusions readily follow by the fact that $d_{-}f(\overline{x},u)$ (and $d_{+}f(\overline{x},u)$, respectively) is the smallest (the greatest) limit of the quotient $(t_n)^{-1}(f(\overline{x}+t_nu_n)-f(\overline{x}))$ when $(t_n)\downarrow 0$, $(u_n)\to u$ are in such a way that the limit exists.

(iii) Suppose that $d_{-}f(\overline{x},u) \in \mathbb{R}$. Take $V := [d_{-}f(\overline{x},u),\infty)$. We know that there are some sequences $(t_n)\downarrow 0$, $(u_n)\to u$ such that

$$d_{-}f(\overline{x},u) = \lim_{n\to\infty}\frac{f(\overline{x}+t_nu_n)-f(\overline{x})}{t_n}.$$

So, one can find a strictly increasing sequence of natural numbers (n_k) such that for all $k \in \mathbb{N}\setminus\{0\}$

$$\frac{f(\overline{x}+t_{n_k}u_{n_k})-f(\overline{x})}{t_{n_k}} < d_{-}f(\overline{x},u)+\frac{1}{k}.$$

Therefore, for all k,

$$f(\overline{x})+t_{n_k}\left(d_{-}f(\overline{x},u)+k^{-1}\right) \in [f(\overline{x}+t_{n_k}u_{n_k}),+\infty),$$

so

$$f(\overline{x})+t_{n_k}[d_{-}f(\overline{x},u)+k^{-1},+\infty) \subset [f(\overline{x}+t_{n_k}u_{n_k}),+\infty) = \operatorname{Epi} f(\overline{x}+t_{n_k}u_{n_k}).$$

Since, clearly $[d_{-}f(\overline{x},u)+k^{-1},+\infty) \overset{(K_{-})}{\to} [d_{-}f(\overline{x},u),\infty)$, we get that

$$[d_{-}f(\overline{x},u),\infty) \in D^{(K_{-})}\operatorname{Epi} f(\overline{x})(u).$$

Suppose now, by way of contradiction that would exist a bigger set in $D^{(K_{-})}\operatorname{Epi} f(\overline{x})(u)$. Then it would exist $v < d_{-}f(\overline{x},u)$ such that $\{v\} \in D^{(K_{-})}\operatorname{Epi} f(\overline{x})(u)$. Whence, by Proposition 3.2, $\{v\} \subset D^{-}\operatorname{Epi} f(\overline{x})(u)$, but the latter set is known to be exactly $[d_{-}f(\overline{x},u),\infty)$ (see [3, Proposition 3.5]).

The case $d_{-}f(\overline{x},u) = -\infty$ is similar. □

Remark 3.8. In fact, it is easy to see that $D^{(K_{-})}f(\overline{x})(u)$ consists of all subsets of the set of limits of the quotient $(t_n)^{-1}(f(\overline{x}+t_nu_n)-f(\overline{x}))$ for $(t_n)\downarrow 0$, $(u_n)\to u$ when this is convergent.

We end this section by proposing a variation of the main concept discussed up to this point. Let again $F: X \to 2^Y\setminus\{\emptyset\}$. For $\overline{x}\in X$ and $u \in X$ we define (see [8]) the Penot-type directional derivative of F at $\overline{x}$ in direction u with respect to (K_-) as

$$D_P^{(K_-)} F(\overline{x})(u) = \left\{ \begin{array}{c} V \in 2^Y \setminus \{\emptyset\} \mid \forall (t_n) \downarrow 0,\ \forall (u_n) \to u,\ \exists (V_n) \overset{(K_-)}{\to} V, \\ \forall n \geq n_0 : F(\overline{x}) + t_n V_n \subset F(\overline{x} + t_n u_n) \end{array} \right\}.$$

This time, if $A, B \in D_P^{(K_-)} F(\overline{x})(u)$, then $A \cup B \in D_P^{(K_-)} F(\overline{x})(u)$. Moreover, this directional derivative can be compared with $D^+ F(\overline{x})(u)$ from Definition 2.5 on the same ideas as before. We refrain ourselves from doing this here, and we let this exercise to the interested reader. However, we reconsider this notion in the next section, in order to deal with constrained problems and in the last section in order to illustrate a possible axiomatic approach in the theory of directional derivatives based on set convergences.

4. Optimality conditions for some efficiency concepts

In this section, we deal with some necessary optimality conditions one can get using the directional derivatives we discussed above.

Proposition 4.1. *If $\overline{x}$ is a local $l-$weak minimum point for F (on X) and* $\operatorname{WMin}(F(\overline{x}), K) \neq \emptyset$, *then for all $u \in X$ and all $V \in D^{(K_-)} \operatorname{Epi} F(\overline{x})(u)$, one has $V \cap (-\operatorname{int} K) = \emptyset$.*

Proof. Suppose, by way of contradiction, that there are $u \in X, V \in D^{(K_-)} \operatorname{Epi} F(\overline{x})(u)$ and $v \in V \cap (-\operatorname{int} K)$. Then, according to the definition, there are $(t_n) \downarrow 0$, $(u_n) \to u$, $(V_n) \overset{(K_-)}{\to} V$, such that for all n,

$$F(\overline{x}) + t_n V_n + K \subset F(\overline{x} + t_n u_n) + K.$$

Since $(V_n) \overset{(K_-)}{\to} V$, there is a sequence $(v_n) \to v$ such that $v_n \in V_n$ for all n. Therefore, for n large enough,

$$\begin{aligned} F(\overline{x}) &\subset F(\overline{x} + t_n u_n) + K - t_n v_n \\ &\subset F(\overline{x} + t_n u_n) + \operatorname{int} K. \end{aligned}$$

Since $\overline{x} + t_n u_n \to \overline{x}$, the local weak minimality of $\overline{x}$ implies

$$F(\overline{x}) \subset F(\overline{x}) + \operatorname{int} K,$$

and this contradicts the assumption $\operatorname{WMin}(F(\overline{x}), K) \neq \emptyset$. Consequently, the conclusion holds. □

Remark 4.2. In view of Proposition 3.5, the above assertion generalizes some optimality conditions from [2] and [3].

Let us consider now, inspired by [4], a slightly different approximate solution concept for the set-optimization problem (P). In order to obtain necessary optimality conditions for this kind of concept the authors in [4] use the classical Bouligand derivative, a construction that is well-suited for vector optimization problems but seems to not fit equally well in the set-optimization setting. This fact is visible in [4, Theorem 5.1] where a strong condition on $F(\overline{x})$ has to be imposed.

Definition 4.3. Let $e \in K \setminus \{0\}$, $\varepsilon > 0$ and $\varphi : X \to [0, \infty)$ such that $\varphi(x) > 0$ if $x \neq 0$. One says that $\overline{x} \in X$ is a $l-$weak $(\varepsilon, e, \varphi)$ minimum point for (P) if $\overline{x}$ is $l-$weak minimum for $F(\cdot) + \varepsilon\varphi(\cdot - \overline{x})\, e$ on M.

Proposition 4.4. *In the notation of Definition 4.3, suppose that* $M = X$, $\overline{x} \in X$ *is a* $l-$*weak* $(\varepsilon, e, \varphi)$ *minimum point for* (P) *and* $\operatorname{WMin}(F(\overline{x}), K) \neq \emptyset$. *Then one has*

$$(V + d_+\varphi(0)(u)\, e) \cap (-\operatorname{int} K) = \emptyset,$$

for all $u \in X$ *with* $d_+\varphi(0)(u) \in \mathbb{R}$ *and for all* $V \in D^{(K_-)} \operatorname{Epi} F(\overline{x})(u)$.

Proof. Consider the set-valued map $G : X \rightrightarrows Y$,

$$G(x) = F(x) + \varepsilon\varphi(x - \overline{x})\, e.$$

Clearly, under our assumptions, $\operatorname{WMin}(G(\overline{x}), K) \neq \emptyset$ because $G(\overline{x}) = F(\overline{x})$. According to Proposition 4.1, one has $V \cap (-\operatorname{int} K) = \emptyset$, for all $u \in X$ and all $V \in D^{(K_-)} \operatorname{Epi} G(\overline{x})(u)$. In order to get the conclusion, it is enough to prove that for all $V \in D^{(K_-)} \operatorname{Epi} F(\overline{x})(u)$,

$$V + d_+\varphi(0)(u)\, e \in D^{(K_-)} \operatorname{Epi} G(\overline{x})(u).$$

Let then $V \in D^{(K_-)} \operatorname{Epi} F(\overline{x})(u)$, meaning that there are some sequences $(t_n) \downarrow 0$, $(u_n) \to u$, $(V_n) \overset{(K_-)}{\to} V$, such that for all n

$$F(\overline{x}) + t_n V_n \subset F(\overline{x} + t_n u_n) + K.$$

But for these sequences, one can find some subsequences $(t_{n_k})_k$ and $(u_{n_k})_k$ such that for all nonzero k,

$$d_+\varphi(0)(u) + \frac{1}{k} > \frac{\varphi(t_{n_k} u_{n_k}) - \varphi(0)}{t_{n_k}},$$

which gives

$$t_{n_k}\left(d_+\varphi(0)(u) + k^{-1}\right) + \varphi(0) > \varphi(t_{n_k} u_{n_k}),$$

that is

$$\varphi(0) + t_{n_k}\left(d_+\varphi(0)(u) + k^{-1}\right) \in (\varphi(t_{n_k} u_{n_k}), +\infty).$$

By multiplication with e, we get

$$\varphi(0)\, e + t_{n_k}\left(d_+\varphi(0)(u) + k^{-1}\right) e \in \varphi(t_{n_k} u_{n_k})\, e + K.$$

Summing the relations for F and φ,

$$\begin{aligned} F(\overline{x}) + \varphi(0)\, e + t_{n_k} V_{n_k} + t_{n_k}\left(d_+\varphi(0)(u) + k^{-1}\right) e \quad \in \quad & F(\overline{x} + t_{n_k} u_{n_k}) \\ & + \varphi(t_{n_k} u_{n_k})\, e + K. \end{aligned}$$

The obvious relation $V_{n_k} + d_+\varphi(0)(u)\, e + k^{-1} \overset{(K_-)}{\to} V + d_+\varphi(0)(u)\, e$ confirms the above claim. The proof is complete. □

Remark 4.5. Notice that the efficiency concept that appears in Definition 4.3 is a natural outcome of a penalization procedure as devised in [2, Proposition 4.8]. Namely, in that result, under certain conditions and for an $e \in K \setminus \{0\}$, a $l-$weak minimum for F on M is shown to be a $l-$weak minimum on X (that is, without constraints) for the set-valued map

$$x \rightrightarrows F(x) + Ld_M(x)\, e,$$

where L is a constant associated to a generalized Lipschitz property of F.

Now, using the same idea as in Proposition 4.1 and the interplay of $\exists$ and $\forall$ one easily gets necessary optimality conditions in the constrained case in terms of $D_P^{(K-)}F(\overline{x})(u)$. As usual, for a nonempty set $M \subset X$ and for $x \in M$, one denotes by $T_B(M, x)$ the Bouligand tangent cone to M at x, that is,

$$T_B(M, x) = \{u \in X \mid \exists (t_n) \downarrow 0,\ \exists (u_n) \to u : x + t_n u_n \in M,\ \forall n\}.$$

Proposition 4.6. *If $\overline{x}$ is a local $l-$weak minimum point for F on X and* $\operatorname{WMin}(F(\overline{x}), K) \neq \emptyset$, *then for all $u \in T_B(M, \overline{x})$ and $V \in D_P^{(K-)} \operatorname{Epi} F(\overline{x})(u)$, one has $V \cap (-\operatorname{int} K) = \emptyset$.*

Similarly, optimality conditions for $l-$weak $(\varepsilon, e, \varphi)$ minimality for (P) can be devised.

5. Towards an axiomatic approach

Of course, (K_-) is just one of many possible set convergences that can be considered: see [10], [11] and the references therein. We denote by $\mathcal{F}$ the class of closed nonempty subsets of Y. Suppose that $\emptyset \neq \mathcal{A} \subset 2^Y$ and denote by c a set convergence on $\mathcal{A}$ in the sense presented in [7, p. 74, items (a), (b), (c)]. As it is well known there are many set convergences that can be considered on subclasses $\mathcal{A}$ of $\mathcal{F}$ or even on $2^Y \setminus \{\emptyset\}$. As before, we consider here only sequences of sets and as a standing assumption we understand that all the involved sets are in $\mathcal{A}$ every time when we speak about a set convergence on $\mathcal{A}$. Then, in this setting, we denote the fact that $(A_n) \subset \mathcal{A}$ is convergent to $A \in \mathcal{A}$ in the sense of c by $A \in c - \lim A_n$ or $(A_n) \xrightarrow{c} A$.

This general discussion and the concrete example of (K_-) open the way to define a whole class of directional derivatives on a similar pattern. Let F take values in $\mathcal{A}$ and c be a set convergence on $\mathcal{A}$. For $\overline{x} \in X$ and $u \in X$ we define the Bouligand-type directional derivative of F at $\overline{x}$ in direction u with respect to c as

$$D^c F(\overline{x})(u) = \left\{ \begin{array}{c} V \in \mathcal{A} \mid \exists (t_n) \downarrow 0,\ \exists (u_n) \to u,\ \exists (V_n) \xrightarrow{c} V, \\ \forall n : F(\overline{x}) + t_n V_n \subset F(\overline{x} + t_n u_n) \end{array} \right\}.$$

Of course, every set convergence on a class $\mathcal{A} \subset 2^Y$ generates a directional derivative and therefore, the above definition subsumes a wide variety of concepts which can be classified by the very classifications of set convergences (see [10] for such a classification).

For instance, besides (K_-) considered above, one can be equally interested in the "lower half" of Hausdorff-Pompeiu convergence, that is denoted by (H_-), namely: $V \in (H_-) - \lim V_n$ if and only if

$$\forall \varepsilon > 0,\ \exists n_\varepsilon, \forall n \geq n_\varepsilon : V \subset (V_n)^\varepsilon := \{y \in Y \mid d(y, V_n) < \varepsilon\}.$$

Remark 5.1. (i) Clearly, (H_-) convergence implies (K_-) convergence, but the converse is not true. Therefore, one has $D^{(H_-)}F(\overline{x})(u) \subset D^{(K_-)}F(\overline{x})(u)$ and one can have $V \in D^{(K_-)}F(\overline{x})(u)$ but $V \notin D^{(H_-)}F(\overline{x})(u)$. For instance, consider that Y is infinite dimensional and take $(x_n) \subset S_X$ such that $\|x_n - x_m\| > 2^{-1}$ for all natural distinct numbers m, n. According to Riesz Lemma, such a sequence exists. Consider the sets $A_n = \{x_k \mid k \leq n\}$ for all n and $A = \{x_n \mid n \in \mathbb{N}\}$. If for $\overline{x}, u \in X$, $(t_n) \downarrow 0$, and $(u_n) \to u$,

$$F(\overline{x}) + t_n A_n \subset F(\overline{x} + t_n u_n),$$

then $A \in D^{(K_-)}F(\overline{x})(u)$, but $A \notin D^{(H_-)}F(\overline{x})(u)$. Similar remarks hold for any two comparable set convergences.

(ii) These two convergences are in a sense "extreme": (K_-) is weak and (H_-) is strong and this is the main reason we considered them here. In general, if a result holds for (K_-) then it holds for a wide range of set convergences.

Remark 5.2. Actually, if one takes stronger convergences that (K_-) in Proposition 3.2, then better conclusion can be devised. For instance, if we consider $c = (H_-)$, in proof of Proposition 3.2, since $(V_n) \overset{H_-}{\to} V$, then one gets that for all $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that for $n \geq n_\varepsilon$,

$$V \subset V_n + B(0, \varepsilon).$$

Therefore, for n large enough,

$$F(\overline{x}) + t_n V \subset F(\overline{x}) + t_n (V_n + B(0,\varepsilon)) = F(\overline{x} + t_n u_n) + t_n B(0,\varepsilon).$$

This implies that $V \subset D^- F(\overline{x})(u)$ with an uniform ε for all $v \in V$.

Similarly one can consider the weaker set convergence (AW_-) (a "half") of Attouch-Wets topology if one restricts it to $\mathcal{F}$ (see [1]). Namely, $A \in (AW_-)-\lim A_n$ if and only if

$$\exists \overline{y} \in Y,\ \exists \rho_0 > 0,\ \forall \rho \geq \rho_0 : e(A \cap D(\overline{y}, \rho), A_n) \to 0.$$

Now, in Proposition 3.2 if $(V_n) \overset{(AW_-)}{\to} V$, then consider $v \in V$ and for ρ big enough, $V \cap D(\overline{y}, \rho) \neq \emptyset$, whence for all $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that for $n \geq n_\varepsilon$,

$$v \in V_n + B(0, \varepsilon),$$

and again $V \subset D^- F(\overline{x})(u)$ with an uniform ε for all v in bounded subsets of V. The same example as in Remark 5.1 (i) shows that the inclusion $D^{(AW_-)}F(\overline{x})(u) \subset D^{(K_-)}F(\overline{x})(u)$ can be strict. Taking the easiest example of subsets of natural numbers $A_n = \{k \in \mathbb{N} \mid k \leq n\}$ for all n and $A = \mathbb{N}$ one gets as well that the inclusion $D^{(H_-)}F(\overline{x})(u) \subset D^{(AW_-)}F(\overline{x})(u)$ can be strict.

Remark 5.3. As in Proposition 4.1, one can prove that if $\overline{x}$ is a $l-$weak minimum point for F (on X) and $\operatorname{WMin}(F(\overline{x}), K) \neq \emptyset$, then for all $u \in X$ and all $V \in D^{(H_-)} \operatorname{Epi} F(\overline{x})(u)$ we get that

$$\forall \varepsilon > 0, \forall v \in V : B(v, \varepsilon) \not\subset -\operatorname{int} K.$$

Remark 5.4. In fact, in the preceding section, in the proof of our results we used only some properties of (K_-) and this could be an impetus to consider an abstract set convergence endowed with some properties. This abstracting could be combined as well with a play of $\exists$ and $\forall$ quantifiers (as we did when we briefly discussed $D_P^{(K_-)} \operatorname{Epi} F(\overline{x})(u)$) in order to get other differentiation objects and corresponding results. Another variation that can be useful in certain problems is to consider, into the definition of directional derivative the reverse of the inclusion

$$F(\overline{x}) + t_n V_n \subset F(\overline{x} + t_n u_n).$$

For instance, the latter modification is useful in dealing with $\preceq_K^u$ order (see [6]) in set-optimization problems.

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