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ON THE PSEUDO-MONOTONICITY OF TENSOR VARIATIONAL INEQUALITIES AND APPLICATIONS

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Abstract. The paper deals with the study of tensor variational inequalities under pseudo-monotonicity assumptions. Several preliminary results are proved in order to show some properties of their solution set. Moreover a supply chain network in which every firm produces several commodities and shares common storage sites or distributors is proposed. The noncooperative behaviour of the firms is expressed by a tensor variational inequality. Thus the existence of the equilibrium distribution is obtained.

1. Introduction

In the paper we analyze some properties of the solution set of tensor variational inequalities under pseudo-monotonicity assumptions. We prove some preliminary results and then we are able to establish, under suitable conditions, not only the existence of solutions but also some useful characterization. Our study is performed within the functional setting of a tensor space (instead of an Euclidean space) endowed with an inner product which makes it a Hilbert space. Another aim of the paper is to present an equilibrium model on a supply chain network. More precisely we consider an economic market in which each firm produces several commodities and shares common storage sites or distributors. Each firm follows a noncooperative behaviour trying to maximize its own profit when the optimal distribution pattern of the others is given. Therefore the equilibrium condition is given by an extension of the Nash principle and is characterized by a suitable tensor variational inequality. Thanks to the tensor variational formulation, existence results can be shown by using the theoretical theorems proved in the first part of the paper.

Tensor variational inequalities are a very useful and powerful tool to investigate and solve some equilibrium problems in economics and game theory. In the last years questions regarding solutions, ill-posedness, inverse problem and numerical methods for tensor variational inequalities have been studied and advances have been made in this direction. Existence results under different assumptions have been investigated in [3] and [4] (see also the reference therein). Furthermore the

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ill-posed problem for tensor variational inequalities, namely when the uniqueness of solution is not guaranteed, have been analyzed in [4]. In that paper a sequence of solutions to regularized tensor variational inequalities is defined and its convergence to a solution to the ill-posed inequality with minimal norm is proved. In [6] and [5] some numerical methods to compute solutions to tensor variational inequalities are presented and the convergence analysis is performed. Finally, in [1] inverse tensor variational inequalities are introduced and some well-posedness characterizations are obtained. Such inverse inequalities are useful to study the policymarker's point of view of the general oligopolistic market equilibrium problem. In the model taxes and incentives are imposed by the policymaker in order to regularize commodity exportations.

Among the many applications of tensor variational inequalities we find the model of a general oligopolistic market equilibrium problem which is the problem of finding a trade equilibrium in a supply-demand market between a finite number of spatially separated firms who produce several commodities and ship such commodities to some demand markets. Different and similar generalizations of this model and Nash equilibrium based model can be found in [3], [12], [11], [7] and [8], [9].

Here we are interested on the study of a general supply chain network game theory model with wage-response productivity since it was observed that recently some international companies are raising wages to attract workers and improves the productivity. This is confirmed by numerous statistical analysis in which raising wages may enhance labor productivity, including that in manufacturing (Karp [10]). Precisely, in our model, we investigate on the impacts of wage-responsive productivity of labor in supply chain networks on product consumer prices and profits of competing firms which produce several commodities and share parts of the supply chain network. The crucial point is to define for each path of the supply chain network a flow conservation equation following the formulation of the traffic equilibrium problem (see for instance [2]). Making use of the game theory, we present the equilibrium condition and express it as a tensor variational inequality. In [11], the author, for the first time incorporates the labor in supply chain networks using the game theory. She considers three different sets of constraints on labor bounds on supply chain network links; a bound on labor across a tier of supply chain links corresponding to production, transportation, etc., or a bound on labor availability in the supply chain network economy. Then, the same author in [12] improves the model assuming that each link productivity factor is an increasing function of the wage on the link (and not fixed) and the amount of labor available on each link is fixed. Moreover an upper bound on the wage on each link is given. In this paper, starting from this improvement, we want to include in the model that each firm produces several commodities and shares parts of the supply chain network with the other firms. Indeed it happens that companies from a specific sector produce not one but several products. Moreover some companies can decide to have common storage sites or distributors. Examples can be found in different industrial sectors: modern fashion factories share same distribution network or firms that sell their product online can share the same distributor/platform.

The paper is organized as follows. In Section 2, we present the setting in which our results are obtained. In particular we give the definition of the tensor Hibert space and some preliminary results very useful in the sequel. In Section 2, we prove the main results regarding the properties of the solution set of tensor variational inequalities with pseudo-monotone tensor fields. Precisely we deduce, under which assumptions, the existence of a solution is guaranteed. Section 4 deals with a general supply chain network game theory model with wage-response productivity and its tensor variational formulation. In Section 5 we illustrate the proposed model on an example and we compute the solution to the governing tensor variational inequality.

2. Preliminaries and notations

Let $N \geq 1$ and $m_1, m_2, \ldots, m_N$ be positive integers. A N-order $(m_1, \ldots, m_N)$-dimensional real tensor $\mathcal{X}$ is a multiple array in $\mathbb{R}^{m_1 \times m_2 \times \cdots \times m_N}$ which elements $x_{i_1,\ldots,i_m} \in \mathbb{R}$, for any $i_k = 1, \ldots, m_k$ and $k = 1, \ldots, N$. We denote by $\mathbb{R}^{[m_1,\ldots,m_N]}$ the set of N-order $(m_1, \ldots, m_N)$-dimensional real tensors. If $m_1 = \cdots = m_n = m$, then $\mathcal{X}$ is said to be an $N-$order $m-$dimensional real tensor and has m^N entries. Let us indicate the set of all the N-order m-dimensional tensors with $\mathbb{R}^{[N,m]}$. Tensors are denoted by an italic capital letter $\mathcal{X}, \mathcal{Y}, \ldots$. We remark that matrices, vectors and scalars are tensors of order two, one and zero, respectively.

We introduce the product between tensors $\langle \cdot, \cdot \rangle$ as follows

$$\langle \mathcal{X}, \mathcal{Y} \rangle = \sum_{i_1=1}^{m} \cdots \sum_{i_N=1}^{m} x_{i_1,\ldots,i_N} y_{i_1,\ldots,i_N}, \quad \forall \mathcal{X}, \mathcal{Y} \in \mathbb{R}^{[N,m]},$$

and observe that $(\mathbb{R}^{[N,m]}, \langle \cdot, \cdot \rangle)$ is a Hilbert space. Now we introduce the tensor variational inequality problem.

Definition 2.1. Let K be a nonempty closed convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a tensor mapping. The tensor variational inequality is the problem of finding $\mathcal{X} \in K$ such that:

$$\langle F(\mathcal{X}), \mathcal{Y} - \mathcal{X} \rangle \geq 0, \quad \forall \mathcal{Y} \in K. \tag{2.1}$$

The solution set of (2.1) is denoted by $\mathbf{Sol}(K, F)$. Let us introduce the maps $\mathbf{G}$ and $\mathbf{H}$ as follows

$$\mathbf{G}(\mathcal{X}) = \mathcal{X} - \Pi_K(\mathcal{X} - F(\mathcal{X})), \quad \forall \mathcal{X} \in K,$$

and

$$\mathbf{H}(\mathcal{X}) = F(\Pi_K(\mathcal{X})) + \mathcal{X} - \Pi_K(\mathcal{X}), \quad \forall \mathcal{X} \in \mathbb{R}^{[N,m]},$$

where Π_K is the projector operator into K. In the vectorial case, the maps $\mathbf{G}$ and $\mathbf{H}$ are known as the natural map and normal map of the pair (K, F), respectively.

We recall the following result which gives an equivalent nonsmooth equation formulation of the problem (2.1) (see [5]).

Proposition 2.2. *Let K be a nonempty closed convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a tensor mapping. It results that:*

$$\mathcal{X} \in \mathbf{Sol}(K, F) \;\Leftrightarrow\; \mathbf{G}(\mathcal{X}) = 0.$$

From Proposition 2.2 , we can derive an alternative nonsmooth equation formulation of tensor variational inequalities.

Proposition 2.3. *Let K be a nonempty closed convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a tensor mapping. A tensor $\mathcal{X}$ is a solution to* (2.1) *if and only if there exists a tensor $\mathcal{Z}$ such that $\mathcal{X} = \Pi_K(\mathcal{Z})$ and $\mathbf{H}(\mathcal{Z}) = 0$.*

Proof. Let us suppose that $\mathcal{X} \in \mathbf{Sol}(K, F)$. Hence $\mathcal{X} = \Pi_K(\mathcal{X} - F(\mathcal{X}))$ by Proposition 2.2. Let us consider $\mathcal{Z} = \mathcal{X} - F(\mathcal{X})$. It is easy to remark $\mathcal{X} = \Pi_K(\mathcal{Z})$ and $\mathbf{H}(\mathcal{Z}) = 0$.

Conversely let us assume that $\mathcal{X} = \Pi_K(\mathcal{Z})$ and $\mathbf{H}(\mathcal{Z}) = 0$, then

$$\mathcal{Z} = \mathcal{X} - F(\mathcal{X}) \quad \text{and} \quad \mathcal{X} = \Pi_K(\mathcal{X} - F(\mathcal{X})).$$

Making use of Proposition 2.2 we deduce that $\mathcal{X}$ is a solution to (2.1). □

Existence and uniqueness results for tensor variational inequalities were first proved in [3] and [4]. For the sake of completeness we recall them here.

Theorem 2.4. *Let K be a nonempty compact convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a continuous tensor mapping. Then the tensor variational inequality problem* (2.1) *admits at least one solution.*

Theorem 2.5. *Let K be a nonempty closed convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a continuous tensor mapping satisfying the coercivity condition*

$$\lim_{\|\mathcal{X}\| \to +\infty} \frac{\langle F(\mathcal{X}) - F(\mathcal{X}_0), \mathcal{X} - \mathcal{X}_0 \rangle}{\|\mathcal{X} - \mathcal{X}_0\|} = +\infty,$$

for some $\mathcal{X}_0 \in K$. Then the tensor variational inequality (2.1) *admits a solution.*

If the set K is unbounded, existence results can be proved adding monotonicity assumption on the function F.

Definition 2.6. Let K be a subset of $\mathbb{R}^{[N,m]}$. A tensor mapping $F : K \to \mathbb{R}^{[N,m]}$ is said to be

- monotone on K if, for every $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{[N,m]}$,
$$\langle F(\mathcal{X}) - F(\mathcal{Y}), \mathcal{X} - \mathcal{Y} \rangle \geq 0;$$
- strictly monotone on K if, for every $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{[N,m]}$, with $\mathcal{X} \neq \mathcal{Y}$,
$$\langle F(\mathcal{X}) - F(\mathcal{Y}), \mathcal{X} - \mathcal{Y} \rangle > 0;$$
- strongly monotone on K if, for every $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{[N,m]}$, there exists $\nu > 0$ such that
$$\langle F(\mathcal{X}) - F(\mathcal{Y}), \mathcal{X} - \mathcal{Y} \rangle \geq \nu \|\mathcal{X} - \mathcal{Y}\|^2.$$
- ξ-monotone on K for some $\xi > 1$ if, for every $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{[N,m]}$, there exists $\nu > 0$ such that
$$\langle F(\mathcal{X}) - F(\mathcal{Y}), \mathcal{X} - \mathcal{Y} \rangle \geq \nu \|\mathcal{X} - \mathcal{Y}\|^{\xi}. \tag{2.2}$$

The following result is valid (see [4]).

Theorem 2.7. *Let K be a nonempty closed convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a tensor mapping. The following statements hold:*

a) *if F is continuous and monotone, then $\mathbf{Sol}(F, K)$ is nonempty closed and convex;*

b) *if F is strictly monotone and there exists a solution to* (2.1)*, then it is unique;*
c) *if F is continuous and strongly monotone, then there exists a unique solution to* (2.1).

In the sequel, we establish the existence results for tensor variational inequalities under conditions based on the degree-theoretic approach. The Brouwer degree of a function is a topological concept that allows us to claim the existence of zeros of the function in a specified open bounded set (see [13]). It is natural to extend this concept to maps on the tensor space.

We establish a new existence result.

Theorem 2.8. *Let K be a nonempty closed convex subset of $\mathbb{R}^{[N,m]}$, D be a subset of K and $F : D \to \mathbb{R}^{[N,m]}$ be a continuous tensor mapping on the open set D. The following statements hold:*

(a) *if there exists a bounded open set U satisfying $\overline{U} \subseteq D$ and such that* $\deg(\mathbf{G}, U)$ *is well defined and nonzero, then* (2.1) *has a solution in U;*
(b) *if there exists a bounded open set U' such that* $\deg(\mathbf{H}, U')$ *is well defined and nonzero, then* (2.1) *has a solution $\mathcal{X}$ such that $\mathcal{X} - F(\mathcal{X}) \in U'$.*

Proof. Both statements follow immediately from the property of the degree. Indeed, if $\deg(\mathbf{G}, U)$ is well defined and nonzero, then $\mathbf{G}$ has a zero in U; but such a zero is also a solution to (2.1) by using Proposition 2.2. For what concerns the statement (b), $\mathbf{H}$ has a zero in U' which we denote by $\mathcal{Z}$. By Proposition 2.2, $\mathcal{X} = \Pi(\mathcal{Z}) \in \mathbf{Sol}(K, F)$. Since $0 = \mathbf{H}(\mathcal{Z}) = F(\mathcal{X}) + \mathcal{Z} - \mathcal{X}$, it follows that $\mathcal{X} - F(\mathcal{X}) \in U'$ as claimed. □

We present a natural generalization of the classic theorem in topology known as the Tietze-Urysohn Extension Theorem.

Lemma 2.9. *Let K be a nonempty closed subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a continuous tensor mapping. Then there exists a continuous extension $\overline{F} : \mathbb{R}^{[N,m]} \to \mathbb{R}^{[N,m]}$ such that $\overline{F}(\mathcal{X}) = F(\mathcal{X})$, for all $\mathcal{X} \in K$.*

Let us fix K a closed subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ a continuous tensor mapping. Let us remark that if $\overline{F} : \mathbb{R}^{[N,m]} \to \mathbb{R}^{[N,m]}$ denotes a continuous extension of F as stipulated by Lemma 2.9, then $\mathbf{Sol}(K, F) = \mathbf{Sol}(K, \overline{F})$. Based on this observation, we next give a very wide sufficient condition under which (2.1) has a solution.

Now we prove the following proposition.

Proposition 2.10. *Let K be a nonempty closed convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a continuous tensor mapping. Let us consider the following statements:*

(a) *there exists a vector $\mathcal{X}_{ref} \in K$ such that the set*

$$L_< = \{\mathcal{X} \in K : \langle F(\mathcal{X}), \mathcal{X} - \mathcal{X}_{ref} \rangle < 0\}$$

is bounded (possibly empty);

(b) *there exist a bounded open set Ω and a tensor $\mathcal{X}_{ref} \in K \cap \Omega$ such that*

$$\langle F(\mathcal{X}), \mathcal{X} - \mathcal{X}_{ref} \rangle \geq 0, \quad \forall \mathcal{X} \in K \cap \partial\Omega; \tag{2.3}$$

(c) (2.1) *has a solution.*

It results that $(a) \Rightarrow (b) \Rightarrow (c)$. *Moreover, if the set*

$$L_{\leq} = \{\mathcal{X} \in K : \langle F(\mathcal{X}), \mathcal{X} - \mathcal{X}_{ref} \rangle \leq 0\},$$

which is nonempty and larger than $L_<$, *is bounded then* $\mathbf{Sol}(K, F)$ *is nonempty and compact.*

Proof. Let us suppose that the statement (a) holds. Let Ω be a bounded open set containing $\{\mathcal{X}_{ref}\} \cup L_<$. Since Ω is open and contains $L_<$, we have $L_< \cap \partial\Omega = \emptyset$. As a result (2.3) holds and (b) follows.

Assume (b) holds. Let $\overline{F} : \mathbb{R}^{[N,m]} \to \mathbb{R}^{[N,m]}$ be the continuous extension of F as in Lemma 2.9. Since $\overline{F}$ and F are equal on K, it follows

$$\langle \overline{F}(\mathcal{X}), \mathcal{X} - \mathcal{X}_{ref} \rangle \geq 0, \quad \forall \mathcal{X} \in K \cap \partial\Omega.$$

For simplicity, we drop the bar on $\overline{F}$ and assume that F is a continuous mapping defined on the entire tensor space $\mathbb{R}^{[N,m]}$. In order to prove that $\mathbf{Sol}(K, F)$ is nonempty, we proceed by contradiction. Let us assume that the solution set is empty. Since the zeros (if any) of $\mathbf{G}$ coincide with the solutions to (2.1) and the latter problem has no solutions by assumption, we have $\mathbf{G}^{-1}(0) \cap \partial\Omega = \emptyset$. As a consequence, $\deg(\mathbf{G}, \Omega)$ is well defined. We claim that this degree is nonzero. Consider the homotopy:

$$\mathcal{G}(\mathcal{X}, t) = \mathcal{X} - \Pi_K(t(\mathcal{X} - F(\mathcal{X})) + (1-t)\mathcal{X}_{ref}), \quad \forall (\mathcal{X}, t) \in \overline{\Omega} \times [0, 1].$$

We deduce $\mathcal{G}(\mathcal{X}, 0) = \mathcal{X} - \mathcal{X}_{ref}$; since $\mathcal{X}_{ref} \in \Omega$, it results that $\deg(\mathcal{G}(\cdot, 0), \Omega)$ is well defined and equal to one. Furthermore, $\mathcal{G}(\mathcal{X}, 1) = \mathbf{G}(\mathcal{X})$. We now show that if $\mathcal{G}(\mathcal{X}, t) = 0$, for some $(\mathcal{X}, t) \in \overline{\Omega} \times (0, 1)$, then $\mathcal{X} \notin \partial\Omega$. Assume $\mathcal{G}(X, t) = 0$ for some $0 < t < 1$. Without loss of generality, we may assume $\mathcal{X} \neq \mathcal{X}_{ref}$. Since $\mathcal{G}(\mathcal{X}, t) = 0$, by the definition of $\mathcal{G}$, we have $\mathcal{X} \in K$ and

$$\langle \mathcal{X} - t(\mathcal{X} - F(\mathcal{X})) - (1-t)\mathcal{X}_{ref}, \mathcal{Y} - \mathcal{X} \rangle \geq 0, \quad \forall \mathcal{Y} \in K.$$

In particular, for $\mathcal{Y} = \mathcal{X}_{ref}$, we get

$$\langle tF(\mathcal{X}) + (1-t)(\mathcal{X} - \mathcal{X}_{ref}), \mathcal{X}_{ref} - \mathcal{X} \rangle \geq 0,$$

which implies

$$\langle F(\mathcal{X}), \mathcal{X}_{ref} - \mathcal{X} \rangle \geq \frac{1-t}{t} \|\mathcal{X} - \mathcal{X}_{ref}\|^2 > 0,$$

since $t \in (0, 1)$ and $\mathcal{X} \neq \mathcal{X}_{ref}$. Thus $\mathcal{X}$ does not belong to $\partial\Omega$. Consequently, by the homotopy invariance property of the degree, we deduce that

$$\deg(\mathbf{G}, \Omega) = \deg(\mathcal{G}(\cdot, 1), \Omega) = \deg(\mathcal{G}(\cdot, 0), \Omega) = 1.$$

Taking into account Theorem 2.8, we have that $\mathbf{Sol}(K, F)$ is nonempty. Consequently we obtain a contradiction. Hence we have shown that $(b) \Rightarrow (c)$. If the set $L_{\leq}$ is bounded, then $L_<$ is also bounded; hence $\mathbf{Sol}(K, F)$ is nonempty. Moreover, observing that $\mathbf{Sol}(K, F)$ is a subset of $L_{\leq}$, we deduce the compactness of $\mathbf{Sol}(K, F)$. □

Proposition 2.10 has some special cases. Let us start with the first fundamental consequence which improves Theorem 2.4.

Corollary 2.11. *Let K be a nonempty compact convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a continuous tensor mapping. Then $\mathbf{Sol}(K, F)$ is nonempty and compact.*

Proof. The set $L_{\leq}$ is trivially compact for every choice of $\mathcal{X}_{ref} \in K$. □

The next result is another consequence of Proposition 2.10, which holds for an unbounded set K.

Corollary 2.12. *Let K be a nonempty closed convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a continuous tensor mapping. If there exists a tensor $\mathcal{X}_{ref} \in K$ such that*

$$\langle F(\mathcal{X}), \mathcal{X} - \mathcal{X}_{ref} \rangle \geq 0, \quad \forall \mathcal{X} \in K,$$

then (2.1) *has a solution.*

Proof. The last assumption trivially implies that the set $L_{<}$ is empty. □

We are able to obtain the following result.

Theorem 2.13. *Let K be a nonempty closed convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a tensor mapping. The following statements hold:*

(a) *if F is continuous and ξ-monotone on K for some $\xi > 1$, then* (2.1) *has an unique solution $\mathcal{X}^*$;*

(b) *if F is defined, Lipschitz continuous, and ξ-monotone on a set $\Omega \supseteq K$ for some $\xi > 1$, then there exists a constant $\nu' > 0$ such that for every tensor $\mathcal{X} \in \Omega$,*

$$\|\mathcal{X} - \mathcal{X}^*\| \leq \nu' \|\mathbf{G}(\mathcal{X})\|^{\frac{1}{\xi - 1}},$$

where $\mathcal{X}^*$ is the unique solution to (2.1).

Proof. If F is continuous and ξ-monotone on K for some $\xi > 1$, the existence of a solution to (2.1) follows by using Proposition 2.10 and the observation noted before the statement of the proposition; the uniqueness of the solution follows from the statement (b) of Theorem 2.7. Thus the statement (a) is achieved.

In order to establish (b), let $\nu > 0$ be such that (2.2) holds. For a given tensor $\mathcal{X} \in \Omega$, we set $\mathcal{V} = \mathbf{G}(\mathcal{X})$. We have

$$\mathcal{X} - \mathcal{V} = \Pi_K(\mathcal{X} - F(\mathcal{X})).$$

By the tensor variational characterization of the projection, it results

$$\langle F(\mathcal{X}) - \mathcal{V}, \mathcal{Y} - \mathcal{X} + \mathcal{V} \rangle \geq 0, \quad \forall\, \mathcal{Y} \in K.$$

In particular, set $\mathcal{Y} = \mathcal{X}^*$, we obtain

$$\langle F(\mathcal{X}) - \mathcal{V}, \mathcal{X}^* - \mathcal{X} + \mathcal{V} \rangle \geq 0.$$

Since $\mathcal{X}^* \in \mathbf{Sol}(K, F)$ and $\mathcal{X} - \mathcal{V} \in K$, it follows

$$\langle F(\mathcal{X}^*), \mathcal{X} - \mathcal{V} - \mathcal{X}^* \rangle \geq 0.$$

Adding the two inequalities and rearranging terms, we deduce

$$\langle F(\mathcal{X}) - F(\mathcal{X}^*), \mathcal{X} - \mathcal{X}^* \rangle \leq \langle F(\mathcal{X}) - F(\mathcal{X}^*), \mathcal{V} \rangle.$$

By the ξ-monotonicity of F on Ω, the left-hand side is not smaller than $\nu\|\mathcal{X}-\mathcal{X}^*\|^\xi$ while the right-hand side is not greater than $L\|\mathcal{V}\|\|\mathcal{X}-\mathcal{X}^*\|$, where $L>0$ is a Lipschitz constant of F on Ω. Consequently, we get

$$\|\mathcal{X}-\mathcal{X}^*\| \leq (\nu^{-1}L)^{\frac{1}{\xi-1}}\|\mathcal{V}\|^{\frac{1}{\xi-1}}.$$

Hence the statement (b) holds, where $\nu' = (\nu^{-1}L)^{\frac{1}{\xi-1}}$. □

3. Existence results for pseudo-monotone tensor variational inequalities

We investigate on the existence of solutions for pseudo-monotone tensor variational inequalities. To this aim we present some definitions.

Definition 3.1. Let K be a nonempty subset of $\mathbb{R}^{[N,m]}$. A tensor mapping $F : K \to \mathbb{R}^{[N,m]}$ is said to be

(a) pseudo-monotone on K if, for all $\mathcal{X},\mathcal{Y} \in K$,

$$\langle F(\mathcal{Y}), \mathcal{X}-\mathcal{Y}\rangle \geq 0 \implies \langle F(\mathcal{X}), \mathcal{X}-\mathcal{Y}\rangle \geq 0;$$

(b) strictly pseudo-monotone on K if, for all $\mathcal{X},\mathcal{Y} \in K$, with $\mathcal{X} \neq \mathcal{Y}$,

$$\langle F(\mathcal{Y}), \mathcal{X}-\mathcal{Y}\rangle \geq 0 \implies \langle F(\mathcal{X}), \mathcal{X}-\mathcal{Y}\rangle > 0;$$

(b) strongly pseudo-monotone on K if there exists a constant $\nu > 0$ such that, for all $\mathcal{X},\mathcal{Y} \in K$,

$$\langle F(\mathcal{Y}), \mathcal{X}-\mathcal{Y}\rangle \geq 0 \implies \langle F(\mathcal{X}), \mathcal{X}-\mathcal{Y}\rangle \geq \nu\|\mathcal{X}-\mathcal{Y}\|.$$

Firstly we prove the following result.

Theorem 3.2. *Let K be a nonempty closed convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a tensor mapping. If F is strictly pseudo-monotone on K and there exists a solution to* (2.1)*, then it is unique.*

Proof. Let us suppose that $\mathcal{X} \neq \mathcal{X}'$ are two distinct solutions to (2.1). It results, for all $\mathcal{Y} \in K$,

$$\langle F(\mathcal{X}), \mathcal{Y}-\mathcal{X}\rangle \geq 0 \quad \text{and} \quad \langle F(\mathcal{X}'), \mathcal{Y}-\mathcal{X}'\rangle \geq 0.$$

Substituting $\mathcal{Y} = \mathcal{X}'$ into the first inequality and $\mathcal{Y} = \mathcal{X}$ in the second one, we deduce

$$\langle F(\mathcal{X}), \mathcal{X}'-\mathcal{X}\rangle \geq 0 \quad \text{and} \quad \langle F(\mathcal{X}'), \mathcal{X}-\mathcal{X}'\rangle \geq 0.$$

By the first inequality and the strict pseudo-monotonicity of F on K, we obtain

$$\langle F(\mathcal{X}'), \mathcal{X}'-\mathcal{X}\rangle > 0,$$

which is in contradiction with the second one. Hence the claim is achieved. □

The strict pseudo-monotonicity of F on K is in general not sufficient so that (2.1) has a solution (see some examples in the vectorial case). It is worth to show the following necessary and sufficient condition for a pseudo-monotone tensor variational inequality to have a solution.

Theorem 3.3. *Let K be a nonempty closed convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a continuous tensor mapping. Let us suppose that F is pseudo-monotone on K. Then the statements (a), (b) and (c) in Proposition 2.10 are equivalent.*

Proof. Taking into account Proposition 2.10, it suffices to prove that (c) implies (a). If (2.1) has a solution, we denote by $\mathcal{X}_{ref}$ such a solution. By the pseudo-monotonicity of F on K, it results

$$\langle F(\mathcal{Y}), \mathcal{Y} - \mathcal{X}_{ref} \rangle \geq 0, \quad \forall \mathcal{Y} \in K.$$

As a consequence the set $L_<$ is empty. □

The previous characterization allows us to establish the following existence result.

Theorem 3.4. *Let K be a nonempty closed convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a strongly pseudo-monotone and continuous tensor mapping. Then (2.1) has a unique solution.*

Proof. Let $\nu > 0$ be the modulus of the strong pseudo-monotonicity of the tensor mapping F. Let us observe that the statement (a) of Theorem 3.2 implies that (2.1) cannot have more than one solution.

Now we prove that $\mathbf{Sol}(K, F)$ is nonempty. To this aim, taking into account Theorem 3.3, it is sufficient to find a tensor $\mathcal{X}_{ref} \in K$ such that the set $L_\leq$ is bounded. Let us fix the tensor $\mathcal{X}_{ref} \in K$. For every $\mathcal{X} \in L_\leq$, it results $\langle F(\mathcal{X}), \mathcal{X}_{ref} - \mathcal{X} \rangle \geq 0$. By the strong pseudo-monotonicity of F we deduce

$$\langle F(\mathcal{X}_{ref}), \mathcal{X}_{ref} - \mathcal{X} \rangle \geq \nu \|\mathcal{X}_{ref} - \mathcal{X}\|^2.$$

Thus, by the Cauchy–Schwarz inequality, one has

$$\begin{aligned} \nu \|\mathcal{X}_{ref} - \mathcal{X}\|^2 &\leq \langle F(\mathcal{X}_{ref}), \mathcal{X}_{ref} - \mathcal{X} \rangle \\ &\leq \|F(\mathcal{X}_{ref})\| \|\mathcal{X}_{ref} - \mathcal{X}\|. \end{aligned}$$

As a consequence, we get

$$\|\mathcal{X}_{ref} - \mathcal{X}\| \leq \frac{\|F(\mathcal{X}_{ref})\|}{\nu}.$$

Therefore, we have that $L_\leq \subseteq \overline{B}\left(\mathcal{X}_{ref}, \frac{\|F(\mathcal{X}_{ref})\|}{\nu}\right)$, where $\overline{B}\left(\mathcal{X}_{ref}, \frac{\|F(\mathcal{X}_{ref})\|}{\nu}\right)$ is the closed ball with center $\mathcal{X}_{ref}$ and radius $\frac{\|F(\mathcal{X}_{ref})\|}{\nu}$. For the boundedness of $L_\leq$, we conclude that $\mathbf{Sol}(K, F)$ is nonempty. □

We introduce the recession cone of a nonempty closed convex subset K of $\mathbb{R}^{[N,m]}$ which is the maximal convex cone whose translate in every tensor of K lies in K:

$$K_\infty = \{\mathcal{X} \in K \,:\, \mathcal{X}_0 + \alpha \mathcal{X} \in K, \quad \forall \alpha \geq 0,\ \forall \mathcal{X}_0 \in K\},$$

and the orthogonal complement of K:

$$K^\perp = \{\mathcal{X} \in K \,:\, \langle \mathcal{X}, \mathcal{Y} \rangle = 0, \quad \forall \mathcal{Y} \in K\}.$$

Theorem 3.5. *Let K be a nonempty closed convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a continuous tensor mapping. Assume that F is pseudo-monotone on K. The following statements are equivalent:*

(a) $\mathbf{Sol}(K, F)$ *is convex;*

(b) *if there exists a tensor* $\mathcal{X}_{ref} \in K$ *such that* $F(\mathcal{X}_{ref}) \in \text{int}(K_\infty)^*$ *then* $\mathbf{Sol}(K, F)$ *is nonempty, convex and compact.*

Proof. Let F be a pseudo-monotone tensor mapping on K. We claim that

$$\mathbf{Sol}(K,F) = \bigcap_{\mathcal{Y}\in K} \{\mathcal{X} \in K : \langle F(\mathcal{Y}), \mathcal{Y} - \mathcal{X}\rangle \geq 0\}. \tag{3.1}$$

Indeed if $\mathcal{X} \in \mathbf{Sol}(K, F)$, we have

$$\langle F(\mathcal{X}), \mathcal{Y} - \mathcal{X}\rangle \geq 0, \quad \forall\, \mathcal{Y} \in K.$$

Making use of the pseudo-monotonicity of F on K, it follows

$$\langle F(\mathcal{Y}), \mathcal{Y} - \mathcal{X}\rangle \geq 0, \quad \forall\, \mathcal{Y} \in K.$$

Consequently $\mathcal{X}$ belongs to the right-hand set in (3.1). Vice versa, let us suppose that $\mathcal{X}$ belongs to $\bigcap_{\mathcal{Y}\in K} \{\mathcal{X} \in K : \langle F(\mathcal{Y}), \mathcal{Y} - \mathcal{X}\rangle \geq 0\}$. Let $\mathcal{Z} \in K$ be arbitrary. The tensor

$$\mathcal{Y} = \tau\mathcal{X} + (1-\tau)\mathcal{Z}, \quad \forall\, \tau \in [0,1],$$

belongs to K. Hence we obtain

$$\langle F(\tau\mathcal{X} + (1-\tau)\mathcal{Z}), \mathcal{Z} - \mathcal{X}\rangle \geq 0, \quad \forall \tau \in (0,1).$$

Passing to the limit as $\tau \to 1$, we get

$$\langle F(\mathcal{X}), \mathcal{Z} - \mathcal{X}\rangle \geq 0, \quad \forall \mathcal{Z} \in K,$$

thus $\mathcal{X} \in \mathbf{Sol}(K, F)$. Therefore, the identity (3.1) holds. Since for each fixed but arbitrary $\mathcal{Y} \in K$, the set

$$\{\mathcal{X} \in K : \langle F(\mathcal{Y}), \mathcal{Y} - \mathcal{X}\rangle \geq 0\}$$

is convex and the intersection of any number of convex sets is convex, then $\mathbf{Sol}(K, F)$ is convex. Then the statement (a) is achieved.

In order to prove the opposite implication, it suffices to show that if the tensor $\mathcal{X}_{ref}$ exists with the above property, then the set

$$L_{\leq} = \{\mathcal{X} \in K : \langle F(\mathcal{X}), \mathcal{X} - \mathcal{X}_{ref}\rangle \leq 0\}$$

is bounded. By the pseudo-monotonicity of F on K, we deduce

$$L_{\leq} \subseteq \{\mathcal{X} \in K : \langle F(\mathcal{X}_{ref}), \mathcal{X}_{ref} - \mathcal{X}\rangle \geq 0\}. \tag{3.2}$$

The set in the right-hand side is closed and convex. If it is unbounded then it must have a nonzero recession direction; namely there exists a nonzero tensor $\mathcal{V} \in K_\infty$ such that $\langle F(\mathcal{X}_{ref}), \mathcal{V}\rangle \leq 0$. Since $F(\mathcal{X}_{ref}) \in \text{int}(K_\infty)^*$, we obtain that $F(\mathcal{X}_{ref}) - \delta\mathcal{V} \in (K_\infty)^*$ for some scalar $\delta > 0$. Hence, it results

$$0 \leq \langle \mathcal{V}, F(\mathcal{X}_{ref}) - \delta\mathcal{V}\rangle \leq -\delta\langle \mathcal{V}, \mathcal{V}\rangle < 0.$$

Such a contradiction shows that the set on the right-hand side of (3.2) is bounded and hence $L_{\leq}$ is bounded too. □

Finally, the following result is valid.

Proposition 3.6. *Let K be a nonempty convex subset of $\mathbb{R}^{[N,m]}$ and $F : K \to \mathbb{R}^{[N,m]}$ be a pseudo-monotone tensor mapping. For any two solutions $\mathcal{X}_1$ and $\mathcal{X}_2$ to* (2.1)*, it results*

$$\langle F(\mathcal{X}_1), \mathcal{X}_1 - \mathcal{X}_2\rangle = \langle F(\mathcal{X}_2), \mathcal{X}_1 - \mathcal{X}_2\rangle = 0; \tag{3.3}$$

consequently,

$$\langle F(\mathcal{X}_1) - F(\mathcal{X}_2), \mathcal{X}_1 - \mathcal{X}_2\rangle = 0 \tag{3.4}$$

and $F(\mathbf{Sol}(K,F)) \subseteq (\mathbf{Sol}(K,F)_\infty)^\perp$.

Proof. Since $\mathcal{X}_1$ and $\mathcal{X}_2$ are both solutions to (2.1), we deduce

$$\langle F(\mathcal{X}_2), \mathcal{X}_1 - \mathcal{X}_2\rangle \geq 0 \quad \text{and} \quad \langle F(\mathcal{X}_1), \mathcal{X}_2 - \mathcal{X}_1\rangle \geq 0$$

By using the pseudo-monotonicity of F on K, the previous inequalities imply

$$\langle F(\mathcal{X}_1), \mathcal{X}_1 - \mathcal{X}_2\rangle \geq 0 \quad \text{and} \quad \langle F(\mathcal{X}_2), \mathcal{X}_2 - \mathcal{X}_1\rangle \geq 0,$$

respectively. As a consequence, the equalities (3.3) hold true. The equality (3.4) is a trivial consequence of (3.3). Finally, if $\mathcal{V} \in \mathbf{Sol}(K,F)_\infty$ and $\mathcal{X} \in \mathbf{Sol}(K,F)$, then we have $\mathcal{X} + \mathcal{V} \in \mathbf{Sol}(K,F)$. Then, it results $\langle F(\mathcal{X}), \mathcal{V}\rangle = 0$, which implies that $F(\mathbf{Sol}(K,F))$ is a subset of the orthogonal complement of $\mathbf{Sol}(K,F)_\infty$. □

4. A general supply chain network game theory model with wage-responsive productivity

In many economic models the labor to supply chain network activities from production to distribution plays an important role.

Here we extend the model in [12] introducing two main novelties which are specifically:

- each firm can produce not only one but several different type of commodities;
- in the model proposed by Nagurney in [12] the supply chain network of the firms had no links in common. In our supply chain network we remove this constraint, namely it can happen that different firms share storage sites or distributors.

The novelty of our model is to take into account these possibilities and to derive a tensor variational formulation of the Nash equilibrium condition.

The supply chain network, we consider, is made up of m firms F_i, $i = 1, \ldots, m$, and n demand markets D_j, $j = 1, \ldots, n$, which are generally spatially separated, as depicted in Figure 1. Each firm F_i produces l different commodities which are indicated by $k = 1, \ldots, l$. The firms compete in a noncooperative manner, namely each one seeking to maximize its profit when the optimal distribution pattern of the others is given. The global supply chain network is made by production links, transportation links, storage links and distribution links which, as said before, the firms can share some of them.

Let $G = [N, L]$ be the graph made up of the set of nodes N and the set of links L. The set of all the paths originating from the firm F_i and ending in a demand market D_j regarding the commodity k is denoted by P^k_{ij}, the set of all the paths from the firm F_i for the commodity k is denoted by P^k_i, the set of all the paths

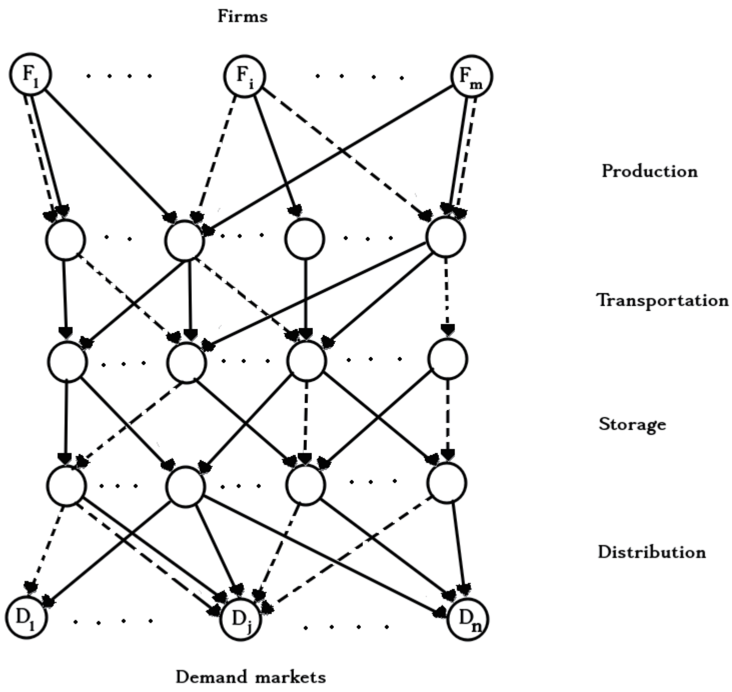

FIGURE 1. The Supply Chain Network Topology

from the firm F_i is denoted by P_i, while the set of all the paths is denoted by P. The number of paths in this sets are

$$|P_{ij}^k| = \overline{p}_{ij}^k, \qquad |P_i^k| = \overline{p}_i^k, \qquad |P_i| = \overline{p}_i, \qquad |P| = \overline{p},$$

respectively. Moreover we denote by L_i^k all the links contained in paths belonging to P_i^k, for $i = 1, \dots, m$ and $k = 1, \dots, l$, while by L all the links contained in paths belonging to P and assume that $|L| = r$.

The product flow of a path p connecting F_i to D_j of a commodity k is denoted by z_p^k, $p \in P_{ij}^k$. Every product path flows z_p^k, with $p \in P_{ij}^k$, for every $i = 1, \dots, m$, $j = 1, \dots, n$ and $k = 1, \dots, l$, must be nonnegative, i.e.

$$z_p^k \geq 0, \quad \forall p \in P_{ij}^k.$$

Let us consider x_{ij}^k the *total* product flow for the commodity k sent from the firm F_i to the demand market D_j, which is given by

$$x_{ij}^k = \sum_{p \in P_{ij}^k} z_p^k,$$

All the firm's product path flows are grouped in the tensor $\mathcal{X} = (x_{ij}^k)$.

The flow conservation laws and the equations relating labor on each link to the product output on the link are presented. To this aim, let d_j^k be the variable expressing the demand for the commodity k of the demand market D_j, $j = 1, \dots, n$, $k = 1, \dots, l$. The demand for each product must be satisfied at each demand market. Consequently the following feasible condition is satisfied:

$$d_j^k = \sum_{i=1}^{n} x_{ij}^k = \sum_{i=1}^{n} \sum_{p \in P_{ij}^k} z_p, \quad \forall j = 1, \dots, n, \ k = 1, \dots, l.$$

Thus the feasible set is given by

$$\mathbb{K} = \Big\{ \mathcal{X} \in \mathbb{R}^{[m,n,l]} : \quad x_{ij}^k \geq 0, \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \ \forall k = 1, \dots, l, \ d_j^k = \sum_{i=1}^{n} x_{ij}^k, \quad \forall j = 1, \dots, n, \ k = 1, \dots, l \Big\}.$$

Moreover we denote by

- $f_{a,i}^k$ the nonnegative flow of each firm P_i of the commodity k on link $a \in L$;
- $l_{a,i}^{k,fixed}$ the fixed amount of labor on link $a \in L$ for the commodity k;
- $w_{a,i}^k$ the wage for a unit of labor on link $a \in L$ for the commodity k per hour the cognizant firm is willing to pay;
- $\alpha_{a,i}^k w_{a,i}^k$ the productivity factor relating input of labor to output of product flow on link $a \in L$ for the commodity k, where $\alpha_{a,i}^k$ is given and positive and is refered to as the wage responsiveness productivity factor;
- $\hat{c}_a$ the total operational cost associated with link $a \in L$, which may depend upon the entire link flow pattern, namely $\hat{c}_a = \hat{c}_a(f)$, $f \in \mathbb{R}^{[n,r,l]}$;
- ρ_j^k the variable expressing the demand price for unity of the commodity k associated to the demand market D_j, $j = 1, \dots, n$, $k = 1, \dots, l$, which may depend upon the entire consumption pattern, namely $\rho_j^k = \rho_j^k(\mathcal{X})$, $\mathcal{X} \in \mathbb{R}^{[m,n,l]}$.

The link flows of each firm P_i, $i = 1, \dots, m$ and for every commodity k are related to the product path flows as follows:

$$f_{a,i}^k = \sum_{j=1}^{n} \sum_{p \in P_{ij}^k} z_p^k = \sum_{j=1}^{n} x_{ij}^k, \quad \forall a \in L_i^k. \tag{4.1}$$

All the link flows $f = (f_{a,i}^k)$ depend on the total product flow $\mathcal{X}$. Moreover we use the following equation to express that the greater is the value of the wage w_a on link a, the more productive is the labor on the link

$$f_{a,i}^k = \alpha_{a,i}^k w_{a,i}^k l_{a,i}^{k,fixed}, \quad \forall a \in L_i^k. \tag{4.2}$$

Now we introduce the utility function of each firm P_i, $i = 1, \dots, m$. The utility function u_i describes the profit of each firm F_i and it is given by the difference between the income (given by the sum of the incomes for every product) and the total operating costs which include fixed costs and all the wages paid for the labor. Hence, the utility u_i is given by

$$u_i = \sum_{k=1}^{l} \sum_{j=1}^{n} \rho_j^k(\mathcal{X}) x_{ij}^k - \sum_{k=1}^{l} \sum_{a \in L_i^k} \hat{c}_a(f) - \sum_{k=1}^{l} \sum_{a \in L_i^k} w_{a,i}^k l_{a,i}^{k,fixed}.$$

Since the total operating costs depend on the entire pattern $\mathcal{X}$ and, due to (4.1) and (4.2), we have

$$w_{a,i}^k l_{a,i}^{k,fixed} = \frac{f_{a,i}^k}{\alpha_{a,i}^k l_{a,i}^{k,fixed}} l_{a,i}^{k,fixed} = \frac{\sum_{j=1}^n x_{ij}^k}{\alpha_{a,i}^k}, \quad \forall a \in L_i^k$$

and then

$$u_i(\mathcal{X}) = \sum_{k=1}^{l} \sum_{j=1}^{n} \rho_j^k(\mathcal{X}) x_{ij}^k - \sum_{k=1}^{l} \sum_{a \in L_i^k} c_a(\mathcal{X}) - \sum_{k=1}^{l} \sum_{a \in L_i^k} \frac{\sum_{j=1}^n x_{ij}^k}{\alpha_{a,i}^k}.$$

Each firm follows a noncooperative behaviour trying to maximize its own profit function considering the optimal distribution pattern of the others. Therefore, the goal of the game is to determine a nonnegative tensor feasible commodity distribution $\mathcal{X}$ for which the m firms and the n demand markets will be in a state of equilibrium according to a general Nash equilibrium principle.

Definition 4.1. A feasible tensor $\mathcal{X}^* \in \mathbb{K}$ is a general supply chain network Nash equilibrium distribution if and only if, for each $i = 1, \dots, m$, it results

$$u_i(\mathcal{X}^*) \geq u_i(X_i, \mathcal{X}^*_{-i}), \tag{4.3}$$

where $\mathcal{X}^*_{-i} = (X_1^*, \dots, X_{i-1}^*, X_{i+1}^*, \dots, X_m^*)$ and X_i is a slice of $\mathcal{X}$ of dimension nl.

For technical reasons, we assume the following:

a) u_i is continuously differentiable, for each $i = 1, \dots, m$;

b) u_i is pseudoconcave[1] with respect to the variables X_i, for each $i = 1, \dots, m$.

Moreover, we indicate with

$$\nabla_D u = \left(\frac{\partial u_i}{\partial x_{ij}^k} \right), \quad i = 1, \dots, m, \ j = 1, \dots, n, \ k = 1, \dots, l.$$

We are able to show the tensor variational formulation of the equilibrium problem.

Theorem 4.2. *A tensor $\mathcal{X}^* \in \mathbb{K}$ is a general supply chain network Nash equilibrium distribution if and only if it is a solution to the following tensor variational inequality*

$$\begin{aligned} &\langle -\nabla_D u(\mathcal{X}^*), \mathcal{X} - \mathcal{X}^* \rangle \\ &= -\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} \frac{\partial u_i(\mathcal{X}^*)}{\partial x_{ij}^k} (x_{ij}^k - (x_{ij}^k)^*) \geq 0, \quad \forall \mathcal{X} \in \mathbb{K}. \end{aligned} \tag{4.4}$$

[1]The profit function $u_i(\mathcal{X})$ is pseudoconcave with respect to the variable $X_i \in \mathbb{R}^{nl}$ if and only if

$$\begin{aligned} &\left\langle \frac{\partial u_i}{\partial x_i}(X_1, \dots, X_i, \dots, X_m), X_i - Y_i \right\rangle \geq 0 \\ &\qquad \Rightarrow u_i(X_1, \dots, X_i, \dots, X_m) \geq u_i(X_1, \dots, Y_i, \dots, X_m). \end{aligned}$$

Proof. Let us start supposing that $\mathcal{X}^* \in \mathbb{K}$ is a solution to (4.4). We prove that it is a general supply chain network Nash equilibrium distribution. By contradiction, we suppose that there exists i^* such that

$$u_{i^*}(\mathcal{X}^*) < u_{i^*}(X_{i^*}, \mathcal{X}^*_{-i^*}).$$

By virtue of the pseudoconcavity of the profit function v_i with respect to the variable $X_i \in \mathbb{R}^{nl}$, for each $i = 1, \dots, m$, we deduce

$$\langle -\nabla_D u(\mathcal{X}^*), \mathcal{X} - \mathcal{X}^* \rangle < 0,$$

which is in contraction with (4.4). The opposite implication follows easily. □

Observing that the feasible set $\mathbb{K}$ is a nonempty closed convex subset of $\mathbb{R}^{[m,n,l]}$ and taking into account Theorem 3.4, the existence of a unique general supply chain network Nash equilibrium distribution is guaranteed assuming that the profit function u is strongly pseudo-monotone and continuous.

5. Numerical example

Let us now consider an economic supply chain network consisting of two firms and two demand markets. Each firm produces two different kind of commodities. The scheme of the production/storage/distribution network is depicted in Figure 2 (precisely dashed and continuous lines depict the two kinds of commodities): the firms share the storage center for the first kind of commodity and the production sites for the second kind of commodity and sell their products at two demand markets.

The total operational link cost functions are:

$$\begin{aligned}
&c_{a_1}(\mathcal{X}) = (x^1_{11})^2\\
&c_{a_2}(\mathcal{X}) = \tfrac{1}{2}(x^1_{12})^2,\\
&c_{b_1}(\mathcal{X}) = 2x^2_{11} - x^2_{22} + 2x^1_{12},\\
&c_{b_2}(\mathcal{X}) = \tfrac{3}{4}(x^1_{21})^2 + x^1_{22}x^1_{21} - 2x^1_{12}x^2_{22} + x^1_{22}x^1_{12} + 3x^2_{22},\\
&c_{c_1}(\mathcal{X}) = (x^2_{11})^2,\\
&c_{c_2}(\mathcal{X}) = x^2_{22},\\
&c_{d_1}(\mathcal{X}) = x^1_{21}x^1_{22} + \tfrac{1}{2}(x^2_{21})^2 + \tfrac{1}{2}x^2_{11}x^2_{21} + \tfrac{1}{2}x^1_{22},\\
&c_{d_2}(\mathcal{X}) = (x^1_{12})^2,\\
&c_{e_1}(\mathcal{X}) = (x^1_{21})^2 + 2(x^2_{22})^2 - 2x^2_{12}x^2_{22} + 2x^1_{12}x^2_{22},\\
&c_{e_2}(\mathcal{X}) = x^2_{11} - (x^2_{12})^2,\\
&c_{f_1}(\mathcal{X}) = 2x^1_{12}x^2_{12} - \tfrac{1}{2}(x^2_{12})^2.
\end{aligned}$$

The demand price functions are:

$$\begin{aligned}
&\rho^1_1(\mathcal{X}) = x^1_{11} + x^1_{21} - 1,\\
&\rho^2_1(\mathcal{X}) = \tfrac{1}{2}x^2_{11} + x^2_{21} + 3,\\
&\rho^1_2(\mathcal{X}) = x^1_{12} + x^1_{22},\\
&\rho^2_2(\mathcal{X}) = 2x^2_{12} + x^2_{22} + 1.
\end{aligned}$$

The $\alpha^k_{a,i}$ parameters, for $a \in L$ are as follows

$$
\begin{array}{llll}
\alpha^1_{a_1,1} = 3, & \alpha^2_{a_1,1} = 5, & \alpha^1_{b_2,2} = 15/2, & \alpha^2_{b_2,2} = 2, \\
\alpha^1_{b_1,1} = 10, & \alpha^2_{b_1,1} = 5, & \alpha^1_{d_1,2} = 10, & \alpha^2_{d_1,2} = 8, \\
\alpha^1_{a_2,1} = 3/2, & \alpha^2_{a_2,1} = 10, & \alpha^1_{c_2,2} = 20, & \alpha^2_{c_2,2} = 8, \\
\alpha^1_{c_2,1} = 10, & \alpha^2_{c_2,1} = 5, & \alpha^1_{e_1,2} = 60, & \alpha^2_{e_1,2} = 4, \\
\alpha^1_{c_1,1} = 10, & \alpha^2_{c_1,1} = 5/2, & \alpha^1_{d_2,2} = 20, & \alpha^2_{d_2,2} = 4, \\
\alpha^1_{f_1,1} = 3, & \alpha^2_{f_1,1} = 10/3, & \alpha^1_{e_2,2} = 15, & \alpha^2_{e_2,2} = 1, \\
\alpha^1_{d_2,1} = 10, & \alpha^2_{d_2,1} = 5/2, & \alpha^1_{c_1,2} = 20, & \alpha^2_{c_1,2} = 2, \\
\alpha^1_{e_2,1} = 3, & \alpha^2_{e_2,1} = 5, & \alpha^1_{f_1,2} = 30, & \alpha^2_{f_1,2} = 4.
\end{array}
$$

The paths in this example are

$$
\begin{array}{l}
p_1 = (a_1, b_1, c_1), \\
p_2 = (a_1, b_1, f_1), \\
p_3 = (d_1, e_1, c_1), \\
p_4 = (d_1, e_1, f_1), \\
p_5 = (a_2, c_2, d_2), \\
p_6 = (a_2, c_2, e_2), \\
p_7 = (b_2, c_2, d_2), \\
p_8 = (b_2, c_2, e_2).
\end{array}
$$

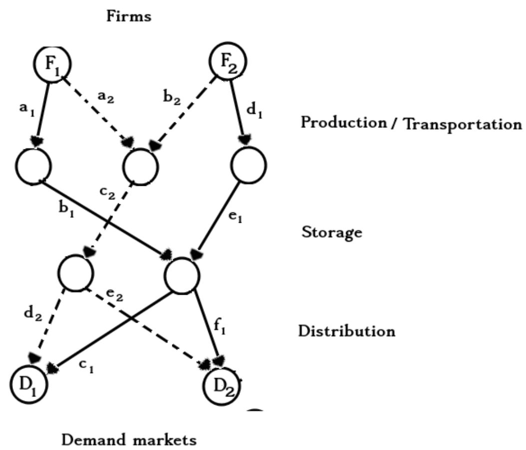

FIGURE 2. The Supply Chain Network Example.

So the utility functions become

$$
\begin{aligned}
u_1(\mathcal{X}) &= x^1_{11}x^1_{21} - 2x^1_{11} - \frac{1}{2}(x^1_{12})^2 + x^1_{12}x^1_{22} - 3x^1_{12} - \frac{1}{2}(x^2_{11})^2 \\
&\quad + x^2_{11}x^2_{21} - 2x^2_{11} + \frac{3}{2}(x^2_{12})^2 + x^2_{12}x^2_{22} - x^2_{12}, \\
u_2(\mathcal{X}) &= x^1_{11}x^1_{21} - \frac{3}{4}(x^1_{21})^2 - x^1_{21}x^1_{22} - \frac{3}{2}x^1_{21} + (x^1_{22})^2 - x^1_{22}x^1_{21} \\
&\quad + \frac{1}{2}(x^2_{21})^2 - x^2_{21} + 2x^2_{12}x^2_{22} - (x^2_{22})^2.
\end{aligned}
$$

Therefore $\nabla_D u$ is given by

$$\begin{aligned}
&\frac{\partial v_1}{x_{11}^1}(\mathcal{X}) = x_{21}^1 - 2, && \frac{\partial v_2}{x_{21}^1}(\mathcal{X}) = x_{11}^1 - \tfrac{3}{2}x_{21}^1 - x_{22}^1 - \tfrac{3}{2},\\
&\frac{\partial v_1}{x_{12}^1}(\mathcal{X}) = -x_{12}^1 + x_{22}^1 - 3, && \frac{\partial v_2}{x_{22}^1}(\mathcal{X}) = 2x_{22}^1 - x_{21}^1,\\
&\frac{\partial v_1}{x_{11}^2}(\mathcal{X}) = -x_{11}^2 + x_{21}^2 - 2, && \frac{\partial v_2}{x_{21}^2}(\mathcal{X}) = x_{21}^2 - 1,\\
&\frac{\partial v_1}{x_{12}^2}(\mathcal{X}) = 3x_{12}^2 + x_{22}^2 - 1, && \frac{\partial v_2}{x_{22}^2}(\mathcal{X}) = 2x_{12}^2 - 2x_{22}^2.
\end{aligned}$$

Taking into account Theorem 4.2, the general supply chain network Nash equilibrium distribution is a solution to the following tensor variational inequality

$$\langle -\nabla_D v(\mathcal{X}^*), \mathcal{X} - \mathcal{X}^* \rangle = -\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} \frac{\partial v_i(\mathcal{X}^*)}{\partial x_{ij}^k}(x_{ij}^k - (x_{ij}^k)^*) \geq 0, \quad \forall \mathcal{X} \in \mathbb{K}.$$

Making use of Corollary 2.1 in [6], the numerical equilibrium distribution can be found solving the following system

$$-\nabla_D v(\mathcal{X}) = 0_{\mathbb{R}^{[2,2,2]}}$$

and verifying that its solution belongs to the interior of the feasible set $\mathbb{K}$. Thus it results

$$\mathcal{X}_1^* = \begin{pmatrix} \frac{11}{2} & 1 \\ 2 & \frac{1}{4} \end{pmatrix}, \quad \mathcal{X}_2^* = \begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{4} \end{pmatrix}.$$

References

[1] F. Anceschi, A. Barbagallo and S. Guarino Lo Bianco, *Inverse tensor variational inequalities and applications*, J. Optim. Theory Appl. **196** (2023), 570–589.

[2] A. Barbagallo, *Degenerate time-dependent variational inequalities with applications to traffic equilibrium problems*, Comput. Methods Appl. Math. **6** (2006) 117–133.

[3] A. Barbagallo and S. Guarino Lo Bianco, *Variational inequalities on a class of structured tensors*, J. Nonlinear Convex Anal. **19** (2018), 711–729.

[4] A. Barbagallo and S. Guarino Lo Bianco, *On ill-posedness and stability of tensor variational inequalities: application to an economic equilibrium*, J. Global Optim. **77** (2020), 125–141.

[5] A. Barbagallo and S. Guarino Lo Bianco, *A new tensor projection method for tensor variational inequalities*, J. Nonlinear Var. Anal. **6** (2022), 213–226.

[6] A. Barbagallo, S. Guarino Lo Bianco and G. Toraldo, *Tensor variational inequalities: theoretical results, numerical methods and applications to an economic equilibrium model*, J. Nonlinear Var. Anal. **4** (2020), 87–105.

[7] A. Barbagallo and P. Mauro, *Inverse Variational Inequality Approach and Applications*, Numer. Funct. Anal. Optim. **35** (2014), 851–867.

[8] C. Chen and L. Zhang, *Finding Nash equilibrium for a class of multi-person noncooperative games via solving tensor complementarity problem*, Appl. Numer. Math. **145** (2019), 458–468.

[9] Z.H. Huang and L.Q. Qi, *Formulating an n-person noncooperative game as a tensor complementarity problem*, Comput. Optim. Appl. **66**, (2017) 557–576.

[10] E. Karp, *The case for raising wages in manufacturing*, Forbes, Match 2. (2021), available at: `https://www.forbes.com/sites/ethankarp/2021/03/02/the-case-for-raising-wages-in-manufacturing/?sh=565d8a755480`.

[11] A. Nagurney, *Supply chain game theory network modeling under labor constraints: Applications to the Covid-19 pandemic*, Eur. J. Oper. Res. **293** (2021), 880–891.

[12] A. Nagurney, *Supply chain networks, wages, and labor productivity: insights from Lagrange. analysis and computations.* J. Global Optim. **83** (2022), 615–638.
[13] M. Väth, *Topological Analysis: From the Basics to the Triple Degree for Nonlinear Fredholm Inclusions*, De Gruyter, Berlin, GE, 2012.

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