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HADAMARD SEMIDIFFERENTIAL OF CONTINUOUS CONVEX FUNCTIONS

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Abstract. The *Hadamard semidifferential calculus* preserves all the operations of the classical differential calculus including the chain rule for a large family of non-differentiable functions including convex functions in the interior of their domain (see [M. C. Delfour, *Hadamard Semidifferential of Functions on an Unstructured Subset of a TVS*, J. Pure Appl. Funct. Anal. **5**, no. 5 (2020), 1039–1072]).

The object of this paper is the Hadamard semidifferential of convex functions which are continuous on the closure of their domain. Such functions are Hadamard semidifferentiable at interior points of their domain in all directions and their semidifferential coincides with the Clarke upper semidifferential. It is shown that, if the function is Lipschitz continuous at a boundary point, then it is Hadamard semidifferentiable for all directions in the adjacent tangent cone at that point. The paper also corrects a missing assumption for the convexity of the functions

$$f_A(x) \overset{\text{def}}{=} \frac{1}{2}\left[\|x\|^2 - d_A(x)^2\right], \quad d_A(x) \overset{\text{def}}{=} \inf_{a\in A} \|x-a\|, \quad \varnothing \neq A \subset X,$$

$$f_{\partial A}(x) \overset{\text{def}}{=} \frac{1}{2}\left[\|x\|^2 - b_A(x)^2\right], \quad b_A(x) \overset{\text{def}}{=} d_A(x) - d_{X\setminus A}(x), \quad \partial A \neq \varnothing.$$

in the statements of parts (iii) and (iv) of Theorems 4.1 and 4.5 in [M. C. Delfour, *Hadamard Semidifferential, Oriented Distance Function, and some Applications*, Commun. Pure Appl. Anal. 21(6) (2022), 1917–1951]. The proofs were provided for a norm on the vector space X arising from an inner product, but this restriction was missing in the statements. Without it, there are finite dimensional counterexamples to the convexity of f_A and $f_{\partial A}$.

1. Introduction

The *Hadamard semidifferential calculus* preserves all the operations of the classical differential calculus including the chain rule for a large family of non-differentiable functions including the convex functions at interior points of their domain. It was extended from the n-dimensional Euclidean space $\mathbb{R}^n$ in [7] to subsets of topological

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vector spaces in [8]. This class of functions includes most function spaces used in *Optimization* and the *Calculus of Variations*, the metric groups used in *Shape and Topological Optimization*, and functions defined on submanifolds.

The object of this paper is the study of the Hadamard semidifferential of convex functions which are continuous on the closure of their domain. Such functions are Hadamard semidifferentiable at interior points of their domain in all directions and their semidifferential coincides with the Clarke upper semidifferential, but this is not necessarily true at boundary points. In section 2, we recall some definitions and results from [8]. In section 3, we compare the Hadamard semidifferential and the Clarke upper semidifferential. In section 4, we specialize the comparison to convex functions. In particular, it is shown that, if a convex function is Lipschitz continuous at a boundary point x of its domain, then it is Hadamard semidifferentiable at all points y in a neighborhood of x for all directions in the adjacent tangent cone at y. In section 5, we correct a missing assumption in the statements of parts (iii) and (iv) of Theorems 4.1 and 4.5 of [9, Thm. 4.1, p. 1928, Thm. 4.5, p. 1933] for the convexity of the functions

$$f_A(x) \stackrel{\text{def}}{=} \frac{1}{2}\left[\|x\|^2 - d_A(x)^2\right], \quad d_A(x) \stackrel{\text{def}}{=} \inf_{a\in A} \|x-a\|, \quad \varnothing \neq A \subset X,$$
$$f_{\partial A}(x) \stackrel{\text{def}}{=} \frac{1}{2}\left[\|x\|^2 - b_A(x)^2\right], \quad b_A(x) \stackrel{\text{def}}{=} d_A(x) - d_{X\setminus A}(x), \quad \partial A \neq \varnothing.$$

The proofs were provided for a norm on the vector space X that arises from an inner product, that is, $\|x\| = \sqrt{x\cdot x}$, but this restriction was missing in the statements of parts (iii) and (iv). Without that assumption, there are finite dimensional counterexamples to the convexity of f_A and $f_{\partial A}$.

2. Semidifferentials of functions defined on a subset of a TVS.

For functions on a smooth embedded submanifold of $\mathbb{R}^n$ of dimension $d < n$ or on an unstructured subset A of a *Topological Vector Space* (TVS) X, the Hadamard semidifferential is the natural choice over the M-semidifferential (see [8, Dfn. 3.4] since it uses semitrajectories in A that do not a priori require some specific structure on A. For a subset A of X, the tangent space at interior points of A is X, but, at the boundary ∂A, the tangent space will generally be only a cone. For instance, for a smooth embedded submanifold of dimension $d < n$ in $\mathbb{R}^n$, $\overline{A} = \partial A$ and all points of A are boundary points where the tangent space is a d-dimensional linear subspace.

For the convenience of the reader, we recall some definitions and theorems from [8, sec. 4].

Definition 2.1. Let A be a non-empty subset of a topological vector space X. An *admissible semitrajectory* at $x \in A$ in A is a function $h : [0,\tau) \to A$, $\tau > 0$, such that

$$h(0) = x \quad \text{and} \quad h'(0^+) \stackrel{\text{def}}{=} \lim_{t\searrow 0} \frac{h(t)-h(0)}{t} \text{ exists in } X.$$

$h'(0^+)$ is the *semitangent* to the trajectory h in A at $h(0) = x$.

Definition 2.2. Let A be a non-empty subset of a topological vector space X. The *adjacent or intermediary tangent cone*[1] to A at $x \in A$ is defined as

$$T^\flat_A(x) \overset{\text{def}}{=} \left\{ v \in X : \forall \{t_n \searrow 0\},\, \exists \{x_n\} \subset A \text{ such that } \lim_{n\to\infty} \frac{x_n - x}{t_n} = v \right\}.$$

If A is convex, $T^\flat_A(x) = \overline{\{\lambda(A-x) : \lambda \geq 0\}}$.

$T^\flat_A(x)$ is related to the notion of admissible semitrajectories at x in A.

Theorem 2.3. *Let A be a subset of a topological vector space X. For $x \in A$,*

$$T^\flat_A(x) = \left\{ h'(0^+) : h \text{ an admissible semitrajectory in } A \text{ at } x \right\}.$$

We now have all the elements to extend the definition of the Hadamard semidifferential to a subset of a TVS.

Definition 2.4. Let X and Y be TVS, A, $\varnothing \neq A \subset X$, and $f : A \to Y$.

(i) The function f is *Hadamard semidifferentiable at $x \in A$ in the direction $v \in T^\flat_A(x)$* if there exists $g(x,v) \in Y$ such that, for all admissible semitrajectories h in A at x such that $h'(0^+) = v$,

$$(f \circ h)'(0^+) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{f(h(t)) - f(h(0))}{t} = g(x,v). \tag{2.1}$$

The element $g(x,v)$ will be denoted $d_H f(x;v)$.

(ii) f is *Hadamard semidifferentiable at $x \in A$* if f is Hadamard semidifferentiable at x in all directions $v \in T^\flat_A(x)$.

(iii) f is *Hadamard differentiable at $x \in A$* if $T^\flat_A(x)$ is a linear subspace, f is Hadamard semidifferentiable at $x \in A$, and the function $v \mapsto d_H f(x;v) : T^\flat_A(x) \to Y$ is linear in which case it will be denoted $Df(x)$. □

The Hadamard semidifferentiability enjoys all the nice properties of the classical finite dimensional differential calculus including the chain rule.

Theorem 2.5. *Let X and Y be topological vector spaces and A, $\varnothing \neq A \subset X$.*

(i) *If $f : A \to Y$ is Hadamard semidifferentiable at $x \in A$ in the direction $v \in T^\flat_A(x)$, then for all admissible semitrajectory h in A such that $h'(0^+) = v$, $f \circ h$ is an admissible trajectory in $f(A)$ such that $(f \circ h)'(0^+) = d_H f(x;v) \in T^\flat_{f(A)}(f(x))$. The positively homogeneous mapping*

$$v \mapsto d_H f(x;v) : T^\flat_A(x) \to T^\flat_{f(A)}(f(x)) \subset Y \tag{2.2}$$

is sequentially continuous for the induced topologies.

(ii) *If $f_1 : A \to Y$ and $f_2 : A \to Y$ are Hadamard semidifferentiable at $x \in A$ in the direction $v \in T^\flat_A(x)$, then for all α and β in $\mathbb{R}$,*

$$d_H(\alpha f_1 + \beta f_2)(x;v) = \alpha\, d_H f_1(x;v) + \beta\, d_H f_2(x;v), \tag{2.3}$$

and $\alpha f_1 + \beta f_2$ is Hadamard semidifferentiable at x in the direction v.

[1]The natural tangent cone associated with a Hadamard semidifferentiable function defined on a subset of a topological vector space (see [8]) is the *adjacent tangent cone* defined in the book of Aubin and Frankowska [2, p. 128] in terms of sequences rather than its equivalent definition [2, Dfn. 4.1.5, p. 127] in a normed vector space via the distance function.

(iii) *(Chain rule) Let X, Y, Z be topological vector spaces, $A \subset X$, $g : A \to Y$, and $f : g(A) \to Z$ be functions such as g is Hadamard semidifferentiable at x in the direction $v \in T^\flat_A(x)$ and f is Hadamard semidifferentiable at $g(x)$ in $g(A)$ in the direction $d_H g(x; v)$. Then $d_H g(x; v) \in T^\flat_{g(A)}(x)$, $f \circ g$ is Hadamard semidifferentiable at x in the direction $v \in T^\flat_A(x)$, and*

$$d_H(f \circ g)(x; v) = d_H f(g(x); d_H g(x; v)). \tag{2.4}$$

The next question is the continuity of a semidifferentiable function.

Theorem 2.6. *Let X and Y be topological vector spaces, $\varnothing \neq A \subset X$, and $f : A \to Y$. Assume that f is Hadamard semidifferentiable at $x \in A$.*

(i) *If there exists a bounded neighborhood $U(0) \in \mathscr{R}$ in X,[2] then f is sequentially continuous[3] at x in A for the induced topology on A.*

(ii) *If X is a Fréchet space,[4] then $v \mapsto d_H f(x; v) : T^\flat_A(x) \to T^\flat_{f(A)}(f(x))$ is positively homogeneous and continuous for the induced topologies. If X and Y are Fréchet spaces, then f is continuous at x.*

Additional operations such as the *lower and upper envelops* of a finite family of real-valued functions are available: for $f_i : X \to \mathbb{R}$, $i = 1, \dots m$,

$$d_H \left(\max_{1\leq i\leq m} f_i \right)(x; v) = \max_{i\in I(x)} d_H f_i(x; v), \quad I(x) = \{i : f_i(x) = \max_{1\leq j\leq m} f_j(x)\}$$

$$d_H \left(\min_{1\leq i\leq m} f_i \right)(x; v) = \min_{i\in J(x)} d_H f_i(x; v), \quad J(x) = \{i : f_i(x) = \min_{1\leq j\leq m} f_j(x)\}.$$

This includes the functions $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \min\{f(x), 0\}$.

All convex (resp. concave) functions on X are Hadamard semidifferentiable in the interior of their domain.

3. Comparison of Hadamard and Clarke semidifferentials

3.1. Strict Differentiability.

Recall the definition of strict differentiability introduced by the school of Bourbaki in the fifties, which is strictly stronger than the M-, Hadamard, and Fréchet differentiabilities.

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[2]Recall that in a *topological vector space* (TVS) over $\mathbb{R}$ there is a *fundamental system* $\mathscr{R}$ of neighborhoods of the origin for which ([11, Dfn. pp. 79–80, Thm. 1, p. 81])
(i) every V in $\mathscr{R}$ is absorbing and balanced, and
(ii) for every $V \in \mathscr{R}$, there exists $U \in \mathscr{R}$ such that $U + U \subset V$.
In this paper we assume that the *neighborhoods* of the origin are the elements of $\mathscr{R}$.
A set A is bounded if, for all $V \in \mathscr{R}$, there exists $\alpha > 0$ such that $A \subset \lambda V$ for all $\lambda \geq \alpha$ ([11, Dfn. 1, p. 108]).

[3]Note the following natural equivalence for the semicontinuity in terms of semitrajectories. Let X and Y be topological spaces and A a subset of X. A function $f : A \to Y$ is *sequentially continuous* at $a \in A$ if and only if for all *semitrajectories* $h : [0, \tau) \to A$

$$\lim_{t\searrow 0} h(t) = a \quad \Rightarrow \quad \lim_{t\searrow 0} f(h(t)) = f(a), \tag{2.5}$$

where A is endowed with the topology induced by X.

[4]A complete, metrizable, locally convex topological space is called a *Fréchet space* ([11, Dfn. 4, p, 136]).

Definition 3.1 (Clarke [5, p. 30–31])**.** Given two Banach spaces X and Y, a function $f : X \to Y$ is *strictly differentiable* at x if there exists a continuous linear function $Df(x) : X \to Y$ such that

$$\forall v \in X, \ \lim_{\substack{t \searrow 0 \\ y \to x}} \frac{f(y+tv) - f(y)}{t} = Df(x)v. \tag{3.1}$$

A strictly differentiable function at x is Lipschitz continuous at x (Clarke [5, Prop. 2.2.1. p. 31]) in the following sense.

Definition 3.2. Let X and Y be normed spaces.

(i) A function $f : X \to Y$ is Lipschitz continuous at $x \in X$ if there exists a constant $c(x) > 0$ and a ball $B_r(x)$ such that

$$\forall y, z \in B_r(x), \quad \|f(y) - f(z)\| \le c(x)\|y - z\|. \tag{3.2}$$

(ii) Given $U \subset X$, a function $f : U \to Y$ is Lipschitz continuous at $x \in U$ if there exists a constant $c(x) > 0$ and a ball $B_r(x)$ such that

$$\forall y, z \in B_r(x) \cap U, \quad \|f(y) - f(z)\| \le c(x)\|y - z\|. \tag{3.3}$$

3.2. **Upper and Lower Semidifferentials.** For real-valued functions $f : X \to \mathbb{R}$ which are Lipschitz continuous at $x \in X$, lower and upper notions of Gateaux, M-, and strict differentiabilty can be obtained by replacing the limit by the lim inf or the lim sup. They are the so called *upper and lower semidifferentials* in the terminology of Cannarsa and Sinestrari [3].

Upper and lower semidifferentials of locally Lipschitz functions are more general, but the basic operations of the differential calculus are lost and one resorts to the notion of *subdifferential* and the tools of *set-valued analysis* to restore some form of calculus. This is a disadvantage over the *Hadamard semidifferential calculus.*

$\underline{d}f(x;v) \overset{\text{def}}{=} \liminf_{t \searrow 0} \frac{f(x+tv) - f(x)}{t}$ lower Gateaux semidifferential at x in the direction v	$\overline{d}f(x;v) \overset{\text{def}}{=} \limsup_{t \searrow 0} \frac{f(x+tv) - f(x)}{t}$ upper Gateaux semidifferential at x in the direction v
$\underline{d}_M f(x;v) \overset{\text{def}}{=} \liminf_{\substack{t \searrow 0 \\ w \to v}} \frac{f(x+tw) - f(x)}{t}$ lower M-semidifferential at x in the direction v	$\overline{d}_M f(x;v) \overset{\text{def}}{=} \limsup_{\substack{t \searrow 0 \\ w \to v}} \frac{f(x+tw) - f(x)}{t}$ upper M-semidifferential at x in the direction v.
$\underline{d}_C f(x;v) \overset{\text{def}}{=} \liminf_{\substack{t \searrow 0 \\ y \to x}} \frac{f(y+tv) - f(y)}{t}$ Clarke lower semidifferential at x in the direction v	$\overline{d}_C f(x;v) \overset{\text{def}}{=} \limsup_{\substack{t \searrow 0 \\ y \to x}} \frac{f(y+tv) - f(y)}{t}$ Clarke upper semidifferential at x in the direction v

The upper notion of strict differentiability $\overline{d}_C f(x;v)$ corresponds to the *generalized directional derivative* developed by Clarke [4] in his thesis in 1973.

3.3. **Examples of Hadamard and Clarke Upper Semidifferentiable Functions.** It is important to observe that the family of Hadamard semidifferentiable functions and the family of Lipschitz continuous functions with a Clarke upper semidifferential have a large intersection but they are distinct.

In general, a function f can be Fréchet differentiable (hence Hadamard semidifferentiable) at a point x, where f is not Lipschitz at x in any neighborhood of x as illustrated in Example 3.3. In this example $\overline{d}_C f(x;v) = +\infty$.

In the other direction, recall that a Lipschitz continuous function at x is Hadamard semidifferentiable at x if and only if

$$\forall v \in X, \quad \lim_{t \searrow 0} \frac{f(x+tv)-f(x)}{t} \text{ exists} \tag{3.4}$$

(see [8, Thm. 3.10. p. 1052]). Example 3.4 gives a Lipschtz continuous function, which does not verify condition (3.4) at $x=0$.

Example 3.3. The function (see Figure 1)

$$f(x) \overset{\text{def}}{=} \begin{Bmatrix} x^{3/2}\sin\dfrac{1}{x}, & x>0 \\ 0, & x \le 0 \end{Bmatrix}, \quad f'(x) = \begin{Bmatrix} \dfrac{3}{2}\, x^{1/2}\sin\dfrac{1}{x} - \dfrac{1}{x^{1/2}}\cos\dfrac{1}{x}, & x>0 \\ 0, & x\le 0 \end{Bmatrix}$$

is differentiable everywhere, but it is not Lipschitz in any neighborhood of $x=0$. To show that its Clarke upper semidifferential is $+\infty$ at $x=0$, choose $v=-1$ and

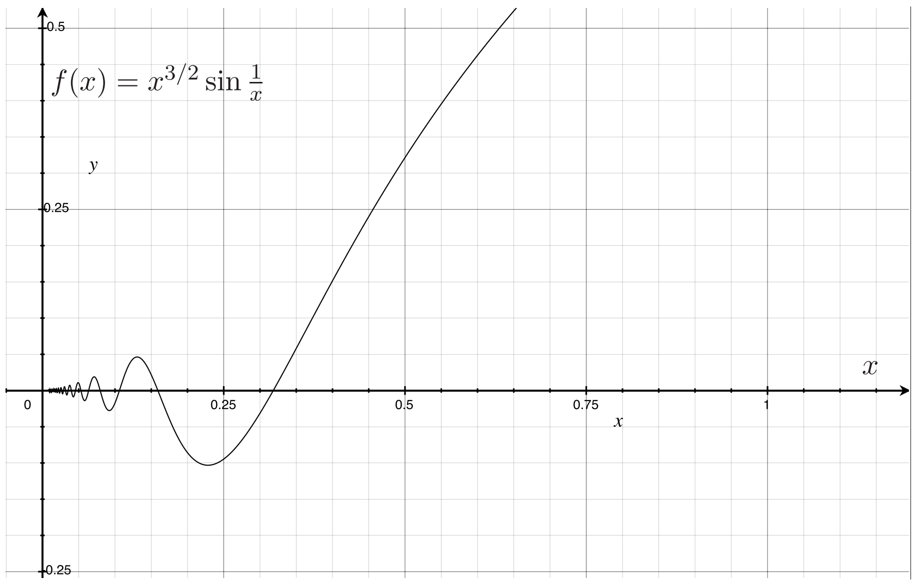

FIGURE 1. f is Hadamard (semi)differentiable at 0 and not Lipschitz continuous in any neighborhood of $x=0$.

the sequences $\{y_n\}$ and $\{t_n\}$ as follows

$$y_n = \frac{1}{2\pi n}, \quad y_n + t_n v = y_n - t_n = \frac{1}{2\pi n + \pi/2} \tag{3.5}$$

$$\Rightarrow t_n = \frac{1}{2\pi n} - \frac{1}{2\pi n + \pi/2} = \frac{\pi}{2} \frac{1}{(2\pi n)(2\pi n + \pi/2)} > 0. \tag{3.6}$$

The strict differential quotient becomes

$$\begin{aligned}
&\frac{f(y_n - t_n) - f(y_n)}{t_n} \\
&= \frac{[(y_n - t_n)^{3/2} - y_n^{3/2}]}{t_n} \sin(1/(y_n - t_n)) + \frac{(y_n^{3/2})}{t_n} [\sin(1/(y_n - t_n)) - \sin(1/y_n)] \\
&= \frac{[(y_n - t_n)^{3/2} - y_n^{3/2}]}{t_n} + \frac{(y_n^{3/2})}{t_n} = \frac{(\frac{1}{2\pi n})^{3/2}}{1/(2\pi n)(2\pi n + \pi/2))} \to +\infty.
\end{aligned}$$

$$\Rightarrow \text{at } x = 0 \text{ and for } v = -1, \quad \limsup_{\substack{y \to 0 \\ t \searrow 0}} \frac{f(y + tv) + f(y)}{t} = +\infty. \tag{3.7}$$

Example 3.4. Define $f : (-\infty, 2] \to \mathbb{R}$ as follows (see Figure 2)

$$f(x) \overset{\text{def}}{=} \begin{Bmatrix} 3\left[\frac{1}{2^n} - x\right], & \frac{3}{2^{n+2}} < x \le \frac{1}{2^n}, & n = -1, 0, 1, 2, \dots \\ 3\left[x - \frac{1}{2^{n+1}}\right], & \frac{1}{2^{n+1}} < x \le \frac{3}{2^{n+2}}, & n = -1, 0, 1, 2, \dots \\ 0, & x \le 0, & \end{Bmatrix}. \tag{3.8}$$

For $v = 1$

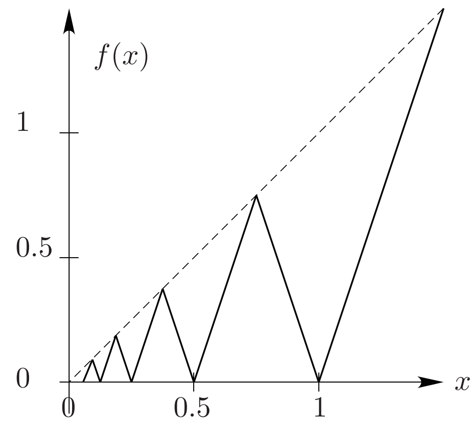

FIGURE 2. A Lipschitz continuous function f which is not Hadamard semidifferentiable at $x = 0$.

$$\liminf_{t \searrow 0} \frac{f(0 + tv) - f(0)}{t} = 0, \quad \limsup_{t \searrow 0} \frac{f(0 + tv) - f(0)}{t} = 1. \tag{3.9}$$

4. Semidifferentials of convex functions

Convex functions f in the interior of their domain dom f are locally Lipschitz and regular ([5, Prop. 2.2.7, p. 36, Def. 2.3.4, p. 39]), that is,

$$\forall v \in X \quad \overline{d}_C f(x;v) = df(x;v) \stackrel{\text{def}}{=} \lim_{\theta \searrow 0} \frac{f(x+\theta v) - f(x)}{\theta}.$$

But a function which is Lipschitz at a point x is Hadamard semidifferentiable at x if and only if $df(x;v)$ exists. Hence, the two semidifferentials coincide:

$$\overline{d}_C f(x;v) = d_H f(x;v). \tag{4.1}$$

For concave functions, we have an analogous identity but with the Clarke lower semidifferential: $\underline{d}_C f(x;v) = d_H f(x;v)$. In particular, for linear combinations of a convex function f and a concave function g, we have

$$\forall \alpha,\, \beta \in \mathbb{R}, \quad d_H(\alpha f + \beta g)(x;v) = \alpha\, d_H f(x;v) + \beta\, d_H g(x;v). \tag{4.2}$$

The calculus is an advantage over the Clarke upper semidifferential.

It is interesting to look at what is happening at the boundary of dom f, where, in general, only directions in the convex tangent cone are admissible. We use Definition 2.4 of a Hadamard semidifferential for a continuous convex function at a boundary point of the convex subset dom f.

Therefore, it is suficient to consider continuous convex functions $f : U \to \mathbb{R}$ defined on a closed convex subset U of X. Think of U as the closure of dom f. We begin with the two examples in dimension one in Figure 3.

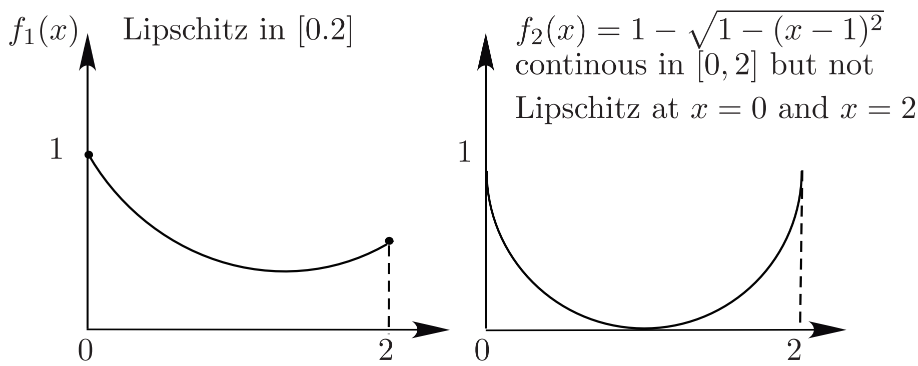

Figure 3. Continuous convex functions f_1 and f_2 in $U = [0,2]$.

Example 4.1. In both examples of Figure 3, the interval $U = [0,2]$ is polyhedral, that is, $T_x^\flat(U) = \mathbb{R}^+(U - x)$. Here $T_0^\flat(U) = [0,+\infty)$ and $T_2^\flat(U) = [0,+\infty)$. For the Lipschitz continuous function f_1 on U, it will be shown in Theorem 4.2 (iv) that

$$\forall v \in T_x^\flat(U), \quad \overline{d}_C f_1(x;v) = d_H f_1(x;v).$$

The continuous convex function

$$x \mapsto f_2(x) = 1 - \sqrt{1-(x-1)^2} : [0,1] \to \mathbb{R} \tag{4.3}$$

is not Lipschitz at the boundary points 0 and 2.

Choose an admissible semitrajectory $h : [0, 1)$ such that

$$h(0) = 0 \text{ and } h'(0^+) = 1. \tag{4.4}$$

Then for $t > 0$ and $h(t)/t \to 1$

$$\frac{f_2(h(t)) - f_2(h(0))}{t} = -\frac{\sqrt{1-(h(t)-1)^2}}{t} = -\sqrt{\frac{2}{t}\frac{h(t)}{t} - \left(\frac{h(t)}{t}\right)^2} \to -\infty.$$

For the strict differential quotient corresponding to $v = 1$

$$\frac{f_2(y+tv) - f_2(y)}{t} = \frac{2y+t-2}{\sqrt{(2-y)y} + \sqrt{(2-y-t)(y+t)}}. \tag{4.5}$$

For $0 < t < 1/2$ and $0 < y < 1/2$

$$\begin{aligned} \frac{f_2(y+t) - f_2(y)}{t} &\le -\frac{1}{2}\frac{1}{\sqrt{(2-y)y} + \sqrt{(2-y-t)(y+t)}} \\ &\le -\frac{1}{2}\frac{1}{2\sqrt{(2(y+t)}}. \end{aligned} \tag{4.6}$$

As $t \searrow 0$ and $y \searrow 0$,

$$\frac{f_2(y+t) - f_2(y)}{t} \to -\infty \quad \Rightarrow \overline{d}_C f(x; v) = -\infty = d_H f(x; v). \tag{4.7}$$

So, if we allow the value $-\infty$ in the definition of $d_H f(x; v)$ and $\overline{d}_C f(x; v)$, f_2 is Hadamard semidifferentiable at $x = 0$ for directions in the cone $T_0^\flat([0, 2]) = [0, \infty)$ and $\overline{d}_C f(x; v) = d_H f(x; v)$.

Theorem 4.2. *Let $f : U \to \mathbb{R}$ be a convex function in a closed convex subset U of a locally convex topological vector space X.*

(i) *For each $x \in U$ and $v \in \mathbb{R}^+(U - x)$, there exists $\theta_0 \in (0, 1)$ such that $x + \theta v \in U$, $0 \le \theta < \theta_0$, the function*

$$\theta \mapsto \frac{f(x+\theta v) - f(x)}{\theta} : (0, \theta_0) \to \mathbb{R} \tag{4.8}$$

is monotone increasing as θ increases, and the limit

$$df(x; v) \overset{\text{def}}{=} \lim_{\theta \searrow 0} \frac{f(x+\theta v) - f(x)}{\theta} \le \frac{f(x+\theta_0 v) - f(x)}{\theta_0}) < +\infty \tag{4.9}$$

is finite or possibly $-\infty$ as shown in Example 3. Moreover, $v \mapsto df(x; v) : \mathbb{R}^+(U - x) \to \mathbb{R} \cup \{-\infty\}$ is convex and positively homogeneous.

(ii) *Let X be a normed vector space. If f is Lipschitz at $x \in U$, that is, there exists $\eta > 0$ and a constant $c(x) > 0$ such that*

$$\forall y_1, y_2 \in B_{2\eta}(x) \cap U, \quad |f(y_2) - f(y_1)| \le c(x)\|y_2 - y_1\|, \tag{4.10}$$

then, for all $y \in B_\eta(x) \cap U$ and $v \in T_y^\flat(U) = \overline{\mathbb{R}^+(U - y)}$, $d_H f(y; v)$ exists and is finite, and for all $y \in B_\eta(x) \cap U$

$$\forall v, w \in T_y(U), \ |d_H f(y; w) - d_H f(y; v)| \le c(x)\|w - v\|. \tag{4.11}$$

Conversely, if there exist $x \in U$, $\eta > 0$, and a constant $c(x) > 0$ such that for all $y \in B_\eta(x) \cap U$ and all $v \in T_y^\flat(U)$, $d_H f(y;v)$ exists and is finite, and (4.11) is verified, then

$$\forall y_1, y_2 \in B_\eta(x) \cap U, \quad |f(y_2) - f(y_1)| \leq c(x)\|y_2 - y_1\|, \tag{4.12}$$

and f is Lipschitz at $x \in U$.

(iii) *Let X be a normed vector space. Assume that f is continuous in U. For all $x \in U$ and $v \in \mathbb{R}^+(U - x)$, by slightly modifying the definition of $\overline{d}_C f(x;v)$ for a continuous convex function f defined only in U,*

$$\overline{d}_C f(x;v) \overset{\text{def}}{=} \limsup_{\substack{y \to x,\, y \in U \\ t \searrow 0}} \frac{f(y+tv) - f(y)}{t} = \lim_{t \searrow 0} \frac{f(x+tv) - f(x)}{t},$$

where the above expressions can be $-\infty$ at a boundary point $x \in \partial U$.

(iv) *Let X be a normed vector space. If $x \in \operatorname{int} U$, then*

$$\forall v \in X, \quad d_H f(x;v) = \overline{d}_C f(x;v) = d_C f(x;v); \tag{4.13}$$

if f is Lipschitz continuous at $x \in \partial U$, then

$$\forall v \in T_x^\flat(U), \quad d_H f(x;v) = \lim_{\substack{w \to v \\ w \in \mathbb{R}^+(U-x)}} \overline{d}_C f(x;w). \tag{4.14}$$

In both cases, the semidifferential is finite.

Remark 4.3. Part (i) is a straightforward extension of the well-known result for an interior point of the domain (see, for instance, [1, Prop. 1, pp. 202-203]), but I didn't find a specific reference to quote when $f(x)$ is finite at a boundary point x of the domain. The proof of part (iii) is based on the one of Aubin [1, Thm. 1, pp. 204–205] for an interior point of the domain of a continuous convex function.

Remark 4.4. The right-hand side of identity (4.14) indicates how the definition of the Clarke upper semidifferential must be modified to include directions in the closure of the cone $\mathbb{R}^+(U - x)$ at a boundary point of U. In contrast, the definition of the Hadamard semidifferential does not require any change and all the operations of the nice Hadamard semidifferential calculus are preserved.

Remark 4.5. Of course, if $x \in \operatorname{int} U$, the Lipschitz continuity at x is equivalent to the continuity at x (see Ekeland-Temam [10, Crl. 2.4, p. 12]). It is also equivalent to the condition given in (ii) (see [6, Thm. 4.7 (ii) to (iv), p. 126] or [7, Thm. 4.7 (ii) to (iv), p. 132] for a proof in finite dimension). But for a boundary point $x \in \partial U$, the continuity at x is not sufficient as shown in Example 4.1.

Remark 4.6. In wiew of part (i) of Theorem 4.2 the conditions for the converse in part (ii) can be weakened as follows: there exists $\eta > 0$ and a constant $c(x) > 0$ such that for all $y \in B_\eta(x) \cap U$

$$\forall v, w \in \mathbb{R}^+(U - y), \quad |df(y;w) - df(y;v)| \leq c(x)\|w - v\|. \tag{4.15}$$

This last inequality is sufficient to conclude that $d_H f(y;v)$ exists and is finite.

Proof. (i) We adapt the proof of Aubin [1, Prop. 1, pp. 202–203]. If $U = \{x\}$ is a singleton, there is nothing to prove. If U is not a singleton, for $x \in U$ and $v \in \mathbb{R}^+(U - x)$, $v \neq 0$, there exists $\lambda > 0$ and $y \in U$ such that $v = \lambda(y - x)$. Therefore,

$$\forall \theta,\ 0 \le \theta < \theta_0 \overset{\text{def}}{=} \min\{1, \lambda^{-1}\}, \quad x + \theta v \in U.$$

Define

$$\varphi(\theta) \overset{\text{def}}{=} \frac{f(x+\theta v) - f(x)}{\theta}, \quad 0 < \theta \le \theta_0, \tag{4.16}$$

and show that φ is monotone decreasing as $\theta \searrow 0$. For all θ_1, θ_2, $0 < \theta_1 < \theta_2 < \theta_0$,

$$x + \theta_1 v = \frac{\theta_1}{\theta_2}(x + \theta_2 v) + \left(1 - \frac{\theta_1}{\theta_2}\right) x \tag{4.17}$$

$$\begin{aligned} f(x+\theta_1 v) - f(x) &= f\left(\frac{\theta_1}{\theta_2}(x + \theta_2 v) + \left(1 - \frac{\theta_1}{\theta_2}\right) x\right) - f(x) \\ &\le \frac{\theta_1}{\theta_2} f(x+\theta_2 v) + \left(1 - \frac{\theta_1}{\theta_2}\right) f(x) - f(x) = \frac{\theta_1}{\theta_2}[f(x+\theta_2 v) - f(x)] \\ &\Rightarrow \varphi(\theta_1) \le \varphi(\theta_2). \end{aligned} \tag{4.18}$$

By monotonicity the following limit exists

$$\lim_{\theta \searrow 0} \frac{f(x+\theta v) - f(x)}{\theta} \le \varphi(\theta_0) < +\infty \tag{4.19}$$

and it can be $-\infty$. The convexity and positive homogeneity of $v \mapsto df(x; v)$ follow from the same arguments as in the proof of [1, Thm. 1, pp. 204]), but $df(x; v)$ can be $-\infty$.

(ii) If $U = \{x\}$ is a singleton, there is nothing to prove. If U is not a singleton, and f is Lipschitz at x with constant $c(x)$

$$-c(x)\|v\| \le \frac{f(x+\theta v) - f(x)}{\theta}$$

and from (4.9) in part (i) the limit $df(x; v)$ is finite. As a result, for each admissible semitrajectoriy h such that $h'(0^+) = v \in \mathbb{R}^+(U - x)$ and θ sufficiently small

$$\frac{f(h(\theta)) - f(x)}{\theta} = \frac{f(x + \theta h'(0^+)) - f(x)}{\theta} + \frac{f(h(\theta)) - f(x + \theta h'(0^+))}{\theta}.$$

The first term on the right-hand side has a finite limit and the second term goes to zero since

$$\left\| \frac{f(h(\theta)) - f(x + \theta h'(0^+))}{\theta} \right\| \le c(x) \left\| \frac{h(\theta) - x}{\theta} - h'(0^+) \right\| \to 0. \tag{4.20}$$

As a result for all $v \in \mathbb{R}^+(U - x)$

$$d_H f(x; v) \overset{\text{def}}{=} \lim_{\theta \searrow 0} \frac{f(h(\theta)) - f(x)}{\theta} = \lim_{\theta \searrow 0} \frac{f(x+\theta v) - f(x)}{\theta} = df(x; v)$$

and f is Hadamard semidifferentiable at x for all $v \in \mathbb{R}^+(U - x)$.

Given v_2 and v_1 in $\mathbb{R}^+(U-x)$ and admissible semitrajectories h_1 and h_2 such that $h_1'(0^+) = v_1$ and $h_2'(0^+) = v_2$, for $\theta > 0$ sufficiently small,

$$\begin{aligned}
&\left\| \frac{f(h_2(\theta)) - f(x)}{\theta} - \frac{f(h_1(\theta)) - f(x)}{\theta} \right\| \\
&\le c(x) \left(\|v_2 - v_2\| + \left\| \frac{h_2(\theta) - x}{\theta} - h_2'(0^+) \right\| + \left\| \frac{h_1(\theta) - x}{\theta} - h_1'(0^+) \right\| \right) \\
&\qquad \Rightarrow |d_H f(x; v_2) - d_H f(x; v_1)| \le c(x) \|v_2 - v_1\|.
\end{aligned}$$

So the function $v \mapsto d_H f(x;v) : \mathbb{R}^+(U-x) \to \mathbb{R}$ is Lipschitz continuous and hence, on its closure $T_x^\flat(U) = \overline{\mathbb{R}^+(U-x)}$. Given $v \in T_x^\flat(U)$, there exists a Cauchy sequence $\{v_n\} \subset \mathbb{R}^+(U-x)$ such that $v_n \to v$ in X. Therefore, $d_H f(x; v_n)$ is a Cauchy sequence in $\mathbb{R}$ which converges to a unique limit

$$L \stackrel{\text{def}}{=} \lim_{n\to\infty} d_H f(x; v_n) \text{ in } \mathbb{R}, \tag{4.21}$$

which is independent of the choice of the sequence $v_n \to v$. Now, let h be an admissible semitrajectory such that $h'(0^+) = v$,

$$\begin{aligned}
&\left\| \frac{f(h(\theta)) - f(x)}{\theta} - L \right\| \\
&\le \left\| \frac{f(h(\theta)) - f(x)}{\theta} - d_H f(x, v_n) \right\| + \|L - d_H f(x, v_n)\| \\
&\le \left\| \frac{f(h_n(\theta)) - f(x)}{\theta} - d_H f(x, v_n) \right\| + \left\| \frac{f(h(\theta)) - f(h_n(\theta))}{\theta} \right\| + \|L - d_H f(x, v_n)\| \\
&\le \left\| \frac{f(h_n(\theta)) - f(x)}{\theta} - d_H f(x, v_n) \right\| + \|L - d_H f(x, v_n)\| \\
&\quad + c(x) \left(\left\| \frac{h(\theta) - x}{\theta} - v \right\| + \left\| \frac{h_n(\theta) - x}{\theta} - v_n \right\| + \|v - v_n\| \right).
\end{aligned}$$

Letting θ go to zero

$$\limsup_{\theta \searrow 0} \left\| \frac{f(h(\theta)) - f(x)}{\theta} - L \right\| \le \|L - d_H f(x, v_n)\| + c(x) \|v - v_n\|.$$

But the right-hand side goes to zero as $n \to \infty$ and, by definition,

$$d_H f(x; v) \stackrel{\text{def}}{=} \lim_{\theta \searrow 0} \frac{f(h(\theta)) - f(x)}{\theta} = L. \tag{4.22}$$

Then $d_H f(x;v)$ exists for all $v \in T_x^\flat(U)$ and the function $v \mapsto d_H f(x;v) : T_x^\flat(U) \to \mathbb{R}$ is Lipschitz continuous with constant $c(x)$.

Since f is Lipchitz at $x \in U$, there exist $\eta > 0$ and $c(x) > 0$ such that

$$\forall y_1, y_2 \in B_{2\eta}(x) \cap U, \quad |f(y_2) - f(y_1)| \le c(x) \|y_2 - y_1\|. \tag{4.23}$$

For $y \in B_\eta(x) \cap U$ and $z_1, z_2 \in B_\eta(y) \cap U$, $z_1, z_2 \in B_{2\eta}(x) \cap U$. Then

$$\forall z_1, z_2 \in B_\eta(y) \cap U, \quad |f(z_2) - f(z_1)| \le c(x) \|z_2 - z_1\| \tag{4.24}$$

and $f : B_\eta(y) \cap U \to \mathbb{R}$ is Lipschitz with constant $c(x)$. So, by the previous arguments, f is Hadamard semidifferentiable at all $y \in B_\eta(x) \cap U$ and

$$\forall y \in B_\eta(x) \cap U,\, \forall v, w \in T_y(U), \quad |d_H f(y; w) - d_H f(y; v)| \le c(x)\|w - v\|.$$

Conversely, note that $d_H f(y; 0) = 0$, for all $y \in B_\eta(x) \cap U$. So, for all $y \in B_\eta(x) \cap U$ and $v \in T_y(U)$

$$|d_H f(y; v)| = |d_H f(y; v) - d_H f(y; 0)| \le c(x)\|v - 0\| = c(x)\|v\|.$$

By convexity, for all $y_1, y_2 \in B_\eta(x) \cap U$

$$f(y_2) - f(y_1) \ge d_H f(y_1; y_2 - y_1) \ge -c(x)\|y_2 - y_1\| \tag{4.25}$$

$$f(y_1) - f(y_2) \ge d_H f(y_2; y_1 - y_2) \ge -c(x)\|y_1 - y_2\| \tag{4.26}$$

$$\Rightarrow \forall y_1, y_2 \in B_\eta(x), \quad |f(y_2) - f(y_1)| \le c(x)\|y_2 - y_1\| \tag{4.27}$$

and f is Lipschitz at $x \in U$.

(iii) Given $x \in U$, let $v = \lambda(y - x) \in \mathbb{R}^+(U - x)$ and $\theta_0 = \min\{\lambda^{-1}, 1\}$ be as in part (i) where it was established that the limit

$$\lim_{t \searrow 0} \frac{f(x + tv) - f(x)}{t}$$

is finite or $-\infty$. By definition,

$$\begin{aligned} \lim_{t \searrow 0} \frac{f(x + tv) - f(x)}{t} &= \limsup_{t \searrow 0} \frac{f(x + tv) - f(x)}{t} \\ &\le \limsup_{\substack{y \to_U x \\ t \searrow 0}} \frac{f(y + tv) - f(y)}{t} = \overline{d}_C f(x; v). \end{aligned} \tag{4.28}$$

In the other direction, given μ such that $0 < 2\mu < \theta_0$, the function

$$(t, y) \mapsto \frac{f(y + tv) - f(y)}{t} \tag{4.29}$$

is continuous at (μ, x). Given $\varepsilon > 0$, there exists α, $0 < \alpha \le \mu$, such that for $t > 0$ such that for $|t - \mu| \le \alpha$ and $\|y - x\| \le \alpha$

$$\left| \frac{f(y + tv) - f(y)}{t} - \frac{f(x + \mu\, v) - f(x)}{\mu} \right| < \varepsilon. \tag{4.30}$$

In particular, for $|t - \mu| \le \alpha$, that is, $0 < \mu - \alpha \le t \le \mu + \alpha < \theta_0$, by monotonicity

$$\frac{f(y + \alpha v) - f(y)}{\alpha} \le \frac{f(y + (\mu + \alpha)v) - f(y)}{\mu + \alpha} \le \frac{f(x + \mu v) - f(x)}{\mu} + \varepsilon \tag{4.31}$$

$$\Rightarrow \sup_{\substack{\|y - x\| \le \alpha \\ y \in U}} \frac{f(y + \alpha v) - f(y)}{\alpha} \le \frac{f(x + \mu v) - f(x)}{\mu} + \varepsilon. \tag{4.32}$$

Always by monotonicity $(0 < \alpha < \mu)$ for $0 < \beta \le \alpha$

$$\sup_{\substack{\|y - x\| \le \alpha \\ y \in U \\ 0 < \beta \le \alpha}} \frac{f(y + \beta v) - f(y)}{\beta} \le \sup_{\substack{\|y - x\| \le \alpha \\ y \in U}} \frac{f(y + \alpha v) - f(y)}{\alpha} \le \frac{f(x + \mu v) - f(x)}{\mu} + \varepsilon.$$

Then,

$$\lim_{\alpha\searrow 0}\sup_{\substack{\|y-x\|\le\alpha\\ y\in U\\ 0<\beta\le\alpha}}\frac{f(y+\beta v)-f(y)}{\beta}\le\frac{f(x+\mu v)-f(x)}{\mu}+\varepsilon$$

$$\overline{d}_C f(x;v)\overset{\text{def}}{=}\limsup_{\substack{y\to x,\,y\in U\\ t\searrow 0}}\frac{f(y+tv)-f(y)}{t}\le\liminf_{\mu\searrow 0}\frac{f(x+\mu v)-f(x)}{\mu}+\varepsilon.$$

Letting ε go to zero and using inequality (4.28), we get

$$\lim_{t\searrow 0}\frac{f(x+tv)-f(x)}{t}\le\overline{d}_C f(x;v)\le\liminf_{t\searrow 0}\frac{f(x+tv)-f(x)}{t}. \tag{4.33}$$

So for all $x\in U$ and $v\in\mathbb{R}^+(U-x)$

$$\lim_{t\searrow 0}\frac{f(x+tv)-f(x)}{t}=\overline{d}_C f(x;v), \tag{4.34}$$

where the limit can be $-\infty$ at a point $x\in\partial U$.

(iv) For $x\in\operatorname{int}U$, the continuous convex function f is Lipschitz at x and the conclusion follows from parts (i) and (ii). For a Lipschitz continuous function f at $x\in\partial U$, the conclusion also follows from parts (i) and (ii). □

5. Missing assumption in [9, Thms. 4.1 and 4.5]

Martin Brokate[5] brought to my attention that there is a missing assumption in parts (iii) and (iv) of Theorems 4.1 and 4.5 of my recent paper [9].

Theorem 4.1 assumes that the space X is a normed vector space. However, in part (iii) the proof of the convexity of the function

$$x\mapsto f_A(x)\overset{\text{def}}{=}\frac{1}{2}\left[\|x\|^2-d_A(x)^2\right]:X\to\mathbb{R},\quad d_A(x)\overset{\text{def}}{=}\inf_{a\in A}\|x-a\|, \tag{5.1}$$

is only given for a space X whose norm arises from an inner product

$$\|x\|=\sqrt{x\cdot x}. \tag{5.2}$$

He points out that the convexity is not true for an arbitrary norm. He gives the following simple example in dimension $n=2$ for the norm

$$\|(x_1,x_2)\|\overset{\text{def}}{=}\max\{|x_1|,|x_2|\} \tag{5.3}$$

and the set $A=\{(1,0)\}$. Indeed,

$$f_A(x_1,x_2)=\frac{1}{2}\left[(\max\{|x_1|,|x_2|\})^2-(\max\{|x_1-1|,|x_2|\})^2\right]. \tag{5.4}$$

Along the line $L=\{(0,x_2):x_2\in\mathbb{R}\}$, the function

$$x_2\mapsto f_A(0,x_2)=\frac{1}{2}\left[|x_2|^2-\max\{1,|x_2|^2\}\right]=\frac{1}{2}\min\{|x_2|^2-1,0\} \tag{5.5}$$

is not convex. Yet, the Hadamard semidifferential exists.

The assumption that the norm arises from an inner product is also required in part (iv) of Theorem 4.1 since it uses part (iii).

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The same criticism applies to the proof that the function

$$f_{\partial A}(x) \stackrel{\text{def}}{=} \frac{1}{2}\left[\|x\|^2 - b_A(x)^2\right], \quad b_A(x) \stackrel{\text{def}}{=} d_A(x) - d_{X\backslash A}(x), \quad \partial A \neq \varnothing, \tag{5.6}$$

is convex in [9, Thms. 4.5 (iii) and (iv)], where the norm is also assumed to arise from an inner product.

Both corrections have no impact on the remainder of the paper [9]. For completeness, we give the correct versions below.

Theorem 5.1 ([9, Thm. 4,1])**.** *Let $A \neq \varnothing$ be a subset of a normed vector space X.*

(i) *For all $x, y \in X$*

$$|d_A(y) - d_A(x)| \leq \|y - x\|.$$

(ii) *For all $x \in X$, $d_A(x) = d_{\overline{A}}(x)$ and $\Pi_{\overline{A}}(x) = \Pi_A(x)$.*

(iii) *The norm $x \mapsto n(x) = \|x\|$ is convex, continuous, Hadamard semidifferentiable at every $x \in X$, and $v \mapsto d_H n(x; v)$ is sublinear and continuous.*

If the norm arises from an inner product ($n(x) = \sqrt{x \cdot x}$), the function

$$f_A(x) \stackrel{\text{def}}{=} \frac{1}{2}\left[\|x\|^2 - d_A(x)^2\right] \tag{5.7}$$

is convex, continuous, Hadamard semidifferentiable at every $x \in X$, and $v \mapsto d_H f_A(x; v)$ is sublinear and continuous. Moreover,

$$\forall v \in X, \quad d_H d_A^2(x; v) = x \cdot v - d_H f_A(x; v), \tag{5.8}$$

d_A^2 is Hadamard semidifferentiable, $v \mapsto d_H d_A^2(x; v)$ is suplinear, and for all $x \in \overline{A}$, d_A^2 is Hadamard differentiable and $d_H d_A^2(x; v) = 0$ for all $v \in X$.

(iv) *If the norm arises from an inner product, d_A is Hadamard semidifferentiable in $X \backslash \partial \overline{A}$, and*

$$\forall v \in X, \quad |d_H d_A(x; v)| \leq \|v\|.$$

Theorem 5.2 ([9, Thm. 4,5])**.** *Let A be a subset of a normed vector space X such that $\partial A \neq \varnothing$.*

(i) *b_A is well-defined, Lipschitz continuous, and*

$$\forall x, y \in X, \quad |b_A(y) - b_A(x)| \leq \|y - x\|. \tag{5.9}$$

(ii) *For $x \in \partial A$, $T^\flat_{\partial A}(x)$ is a closed cone at 0 and*

$$\begin{aligned} T^\flat_{\partial A}(x) &= \left\{ v \in X : \lim_{t \searrow 0} \frac{d_{\partial A}(x + tv)}{t} = 0 \right\} = \{v \in X : d_H d_{\partial A}(x; v) = 0\} \\ &= \left\{ v \in X : \lim_{t \searrow 0} \frac{b_A(x + tv)}{t} = 0 \right\} = \{v \in X : d_H b_A(x; v) = 0\}. \end{aligned} \tag{5.10}$$

(iii) *If the norm arises from an inner product ($n(x) = \sqrt{x \cdot x}$), the function*

$$f_{\partial A}(x) \stackrel{\text{def}}{=} \frac{1}{2}\left[\|x\|^2 - b_A(x)^2\right] \tag{5.11}$$

is convex, continuous, Hadamard semidifferentiable at every $x \in X$, and $v \mapsto d_H f_{\partial A}(x; v)$ is sublinear. Moreover, for all $v \in X$

$$d_H b_A^2(x; v) = x \cdot v - d_H f_{\partial A}(x; v), \tag{5.12}$$

b_A^2 is Hadamard semidifferentiable, $v \mapsto d_H b_A^2(x;v)$ is suplinear, and for all $x \in \partial A$, $b_A(x)$ is Hadamard differentiable and $d_H b_A^2(x;v) = 0$ for all $v \in X$.

(iv) *If the norm arises from an inner product, b_A is Hadamard semidifferentiable in $X \backslash \partial(\partial A)$, and*

$$\forall v \in X, \quad |d_H b_A(x;v)| \leq \|v\|. \tag{5.13}$$

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