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PROXIMAL ITERATIVE ALGORITHM FOR FIXED POINT PROBLEMS OF TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS AND SYSTEM OF GENERALIZED NONLINEAR VARIATIONAL-LIKE INEQUALITIES

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Abstract. This paper aims at investigating the problem of finding a common element of the set of fixed points of a total asymptotically nonexpansive mapping and the set of solutions of a system of generalized nonlinear variational-like inequalities. With the goal of finding such an point, we apply the notion of P-η-proximal mapping and propose a proximal iterative algorithm. At the end of the paper, under suitable assumptions imposed on the parameters and mappings, the strong convergence of the sequence generated by our suggested iterative algorithm to a common element of the two sets mentioned above is proved.

1. Introduction

The theory of variational inequalities, which the starting point of its study dates back to around the middle of 60's with the pioneer work of Fichera [24] and Stampacchia [40], independently, has been widely studied and continues to be an active topic for research. This is mainly because of its extraordinary utility and broad applicability in many area of science, engineering, social science, and management, see, for example [4, 5, 9, 18, 23]. Due to the fact that many problems occurring in optimization, transportation, economics, elasticity and applied sciences can be formulated in the form of variational inequalities (see, for example, [25, 26]), the introduction of different generalizations of variational inequalities has received a great deal of interest from the mathematics community and many interesting extensions of them in different contexts have been appeared in the literature during the last decades.

With inspiration and motivation from the concept of invexity which was first introduced by Hanson in 1981 [29], Parida et al. [37] and Yang and Chen [44] proceeded, independently, to introduce an extension of variational inequalities that have been emerged in the literature entitled variational-like inequalities or pre-variational inequalities. It is significant to emphasize that there is a difference between the class of variational inequalities and the class of variational-like inequalities; indeed,

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in the formulation of variational inequalities the linear term $y - x$ has been replaced by a vector-valued term $\eta(y, x)$, where η is a vector-valued bifunction. In the last decades, there has been substantial progress made by authors in developing efficient and implementable iterative methods for solving a wide class of variant variational inequality problems, which for example one can refer to linear approximation method, descent and Newton's methods, inertial proximal-point method, projection method and its variant forms, fixed point iteration method, extragradient method, the method based on KKM technique, Wiener-Hopf technique and the auxiliary principle technique, Halpern's iterative method, and so forth. The presence of the vector-valued term $\eta(y, x)$ in the formulation of the variational-like inequalities makes us a restriction in using of most of the solution methods mentioned above. For solving different classes of variational-like inequalities, among these methods, the method based on the auxiliary principle technique and proximal-point method are the most popular ones. Indeed, due to the need to solve various kinds of variational-like inequalities in the framework of different spaces, the introduction of resolvent operators has attracted and continues to attract the interest of many authors. For instance, Ding and Luo [20] and Lee et al. [32] introduced, independently, the concepts of η-subdifferential and η-proximal point mappings of a proper functional and with the aid of them found the solutions of classes of variational-like inequalities in the setting of Hilbert space. Two years later, Ding and Xia [22] succeeded to introduce the notion of J-proximal mapping associated with a lower semicontinuous subdifferentiable proper (not necessarily convex) functional on reflexive Banach spaces. In another successfully attempt in this direction, other class of resolvent operators the so-called J^η-proximal (also referred to as P-η-proximal) mappings associated with a proper, lower semicontinuous and η-subdifferentiable (not necessarily convex) functional, which is essentially wider than the class of J-proximal mappings, was introduced by Ahmad et al. [2] and Kazmi and Bhat [30] independently. Under suitable hypotheses, the existence and Lipschitz continuity of such proximal mappings were proved. They employed the concept of J^η-proximal mapping and proposed some iterative algorithms for finding approximate solutions of classes of generalized multivalued nonlinear variational-like inequality problems in the Banach space setting. Besides, they studied the convergence analysis of the sequences generated by their iterative algorithms under some appropriate conditions.

Since the appearance of the theory of fixed points in the beginning of 20's with the admired Banach fixed point theorem, it has been revealed as a major, important and interesting tool in the study of nonlinear phenomena and can be applied in various disciplines of mathematics and mathematical sciences like economics, optimization theory, approximation theory, etc. Because of the existence of a strong connection between solving various kinds of mathematical problems and fixed point problems, there is no doubt that nowadays fixed point theory is one of the most powerful tools of modern mathematics. Indeed, in a wide range of mathematical problems the existence of a solution is equivalent to the existence of a fixed point for a suitable mapping. This is the reason why this topic has grown very rapidly and has influenced some other branches of mathematics such as differential equations, topology, game theory, optimal control, functional analysis, and so forth. In view of the fact that the variational inequality problems and the fixed point problems are very closely related, the investigation on the problem of finding a common point that lies in the solution set of a variational inequality problem and the set of fixed points of a given mapping is being the focus of attention of researchers in recent years. We refer the reader to [1, 3, 7, 8, 11–13, 35, 36, 41, 43, 46, 47] for more details and further information.

On the other hand, the study of nonexpansive mapping, which its origin dates back to the sixties, is a very interesting research area in fixed point theory. It is worthwhile to stress that the study of monotone and accretive operators, two classes of operators which arise naturally in the theory of differential equations has led to the emergence of nonexpansive mappings. In recent decades, the important role and many diverse applications of nonexpansive mapping in the theory of fixed points was a strong motivation for many researchers to extend and generalize it in different contexts. For instance, in 1972, Goebel and Kirk [27] introduced a class of generalized nonexpansive mappings the so-called asymptotically nonexpansive mappings. Another successfully effort was made by Sahu [38] in 2005 and the class of nearly asymptotically nonexpansive mappings as a generalization of asymptotically nonexpansive mappings was introduced. The attempts have been continued and one year later another important class of generalized nonexpansive mappings the so-called total asymptotically nonexpansive mappings, which is essentially broader than the classes of nearly asymptotically nonexpansive mappings and asymptotically nonexpansive mappings, was introduced by Alber et al. [6]. They studied the approximation methods of fixed points for this kind of mappings. Further generalizations of nonexpansive mappings along with relevant commentaries can be found in [6, 10, 14, 16, 27, 38, 39, 42] and the references contained therein.

Motivated by these advances, this paper is devoted to the investigation of the problem of finding a point in the intersection of the solutions set of a system of generalized nonlinear variational-like inequalities (for short, SGNVLI) and the fixed points set of a total asymptotically nonexpansive mapping. Under suitable hypotheses imposed on the parameters and mappings, the existence of a unique solution for the SGNVLI is proved and using the concept of P-η-proximal mapping, a proximal iterative algorithm is constructed. Finally, the strong convergence of the sequence generated by our proposed iterative algorithm to a common element of the set of fixed points of the total asymptotically nonexpansive mapping and the set of solutions of the SGNVLI is demonstrated.

2. Notation, basic definitions and fundamental properties

Let X be a real Banach space with its dual space X^* and $\langle .,. \rangle$ be the dual pairing between X and X^*. With slight abuse of notation we will use the same symbol $\|.\|$ for the norm in X and X^*. As usual, w^* will stand for the weak star topology in X^*.

For any given function $f : X \to \mathbb{R} \cup \{\pm\infty\}$, $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ is called the *effective domain* of f. Such a function is said to be *proper* if its effective domain is nonempty and it is real-valued on its effective domain, what is equivalent, f is *proper* if $f(x) > -\infty$ for all $x \in X$ and $f(x) < +\infty$ for at least one $x \in X$.

We say that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is *lower semicontinuous* at $x_0 \in X$, provided that $f(x_0) \leq \liminf_n f(x_n)$, for every sequence $\{x_n\} \subset X$ satisfying

$\lim_n x_n = x_0$. If the property holds for every point $x_0 \in X$ we say that f is *lower semicontinuous* on X.

Definition 2.1. The function $f : X \times X \to \mathbb{R} \cup \{+\infty\}$ is called lower semicontinuous in the second argument on X if for each $x \in X$, the function $f(x, .) : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous on X.

Similarly, one can define the lower semicontinuity of the function f in the first argument.

Recall that the function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *convex* if the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

holds for every $\lambda \in [0, 1]$ and all $x, y \in X$, for which the right-hand side is meaningful.

Definition 2.2 ([48])**.** An extended real-valued functional $f : (x, y) \in X \times X \to f(x, y) \in \mathbb{R} \cup \{\pm\infty\}$ is said to be 0-diagonally quasi-concave (in short, 0-DQCV)

(i) in the first argument (or with respect to x), if for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of X and any $\hat{x} \in \mathrm{Co}(\{x_1, x_2, \ldots, x_n\})$, we have

$$\min_{1\leq i\leq n} f(x_i, \hat{x}) \leq 0,$$

where for any given set $A \subset X$, $\mathrm{Co}(A)$ denotes the closed convex hull of A consisting of all vectors of the form $\sum_{i=1}^{n} \lambda_i u_i$ with $u_i \in A_i$, $\lambda_i \in \mathbb{R}_+ = [0, +\infty)$ and $\sum_{i=1}^{n} \lambda_i = 1$;

(ii) in the second argument (or with respect to y), if for any finite subset $\{y_1, y_2, \ldots, y_n\}$ of X and any $\hat{y} \in \mathrm{Co}(\{y_1, y_2, \ldots, y_n\})$, we have

$$\min_{1\leq i\leq n} f(\hat{y}, y_i) \leq 0.$$

Lemma 2.3 ([21])**.** *Let D be a nonempty convex subset of a topological vector space and let $f : D \times D \to \mathbb{R} \cup \{\pm\infty\}$ be an extended real-valued functional such that*

(i) *f is lower semicontinuous in the second argument on every nonempty compact subset of D;*

(ii) *f is 0-DQCV in the first argument;*

(iii) *there exists a nonempty compact convex subset D_0 of D and a nonempty compact subset K of D such that for each $y \in D \backslash K$, there is an $x \in \mathrm{Co}(D_0 \cup \{y\})$ satisfying $f(x, y) > 0$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in D$.

In 2000, Lee et al. [32] and Ding and Luo [20] independently introduced the notion of η-subdifferential in a more general setting than that given in [45] as follows.

Definition 2.4 ([20, 32])**.** For a given vector-valued mapping $\eta : X \times X \to X$, the proper functional $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is said to be η-subdifferentiable at a point $x \in X$ if there exists a point $x^* \in X^*$ such that

$$\langle x^*, \eta(y, x)\rangle \leq \phi(y) - \phi(x), \quad \forall y \in X.$$

Such a point x^* is called η-subgradient of ϕ at x. The set of all η-subgradients of ϕ at x is denoted by $\partial_\eta\phi(x)$. We can associate with each ϕ the η-subdifferential mapping $\partial_\eta\phi$ defined by

$$\partial_\eta\phi(x) = \begin{cases} \{x^* \in X^* : \langle x^*, \eta(y,x)\rangle \leq \phi(y) - \phi(x), \forall y \in X\}, & x \in \operatorname{dom}\phi, \\ \emptyset, & x \notin \operatorname{dom}\phi. \end{cases}$$

For $x \in \operatorname{dom}\phi$, $\partial_\eta\phi(x)$ is called the η-subdifferential of ϕ at x.

It is significant to emphasize that in the definition of η-subdifferential in the sense of Yang and Craven [45], the function ϕ needs to be local Lipschitz and cannot take the value $+\infty$. The following new example illustrates that the concept of η-subdifferential introduced in [20, 32] is more general than that given in [45].

Example 2.5. Let X be the set of all real numbers endowed with the Euclidean norm $\|.\| = |.|$ and let the mappings $\phi : X \to \mathbb{R} \cup \{+\infty\}$ and $\eta : X \times X \to X$ be defined, respectively, by

$$\phi(x) = \begin{cases} \lambda\big(x^p|x| + \sum_{n=1}^{\frac{k-1}{2}} a_n \sqrt[2n+1]{x|x|}\big) + \theta, & x \leq 0, \\ +\infty, & x > 0, \end{cases}$$

and

$$\eta(x,y) = \varsigma\big(x^p|x| + \sum_{n=1}^{\frac{k-1}{2}} a_n \sqrt[2n+1]{x|x|}\big) + \zeta\big(y^p|y| + \sum_{n=1}^{\frac{k-1}{2}} a_n \sqrt[2n+1]{y|y|}\big), \ \forall x, y \in X,$$

where k and p are arbitrary but fixed odd natural numbers, and $\lambda, \varsigma, \xi, a_n > 0$ ($n = 1, 2, \ldots, \frac{k-1}{2}$) and $\theta \in \mathbb{R}$ are arbitrary constants. We now show that for given $x \in \operatorname{dom}\phi$, $\partial_\eta\phi(x) = [\frac{\lambda}{\varsigma}, +\infty)$. For this end, take $x \in \operatorname{dom}\phi$ arbitrarily but fixed. Then, we have

$$\phi(x) = \lambda\big(x^p|x| + \sum_{n=1}^{\frac{k-1}{2}} a_n \sqrt[2n+1]{x|x|}\big) + \theta \quad \text{and} \quad x \leq 0.$$

If $\omega \in \partial_\eta\phi(x)$, then we get

$$\begin{aligned} &\omega\Big(\varsigma\big(y^p|y| + \sum_{n=1}^{\frac{k-1}{2}} a_n \sqrt[2n+1]{y|y|}\big) + \zeta\big(x^p|x| + \sum_{n=1}^{\frac{k-1}{2}} a_n \sqrt[2n+1]{x|x|}\big)\Big) \\ &\leq \phi(y) - \lambda\big(x^p|x| + \sum_{n=1}^{\frac{k-1}{2}} a_n \sqrt[2n+1]{x|x|}\big) - \theta, \quad \forall y \in X. \end{aligned}$$

Owing to the fact that $\phi(y) = +\infty$ for all $y > 0$, we deduce that

$$\begin{aligned} &\omega\Big(\varsigma\big(y^p|y| + \sum_{n=1}^{\frac{k-1}{2}} a_n \sqrt[2n+1]{y|y|}\big) + \zeta\big(x^p|x| + \sum_{n=1}^{\frac{k-1}{2}} a_n \sqrt[2n+1]{x|x|}\big)\Big) \\ &\leq \lambda\Big(y^p|y| - x^p|x| + \sum_{n=1}^{\frac{k-1}{2}} a_n\big(\sqrt[2n+1]{y|y|} - \sqrt[2n+1]{x|x|}\big)\Big), \quad \forall y \leq 0. \end{aligned} \tag{2.1}$$

If $x=0$, then utilizing (2.1), it follows that

$$\omega\varsigma\big(y^p|y|+\sum_{n=1}^{\frac{k-1}{2}}a_n\sqrt[2n+1]{y|y|}\big)\leq\lambda\big(y^p|y|+\sum_{n=1}^{\frac{k-1}{2}}a_n\sqrt[2n+1]{y|y|}\big),\quad\forall y\leq 0,$$

from which yields $\omega\geq\frac{\lambda}{\varsigma}$. For the case when $x<0$, relying on the fact that

$$\varsigma\big(y^p|y|+\sum_{n=1}^{\frac{k-1}{2}}a_n\sqrt[2n+1]{y|y|}\big)+\zeta\big(x^p|x|+\sum_{n=1}^{\frac{k-1}{2}}a_n\sqrt[2n+1]{x|x|}\big)<0,$$

recalling (2.1), we derive that for all $y\leq 0$,

$$\omega\geq\frac{\lambda\big(y^p|y|-x^p|x|+\sum_{n=1}^{\frac{k-1}{2}}a_n\big(\sqrt[2n+1]{y|y|}-\sqrt[2n+1]{x|x|}\big)\big)}{\varsigma\big(y^p|y|+\sum_{n=1}^{\frac{k-1}{2}}a_n\sqrt[2n+1]{y|y|}\big)+\zeta\big(x^p|x|+\sum_{n=1}^{\frac{k-1}{2}}a_n\sqrt[2n+1]{x|x|}\big)}.\tag{2.2}$$

Passing to the limit for $y\to-\infty$ in (2.2), it follows that $\omega\geq\frac{\lambda}{\varsigma}$. Thus, in any case, we note that $\omega\geq\frac{\lambda}{\varsigma}$ and so $\partial_\eta\phi(x)\subseteq[\frac{\lambda}{\varsigma},+\infty)$ for all $x\leq 0$. To prove $\partial_\eta\phi(x)=[\frac{\lambda}{\varsigma},+\infty)$, it is sufficient to show that $[\frac{\lambda}{\varsigma},+\infty)\subseteq\partial_\eta\phi(x)$ for all $x\leq 0$. Take $\omega\in[\frac{\lambda}{\varsigma},+\infty)$ arbitrarily but fixed and assume, on the contrary, that $\omega\notin\partial_\eta\phi(x_0)$ for some $x_0\leq 0$. Then there exists $y_0\leq 0$ such that

$$\begin{aligned}&\omega\big(\varsigma\big(y_0^p|y_0|+\sum_{n=1}^{\frac{k-1}{2}}a_n\sqrt[2n+1]{y_0|y_0|}\big)+\zeta\big(x_0^p|x_0|+\sum_{n=1}^{\frac{k-1}{2}}a_n\sqrt[2n+1]{x_0|x_0|}\big)\big)\\&>\lambda\big(y_0^p|y_0|-x_0^p|x_0|+\sum_{n=1}^{\frac{k-1}{2}}a_n\big(\sqrt[2n+1]{y_0|y_0|}-\sqrt[2n+1]{x_0|x_0|}\big)\big).\end{aligned}\tag{2.3}$$

Evidently, the case where $x_0=y_0=0$ cannot happen. If $x_0=0$ and $y_0<0$, then making use of (2.3) we conclude that $\omega<\frac{\lambda}{\varsigma}$, which is a contradiction. For the case when $x_0<0$ and $y_0=0$, then utilizing (2.3) it follows that $\omega<-\frac{\lambda}{\zeta}$, which leads to a contradiction. Finally, if $x_0,y_0<0$, then using (2.3) and in virtue of the fact that

$$\varsigma\big(y_0^p|y_0|+\sum_{n=1}^{\frac{k-1}{2}}a_n\sqrt[2n+1]{y_0|y_0|}\big)+\zeta\big(x_0^p|x_0|+\sum_{n=1}^{\frac{k-1}{2}}a_n\sqrt[2n+1]{x_0|x_0|}\big)<0,$$

we obtain

$$\frac{\lambda}{\varsigma}\leq\omega<\frac{\lambda\big(y_0^p|y_0|-x_0^p|x_0|+\sum_{n=1}^{\frac{k-1}{2}}a_n\big(\sqrt[2n+1]{y_0|y_0|}-\sqrt[2n+1]{x_0|x_0|}\big)\big)}{\varsigma\big(y_0^p|y_0|+\sum_{n=1}^{\frac{k-1}{2}}a_n\sqrt[2n+1]{y_0|y_0|}\big)+\zeta\big(x_0^p|x_0|+\sum_{n=1}^{\frac{k-1}{2}}a_n\sqrt[2n+1]{x_0|x_0|}\big)},$$

which implies that

$$\lambda(\zeta+\varsigma)\big(x_0^p|x_0|+\sum_{n=1}^{\frac{k-1}{2}}a_n\sqrt[2n+1]{x_0|x_0|}\big)>0.\tag{2.4}$$

Since $\lambda,\varsigma,\zeta>0$, and k,p are odd natural numbers, from (2.4) it follows that $x_0>0$ which is also a contradiction. In the light of the arguments mentioned above, we infer that $[\frac{\lambda}{\varsigma}, +\infty) \subseteq \partial_\eta\phi(x)$, for all $x \leq 0$. Therefore, $\partial_\eta\phi(x) = [\frac{\lambda}{\varsigma}, +\infty)$ for all $x \leq 0$.

Definition 2.6 ([2,30])**.** Let $\eta : X \times X \to X$ be a vector-valued mapping, $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be an η-subdifferentiable (not necessarily convex) proper functional and $P : X \to X^*$ be a single-valued mapping. If for any given point $x^* \in X^*$ and $\rho > 0$, there exists a unique point $x \in X$ satisfying

$$\langle P(x) - x^*, \eta(y, x)\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \quad \forall y \in X,$$

then the mapping $x^* \to x$, denoted by $R^{\partial_\eta\phi}_{\rho,P}$, is called P-η-proximal mapping of ϕ. Clearly, in the light of Definition 2.4, we have $x^* - P(x) \in \rho\partial_\eta\phi(x)$ and then it follows that

$$x = R^{\partial_\eta\phi}_{\rho,P}(x^*) = (P + \rho\partial_\eta\phi)^{-1}(x^*).$$

Definition 2.7. Let $P : X \to X^*$ and $\eta : X \times X \to X$ be the mappings. We say that the mapping P is

(i) k-strongly η-monotone if there exists a constant $k > 0$ such that

$$\langle P(x) - P(y), \eta(x, y)\rangle \geq k\|x - y\|^2, \quad \forall x, y \in X;$$

(ii) μ-Lipschitz continuous if there exists a constant $\mu > 0$ such that

$$\|P(x) - P(y)\| \leq \mu\|x - y\|, \quad \forall x, y \in X.$$

Definition 2.8. The vector-valued mapping $\eta : X \times X \to X$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that $\|\eta(x, y)\| \leq \tau\|x - y\|$, for all $x, y \in X$.

In view of the aforementioned arguments, a natural question to ask is whether for given mappings $\eta : X \times X \to X$ and $P : X \to X^*$, an η-subdifferentiable (not necessarily convex) proper functional $\phi : X \to \mathbb{R} \cup \{+\infty\}$ and an arbitrary real constant $\rho > 0$, the P-η-proximal mapping associated with the mappings P, η, ϕ and the constant ρ is well defined necessarily? Answering this question is not a trivial matter. The answer provided by Ahmad et al. [2] and Kazmi et al. [30] reads as follows.

Theorem 2.9 ([2,30])**.** *Let X be a reflexive Banach space, $\eta : X \times X \to X$ be a τ-Lipschitz continuous mapping such that $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in X$, and $P : X \to X^*$ be a γ-strongly η-monotone continuous mapping. Suppose that for any given $x^* \in X^*$, the function $h : (y, x) \in X \times X \to h(y, x) = \langle x^* - P(x), \eta(y, x)\rangle \in \mathbb{R} \cup \{+\infty\}$ is* 0-DQCV *in the first argument. Furthermore, let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous η-subdifferentiable proper functional on X, which may not be convex. Then for any given $\rho > 0$ and $x^* \in X^*$, there exists a unique point $x \in X$ such that*

$$\langle P(x) - x^*, \eta(y, x)\rangle \geq \rho\phi(x) - \rho\phi(y), \quad \forall y \in X,$$

that is, $x = R^{\partial_\eta\phi}_{\rho,P}(x^*)$ and so the P-η-proximal mapping associated with ϕ, P, η and $\rho > 0$ is well defined.

By a careful reading the proof of Theorem 3.1 in [30], we found that there are some errors in its proof and context. Firstly, in the light of the proof of [30, Theorem 3.1] and comparing it with the hypotheses appeared in its context, we inferred that the mapping η must be τ-Lipschitz continuous. In fact, the continuity condition of the mapping η in the context of [30, Theorem 3.1] must be replaced by the τ-Lipschitz continuity condition, as we have done in the context of Theorem 2.9. Secondly, there is an error on page 171, line 1 of [30] which must be resolved. In the proof of Theorem 3.1 of [30], the authors employed proof by contradiction and concluded that for any given $P : X \to X^*$, $\rho > 0$, $x \in X^*$ and $\phi : X \to \mathbb{R} \cup \{+\infty\}$, the functional $f : X \times X \to \mathbb{R} \cup \{+\infty\}$ defined, for all $(y, u) \in X \times X$, by

$$f(y, x) = \langle x - P(u), \eta(y, u)\rangle + \rho\phi(u) - \rho\phi(y)$$

satisfies Lemma 2.3(ii). In the process of achieving a contradiction, but there is an error. In fact, they assumed, on the contrary, that there exist a finite set $\{y_1, y_2, \ldots, y_m\} \subset X$ and $u_0 = \sum_{i=1}^m \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$ such that

$$\langle x - P(u_0), \eta(y_i, u_0)\rangle + \rho\phi(u_0) - \rho\phi(y_i) > 0, \quad i = 1, 2, \ldots, m. \tag{2.5}$$

Using property of η-subdifferentiability of ϕ at $u_0 \in X$, they deduced the existence of a point $f^* \in X^*$ satisfying the following relation:

$$\rho\phi(y_i) - \rho\phi(u_0) \geq \rho\langle f^*, \eta(y_i, u_0)\rangle, \quad i = 1, 2, \ldots, m. \tag{2.6}$$

Making use of (2.5) and (2.6), then they derived the following inequality:

$$\langle x - P(u_0) - \rho f^*, \eta(y_i, u_0)\rangle > 0, \quad i = 1, 2, \ldots, m. \tag{2.7}$$

Finally, applying (2.7) and taking into account that the functional $h : (y, u_0) \in X \times X \to h(y, u_0) = \langle x - P(u_0) - \rho f^*, \eta(y, u_0)\rangle \in \mathbb{R} \cup \{+\infty\}$ is 0-DQCV in y (in the first argument) and $\eta(u_0, u_0) = 0$, they obtained a contradiction as follows:

$$\begin{aligned} 0 &< \sum_{i=1}^m \lambda_i \langle x - P(u_0) - \rho f^*, \eta(y_i, u_0)\rangle \\ &= \langle x - P(u_0) - \rho f^*, \eta(u_0, u_0)\rangle = 0. \end{aligned} \tag{2.8}$$

In virtue of (2.8) and using the fact that $\sum_{i=1}^m \eta(y_i, u_0) = \eta(\sum_{i=1}^m y_i, u_0) = \eta(u_0, u_0) = 0$, that is, the property of linearity of η, they got the required contradiction. But, in view of the assumptions mentioned in the context of [30, Theorem 3.1], η is not linear necessarily and so $\sum_{i=1}^m \eta(y_i, u_0) = \eta(\sum_{i=1}^m y_i, u_0)$ does not hold necessarily. A correct proof in order to achieving a contradiction can be found on page 300 of [2].

We now provide a new example in which the existence of the two mappings $\eta : X \times X \to X$ and $P : X \to X^*$ satisfying all the conditions of Theorem 2.9 is shown.

Example 2.10. Let X be the set of all real numbers endowed with the Euclidean norm $\|.\| = |.|$ and let the mappings $\eta : X \times X \to X$ and $P : X \to X^*$ be defined, respectively, by

$$\eta(x,y)=\begin{cases}\mu(|xy|^pLn(\theta+|xy|^q)+1)(x-y), & \text{if } |xy|<s,\\ \varrho(|xy|^k-\varsigma Log_m|xy|)(x-y), & \text{if } s\leq|xy|<t,\\ (\frac{\gamma}{|xy|^l\alpha^{\beta|xy|}}+\sigma)(x-y), & \text{if } t\leq|xy|,\end{cases}$$

and $P(x)=\xi x$ for all $x,y\in X$, where $\mu,\varrho,\gamma,\sigma,\varsigma,\xi,k,l,p,q,\beta$ are arbitrary constants that are strictly bigger than zero, and θ,s,t,α,m are arbitrary real constants such that $\theta\geq 1$ and $0<s<t\leq 1<\alpha,m$. It can be easily seen that $\eta(x,y)+\eta(y,x)=0$ for all $x,y\in X$.

For all $x,y\in X$, we have

$$|\eta(x,y)|=\begin{cases}\mu(|xy|^pLn(\theta+|xy|^q)+1)|x-y|, & \text{if } |xy|<s,\\ \varrho(|xy|^k-\varsigma Log_m|xy|)|x-y|, & \text{if } s\leq|xy|<t,\\ (\frac{\gamma}{|xy|^l\alpha^{\beta|xy|}}+\sigma)|x-y|, & \text{if } t\leq|xy|.\end{cases}$$

Taking into account that $\theta\geq 1$ and $q>0$, it follows that $Ln(\theta+|xy|^q)\geq 0$ and so for all $x,y\in X$ with $|xy|<s$, we have

$$1<|xy|^pLn(\theta+|xy|^q)+1<s^pLn(\theta+k^p)+1. \tag{2.9}$$

The fact that $0<s<t\leq 1$ implies that for all $x,y\in X$ with $s\leq|xy|<t$,

$$0<s^k-\varsigma Log_mt<|xy|^k-\varsigma Log_m|xy|<t^k-\varsigma Log_ms. \tag{2.10}$$

Since $l,\beta,\gamma,\sigma>0$ and $\alpha>1$, for all $x,y\in X$ with $|xy|\geq t$, we yield

$$\sigma<\frac{\gamma}{|xy|^l\alpha^{\beta|xy|}}+\sigma\leq\frac{\gamma}{t^l\alpha^{\beta t}}+\sigma,. \tag{2.11}$$

Making use of (2.9)–(2.11) and in view of the fact that $\mu,\varrho>0$, we deduce that for all $x,y\in X$,

$$|\eta(x,y)|\leq\max\Big\{\mu(s^pLn(\theta+k^p)+1),\varrho(t^k-\varsigma Log_ms),\frac{\gamma}{t^l\alpha^{\beta t}}+\sigma\Big\}|x-y|,$$

which means that η is $\max\Big\{\mu(s^pLn(\theta+k^p)+1),\varrho(t^k-\varsigma Log_ms),\frac{\gamma}{t^l\alpha^{\beta t}}+\sigma\Big\}$-Lipschitz continuous.

Let us now define, associated with each $z\in X$, a correspondence $h_z:X\times X\to\mathbb{R}\cup\{+\infty\}$ for each $(y,x)\in X\times X$ by

$$h_z(y,x)=\langle z-P(x),\eta(y,x)\rangle=(z-P(x))\eta(y,x).$$

We claim that the function h_z is 0-DQCV in the first argument. In order to prove our claim, suppose, on the contrary, that there exist a finite set $\{y_1,y_2,\dots,y_n\}$ and $u=\sum_{i=1}^n\lambda_iy_i$ with $\lambda_i\geq 0$ and $\sum_{i=1}^n\lambda_i=1$ such that for each $i\in\{1,2,\dots,n\}$,

$$0<h_z(y_i,u)=\begin{cases}\mu(z-\xi u)(|y_iu|^pLn(\theta+|y_iu|^q)+1)(y_i-u), & \text{if } |y_iu|<s,\\ \varrho(z-\xi u)(|y_iu|^k-\varsigma Log_m|y_iu|)(y_i-u), & \text{if } s\leq|y_iu|<t,\\ (z-\xi u)(\frac{\gamma}{|y_iu|^l\alpha^{\beta|y_iu|}}+\sigma)(y_i-u), & \text{if } t\leq|y_iu|.\end{cases}$$

In the light of this fact we conclude that for each $i \in \{1, 2, \ldots, n\}$, $(z-\xi u)(y_i-u) > 0$ and so

$$0 < \sum_{i=1}^{n} \lambda_i(z - \xi u)(y_i - u) = (z - \xi u)(\sum_{i=1}^{n} \lambda_i y_i - \sum_{i=1}^{n} \lambda_i u)$$
$$= (z - \xi u)(u - u) = 0,$$

which leads to a contradiction. Therefore, for any given $z \in X$, the function h_z is 0-DQCV in the first argument. In virtue of the facts that for all $x, y \in X$,

$$\langle P(x) - P(y), \eta(x, y)\rangle = \begin{cases} \mu\xi(|xy|^p Ln(\theta + |xy|^q) + 1)|x - y|^2, & \text{if } |xy| < s, \\ \varrho\xi(|xy|^k - \varsigma Log_m|xy|)|x - y|^2, & \text{if } s \leq |xy| < t, \\ \xi(\frac{\gamma}{|xy|^l \alpha^{\beta|xy|}} + \sigma)|x - y|^2, & \text{if } t \leq |xy|, \end{cases}$$

$\theta \geq 1$, $0 < s < t \leq 1 < \alpha, m$, and $\beta, k, l, \sigma, \gamma, \varsigma, \varrho, p, q, \mu > 0$, we deduce that

$$\langle P(x) - P(y), \eta(x, y)\rangle \geq \mu\xi|x - y|^2, \quad \forall x, y \in X \text{ with } |xy| \in [0, s),$$

$$\langle P(x) - P(y), \eta(x, y)\rangle \geq \varrho\xi(s^k - \varsigma Log_m t)|x - y|^2, \quad \forall x, y \in X \text{ with } |xy| \in [s, t)$$

and

$$\langle P(x) - P(y), \eta(x, y)\rangle \geq \xi\sigma|x - y|^2, \quad \forall x, y \in X \text{ with } |xy| \in [t, +\infty).$$

Hence,

$$\langle P(x) - P(y), \eta(x, y)\rangle \geq \min\Big\{\mu\xi, \varrho\xi(s^k - \varsigma Log_m t), \xi\sigma\Big\}|x - y|^2, \quad \forall x, y \in X,$$

which means that P is a $\min\Big\{\mu\xi, \varrho\xi(s^k - \varsigma Log_m t), \xi\sigma\Big\}$-strongly η-monotone mapping. Thereby, the mappings η and P are satisfied all the conditions of Theorem 2.9.

We now close this section by the following theorem due to Ahmad et al. [2] and Kazmi and Bhat [30] in which the suitable conditions for the P-η-proximal mapping $R^{\partial_\eta \phi}_{\rho,P}$ associated with the mappings ϕ, P, η and the constant $\rho > 0$ to be Lipschitz continuous are stated and an estimate of its Lipschitz constant is computed.

Theorem 2.11 ([2,30])**.** *Let X be a reflexive Banach space with the dual space X^*, $\eta : X \times X \to X$ be a τ-Lipschitz continuous mapping such that $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in X$, and let $P : X \to X^*$ be a γ-strongly η-monotone continuous mapping. Suppose that for any given $x^* \in X^*$, the function $h : (y, x) \in X \times X \to h(y, x) = \langle x^* - P(x), \eta(y, x)\rangle \in \mathbb{R} \cup \{+\infty\}$ is* 0-DQCV *in the first argument, $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous η-subdifferentiable proper functional on X and $\rho > 0$ is an arbitrary real constant. Then, the P-η-proximal mapping $R^{\partial_\eta \phi}_{\rho,P} : X^* \to X$ associated with ϕ, P, η and $\rho > 0$ is $\frac{\tau}{\gamma}$-Lipschitz continuous, i.e.,*

$$\|R^{\partial_\eta \phi}_{\rho,P}(x^*) - R^{\partial_\eta \phi}_{\rho,P}(y^*)\| \leq \frac{\tau}{\gamma}\|x^* - y^*\|, \qquad \forall x^*, y^* \in X^*.$$

3. Formulation of the problem and existence theorem of solution

For $i = 1, 2, \ldots, p$, let X_i be real Banach spaces with the dual spaces X_i^*, and $\langle ., . \rangle_i$ be the dual pairing between X_i and its dual space X_i^*. Let for $i = 1, 2, \ldots, p$, $T_i : \prod_{j=1}^p X_j \to X_i^*$, $F : X_i \to X_i^*$, $g_i : X_i \to X_i$ and $\eta_i : X_i \times X_i \to X_i$ be the mappings. Suppose further that for $i = 1, 2, \ldots, p$, $\varphi_i : X_i \times X_i \to \mathbb{R} \cup \{+\infty\}$ are extended real-valued bifunctions such that for each fixed $z_i \in X_i$, $\varphi_i(., z_i) : X_i \to \mathbb{R} \cup \{+\infty\}$ is a proper, lower semicontinuous and η_i-subdifferentiable functional on X_i with $g_i(X_i) \cap \operatorname{dom} \partial_{\eta_i} \varphi_i(., z_i) \neq \emptyset$. We consider the problem of finding $(x_1, x_2, \ldots, x_p) \in \prod_{j=1}^p X_j$ such that $g_i(x_i) \in \operatorname{dom} \partial_{\eta_i} \varphi_i(., x_i)$ for each $i \in \{1, 2, \ldots, p\}$ and

$$\begin{aligned} &\langle T_i(x_1, x_2, \ldots, x_p) - F_i(x_i), \eta_i(y_i, g_i(x_i)) \rangle_i \\ &\geq \varphi_i(g_i(x_i), x_i) - \varphi_i(y_i, x_i), \quad \forall y_i \in X_i, \end{aligned} \tag{3.1}$$

which is called a *system of generalized nonlinear variational-like inequalities* (SGNVLI).

If $p = 1$, $X_1 = \mathcal{H}$ is a real Hilbert space, $T_1 = T$, $A_1 = A$, $\eta_1 = \eta$, $g_1 = g$ and $\varphi_1 = \varphi$, then the problem (3.1) collapses to the problem of finding $x \in \mathcal{H}$ such that $g(x) \in \operatorname{dom} \partial_\eta \varphi(., x)$ and

$$\langle T(x) - A(x), \eta(y, g(x)) \rangle \geq \varphi(g(x), x) - \varphi(y, x), \quad \forall y \in \mathcal{H},$$

which was introduced and studied by Ding and Luo [20].

The following conclusion that tells the problem (3.1) is equivalent to a fixed point problem provides us a characterization of its solution and plays a crucial role in the sequel.

Lemma 3.1. *Assume that $X_i, T_i, A_i, \varphi_i, \eta_i, g_i$ $(i = 1, 2, \ldots, p)$ are the same as in the* SGNVLI (3.1) *such that for each $i \in \{1, 2, \ldots, p\}$, X_i is reflexive and η_i is a τ_i-Lipschitz continuous mapping with $\eta_i(x_i', y_i') + \eta_i(y_i', x_i') = 0$ for all $x_i', y_i' \in X_i$. Let for each $i \in \{1, 2, \ldots, p\}$, $P_i : X_i \to X_i^*$ be a γ_i-strongly η_i-monotone continuous mapping such that $g_i(X_i) \cap \operatorname{dom}(P_i) \neq \emptyset$. For each $i \in \{1, 2, \ldots, p\}$ and for any $x_i^* \in X_i^*$, suppose that the function $h_i : (y_i', x_i') \in X_i \times X_i \to h_i(y_i', x_i') = \langle x_i^* - P_i(x_i'), \eta_i(y_i', x_i') \rangle_i \in \mathbb{R} \cup \{+\infty\}$ is* 0-DQCV *in the first argument. Then, $(x_1, x_2, \ldots, x_p) \in \prod_{j=1}^p X_j$ is a solution of the* SGNVLI (3.1) *if and only if $g(x_i) \in$* $\operatorname{dom}(P_i)$ *for each $i \in \{1, 2, \ldots, p\}$ and*

$$g_i(x_i) = R_{\lambda_i, P_i}^{\partial_{\eta_i} \varphi_i(., x_i)} [(P_i \circ g_i)(x_i) - \lambda_i (T_i(x_1, x_2, \ldots, x_p) - F_i(x_i))],$$

where for $i = 1, 2, \ldots, p$, $\lambda_i > 0$ are constants, $P_i \circ g_i$ denotes P_i composition g_i, and $R_{\lambda_i, P_i}^{\partial_{\eta_i} \varphi_i(., x_i)} = (P_i + \lambda_i \partial_{\eta_i} \varphi_i(., x_i))^{-1}$ is P_i-η_i-proximal mapping of $\varphi_i(., x_i)$.

Proof. Utilizing Definitions 2.4 and 2.6, we deduce that $(x_1, x_2, \ldots, x_p) \in \prod_{j=1}^p X_j$ is a solution of the SGNVLI (3.1) if and only if for $i = 1, 2, \ldots, p$,

$$\begin{aligned} &\varphi_i(y_i, x_i) - \varphi_i(g_i(x_i), x_i) \geq \langle -(T_i(x_1, x_2, \ldots, x_p) - F_i(x_i)), \eta_i(y_i, g_i(x_i)) \rangle_i, \ \forall y_i \in X_i, \\ &\Leftrightarrow -(T_i(x_1, x_2, \ldots, x_p) - F_i(x_i)) \in \partial_{\eta_i} \varphi_i(g_i(x_i), x_i) \\ &\Leftrightarrow (P_i \circ g_i)(x_i) - \lambda_i (T_i(x_1, x_2, \ldots, x_p) - F_i(x_i)) \in (P_i + \lambda_i \partial_{\eta_i} \varphi_i(., x_i))(g_i(x_i)) \\ &\Leftrightarrow g_i(x_i) = R_{\lambda_i, P_i}^{\partial_{\eta_i} \varphi_i(., x_i)} [(P_i \circ g_i)(x_i) - \lambda_i T_i(x_1, x_2, \ldots, x_p) - F_i(x_i))], \end{aligned}$$

where for $i = 1, 2, \dots, p$, $R^{\partial_{\eta_i}\varphi_i(.,x_i)}_{\lambda_i,P_i} = (P_i + \lambda_i \partial_{\eta_i}\varphi_i(., x_i))^{-1}$. □

Before turning to the main results of this paper, let us recall some needed concepts and an efficient and useful lemma.

Recall that a normed space X is called *strictly convex* if S_X, the unit sphere in X, is strictly convex, that is, the inequality $\|x + y\| < 2$ holds for all distinct unit vectors x and y in X. It is said to be *smooth* if, for every vector x in B_X, the unit ball in X, there exists a unique $x^* \in X^*$ such that $\|x^*\| = x^*(x) = 1$. It is known that X is smooth if X^* is strictly convex, and that X is strictly convex if X^* is smooth.

Definition 3.2. A normed space X is said to be

(i) uniformly convex if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in B_X$ with $\|x - y\| \geq \varepsilon$ the inequality $\|x + y\| \leq 2(1 - \delta)$ holds;
(ii) uniformly smooth if for any given $\varepsilon > 0$ there exists $\tau > 0$ such that for all $x, y \in B_X$ with $\|x - y\| \leq 2\tau$, the inequality $\|x + y\| \geq 2(1 - \varepsilon\tau)$ holds.

The functions $\delta : [0, 2] \to [0, 1]$ and $\rho_X : [0, +\infty) \to [0, +\infty)$ defined by the formulae

$$\delta_X(\varepsilon) = \inf\{1 - \frac{\|x + y\|}{2} : x, y \in B_X, \|x - y\| \geq \varepsilon\}$$

and

$$\rho_X(\tau) = \sup\{\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\|) - 1 : x, y \in B_X\}$$

are called the *modulus of convexity and smoothness* of X, respectively.

Remark 3.3. It should be pointed out that

(i) in the definition of $\delta_X(\varepsilon)$ we can as well take the infimum over all vectors $x, y \in S_X$ with $\|x - y\| = \varepsilon$, see for example [17];
(ii) in the definition of $\rho_X(\tau)$, we can as well take the supremum over all vectors $x, y \in S_X$;
(iii) the functions δ_X and ρ_X are continuous and increasing on the intervals $[0, 2]$ and $[0, +\infty)$, respectively, and $\delta_X(0) = \rho_X(0) = 0$. In addition, ρ_X is a convex function on the interval $[0, +\infty)$ and $\rho_X(\tau) \leq \tau$ for all $\tau \geq 0$;
(iv) in the light of definitions of the functions δ_X and ρ_X, a normed space X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$, and is uniformly smooth if and only if $\lim_{\tau\to 0} \frac{\rho_X(\tau)}{\tau} = 0$;
(v) any uniformly convex and any uniformly smooth Banach space is reflexive;
(vi) a Banach space X is uniformly convex (resp., uniformly smooth) if and only if X^* is uniformly smooth (resp., uniformly convex);
(vii) the spaces l^p, L^p and W^p_m, $1 < p < \infty$, $m \in \mathbb{N}$, are uniformly convex as well as uniformly smooth, see [19, 28, 33]. At the same time, the modulus of convexity and smoothness of a Hilbert space and the spaces l^p, L^p and W^p_m, $1 < p < \infty$, $m \in \mathbb{N}$ can be found in [19, 28, 33].

The function $J : X \to 2^{X^*}$ defined by the formula

$$J(x) := \{x^* \in X^* : \langle x^*, x\rangle = \|x^*\|\|x\|, \|x^*\| = \|x\|\}, \quad \forall x \in X,$$

is called the *normalized duality mapping* from X into X^*. We observe immediately that if $X = \mathcal{H}$, a Hilbert space, then J is the identity mapping on $\mathcal{H}$. In the meanwhile, it is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$.

Definition 3.4. For $i = 1, 2, \dots, p$, let X_i be real Banach spaces with the topological dual spaces X_i^*. The mapping $T_i : \prod_{j=1}^{p} X_j \to X_i^*$ is said to be $\mu_{i,j}$-Lipschitz continuous in the jth argument if, there exists a constant $\mu_{i,j} > 0$ such that

$$\begin{aligned}&\|T_i(x_1, x_2, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_p) - T_i(x_1, x_2, \dots, x_{j-1}, x_j', x_{j+1}, \dots, x_p)\|_i \\ &\le \mu_{i,j}\|x_j - x_j'\|_j, \quad x_j, x_j' \in X_j, x_i \in X_i (i = 1, 2, \dots, p; i \neq j).\end{aligned}$$

Definition 3.5. Let X be a real uniformly smooth Banach space with the dual space X^*, and J be the normalized duality mapping from X into X^*. A mapping $g : X \to X$ is said to be k-strongly accretive if there exists a constant $k > 0$ such that

$$\langle J(x - y), g(x) - g(y)\rangle \ge k\|x - y\|^2, \quad \forall x, y \in X.$$

Lemma 3.6 ([15])**.** *Let X be a real uniformly smooth Banach space with the dual space X^* and J be the normalized duality mapping from X into X^*. Then for all $x, y \in X$, we have*

(i) $\|x + y\|^2 \le \|x\|^2 + 2\langle J(x + y), y\rangle$;

(ii) $\langle J(x) - J(y), x - y\rangle \le 2d^2(x, y)\rho_X(\frac{4\|x-y\|}{d(x,y)})$, *where* $d(x, y) = \sqrt{\frac{\|x\|^2+\|y\|^2}{2}}$ *for all* $x, y \in X$.

It should be remarked that in the original version of the above lemma, d is used instead of $d(x, y)$. But, taking into account that for any given $x, y \in X$, d depends on x and y, it must be replaced by $d(x, y)$, as we have done in part (ii) of Lemma 3.6.

We are now in a position to prove the existence of a unique solution for the problem (3.1). The sufficient conditions which guarantee the existence of a unique solution for the SGNVLI (3.1) are stated in the next theorem.

Theorem 3.7. *Suppose that, for each $i \in \Gamma = \{1, 2, \dots, p\}$, X_i is a real uniformly smooth Banach space with the dual space X_i^* and $\rho_{X_i}(t) \le C_i t^2$ for some $C_i > 0$, and $\langle ., .\rangle_i$ is the dual pair between X_i and X_i^*. Let for each $i \in \Gamma$, the mapping $g_i : X_i \to X_i$ be a k_i-strongly accretive and δ_i-Lipschtiz continuous mapping, $F_i : X_i \to X_i^*$ be ς_i-Lipschitz continuous and the mapping $T_i : \prod_{j=1}^{p} X_j \to X_i^*$ be $\sigma_{i,j}$-Lipschitz continuous in the jth argument ($j \in \Gamma, i \neq j$). Assume that for each $i \in \Gamma$, $\eta_i : X_i \times X_i \to X_i$ is a τ_i-Lipschitz continuous mapping such that $\eta_i(x_i', y_i') + \eta_i(y_i', x_i') = 0$ for all $x_i', y_i' \in X_i$, and $P_i : X_i \to X_i^*$ is a γ_i-strongly η_i-monotone and ϱ_i-Lipschitz continuous mapping such that $g_i(X_i) \subseteq \mathrm{dom}(P_i)$. For each $i \in \Gamma$ and for any $x_i^* \in X_i^*$, let the function $h_i : (y_i', x_i') \in X_i \times X_i \to h_i(y_i', x_i') = \langle x_i^* - P_i(x_i'), \eta_i(y_i', x_i')\rangle_i \in \mathbb{R} \cup \{+\infty\}$ be* 0-DQCV *in the first argument. Suppose that for each $i \in \Gamma$, $\varphi_i : X_i \times X_i \to \mathbb{R} \cup \{+\infty\}$ is an extended real-valued bifunction such that for each fixed point $z_i \in X_i$, $\varphi_i(., z_i) : X_i \to \mathbb{R} \cup \{+\infty\}$ is a proper, lower semicontinuous and η_i-subdifferentiable functional on X_i with $g_i(X_i) \cap \partial_{\eta_i}\varphi_i(., z_i) \neq$*

$\emptyset$. *Furthermore, let for each $i \in \Gamma$, there exist constants $\xi_i, \lambda_i > 0$ such that*

$$\|R^{\partial_{\eta_i}\varphi_i(.,u_i)}_{\lambda_i,P_i}(w_i) - R^{\partial_{\eta_i}\varphi_i(.,v_i)}_{\lambda_i,P_i}(w_i)\|_i \leq \xi_i\|u_i - v_i\|_i, \tag{3.2}$$

for all $u_i, v_i, w_i \in X_i$ and

$$\begin{cases} \sqrt{1 - 2\kappa_i + 64C_i\delta_i^2} + \xi_i + \frac{\tau_i}{\gamma_i}(\varrho_i\delta_i + \lambda_i\varsigma_i) \\ + \sum_{k\in\Gamma\setminus\{i\}} \frac{\tau_k\lambda_k}{\gamma_k}\sigma_{k,i} < 1, \ 2k_i < 1 + 64C_i\delta_i^2. \end{cases} \tag{3.3}$$

Then, the SGNVLI (3.1) *has a unique solution.*

Proof. For given $\lambda_i > 0$ $(i = 1, 2, \dots, p)$, define the mappings $N_{\lambda_i} : \prod_{j=1}^p X_j \to X_i$ and $M_{\lambda_1,\lambda_2,\dots,\lambda_p} : \prod_{j=1}^p X_j \to \prod_{j=1}^p X_j$ by

$$\begin{aligned} N_{\lambda_i}(x_1, x_2, \dots, x_p) = x_i - g_i(x_i) + R^{\partial_{\eta_i}\varphi_i(.,x_i)}_{\lambda_i,P_i}[(P_i \circ g_i)(x_i) \\ - \lambda_i(T_i(x_1, x_2, \dots, x_p) - F_i(x_i))] \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} M_{\lambda_1,\lambda_2,\dots,\lambda_p}(x_1, \dots, x_p) = \big(N_{\lambda_1}(x_1, \dots, x_p), \\ N_{\lambda_2}(x_1, \dots, x_p), \dots, N_{\lambda_p}(x_1, \dots, x_p)\big), \end{aligned} \tag{3.5}$$

for all $(x_1, x_2, \dots, x_p) \in \prod_{j=1}^p X_j$.

Applying (3.2), (3.4) and Theorem 2.11, for any given $(x_1, x_2, \dots, x_p), (\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_p) \in \prod_{j=1}^p X_j$ and for each $i \in \Gamma$, we obtain

$$\begin{aligned}
&\|N_{\lambda_i}(x_1, x_2, \dots, x_p) - N_{\lambda_i}(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_p)\|_i \\
&= \|x_i - g_i(x_i) + R^{\partial_{\eta_i}\varphi_i(.,x_i)}_{\lambda_i,P_i}[(P_i \circ g_i)(x_i) \\
&\quad - \lambda_i(T_i(x_1, x_2, \dots, x_p) - F_i(x_i))] \\
&\quad - \big(\widetilde{x}_i - g_i(\widetilde{x}_i) + R^{\partial_{\eta_i}\varphi_i(.,\widetilde{x}_i)}_{\lambda_i,P_i}[(P_i \circ g_i)(\widetilde{x}_i) \\
&\quad - \lambda_i(T_i(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_p) - F_i(\widetilde{x}_i))]\big)\|_i \\
&\leq \|x_i - \widetilde{x}_i - (g_i(x_i) - g_i(\widetilde{x}_i))\|_i \\
&\quad + \|R^{\partial_{\eta_i}\varphi_i(.,x_i)}_{\lambda_i,P_i}[(P_i \circ g_i)(x_i) - \lambda_i(T_i(x_1, x_2, \dots, x_p) - F_i(x_i))] \\
&\quad - R^{\partial_{\eta_i}\varphi_i(.,\widetilde{x}_i)}_{\lambda_i,P_i}[(P_i \circ g_i)(\widetilde{x}_i) - \lambda_i(T_i(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_p) - F_i(\widetilde{x}_i))]\|_i \\
&\leq \|x_i - \widetilde{x}_i - (g_i(x_i) - g_i(\widetilde{x}_i))\|_i \\
&\quad + \|R^{\partial_{\eta_i}\varphi_i(.,x_i)}_{\lambda_i,P_i}[(P_i \circ g_i)(x_i) - \lambda_i(T_i(x_1, x_2, \dots, x_p) - F_i(x_i))] \\
&\quad - R^{\partial_{\eta_i}\varphi_i(.,\widetilde{x}_i)}_{\lambda_i,P_i}[(P_i \circ g_i)(x_i) - \lambda_i T_i(x_1, x_2, \dots, x_p) - F_i(x_i))]\|_i \\
&\quad + \|R^{\partial_{\eta_i}\varphi_i(.,\widetilde{x}_i)}_{\lambda_i,P_i}[(P_i \circ g_i)(x_i) - \lambda_i(T_i(x_1, x_2, \dots, x_p) - F_i(x_i))] \\
&\quad - R^{\partial_{\eta_i}\varphi_i(.,\widetilde{x}_i)}_{\lambda_i,P_i}[(P_i \circ g_i)(\widetilde{x}_i) - \lambda_i T_i(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_p) - F_i(\widetilde{x}_i))]\|_i \\
&\leq \|x_i - \widetilde{x}_i - (g_i(x_i) - g_i(\widetilde{x}_i))\|_i + \xi_i\|x_i - \widetilde{x}_i\|_i \\
&\quad + \frac{\tau_i}{\gamma_i}\big(\|(P_i \circ g_i)(x_i) - (P_i \circ g_i)(\widetilde{x}_i)\|_i
\end{aligned}$$

$$
\begin{aligned}
&+ \lambda_i\|T_i(x_1,x_2,\dots,x_p) - T_i(\widetilde{x}_1,\widetilde{x}_2,\dots,\widetilde{x}_p)\|_i \\
&+ \lambda_i\|F_i(x_i) - F_i(\widetilde{x}_i)\|_i\big) \qquad (3.6)\\
\leq\ & \|x_i - \widetilde{x}_i - (g_i(x_i) - g_i(\widetilde{x}_i))\|_i + \xi_i\|x_i - \widetilde{x}_i\|_i \\
&+ \frac{\tau_i}{\gamma_i}\big(\|(P_i\circ g_i)(x_i) - (P_i\circ g_i)(\widetilde{x}_i)\|_i \\
&+ \lambda_i\big(\|T_i(x_1,x_2,\dots,x_{i-1},\widetilde{x}_i,x_{i+1},\dots,x_p) \\
&- T_i(\widetilde{x}_1,x_2,\dots,x_{i-1},\widetilde{x}_i,x_{i+1},\dots,x_p)\|_i \\
&+ \|T_i(\widetilde{x}_1,x_2,\dots,x_{i-1},\widetilde{x}_i,x_{i+1},\dots,x_p) \\
&- T_i(\widetilde{x}_1,\widetilde{x}_2,\dots,x_{i-1},\widetilde{x}_i,x_{i+1},\dots,x_p)\|_i \\
&+ \dots + \|T_i(\widetilde{x}_1,\widetilde{x}_2,\dots,\widetilde{x}_{i-1},\widetilde{x}_i,\widetilde{x}_{i+1},\dots,\widetilde{x}_{p-1},x_p) \\
&- T_i(\widetilde{x}_1,\widetilde{x}_2,\dots,\widetilde{x}_{i-1},\widetilde{x}_i,\widetilde{x}_{i+1},\dots,\widetilde{x}_{p-1},\widetilde{x}_p)\|_i\big) \\
&+ \lambda_i\|F_i(x_i) - F_i(\widetilde{x}_i)\|_i\big) \\
=\ & \|x_i - \widetilde{x}_i - (g_i(x_i) - g_i(\widetilde{x}_i))\|_i + \xi_i\|x_i - \widetilde{x}_i\|_i \\
&+ \frac{\tau_i}{\gamma_i}\big(\|(P_i\circ g_i)(x_i) - (P_i\circ g_i)(\widetilde{x}_i)\|_i \\
&+ \lambda_i\sum_{j\in\Gamma\setminus\{i\}}\|T_i(\widetilde{x}_1,\widetilde{x}_2,\dots,\widetilde{x}_{j-1},x_j,x_{j+1},\dots,x_{i-1},\widetilde{x}_i,x_{i+1},\dots,x_p) \\
&- T_i(\widetilde{x}_1,\widetilde{x}_2,\dots,\widetilde{x}_{j-1},\widetilde{x}_j,x_{j+1},\dots,x_{i-1},\widetilde{x}_i,x_{i+1},\dots,x_p)\|_i \\
&+ \lambda_i\|F_i(x_i) - F_i(\widetilde{x}_i)\|_i\big).
\end{aligned}
$$

Taking into account that for each $i\in\Gamma$, the mappings P_i, F_i and g_i are Lipschitz continuous with constants ϱ_i, ς_i and δ_i, respectively, and the mapping T_i is $\sigma_{i,j}$-Lipschitz continuous in the jth argument ($j\in\Gamma, j\neq i$), it follows that for each $i\in\Gamma$,

$$\|F_i(x_i) - F_i(\widetilde{x}_i)\|_i \leq \varsigma_i\|x_i - \widetilde{x}_i\|_i, \tag{3.7}$$

$$\|(P_i\circ g_i)(x_i) - (P_i\circ g_i)(\widetilde{x}_i)\|_i \leq \varrho_i\delta_i\|x_i - \widetilde{x}_i\|_i \tag{3.8}$$

and

$$
\begin{aligned}
&\|T_i(\widetilde{x}_1,\widetilde{x}_2,\dots,\widetilde{x}_{j-1},x_j,x_{j+1},\dots,x_{i-1},\widetilde{x}_i,x_{i+1},\dots,x_p) \\
&- T_i(\widetilde{x}_1,\widetilde{x}_2,\dots,\widetilde{x}_{j-1},\widetilde{x}_j,x_{j+1},\dots,x_{i-1},\widetilde{x}_i,x_{i+1},\dots,x_p)\|_i \\
&\leq \sigma_{i,j}\|x_j - \widetilde{x}_j\|_j.
\end{aligned} \tag{3.9}
$$

Since for each $i\in\Gamma$, the mapping g_i is k_i-strongly accretive and δ_i-Lipschitz continuous, and X_i is a uniformly smooth Banach space with $\rho_{X_i}(t)\leq C_it^2$ for some $C_i>0$, by Lemma 3.6, yields

$$
\begin{aligned}
&\|x_i - \widetilde{x}_i - (g_i(x_i) - g_i(\widetilde{x}_i))\|_i^2 \\
&\leq \|x_i - \widetilde{x}_i\|_i^2 + 2\langle J_i(x_i - \widetilde{x}_i - (g_i(x_i) - g_i(\widetilde{x}_i)), -(g_i(x_i) - g_i(\widetilde{x}_i))\rangle_i \\
&= \|x_i - \widetilde{x}_i\|_i^2 - 2\langle J_i(x_i - \widetilde{x}_i), g_i(x_i) - g_i(\widetilde{x}_i)\rangle_i \\
&\quad + 2\langle J_i(x_i - \widetilde{x}_i - (g_i(x_i) - g_i(\widetilde{x}_i))) - J_i(x_i - \widetilde{x}_i), -(g_i(x_i) - g_i(\widetilde{x}_i))\rangle_i \\
&\leq \|x_i - \widetilde{x}_i\|_i^2 - 2k_i\|x_i - \widetilde{x}_i\|_i + 4d_i^2(x_i - \widetilde{x}_i - (g_i(x_i) - g_i(\widetilde{x}_i)), x_i - \widetilde{x}_i)
\end{aligned}
$$

$$\times \rho_{X_i}\Big(\frac{4\|g_i(x_i)-g_i(\widetilde{x}_i)\|_i}{d_i(x_i-\widetilde{x}_i-(g_i(x_i)-g_i(\widetilde{x}_i)),x_i-\widetilde{x}_i)}\Big)$$
$$\leq (1-2k_i+64C_i\delta_i^2)\|x_i-\widetilde{x}_i\|_i^2,$$

where for each $i\in\Gamma$, J_i is the normalized duality mapping from X_i into X_i^*.

The last inequality implies that for each $i\in\Gamma$,

$$\|x_i-\widetilde{x}_i-(g_i(x_i)-g_i(\widetilde{x}_i))\|_i\leq\sqrt{1-2k_i+64C_i\delta_i^2}\|x_i-\widetilde{x}_i\|_i. \tag{3.10}$$

Substituting (3.7)–(3.10) into (3.6), we derive for each $i\in\Gamma$ that

$$\begin{aligned}
&\|N_{\lambda_i}(x_1,x_2,\dots,x_p)-N_{\lambda_i}(\widetilde{x}_1,\widetilde{x}_2,\dots,\widetilde{x}_p)\|_i\\
&\leq\big(\sqrt{1-2k_i+64C_i\delta_i^2}+\xi_i+\frac{\tau_i}{\gamma_i}(\varrho_i\delta_i+\lambda_i\delta_i)\big)\|x_i-\widetilde{x}_i\|_i\\
&\quad+\frac{\tau_i\lambda_i}{\gamma_i}\sum_{j\in\Gamma\setminus\{i\}}\sigma_{i,j}\|x_j-\widetilde{x}_j\|_j\\
&=\vartheta_i\|x_i-\widetilde{x}_i\|_i+\frac{\tau_i\lambda_i}{\gamma_i}\sum_{j\in\Gamma\setminus\{i\}}\sigma_{i,j}\|x_j-\widetilde{x}_j\|_j,
\end{aligned}\tag{3.11}$$

where for each $i\in\Gamma$,

$$\vartheta_i=\sqrt{1-2k_i+64C_i\delta_i^2}+\xi_i+\frac{\tau_i}{\gamma_i}(\varrho_i\delta_i+\lambda_i\varsigma_i).$$

Let us now define a norm $\|.\|_*$ on $\prod_{i=1}^p X_i$ by

$$\|(x_1,x_2,\dots,x_p)\|_*=\sum_{i=1}^p\|x_i\|_i,\quad\forall(x_1,x_2,\dots,x_p)\in\prod_{i=1}^p X_i. \tag{3.12}$$

It can be easily seen that $(\prod_{i=1}^p X_i,\|.\|_*)$ is a Banach space. Then, recalling (3.6) and (3.11), we conclude that

$$\begin{aligned}
&\|M_{\lambda_1,\lambda_2,\dots,\lambda_p}(x_1,\dots,x_p)-M_{\lambda_1,\lambda_2,\dots,\lambda_p}(\widetilde{x}_1,\widetilde{x}_2,\dots,\widetilde{x}_p)\|_*\\
&=\sum_{i=1}^p\|N_{\lambda_i}(x_1,x_2,\dots,x_p)-N_{\lambda_i}(\widetilde{x}_1,\widetilde{x}_2,\dots,\widetilde{x}_p)\|_i\\
&\leq\sum_{i=1}^p\big(\vartheta_i\|x_i-\widetilde{x}_i\|_i+\frac{\tau_i\lambda_i}{\gamma_i}\sum_{j\in\Gamma\setminus\{i\}}\sigma_{i,j}\|x_j-\widetilde{x}_j\|_j\big)\\
&=(\vartheta_1+\sum_{k=2}^p\frac{\tau_k\lambda_k}{\gamma_k}\sigma_{k,1})\|x_1-\widetilde{x}_1\|_1\\
&\quad+(\vartheta_2+\sum_{k\in\Gamma\setminus\{2\}}\frac{\tau_k\lambda_k}{\gamma_k}\sigma_{k,2})\|x_2-\widetilde{x}_2\|_2\\
&\quad+\cdots+(\vartheta_p+\sum_{k=1}^{p-1}\frac{\tau_k\lambda_k}{\gamma_k}\sigma_{k,p})\|x_p-\widetilde{x}_p\|_p
\end{aligned}\tag{3.13}$$

$$\leq \theta \sum_{i=1}^{p} \|x_i - \widetilde{x}_i\|_i = \theta\|(x_1, x_2, \dots, x_p) - (\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_p)\|_*,$$

where

$$\theta = \max\big\{\vartheta_i + \sum_{k\in\Gamma\setminus\{i\}} \frac{\tau_k\lambda_k}{\gamma_k}\sigma_{k,i} : i = 1, 2, \dots, p\big\}.$$

Evidently, (3.3) ensures that $0 \leq \theta < 1$ and so from (3.2) it follows that $M_{\lambda_1,\lambda_2,\dots,\lambda_p}$ is a contraction mapping. The Banach fixed point theorem guarantees the existence of a unique point $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p) \in \prod_{i=1}^{p} X_i$ such that

$$M_{\lambda_1,\lambda_2,\dots,\lambda_p}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p) = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p).$$

Then, making use of (3.4) and (3.5), we infer that for $i = 1, 2, \dots, p$,

$$g_i(\bar{x}_i) = R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[(P_i \circ g_i)(\bar{x}_i) - \lambda_i(T_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p) - F_i(\bar{x}_i))].$$

Now, Lemma 3.1 implies that $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p) \in \prod_{i=1}^{p} X_i$ is a unique solution of the SGNVLI (3.1). This completes the proof. □

It is known that nonexpansive mapping is that which has Lipschitz's constant equal to 1. In other words, for a given real normed space X with a norm $\|.\|$, a mapping $T : X \to X$ is called *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in X$. As we know, fixed point theory is an immensely active area of research due to its applications in multiple fields. In fact, it consists of many fields of mathematics such as mathematical analysis, general topology and functional analysis. Since 1965 considerable efforts have been aimed to study the fixed point theory for nonexpansive mappings in the setting of different spaces. At the same time, because of the existence of a strong connection between the classes of monotone and accretive mappings, and the class of nonexpansive mappings, the theory of nonexpansive mappings has increasingly received much attentions, and has been greatly extended and generalized in different contexts. For example, in 1972, Goebel and Kirk [27] succeeded to introduce the class of asymptotically nonexpansive mappings as an interesting generalization of the class of nonexpansive mappings as follows.

Definition 3.8 ([27]). The mapping $T : X \to X$ is said to be asymptotically nonexpansive if, there exists a sequence $\{a_n\} \subset (0, +\infty)$ with $\lim_{n\to\infty} a_n = 0$ such that for each $n \in \mathbb{N}$,

$$\|T^n(x) - T^n(y)\| \leq (1 + a_n)\|x - y\|, \quad \forall x, y \in X.$$

Equivalently, we say that T is asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for each $n \in \mathbb{N}$,

$$\|T^n(x) - T^n(y)\| \leq k_n\|x - y\|, \quad \forall x, y \in X.$$

The introduction of the notion of total asymptotically nonexpansive mapping was first made by Alber et al. [6] in 2006 in order to present a unifying framework for generalized nonexpansive mappings existing in the literature and to prove a general convergence theorem applicable to all these classes of mappings as follows.

Definition 3.9 ([6])**.** A mapping $T : X \to X$ is said to be total asymptotically nonexpansive (also referred to as $(\{a_n\}, \{b_n\}, \phi)$-total asymptotically nonexpansive) if, there exist nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $a_n, b_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in X$,

$$\|T^n(x) - T^n(y)\| \leq \|x - y\| + a_n\phi(\|x - y\|) + b_n, \quad \forall n \in \mathbb{N}.$$

It is important to emphasize that under some suitable conditions and using a modified Mann iteration process, the iterative approximation of fixed points of total asymptotically nonexpansive mappings is also studied in [6]. It is noteworthy that, in particular, every asymptotically nonexpansive mapping is total asymptotically nonexpansive with $b_n = 0$ (or equivalently $b_n = 0$ and $a_n = k_n - 1$) for all $n \in \mathbb{N}$ and $\phi(t) = t$ for all $t \geq 0$, but the converse need not be true. The following example shows that the class of total asymptotically nonexpansive mappings is broader than the class of asymptotically nonexpansive mappings.

Example 3.10. For $1 \leq p < \infty$, consider the classical space

$$l^p = \big\{x = \{x_n\}_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty, x_n \in \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}\big\},$$

consisting of all p-power summable sequences, with the p-norm $\|.\|_p$ defined on it by

$$\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}, \quad \forall x = \{x_n\}_{n\in\mathbb{N}} \in l^p.$$

Suppose further that B denote the closed unit ball in the Banach space l^p and consider $X := [0, \varsigma] \times B$, where $\varsigma \in (0, 1]$ is an arbitrary real constant. Furthermore, let the mapping $T : X \to X$ be defined by

$$T(u, x) = \begin{cases} \gamma(u, \widetilde{x}), & \text{if } u \in [0, \varsigma), \\ (0, \gamma\widetilde{x}), & \text{if } u = \varsigma, \end{cases}$$

where

$$\widetilde{x} = (\underbrace{0, 0, \ldots, 0}_{m \ times}, \alpha|x_1|^{\lambda_1}, 0, \alpha \sin^{q_1} |x_2|, 0, \alpha|x_3|^{\lambda_2}, 0, \alpha \sin^{q_2} |x_4|,$$
$$\ldots, \alpha|x_k|^{\lambda_{\frac{k+1}{2}}}, 0, \alpha \sin^{q_{\frac{k+1}{2}}} |x_{k+1}|, 0, \alpha x_{k+2}, 0, \alpha x_{k+3}, \ldots),$$

$\alpha, \gamma \in (0, 1)$ are arbitrary constants, k is an arbitrary but fixed odd natural number, and $m \geq k+1$ and $\lambda_i, q_i \in \mathbb{N}\backslash\{1\}$ $(i = 1, 2, \ldots, \frac{k+1}{2})$ are arbitrary but fixed natural numbers. In fact, $\widetilde{x} = \{\widetilde{x}_n\}_{n=1}^{\infty}$, where $\widetilde{x}_i = 0$ for all $1 \leq i \leq m$, $\widetilde{x}_{m+2i} = 0$ for all $i \in \mathbb{N}$,

$$\widetilde{x}_{m+2i-1} = \begin{cases} \alpha|x_i|^{\lambda_{\frac{i+1}{2}}}, & \text{if } i \in \{2s - 1 | s = 1, 2, \ldots, \frac{k+1}{2}\}, \\ \alpha \sin^{q_{\frac{i}{2}}} |x_i|, & \text{if } i \in \{2s | s = 1, 2, \ldots, \frac{k+1}{2}\}, \end{cases}$$

and $\widetilde{x}_{m+2i-1} = \alpha x_i$ for all $i \geq k+2$. It is plain that the mapping T is discontinuous at the points (ς, x) for all $x \in B$. This fact implies that T is not Lipschitzian and so it is not an asymptotically nonexpansive mapping. For all $(u,x),(v,y)\in[0,\varsigma)\times B$, we obtain

$$\begin{aligned}
&\|T(u,x)-T(v,y)\|_X=\|\gamma\big(u-v,\widetilde{x}-\widetilde{y}\big)\|_X\\
&=\|\gamma\big(u-v,(\underbrace{0,0,\dots,0}_{m\ times},\alpha(|x_1|^{\lambda_1}-|y_1|^{\lambda_1}),0,\alpha(\sin^{q_1}|x_2|-\sin^{q_1}|y_2|),0,\\
&\quad\alpha(|x_3|^{\lambda_2}-|y_3|^{\lambda_2}),0,\alpha(\sin^{q_2}|x_4|-\sin^{q_2}|y_4|),\dots,0,\alpha(|x_k|^{\lambda_{\frac{k+1}{2}}}-|y_k|^{\lambda_{\frac{k+1}{2}}}),0,\\
&\quad\alpha(\sin^{q_{\frac{k+1}{2}}}|x_{k+1}|-\sin^{q_{\frac{k+1}{2}}}|y_{k+1}|),0,\alpha(x_{k+2}-y_{k+2}),0,\alpha(x_{k+3}-y_{k+3}),\dots)\big)\|_X\\
&=\gamma\big(|u-v|+\big(\alpha^p\sum_{i=1}^{\frac{k+1}{2}}\big||x_{2i-1}|^{\lambda_i}-|y_{2i-1}|^{\lambda_i}\big|^p\\
&\quad+\alpha^p\sum_{i=1}^{\frac{k+1}{2}}\big|\sin^{q_i}|x_{2i}|-\sin^{q_i}|y_{2i}|\big|^p+\alpha^p\sum_{i=k+2}^{\infty}|x_i-y_i|^p\big)^{\frac{1}{p}}\big)\\
&\le|u-v|+\alpha\big(\sum_{i=1}^{\frac{k+1}{2}}\big(\sum_{j=1}^{\lambda_i}|x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1}\big)^p|x_{2i-1}-y_{2i-1}|^p\\
&\quad+\sum_{i=1}^{\frac{k+1}{2}}\big(\sum_{r=1}^{q_i}|x_{2i}|^{q_i-r}|y_{2i}|^{r-1}\big)^p|x_{2i}-y_{2i}|^p+\sum_{i=k+2}^{\infty}|x_i-y_i|^p\big)^{\frac{1}{p}}\big).
\end{aligned}\tag{3.14}$$

Since $x,y\in B$, it follows that $0\le|x_{2i-1}|^{\lambda_i-j},|y_{2i-1}|^{j-1}\le1$ for each $j\in\{1,2,\dots,\lambda_i\}$, and $0\le|x_{2i}|^{q_i-r},|y_{2i}|^{r-1}\le1$ for each $r\in\{1,2,\dots,q_i\}$ and $i\in\{1,2,\dots,\frac{k+1}{2}\}$. These facts imply that $0\le\sum_{j=1}^{\lambda_i}|x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1}\le\lambda_i$ and $0\le\sum_{r=1}^{q_i}|x_{2i}|^{q_i-r}|y_{2i}|^{r-1}\le q_i$ for each $i\in\{1,2,\dots,\frac{k+1}{2}\}$. Then, making use of (3.14) we conclude that for all $(u,x),(v,y)\in[0,\varsigma)\times B$,

$$\begin{aligned}
&\|T(u,x)-T(v,y)\|_X\\
&\le|u-v|+\alpha\big(\max\big\{\big(\sum_{j=1}^{\lambda_i}|x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1}\big)^p,\\
&\quad\big(\sum_{r=1}^{q_i}|x_{2i}|^{q_i-r}|y_{2i}|^{r-1}\big)^p,1:i=1,2,\dots,\frac{k+1}{2}\big\}\sum_{i=1}^{\infty}|x_i-y_i|^p\big)^{\frac{1}{p}}\\
&=|u-v|+\alpha\max\big\{\sum_{j=1}^{\lambda_i}|x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1},\\
&\quad\sum_{r=1}^{q_i}|x_{2i}|^{q_i-r}|y_{2i}|^{r-1},1:i=1,2,\dots,\frac{k+1}{2}\big\}\big(\sum_{i=1}^{\infty}|x_i-y_i|^p\big)^{\frac{1}{p}}\\
&=|u-v|+\alpha\max\big\{\sum_{j=1}^{\lambda_i}|x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1},
\end{aligned}\tag{3.15}$$

$$\sum_{r=1}^{q_i}|x_{2i}|^{q_i-r}|y_{2i}|^{r-1},1:i=1,2,\dots,\frac{k+1}{2}\}\|x-y\|_p.$$

If $u\in[0,\varsigma)$ and $v=\varsigma$, then in a similar fashion to the preceding analysis, in virtue of the fact that $0<|u-v|\leq\varsigma\leq 1$, one can show that

$$\begin{aligned}
\|T(u,x)-T(v,y)\|_X &= \|\gamma(u,\widetilde{x})-(0,\gamma\widetilde{y})\|_X=\gamma\|(u,\widetilde{x}-\widetilde{y})\|_X\\
&\leq\gamma\big(|u|+\alpha\max\{\sum_{j=1}^{\lambda_i}|x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1},\\
&\quad\sum_{r=1}^{q_i}|x_{2i}|^{q_i-r}|y_{2i}|^{r-1},1:i=1,2,\dots,\frac{k+1}{2}\}\|x-y\|_p\big)\\
&=\gamma\big(u+\alpha\max\{\sum_{j=1}^{\lambda_i}|x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1},\\
&\quad\sum_{r=1}^{q_i}|x_{2i}|^{q_i-r}|y_{2i}|^{r-1},1:i=1,2,\dots,\frac{k+1}{2}\}\|x-y\|_p\big)\\
&\leq\gamma\big(1+\alpha\max\{\sum_{j=1}^{\lambda_i}|x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1},\\
&\quad\sum_{r=1}^{q_i}|x_{2i}|^{q_i-r}|y_{2i}|^{r-1},1:i=1,2,\dots,\frac{k+1}{2}\}\|x-y\|_p\big)\\
&<|u-v|+\alpha\max\{\sum_{j=1}^{\lambda_i}|x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1},\\
&\quad\sum_{r=1}^{q_i}|x_{2i}|^{q_i-r}|y_{2i}|^{r-1},1:i=1,2,\dots,\frac{k+1}{2}\}\|x-y\|_p+\gamma.
\end{aligned}\tag{3.16}$$

If $u=v=\varsigma$, then by the same arguments as used in (3.14)-(3.16), it follows that for all $x\in B$,

$$\begin{aligned}
\|T(u,x)-T(v,y)\|_X &= \|(0,\gamma\widetilde{x})-(0,\gamma\widetilde{y})\|_X=\gamma\|(0,\widetilde{x}-\widetilde{y})\|_X\\
&=\gamma\alpha\max\{\sum_{j=1}^{\lambda_i}|x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1},\\
&\quad\sum_{r=1}^{q_i}|x_{2i}|^{q_i-r}|y_{2i}|^{r-1},1:i=1,2,\dots,\frac{k+1}{2}\}\|x-y\|_p\\
&\leq|u-v|+\alpha\max\{\sum_{j=1}^{\lambda_i}|x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1},\\
&\quad\sum_{r=1}^{q_i}|x_{2i}|^{q_i-r}|y_{2i}|^{r-1},1:i=1,2,\dots,\frac{k+1}{2}\}\|x-y\|_p+\gamma.
\end{aligned}\tag{3.17}$$

Making use of (3.15)-(3.17), for all $(u,x),(v,y)\in X$, we yield

$$\begin{aligned}\|T(u,x)-T(v,y)\|_X &\leq |u-v|+\alpha\max\Big\{\sum_{j=1}^{\lambda_i}|x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1},\\ &\quad \sum_{r=1}^{q_i}|x_{2i}|^{q_i-r}|y_{2i}|^{r-1},1:i=1,2,\dots,\frac{k+1}{2}\Big\}\|x-y\|_p+\gamma\\ &\leq |u-v|+\|x-y\|_p+\alpha\max\Big\{\sum_{j=1}^{\lambda_i}|x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1},\\ &\quad \sum_{r=1}^{q_i}|x_{2i}|^{q_i-r}|y_{2i}|^{r-1},1:i=1,2,\dots,\frac{k+1}{2}\Big\}(|u-v|+\|x-y\|_p)\\ &\quad +\gamma.\end{aligned}$$

For all $n\geq 2$ and $(u,x)\in[0,\varsigma)\times B$, we have

$$\begin{aligned}T^n(u,x)=\gamma^n\big(u,(&\underbrace{0,0,\dots,0}_{(2^n-1)m\ times},\alpha^n|x_1|^{\lambda_1},\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\alpha^n\sin^{q_1}|x_2|,\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\\ &\alpha^n|x_3|^{\lambda_2},\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\alpha^n\sin^{q_2}|x_4|,\dots,\alpha^n|x_k|^{\lambda_{\frac{k+1}{2}}},\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\\ &\alpha^n\sin^{q_{\frac{k+1}{2}}}|x_{k+1}|,\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\alpha^n x_{k+2},\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\alpha^n x_{k+3},\dots)\big).\end{aligned}$$

Then, for all $(u,x),(v,y)\in[0,\varsigma)\times B$ and $n\geq 2$, by using the same arguments as for (3.14) and (3.15), one can prove that

$$\begin{aligned}\|T^n(u,x)-T^n(v,y)\|_X &\leq |u-v|+\alpha^n\max\Big\{\sum_{j=1}^{\lambda_i}|x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1},\\ &\quad \sum_{r=1}^{q_i}|x_{2i}|^{q_i-r}|y_{2i}|^{r-1},1:i=1,2,\dots,\frac{k+1}{2}\Big\}\|x-y\|_p.\end{aligned}\tag{3.18}$$

If $u\in[0,\varsigma)$ and $v=\varsigma$, then for each $x\in B$ and $n\geq 2$, we have $T^n(u,x)=\gamma^n(u,\hat{x})$ and $T^n(v,x)=(0,\gamma^n\hat{x})=\gamma^n(0,\hat{x})$, where

$$\begin{aligned}\hat{x}=(&\underbrace{0,0,\dots,0}_{(2^n-1)m\ times},\alpha^n|x_1|^{\lambda_1},\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\alpha^n\sin^{q_1}|x_2|,\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\\ &\alpha^n|x_3|^{\lambda_2},\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\alpha^n\sin^{q_2}|x_4|,\dots,\alpha^n|x_k|^{\lambda_{\frac{k+1}{2}}},\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\\ &\alpha^n\sin^{q_{\frac{k+1}{2}}}|x_{k+1}|,\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\alpha^n x_{k+2},\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\alpha^n x_{k+3},\dots).\end{aligned}$$

Thanks to the fact that $0 < |u - v| \leq \varsigma \leq 1$, by an argument analogous to the previous one, for all $n \geq 2$ and $x, y \in B$, one can show that

$$\begin{aligned}
&\|T^n(u,x) - T^n(v,y)\|_X \\
&\leq \gamma^n\big(|u| + \alpha^n \max\{\sum_{j=1}^{\lambda_i} |x_{2i-1}|^{\lambda_i - j}|y_{2i-1}|^{j-1}, \\
&\quad \sum_{r=1}^{q_i} |x_{2i}|^{q_i - r}|y_{2i}|^{r-1}, 1 : i = 1, 2, \ldots, \frac{k+1}{2}\}\|x - y\|_p\big) \\
&\leq \gamma^n\big(\varsigma + \alpha^n \max\{\sum_{j=1}^{\lambda_i} |x_{2i-1}|^{\lambda_i - j}|y_{2i-1}|^{j-1}, \\
&\quad \sum_{r=1}^{q_i} |x_{2i}|^{q_i - r}|y_{2i}|^{r-1}, 1 : i = 1, 2, \ldots, \frac{k+1}{2}\}\|x - y\|_p\big) \\
&\leq \gamma^n(1 + \alpha^n \max\{\sum_{j=1}^{\lambda_i} |x_{2i-1}|^{\lambda_i - j}|y_{2i-1}|^{j-1}, \\
&\quad \sum_{r=1}^{q_i} |x_{2i}|^{q_i - r}|y_{2i}|^{r-1}, 1 : i = 1, 2, \ldots, \frac{k+1}{2}\}\|x - y\|_p\big) \\
&\leq |u - v| + \alpha^n \max\{\sum_{j=1}^{\lambda_i} |x_{2i-1}|^{\lambda_i - j}|y_{2i-1}|^{j-1}, \\
&\quad \sum_{r=1}^{q_i} |x_{2i}|^{q_i - r}|y_{2i}|^{r-1}, 1 : i = 1, 2, \ldots, \frac{k+1}{2}\}\|x - y\|_p + \gamma^n.
\end{aligned} \tag{3.19}$$

For the case when $u = v = \varsigma$, for all $x \in B$ and $n \geq 2$, we have $T^n(u,x) = T^n(v,y) = (0, \gamma^n \hat{x}) = \gamma^n(0, \hat{x})$ and

$$\begin{aligned}
\|T^n(u,x) - T^n(v,y)\|_X &\leq \gamma^n \alpha^n \max\{\sum_{j=1}^{\lambda_i} |x_{2i-1}|^{\lambda_i - j}|y_{2i-1}|^{j-1}, \\
&\quad \sum_{r=1}^{q_i} |x_{2i}|^{q_i - r}|y_{2i}|^{r-1}, 1 : i = 1, 2, \ldots, \frac{k+1}{2}\}\|x - y\|_p \\
&\leq |u - v| + \alpha^n \max\{\sum_{j=1}^{\lambda_i} |x_{2i-1}|^{\lambda_i - j}|y_{2i-1}|^{j-1}, \\
&\quad \sum_{r=1}^{q_i} |x_{2i}|^{q_i - r}|y_{2i}|^{r-1}, 1 : i = 1, 2, \ldots, \frac{k+1}{2}\}\|x - y\|_p \\
&\leq |u - v| + \alpha^n \max\{\sum_{j=1}^{\lambda_i} |x_{2i-1}|^{\lambda_i - j}|y_{2i-1}|^{j-1},
\end{aligned} \tag{3.20}$$

$$\sum_{r=1}^{q_i} |x_{2i}|^{q_i-r}|y_{2i}|^{r-1}, 1 : i = 1, 2, \dots, \frac{k+1}{2}\}\|x - y\|_p + \gamma^n.$$

Relying on the fact that $0 < \gamma < 1$, using (3.18)-(3.20), we deduce that for all $(u, x), (v, y) \in X$ and $n \geq 2$,

$$\begin{aligned} &\|T^n(u,x) - T^n(v,y)\|_X \\ &\leq |u - v| + \alpha^n \max\{\sum_{j=1}^{\lambda_i} |x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1}, \\ &\quad \sum_{r=1}^{q_i} |x_{2i}|^{q_i-r}|y_{2i}|^{r-1}, 1 : i = 1, 2, \dots, \frac{k+1}{2}\}\|x - y\|_p + \gamma^n \\ &\leq |u - v| + \|x - y\|_p + \alpha^n \max\{\sum_{j=1}^{\lambda_i} |x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1}, \\ &\quad \sum_{r=1}^{q_i} |x_{2i}|^{q_i-r}|y_{2i}|^{r-1}, 1 : i = 1, 2, \dots, \frac{k+1}{2}\}(|u - v| \\ &\quad + \|x - y\|_p) + \gamma^n. \end{aligned} \tag{3.21}$$

Employing (3.18) and (3.21) and taking into account that for each $i \in \{1, 2, \dots, \frac{k+1}{2}\}$, $0 \leq \sum_{j=1}^{\lambda_i} |x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1} \leq \lambda_i$ and $0 \leq \sum_{r=1}^{q_i} |x_{2i}|^{q_i-r}|y_{2i}|^{r-1} \leq q_i$, it follows that for all $(u, x), (v, y) \in X$ and $n \in \mathbb{N}$,

$$\begin{aligned} &\|T^n(u,x) - T^n(v,y)\|_X \\ &\leq |u - v| + \|x - y\|_p + \alpha^n \max\{\sum_{j=1}^{\lambda_i} |x_{2i-1}|^{\lambda_i-j}|y_{2i-1}|^{j-1}, \\ &\quad \sum_{r=1}^{q_i} |x_{2i}|^{q_i-r}|y_{2i}|^{r-1}, 1 : i = 1, 2, \dots, \frac{k+1}{2}\}(|u - v| + \|x - y\|_p) + \gamma^n \\ &\leq \|(u,x) - (v,y)\|_X + \alpha^n\theta\|(u,x) - (v,y)\|_X + \gamma^n, \end{aligned} \tag{3.22}$$

where $\theta = \max\{\lambda_i, q_i : i = 1, 2, \dots, \frac{k+1}{2}\}$. Taking $\mu_n = \alpha^n$ and $b_n = \gamma^n$ for each $n \in \mathbb{N}$, we have $b_n, \mu_n \to 0$ as $n \to \infty$ because $\alpha, \gamma \in (0, 1)$.

Let us now define the mapping $\phi : [0, +\infty) \to [0, +\infty)$ as $\phi(t) = \theta t$ for all $t \in [0, +\infty)$. Then, using (3.22), for all $(u, x), (v, y) \in X$ and $n \in \mathbb{N}$, we obtain

$$\|T^n(u,x) - T^n(v,y)\|_X \leq \|(u,x) - (v,y)\|_X + \mu_n\phi(\|(u,x) - (v,y)\|_X) + b_n,$$

which means that T is an $(\{\alpha^n\}, \{\gamma^n\}, \phi)$-total asymptotically nonexpansive mapping.

Lemma 3.11. *Let, for each $i \in \{1, 2 \dots, p\}$, X_i be a real Banach space with a norm $\|.\|_i$, and let $S_i : X_i \to X_i$ be an $(\{a_{n,i}\}_{n=1}^{\infty}, \{b_{n,i}\}_{n=1}^{\infty}, \phi_i)$-total asymptotically nonexpansive mapping. Suppose further that Q and ϕ are self-mappings of $\prod_{i=1}^{p} X_i$ and $\mathbb{R}^+$, respectively, defined by*

$$Q(x_1, x_2, \dots, x_p) = (S_1x_1, S_2x_2, \dots, S_px_p), \tag{3.23}$$

for all $(x_1, x_2, \dots, x_p) \in \prod_{i=1}^{p} X_i$ *and*

$$\phi(t) = \max\{\phi_i(t) : i = 1, 2, \dots, p\}, \quad \forall t \in \mathbb{R}^+. \tag{3.24}$$

Then Q *is a* $\big(\{\sum_{i=1}^{p} a_{n,i}\}_{n=1}^{\infty}, \{\sum_{i=1}^{p} b_{n,i}\}_{n=1}^{\infty}, \phi\big)$*-total asymptotically nonexpansive mapping.*

Proof. Since for each $i \in \{1, 2, \dots, l\}$, S_i is an $\big(\{a_{n,i}\}_{n=1}^{\infty}, \{b_{n,i}\}_{n=1}^{\infty}, \phi_i\big)$-total asymptotically nonexpansive mapping and $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function, for all $(x_1, x_2, \dots, x_p), (y_1, y_2, \dots, y_p) \in \prod_{i=1}^{p} X_i$ and $n \in \mathbb{N}$, yields

$$\begin{aligned}
&\|Q^n(x_1, x_2, \dots, x_p) - Q^n(y_1, y_2, \dots, y_p)\|_* \\
&= \|(S_1^n x_1, S_2^n x_2, \dots, S_p^n x_p) - (S_1^n y_1, S_2^n y_2, \dots, S_p^n y_p)\|_* \\
&= \|(S_1^n x_1 - S_1^n y_1, S_2^n x_2 - S_2^n y_2, \dots, S_p^n x_p - S_p^n y_p)\|_* \\
&= \sum_{i=1}^{p} \|S_i^n x_i - S_i^n y_i\|_i \\
&\leq \sum_{i=1}^{p} \big(\|x_i - y_i\|_i + a_{n,i}\phi_i(\|x_i - y_i\|_i) + b_{n,i}\big) \\
&\leq \sum_{i=1}^{p} \|x_i - y_i\|_i + \sum_{i=1}^{p} a_{n,i}\phi(\|x_i - y_i\|_i) + \sum_{i=1}^{p} b_{n,i} \\
&\leq \sum_{i=1}^{p} \|x_i - y_i\|_i + \sum_{i=1}^{p} a_{n,i}\phi(\sum_{j=1}^{p} \|x_j - y_j\|_j) + \sum_{i=1}^{p} b_{n,i} \\
&= \|(x_1, x_2, \dots, x_p) - (y_1, y_2, \dots, y_p)\|_* \\
&\quad + \sum_{i=1}^{p} a_{n,i}\phi(\|(x_1, x_2, \dots, x_p) - (y_1, y_2, \dots, y_p)\|_*) + \sum_{i=1}^{p} b_{n,i},
\end{aligned} \tag{3.25}$$

where $\|.\|_*$ is a norm on $\prod_{i=1}^{p} X_i$ defined by (3.12). Clearly, (3.25) gives the desired result. □

4. Iterative algorithm and convergence theorem

For each $i \in \Gamma = \{1, 2, \dots, p\}$, let X_i be a real reflexive Banach space with the norm $\|.\|_i$ and dual space X_i^*, and $S_i : X_i \to X_i$ be an $(\{a_{n,i}\}_{n=1}^{\infty}, \{b_{n,i}\}_{n=1}^{\infty}, \phi_i)$-total asymptotically nonexpansive mapping. Assume further that Q is a self-mapping of $\prod_{i=1}^{p} X_i$ defined by (3.23). Denote by $\mathrm{Fix}(S_i)$ $(i = 1, 2, \dots, p)$ and $\mathrm{Fix}(Q)$, respectively, the sets of all the fixed points of S_i $(i = 1, 2, \dots, p)$ and Q. Furthermore, denote the set of all the solutions of SGNVLI (3.1) by Φ_{SGNVLI}. Then, using (3.23), we deduce that for any $(x_1, x_2, \dots, x_p) \in \prod_{j=1}^{p} X_j$, $(x_1, x_2, \dots, x_p) \in \mathrm{Fix}(Q)$ if and only if $x_i \in \mathrm{Fix}(S_i)$ for each $i \in \Gamma$, i.e., $\mathrm{Fix}(Q) = \mathrm{Fix}(S_1, S_2, \dots, S_p) = \prod_{i=1}^{p} \mathrm{Fix}(S_i)$. If $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p) \in \mathrm{Fix}(Q) \cap \Phi_{\mathrm{SGNVLI}}$, then from Lemma 3.1 it follows that for each

$n \in \mathbb{N}$,

$$
\begin{aligned}
\bar{x}_i = S_i^n \bar{x}_i &= \bar{x}_i - g_i(\bar{x}_i) + R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[(P_i \circ g_i)(\bar{x}_i) \\
&\quad - \lambda_i(T_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p) - F_i(\bar{x}_i))] \\
&= S_i^n\big(\bar{x}_i - g_i(\bar{x}_i) + R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[(P_i \circ g_i)(\bar{x}_i) \\
&\quad - \lambda_i(T_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p) - F_i(\bar{x}_i))]\big).
\end{aligned} \tag{4.1}
$$

In the light of the fixed point formulation (4.1), we are able to construct the following perturbed q-step iterative algorithm with mixed errors for finding a common element of the two sets of $\mathrm{Fix}(Q) = \mathrm{Fix}(S_1, S_2, \dots, S_p)$ and Φ_{SGNVLI}.

Algorithm 4.1. Let $X_i, F_i, T_i, \varphi_i, \eta_i, P_i, g_i, h_i$ $(i = 1, 2, \dots, p)$ be the same as in Lemma 3.1 such that for each $i \in \Gamma$, $g_i(X_i) \subseteq \mathrm{dom}(P_i)$. Assume further that for each $i \in \Gamma$, $S_i : X_i \to X_i$ is an $(\{a_{n,i}\}_{n=0}^{\infty}, \{b_{n,i}\}_{n=0}^{\infty}, \phi_i)$-total asymptotically nonexpansive mapping. For any given $(x_{0,1}, x_{0,2}, \dots, x_{0,p}) \in \prod_{i=1}^{p} X_i$, define the iterative sequence $\{(x_{n,1}, x_{n,2}, \dots, x_{n,p})\}_{n=0}^{\infty}$ in $\prod_{i=1}^{p} X_i$ by the iterative schemes

$$
\begin{cases}
x_{n+1,i} &= (1-\alpha_{n,1})x_{n,i} + \alpha_{n,1} S_i^n\{z_{n,i}^{(1)} - g_i(z_{n,i}^{(1)}) \\
&\quad + R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,z_{n,i}^{(1)})}[G_i(z_{n,1}^{(1)}, z_{n,2}^{(1)}, \dots, z_{n,p}^{(1)})]\} \\
&\quad + \alpha_{n,1} e_{n,i}^{(1)} + r_{n,i}^{(1)}, \quad i = 1, 2, \dots, p, \\
z_{n,i}^{(j)} &= (1-\alpha_{n,j+1})x_{n,i} + \alpha_{n,j+1} S_i^n\{z_{n,i}^{(j+1)} - g_i(z_{n,i}^{(j+1)}) \\
&\quad + R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,z_{n,i}^{(j+1)})}[G_i(z_{n,1}^{(j+1)}, z_{n,2}^{(j+1)}, \dots, z_{n,p}^{(j+1)})]\} \\
&\quad + \alpha_{n,j+1} e_{n,i}^{(j+1)} + r_{n,i}^{(j+1)}, \quad j = 1, 2, \dots, q-2, \\
z_{n,i}^{(q-1)} &= (1-\alpha_{n,q})x_{n,i} + \alpha_{n,q} S_i^n\{x_{n,i} - g_i(x_{n,i}) \\
&\quad + R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,x_{n,i})}[G_i(x_{n,1}, x_{n,2}, \dots, x_{n,p})]\} \\
&\quad + \alpha_{n,q} e_{n,i}^{(q)} + r_{n,i}^{(q)}, \quad i = 1, 2, \dots, p,
\end{cases} \tag{4.2}
$$

where for $i = 1, 2, \dots, p$; $j = 1, 2, \dots, q-1$ and for all $n \in \mathbb{N} \cup \{0\}$,

$$
\begin{cases}
G_i(z_{n,1}^{(j)}, z_{n,2}^{(j)}, \dots, z_{n,p}^{(j)}) = (P_i \circ g_i)(z_{n,i}^{(j)}) - \lambda_i(T_i(z_{n,1}^{(j)}, z_{n,2}^{(j)}, \dots, z_{n,p}^{(j)}) - F_i(z_{n,i}^{(j)})), \\
G_i(x_{n,1}, x_{n,2}, \dots, x_{n,p}) = (P_i \circ g_i)(x_{n,i}) - \lambda_i(T_i(x_{n,1}, x_{n,2}, \dots, x_{n,p}) - F_i(x_{n,i})),
\end{cases}
$$

$\lambda_i > 0$ $(i = 1, 2, \dots p)$ are p constants, $\{\alpha_{n,j}\}_{n=0}^{\infty}$ $(j = 1, 2, \dots, q)$ are q sequences in $[0, 1]$ such that $\sum_{n=0}^{\infty} \prod_{j=1}^{q} \alpha_{n,j} = \infty$, and $\{e_{n,i}^{(j)}\}_{n=0}^{\infty}$, $\{r_{n,i}^{(j)}\}_{n=0}^{\infty}$ $(i = 1, 2, \dots, p; j = 1, 2, \dots, q)$ are $2pq$ sequences to take into account a possible inexact computation of the proximal-point mapping point satisfying the following conditions: For $j = 1, 2, \dots, q$, $\{(e_{n,1}^{(j)}, e_{n,2}^{(j)}, \dots, e_{n,p}^{(j)})\}_{n=0}^{\infty}$ and $\{(r_{n,1}^{(j)}, r_{n,2}^{(j)}, \dots, r_{n,p}^{(j)})\}_{n=0}^{\infty}$ are $2q$ in $\prod_{i=1}^{p} X_i$ such that for all $n \geq 0$, $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$,

$$
\begin{cases}
e_{n,i}^{(j)} = e_{n,i}^{\prime(j)} + e_{n,i}^{\prime\prime(j)}, \\
\lim_{n\to\infty} \|(e_{n,1}^{\prime(j)}, e_{n,2}^{\prime(j)}, \dots, e_{n,p}^{\prime(j)})\|_* = 0, \\
\sum_{n=0}^{\infty} \|(e_{n,1}^{\prime\prime(j)}, e_{n,2}^{\prime\prime(j)}, \dots, e_{n,p}^{\prime\prime(j)})\|_* < \infty, \\
\sum_{n=0}^{\infty} \|(r_{n,1}^{(j)}, r_{n,2}^{(j)}, \dots, r_{n,p}^{(j)})\|_* < \infty.
\end{cases} \tag{4.3}
$$

If for each $i \in \Gamma$, $S_i \equiv I_i$, the identity mapping on X_i, $q = 1$, $e^{(1)}_{n,i} = e_{n,i}$, $r^{(1)}_{n,i} = r_{n,i}$ and $\alpha_{n,1} = \alpha_n$ for all $n \geq 0$, then Algorithm 4.1 collapses to the following iterative algorithm.

Algorithm 4.2. Let $X_i, F_i, T_i, \varphi_i, \eta_i, P_i, g_i, h_i$ $(i = 1, 2, \dots, p)$ be the same as in Lemma 3.1 such that for each $i \in \Gamma$, $g_i(X_i) \subseteq \mathrm{dom}(P_i)$. For any given $(x_{0,1}, x_{0,2}, \dots, x_{0,p}) \in \prod_{i=1}^{p} X_i$, compute the iterative sequence $\{(x_{n,1}, x_{n,2}, \dots, x_{n,p})\}_{n=0}^{\infty}$ in $\prod_{i=1}^{p} X_i$ in the following way:

$$\begin{aligned} x_{n+1,i} = (1-\alpha_n)x_{n,i} &+ \alpha_n\{x_{n,i} - g_i(x_{n,i}) \\ &+ R^{\partial_{\eta_i}\varphi_i(.,x_{n,i})}_{\lambda_i,P_i}[G_i(x_{n,1}, x_{n,2}, \dots, x_{n,p})]\} \\ &+ \alpha_n e_{n,i} + r_{n,i}, \quad i = 1, 2, \dots, p, \end{aligned}$$

where $n = 0, 1, 2, \dots$; $\lambda_i > 0$ $(i = 1, 2, \dots, p)$ are p constants, $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$, and for each $i \in \Gamma$, $\{e_{n,i}\}_{n=0}^{\infty}$ and $\{r_{n,i}\}_{n=0}^{\infty}$ are two sequences in X_i, to take into account a possible inexact computation of the proximal-point mapping point satisfying the following conditions:

$$\begin{cases} e_{n,i} = e'_{n,i} + e''_{n,i}, \\ \lim_{n\to\infty} \|(e'_{n,1}, e'_{n,2}, \dots, e'_{n,p})\|_* = 0, \\ \sum_{n=0}^{\infty} \|(e''_{n,1}, e''_{n,2}, \dots, e''_{n,p})\|_* < \infty, \\ \sum_{n=0}^{\infty} \|(r_{n,1}, r_{n,2}, \dots, r_{n,p})\|_* < \infty. \end{cases}$$

Before dealing with the convergence analysis of Algorithm 4.1 for computation of a common element of the two sets of $\mathrm{Fix}(Q)$ and Φ_{SGNVLI}, we need to recall the following lemma.

Lemma 4.3. *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences satisfying the following conditions: there exists a natural number n_0 such that*

$$a_{n+1} \leq (1-t_n)a_n + b_n t_n + c_n, \quad \forall n \geq n_0,$$

where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Proof. The proof follows directly from Lemma 2 in [34]. □

Theorem 4.4. *Let $X_i, F_i, T_i, P_i, \varphi_i, \eta_i, g_i, h_i$ $(i = 1, 2, \dots, p)$ be the same as in Theorem 3.7 and let all the conditions of Theorem 3.7 hold. Suppose that for each $i \in \Gamma$, $S_i : X_i \to X_i$ is an $(\{a_{n,i}\}_{n=0}^{\infty}, \{b_{n,i}\}_{n=0}^{\infty}, \phi_i)$-total asymptotically nonexpansive mapping and Q is a self-mapping of $\prod_{i=1}^{p} X_i$ defined by (3.23) such that $\mathrm{Fix}(Q) \cap \Phi_{\mathrm{SGNVLI}} \neq \emptyset$. Moreover, let there exists a constant $\alpha > 0$ such that $\prod_{j=1}^{q} \alpha_{n,j} \geq \alpha$ for all $n \in \mathbb{N} \cup \{0\}$. Then, the iterative sequence $\{(x_{n,1}, x_{n,2}, \dots, x_{n,p})\}_{n=0}^{\infty}$ generated by Algorithm 4.1 converges strongly to the only element $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)$ of $\mathrm{Fix}(Q) \cap \Phi_{\mathrm{SGNVLI}}$.*

Proof. In view of the fact that all the conditions of Theorem 3.7 hold, the existence of a unique solution $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p) \in \prod_{i=1}^{p} X_i$ for the SGNVLI (3.1) ensures by Theorem 3.7. Then from Lemma 3.1 it follows that for $i = 1, 2, \dots, p$,

$$\begin{aligned} \bar{x}_i = \bar{x}_i - g_i(\bar{x}_i) &+ R^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}_{\lambda_i,P_i}[(P_i \circ g_i)(\bar{x}_i) \\ &- \lambda_i(T_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p) - F_i(\bar{x}_i))]. \end{aligned} \tag{4.4}$$

Since Φ_{SGNVLI} is a singleton set and $\text{Fix}(Q) \cap \Phi_{\text{SGNVLI}} \neq \emptyset$, we deduce that $\bar{x}_i \in \text{Fix}(S_i)$ for $i = 1, 2, \dots, p$. In the light of this fact, (4.4) can be written as follows:

$$\begin{aligned} \bar{x}_i = (1-\alpha_{n,j})\bar{x}_i + \alpha_{n,j} S_i^n \big\{ \bar{x}_i - g_i(\bar{x}_i) \\ + R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[G_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)] \big\}, \end{aligned} \tag{4.5}$$

where for $j = 1, 2, \dots, q$, the sequences $\{\alpha_{n,j}\}_{n=0}^{\infty}$ are the same as in Algorithm 4.1 and for $i = 1, 2, \dots, p$,

$$G_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p) = (P_i \circ g_i)(\bar{x}_i) - \lambda_i(T_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p) - F_i(\bar{x}_i)).$$

Making use of (4.2), (4.5), Theorem 2.11 and the assumptions, for each $i \in \Gamma$ and $n \geq 0$, yields

$$\begin{aligned} \|x_{n+1,i} - \bar{x}_i\|_i &= \|(1-\alpha_{n,1})x_{n,i} + \alpha_{n,1} S_i^n \big\{ z_{n,i}^{(1)} - g_i(z_{n,i}^{(1)}) \\ &\quad + R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,z_{n,i}^{(1)})}[G_i(z_{n,1}^{(1)}, z_{n,2}^{(1)}, \dots, z_{n,p}^{(1)})] \big\} \\ &\quad + \alpha_{n,1} e_{n,i}^{(1)} + r_{n,i}^{(1)} - \big((1-\alpha_{n,1})\bar{x}_i + \alpha_{n,1} S_i^n \big\{ \bar{x}_i - g_i(\bar{x}_i) \\ &\quad + R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[G_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)] \big\} \big) \|_i \\ &\leq (1-\alpha_{n,1})\|x_{n,i} - \bar{x}_i\|_i + \alpha_{n,1} \| S_i^n \big\{ z_{n,i}^{(1)} - g_i(z_{n,i}^{(1)}) \\ &\quad + R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,z_{n,i}^{(1)})}[G_i(z_{n,1}^{(1)}, z_{n,2}^{(1)}, \dots, z_{n,p}^{(1)})] \big\} \\ &\quad - S_i^n \big\{ \bar{x}_i - g_i(\bar{x}_i) + R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[G_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)] \big\} \|_i \\ &\quad + \alpha_{n,1} \|e_{n,i}^{(1)}\|_i + \|r_{n,i}^{(1)}\|_i \\ &\leq (1-\alpha_{n,1})\|x_{n,i} - \bar{x}_i\|_i + \alpha_{n,1} \big(\| z_{n,i}^{(1)} - g_i(z_{n,i}^{(1)}) \\ &\quad + R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,z_{n,i}^{(1)})}[G_i(z_{n,1}^{(1)}, z_{n,2}^{(1)}, \dots, z_{n,p}^{(1)})] \\ &\quad - \big(\bar{x}_i - g_i(\bar{x}_i) + R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[G_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)] \big) \|_i \\ &\quad + a_{n,i} \phi_i \big(\| z_{n,i}^{(1)} - g_i(z_{n,i}^{(1)}) + R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,z_{n,i}^{(1)})}[G_i(z_{n,1}^{(1)}, z_{n,2}^{(1)}, \dots, z_{n,p}^{(1)})] \\ &\quad - \big(\bar{x}_i - g_i(\bar{x}_i) + R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[G_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)] \big) \|_i \big) \\ &\quad + b_{n,i} \big) + \alpha_{n,1} \|e_{n,i}^{(1)}\|_i + \|r_{n,i}^{(1)}\|_i \\ &\leq (1-\alpha_{n,1})\|x_{n,i} - \bar{x}_i\|_i + \alpha_{n,1} \big(\| z_{n,i}^{(1)} - \bar{x}_i - (g_i(z_{n,i}^{(1)}) - g_i(\bar{x}_i)) \|_i \\ &\quad + \| R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,z_{n,i}^{(1)})}[G_i(z_{n,1}^{(1)}, z_{n,2}^{(1)}, \dots, z_{n,p}^{(1)})] \\ &\quad - R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[G_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)] \|_i \\ &\quad + a_{n,i} \phi_i \big(\| z_{n,i}^{(1)} - \bar{x}_i - (g_i(z_{n,i}^{(1)}) - g_i(\bar{x}_i)) \|_i \\ &\quad + \| R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,z_{n,i}^{(1)})}[G_i(z_{n,1}^{(1)}, z_{n,2}^{(1)}, \dots, z_{n,p}^{(1)})] \\ &\quad - R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[G_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)] \|_i \big) + b_{n,i} \big) + \alpha_{n,1} \|e_{n,i}^{(1)}\|_i + \|r_{n,i}^{(1)}\|_i \end{aligned} \tag{4.6}$$

$$
\begin{aligned}
&\leq (1-\alpha_{n,1})\|x_{n,i}-\bar{x}_i\|_i + \alpha_{n,1}\big(\|z_{n,i}^{(1)}-\bar{x}_i-(g_i(z_{n,i}^{(1)})-g_i(\bar{x}_i))\|_i \\
&\quad + \|R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,z_{n,i}^{(1)})}[G_i(z_{n,1}^{(1)},z_{n,2}^{(1)},\dots,z_{n,p}^{(1)})] - R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[G_i(z_{n,1}^{(1)},z_{n,2}^{(1)},\dots,z_{n,p}^{(1)})]\|_i \\
&\quad + \|R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[G_i(z_{n,1}^{(1)},z_{n,2}^{(1)},\dots,z_{n,p}^{(1)})] - R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[G_i(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)]\|_i \\
&\quad + a_{n,i}\phi_i\big(\|z_{n,i}^{(1)}-\bar{x}_i-(g_i(z_{n,i}^{(1)})-g_i(\bar{x}_i))\|_i \\
&\quad + \|R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,z_{n,i}^{(1)})}[G_i(z_{n,1}^{(1)},z_{n,2}^{(1)},\dots,z_{n,p}^{(1)})] - R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[G_i(z_{n,1}^{(1)},z_{n,2}^{(1)},\dots,z_{n,p}^{(1)})]\|_i \\
&\quad + \|R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[G_i(z_{n,1}^{(1)},z_{n,2}^{(1)},\dots,z_{n,p}^{(1)})] - R_{\lambda_i,P_i}^{\partial_{\eta_i}\varphi_i(.,\bar{x}_i)}[G_i(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)]\|_i\big) \\
&\quad + b_{n,i}\big) + \alpha_{n,1}(\|e_{n,i}^{\prime(1)}\|_i + \|e_{n,i}^{\prime\prime(1)}\|_i) + \|r_{n,i}^{(1)}\|_i \\
&\leq (1-\alpha_{n,1})\|x_{n,i}-\bar{x}_i\|_i + \alpha_{n,1}\big(\|z_{n,i}^{(1)}-\bar{x}_i-(g_i(z_{n,i}^{(1)})-g_i(\bar{x}_i))\|_i \\
&\quad + \xi_i\|z_{n,i}^{(1)}-\bar{x}_i\|_i + \frac{\tau_i}{\gamma_i}\|G_i(z_{n,1}^{(1)},z_{n,2}^{(1)},\dots,z_{n,p}^{(1)}) - G_i(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_i \\
&\quad + a_{n,i}\phi_i\big(\|z_{n,i}^{(1)}-\bar{x}_i-(g_i(z_{n,i}^{(1)})-g_i(\bar{x}_i))\|_i + \xi_i\|z_{n,i}^{(1)}-\bar{x}_i\|_i \\
&\quad + \frac{\tau_i}{\gamma_i}\|G_i(z_{n,1}^{(1)},z_{n,2}^{(1)},\dots,z_{n,p}^{(1)}) - G_i(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_i\big) \\
&\quad + b_{n,i}\big) + \alpha_{n,1}\|e_{n,i}^{\prime(1)}\|_i + \|e_{n,i}^{\prime\prime(1)}\|_i + \|r_{n,i}^{(1)}\|_i \\
&\leq (1-\alpha_{n,1})\|x_{n,i}-\bar{x}_i\|_i + \alpha_{n,1}\big(\|z_{n,i}^{(1)}-\bar{x}_i-(g_i(z_{n,i}^{(1)})-g_i(\bar{x}_i))\|_i \\
&\quad + \xi_i\|z_{n,i}^{(1)}-\bar{x}_i\|_i + \frac{\tau_i}{\gamma_i}\big(\|(P_i\circ g_i)(z_{n,i}^{(1)}) - (P_i\circ g_i)(\bar{x}_i)\|_i \\
&\quad + \lambda_i\|T_i(z_{n,1}^{(1)},z_{n,2}^{(1)},\dots,z_{n,p}^{(1)}) - T_i(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_i + \lambda_i\|F_i(z_{n,i}^{(1)})-F_i(\bar{x}_i)\|_i\big) \\
&\quad + a_{n,i}\phi_i\big(\|z_{n,i}^{(1)}-\bar{x}_i-(g_i(z_{n,i}^{(1)})-g_i(\bar{x}_i))\|_i + \xi_i\|z_{n,i}^{(1)}-\bar{x}_i\|_i \\
&\quad + \frac{\tau_i}{\gamma_i}\big(\|(P_i\circ g_i)(z_{n,i}^{(1)}) - (P_i\circ g_i)(\bar{x}_i)\|_i \\
&\quad + \lambda_i\|T_i(z_{n,1}^{(1)},z_{n,2}^{(1)},\dots,z_{n,p}^{(1)}) - T_i(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_i \\
&\quad + \lambda_i\|F_i(z_{n,i}^{(1)})-F_i(\bar{x}_i)\|_i\big)\big) + b_{n,i}\big) + \alpha_{n,1}\|e_{n,i}^{\prime(1)}\|_i + \|e_{n,i}^{\prime\prime(1)}\|_i + \|r_{n,i}^{(1)}\|_i \\
&\leq (1-\alpha_{n,1})\|x_{n,i}-\bar{x}_i\|_i + \alpha_{n,1}\big(\|z_{n,i}^{(1)}-\bar{x}_i-(g_i(z_{n,i}^{(1)})-g_i(\bar{x}_i))\|_i \\
&\quad + \xi_i\|z_{n,i}^{(1)}-\bar{x}_i\|_i + \frac{\tau_i}{\gamma_i}\big(\|(P_i\circ g_i)(z_{n,i}^{(1)}) - (P_i\circ g_i)(\bar{x}_i)\|_i \\
&\quad + \lambda_i\big(\|T_i(z_{n,1}^{(1)},z_{n,2}^{(1)},\dots,z_{n,i-1}^{(1)},\bar{x}_i,z_{n,i+1}^{(1)},\dots,z_{n,p}^{(1)}) \\
&\quad - T_i(\bar{x}_1,z_{n,2}^{(1)},\dots,z_{n,i-1}^{(1)},\bar{x}_i,z_{n,i+1}^{(1)},\dots,z_{n,p}^{(1)})\|_i \\
&\quad + \|T_i(\bar{x}_1,z_{n,2}^{(1)},\dots,z_{n,i-1}^{(1)},\bar{x}_i,z_{n,i+1}^{(1)},\dots,z_{n,p}^{(1)}) \\
&\quad - T_i(\bar{x}_1,\bar{x}_2,\dots,z_{n,i-1}^{(1)},\bar{x}_i,z_{n,i+1}^{(1)},\dots,z_{n,p}^{(1)})\|_i \\
&\quad + \cdots + \|T_i(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_{i-1},\bar{x}_i,\bar{x}_{i+1},\dots,\bar{x}_{p-1},z_{n,p}^{(1)}) \\
&\quad - T_i(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_{i-1},\bar{x}_i,\bar{x}_{i+1},\dots,\bar{x}_{p-1},\bar{x}_p)\|_i\big) \\
&\quad + \lambda_i\|F_i(z_{n,i}^{(1)})-F_i(\bar{x}_i)\|_i\big) + a_{n,i}\phi_i\big(\|z_{n,i}^{(1)}-\bar{x}_i-(g_i(z_{n,i}^{(1)})-g_i(\bar{x}_i))\|_i \\
&\quad + \xi_i\|z_{n,i}^{(1)}-\bar{x}_i\|_i + \frac{\tau_i}{\gamma_i}\big(\|(P_i\circ g_i)(z_{n,i}^{(1)}) - (P_i\circ g_i)(\bar{x}_i)\|_i
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda_i\big(\|T_i(z_{n,1}^{(1)}, z_{n,2}^{(1)}, \dots, z_{n,i-1}^{(1)}, \bar{x}_i, z_{n,i+1}^{(1)}, \dots, z_{n,p}^{(1)})\\
&-T_i(\bar{x}_1, z_{n,2}^{(1)}, \dots, z_{n,i-1}^{(1)}, \bar{x}_i, z_{n,i+1}^{(1)}, \dots, z_{n,p}^{(1)})\|_i\\
&+\|T_i(\bar{x}_1, z_{n,2}^{(1)}, \dots, z_{n,i-1}^{(1)}, \bar{x}_i, z_{n,i+1}^{(1)}, \dots, z_{n,p}^{(1)})\\
&-T_i(\bar{x}_1, \bar{x}_2, \dots, z_{n,i-1}^{(1)}, \bar{x}_i, z_{n,i+1}^{(1)}, \dots, z_{n,p}^{(1)})\|_i\\
&+\dots+\|T_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \dots, \bar{x}_{p-1}, z_{n,p}^{(1)})\\
&-T_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \dots, \bar{x}_{p-1}, \bar{x}_p)\|_i\big)\\
&+\lambda_i\|F_i(z_{n,i}^{(1)}) - F_i(\bar{x}_i)\|_i)\big) + b_{n,i}\big) + \alpha_{n,1}\|e_{n,i}'^{(1)}\|_i + \|e_{n,i}''^{(1)}\|_i + \|r_{n,i}^{(1)}\|_i\\
=\;&(1-\alpha_{n,1})\|x_{n,i} - \bar{x}_i\|_i + \alpha_{n,1}\big(\|z_{n,i}^{(1)} - \bar{x}_i - (g_i(z_{n,i}^{(1)}) - g_i(\bar{x}_i))\|_i\\
&+\xi_i\|z_{n,i}^{(1)} - \bar{x}_i\|_i + \frac{\tau_i}{\gamma_i}\big(\|(P_i\circ g_i)(z_{n,i}^{(1)}) - (P_i\circ g_i)(\bar{x}_i)\|_i\\
&+\lambda_i\sum_{j\in\Gamma\setminus\{i\}}\|T_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{j-1}, z_{n,j}^{(1)}, z_{n,j+1}^{(1)}, \dots, z_{n,i-1}^{(1)}, \bar{x}_i, z_{n,i+1}^{(1)}, \dots, z_{n,p}^{(1)})\\
&-T_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{j-1}, \bar{x}_j, z_{n,j+1}^{(1)}, \dots, z_{n,i-1}^{(1)}, \bar{x}_i, z_{n,i+1}^{(1)}, \dots, z_{n,p}^{(1)})\|_i\\
&+\lambda_i\|F_i(z_{n,i}^{(1)}) - F_i(\bar{x}_i)\|_i\big)a_{n,i}\phi_i\big(\|z_{n,i}^{(1)} - \bar{x}_i - (g_i(z_{n,i}^{(1)}) - g_i(\bar{x}_i))\|_i\\
&+\xi_i\|z_{n,i}^{(1)} - \bar{x}_i\|_i + \frac{\tau_i}{\gamma_i}\big(\|(P_i\circ g_i)(z_{n,i}^{(1)}) - (P_i\circ g_i)(\bar{x}_i)\|_i\\
&+\lambda_i\sum_{j\in\Gamma\setminus\{i\}}\|T_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{j-1}, z_{n,j}^{(1)}, z_{n,j+1}^{(1)}, \dots, z_{n,i-1}^{(1)}, \bar{x}_i, z_{n,i+1}^{(1)}, \dots, z_{n,p}^{(1)})\\
&-T_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{j-1}, \bar{x}_j, z_{n,j+1}^{(1)}, \dots, z_{n,i-1}^{(1)}, \bar{x}_i, z_{n,i+1}^{(1)}, \dots, z_{n,p}^{(1)})\|_i\\
&+\lambda_i\|F_i(z_{n,i}^{(1)}) - F_i(\bar{x}_i)\|_i)\big) + b_{n,i}\big) + \alpha_{n,1}\|e_{n,i}'^{(1)}\|_i + \|e_{n,i}''^{(1)}\|_i + \|r_{n,i}^{(1)}\|_i.
\end{aligned}
$$

By the arguments similar to those of proofs of (3.7)–(3.10), with the help of the assumptions and (4.6), one can prove that for each $i \in \Gamma$ and $n \geq 0$,

$$
\begin{aligned}
\|x_{n+1,i} - \bar{x}_i\|_i \leq\;& (1-\alpha_{n,1})\|x_{n,i} - \bar{x}_i\|_i + \alpha_{n,1}\big(\vartheta_i\|z_{n,i}^{(1)} - \bar{x}_i\|_i\\
&+\frac{\tau_i\lambda_i}{\gamma_i}\sum_{j\in\Gamma\setminus\{i\}}\sigma_{i,j}\|z_{n,j}^{(1)} - \bar{x}_j\|_j + a_{n,i}\phi_i\big(\vartheta_i\|z_{n,i}^{(1)} - \bar{x}_i\|_i\\
&+\frac{\tau_i\lambda_i}{\gamma_i}\sum_{j\in\Gamma\setminus\{i\}}\sigma_{i,j}\|z_{n,j}^{(1)} - \bar{x}_j\|_j\big) + b_{n,i}\big)\\
&+\alpha_{n,1}\|e_{n,i}'^{(1)}\|_i + \|e_{n,i}''^{(1)}\|_i + \|r_{n,i}^{(1)}\|_i,
\end{aligned}
\tag{4.7}
$$

where for each $i \in \Gamma$, ϑ_i is the same as in (3.10). Employing (4.7), for all $n \geq 0$ we deduce that

$$
\|(x_{n+1,1}, x_{n+1,2}, \dots, x_{n+1,p}) - (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)\|_* = \sum_{i=1}^{p}\|x_{n+1,i} - \bar{x}_i\|_i
$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{p}\big((1-\alpha_{n,1})\|x_{n,i}-\bar{x}_i\|_i+\alpha_{n,1}\big(\vartheta_i\|z_{n,i}^{(1)}-\bar{x}_i\|_i\\
&\quad+\frac{\tau_i\lambda_i}{\gamma_i}\sum_{j\in\Gamma\setminus\{i\}}\sigma_{i,j}\|z_{n,j}^{(1)}-\bar{x}_j\|_j+a_{n,i}\phi_i\big(\vartheta_i\|z_{n,i}^{(1)}-\bar{x}_i\|_i\\
&\quad+\frac{\tau_i\lambda_i}{\gamma_i}\sum_{j\in\Gamma\setminus\{i\}}\sigma_{i,j}\|z_{n,j}^{(1)}-\bar{x}_j\|_j\big)+b_{n,i}\big)\\
&\quad+\alpha_{n,1}\|e_{n,i}'^{(1)}\|_i+\|e_{n,i}''^{(1)}\|_i+\|r_{n,i}^{(1)}\|_i\big)\\
&=(1-\alpha_{n,1})\sum_{i=1}^{p}\|x_{n,i}-\bar{x}_i\|_i+\alpha_{n,1}\big(\sum_{i=1}^{p}\big(\vartheta_i\|z_{n,i}^{(1)}-\bar{x}_i\|_i\\
&\quad+\frac{\tau_i\lambda_i}{\gamma_i}\sum_{j\in\Gamma\setminus\{i\}}\sigma_{i,j}\|z_{n,j}^{(1)}-\bar{x}_j\|_j\big)+\sum_{i=1}^{p}a_{n,i}\phi_i\big(\vartheta_i\|z_{n,i}^{(1)}-\bar{x}_i\|_i\\
&\quad+\frac{\tau_i\lambda_i}{\gamma_i}\sum_{j\in\Gamma\setminus\{i\}}\sigma_{i,j}\|z_{n,j}^{(1)}-\bar{x}_j\|_j\big)+\sum_{i=1}^{p}b_{n,i}\big)\\
&\quad+\alpha_{n,1}\sum_{i=1}^{p}\|e_{n,i}'^{(1)}\|_i+\sum_{i=1}^{p}\|e_{n,i}''^{(1)}\|_i+\sum_{i=1}^{p}\|r_{n,i}^{(1)}\|_i\\
&\leq(1-\alpha_{n,1})\sum_{i=1}^{p}\|x_{n,i}-\bar{x}_i\|_i+\alpha_{n,1}\big((\vartheta_1+\sum_{k=2}^{p}\frac{\tau_k\lambda_k}{\gamma_k}\sigma_{k,1})\|z_{n,1}^{(1)}-\bar{x}_1\|_1\\
&\quad+(\vartheta_2+\sum_{k\in\Gamma\setminus\{2\}}\frac{\tau_k\lambda_k}{\gamma_k}\sigma_{k,2})\|z_{n,2}^{(1)}-\bar{x}_2\|_2+\cdots+(\vartheta_p+\sum_{k=1}^{p-1}\frac{\tau_k\lambda_k}{\gamma_k}\sigma_{k,p})\|z_{n,p}^{(1)}-\bar{x}_p\|_p\\
&\quad+\sum_{i=1}^{p}a_{n,i}\phi_i\big(\vartheta_i\|z_{n,i}^{(1)}-\bar{x}_i\|_i+\frac{\tau_i\lambda_i}{\gamma_i}\sum_{j\in\Gamma\setminus\{i\}}\sigma_{i,j}\|z_{n,j}^{(1)}-\bar{x}_j\|_j\big)+\sum_{i=1}^{p}b_{n,i}\big)\\
&\quad+\alpha_{n,1}\sum_{i=1}^{p}\|e_{n,i}'^{(1)}\|_i+\sum_{i=1}^{p}\|e_{n,i}''^{(1)}\|_i+\sum_{i=1}^{p}\|r_{n,i}^{(1)}\|_i\\
&\leq(1-\alpha_{n,1})\sum_{i=1}^{p}\|x_{n,i}-\bar{x}_i\|_i+\alpha_{n,1}\theta\sum_{i=1}^{p}\|z_{n,i}^{(1)}-\bar{x}_i\|_i\\
&\quad+\alpha_{n,1}\sum_{i=1}^{p}a_{n,i}\phi(\theta\sum_{j=1}^{p}\|z_{n,j}^{(1)}-\bar{x}_j\|_j)+\alpha_{n,1}\sum_{i=1}^{p}b_{n,i}\\
&\quad+\alpha_{n,1}\sum_{i=1}^{p}\|e_{n,i}'^{(1)}\|_i+\sum_{i=1}^{p}\|e_{n,i}''^{(1)}\|_i+\sum_{i=1}^{p}\|r_{n,i}^{(1)}\|_i\\
&=(1-\alpha_{n,1})\|(x_{n,1},x_{n,2},\ldots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*\\
&\quad+\alpha_{n,1}\theta\|(z_{n,1}^{(1)},z_{n,2}^{(1)},\ldots,z_{n,p}^{(1)})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*\\
&\quad+\alpha_{n,1}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z_{n,1}^{(1)},z_{n,2}^{(1)},\ldots,z_{n,p}^{(1)})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*)
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha_{n,1}\sum_{i=1}^{p}b_{n,i}+\alpha_{n,1}\|(e_{n,1}'^{(1)},e_{n,2}'^{(1)},\dots,e_{n,p}'^{(1)})\|_*\\
&+\|(e_{n,1}''^{(1)},e_{n,2}''^{(1)},\dots,e_{n,p}''^{(1)})\|_*+\|(r_{n,1}^{(1)},r_{n,2}^{(1)},\dots,r_{n,p}^{(1)})\|_*,
\end{aligned}\tag{4.8}
$$

where ϕ is a self-mapping of $\mathbb{R}^+$ defined by (3.24) and θ is the same as in (3.13).

By following a similar argument as in the proof of (4.8), for $j=1,2,\dots,q-2$, we can show that

$$
\begin{aligned}
&\|(z_{n,1}^{(j)},z_{n,2}^{(j)},\dots,z_{n,p}^{(j)})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*\\
&\le(1-\alpha_{n,j+1})\|(x_{n,1},x_{n,2},\dots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*\\
&\quad+\alpha_{n,j+1}\theta\|(z_{n,1}^{(j+1)},z_{n,2}^{(j+1)},\dots,z_{n,p}^{(j+1)})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*\\
&\quad+\alpha_{n,j+1}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z_{n,1}^{(j+1)},z_{n,2}^{(j+1)},\dots,z_{n,p}^{(j+1)})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*)\\
&\quad+\alpha_{n,j+1}\sum_{i=1}^{p}b_{n,i}+\alpha_{n,j+1}\|(e_{n,1}'^{(j+1)},e_{n,2}'^{(j+1)},\dots,e_{n,p}'^{(j+1)})\|_*\\
&\quad+\|(e_{n,1}''^{(j+1)},e_{n,2}''^{(j+1)},\dots,e_{n,p}''^{(j+1)})\|_*+\|(r_{n,1}^{(j+1)},r_{n,2}^{(j+1)},\dots,r_{n,p}^{(j+1)})\|_*
\end{aligned}\tag{4.9}
$$

and

$$
\begin{aligned}
&\|(z_{n,1}^{(q-1)},z_{n,2}^{(q-1)},\dots,z_{n,p}^{(q-1)})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*\\
&\le(1-\alpha_{n,q})\|(x_{n,1},x_{n,2},\dots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*\\
&\quad+\alpha_{n,q}\theta\|(x_{n,1},x_{n,2},\dots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*\\
&\quad+\alpha_{n,q}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(x_{n,1},x_{n,2},\dots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*)\\
&\quad+\alpha_{n,q}\sum_{i=1}^{p}b_{n,i}+\alpha_{n,q}\|(e_{n,1}'^{(q)},e_{n,2}'^{(q)},\dots,e_{n,p}'^{(q)})\|_*\\
&\quad+\|(e_{n,1}''^{(q)},e_{n,2}''^{(q)},\dots,e_{n,p}''^{(q)})\|_*+\|(r_{n,1}^{(q)},r_{n,2}^{(q)},\dots,r_{n,p}^{(q)})\|_*.
\end{aligned}\tag{4.10}
$$

Applying (4.9) and (4.10), we conclude that for all $n\ge 0$,

$$
\begin{aligned}
&\|(z_{n,1}^{(1)},z_{n,2}^{(1)},\dots,z_{n,p}^{(1)})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*\\
&\le(1-\alpha_{n,2})\|(x_{n,1},x_{n,2},\dots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*\\
&\quad+\alpha_{n,2}\theta\|(z_{n,1}^{(2)},z_{n,2}^{(2)},\dots,z_{n,p}^{(2)})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*\\
&\quad+\alpha_{n,2}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z_{n,1}^{(2)},z_{n,2}^{(2)},\dots,z_{n,p}^{(2)})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*)\\
&\quad+\alpha_{n,2}\sum_{i=1}^{p}b_{n,i}+\alpha_{n,2}\|(e_{n,1}'^{(2)},e_{n,2}'^{(2)},\dots,e_{n,p}'^{(2)})\|_*\\
&\quad+\|(e_{n,1}''^{(2)},e_{n,2}''^{(2)},\dots,e_{n,p}''^{(2)})\|_*+\|(r_{n,1}^{(2)},r_{n,2}^{(2)},\dots,r_{n,p}^{(2)})\|_*\\
&\le(1-\alpha_{n,2})\|(x_{n,1},x_{n,2},\dots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha_{n,2}\theta[(1-\alpha_{n,3})\|(x_{n,1},x_{n,2},\dots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_* \\
&+ \alpha_{n,3}\theta\|(z^{(3)}_{n,1},z^{(3)}_{n,2},\dots,z^{(3)}_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_* \\
&+ \alpha_{n,3}\sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z^{(3)}_{n,1},z^{(3)}_{n,2},\dots,z^{(3)}_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*) \\
&+ \alpha_{n,3}\sum_{i=1}^{p} b_{n,i} + \alpha_{n,3}\|(e'^{(3)}_{n,1},e'^{(3)}_{n,2},\dots,e'^{(3)}_{n,p})\|_* \\
&+ \|(e''^{(3)}_{n,1},e''^{(3)}_{n,2},\dots,e''^{(3)}_{n,p})\|_* + \|(r^{(3)}_{n,1},r^{(3)}_{n,2},\dots,r^{(3)}_{n,p})\|_*] \\
&+ \alpha_{n,2}\sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z^{(2)}_{n,1},z^{(2)}_{n,2},\dots,z^{(2)}_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*) \\
&+ \alpha_{n,2}\sum_{i=1}^{p} b_{n,i} + \alpha_{n,2}\|(e'^{(2)}_{n,1},e'^{(2)}_{n,2},\dots,e'^{(2)}_{n,p})\|_* \\
&+ \|(e''^{(2)}_{n,1},e''^{(2)}_{n,2},\dots,e''^{(2)}_{n,p})\|_* + \|(r^{(2)}_{n,1},r^{(2)}_{n,2},\dots,r^{(2)}_{n,p})\|_* \\
= {}& (1-\alpha_{n,2}+\alpha_{n,2}(1-\alpha_{n,3})\theta)\|(x_{n,1},x_{n,2},\dots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_* \\
&+ \alpha_{n,2}\alpha_{n,3}\theta^2\|(z^{(3)}_{n,1},z^{(3)}_{n,2},\dots,z^{(3)}_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_* \\
&+ \alpha_{n,2}\alpha_{n,3}\theta\sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z^{(3)}_{n,1},z^{(3)}_{n,2},\dots,z^{(3)}_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*) \\
&+ \alpha_{n,2}\sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z^{(2)}_{n,1},z^{(2)}_{n,2},\dots,z^{(2)}_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*) \\
&+ (\alpha_{n,2}\alpha_{n,3}\theta+\alpha_{n,2})\sum_{i=1}^{p} b_{n,i} + \alpha_{n,2}\alpha_{n,3}\theta\|(e'^{(3)}_{n,1},e'^{(3)}_{n,2},\dots,e'^{(3)}_{n,p})\|_* \\
&+ \alpha_{n,2}\|(e'^{(2)}_{n,1},e'^{(2)}_{n,2},\dots,e'^{(2)}_{n,p})\|_* + \alpha_{n,2}\theta\|(e''^{(3)}_{n,1},e''^{(3)}_{n,2},\dots,e''^{(3)}_{n,p})\|_* \\
&+ \|(e''^{(2)}_{n,1},e''^{(2)}_{n,2},\dots,e''^{(2)}_{n,p})\|_* + \alpha_{n,2}\theta\|(r^{(3)}_{n,1},r^{(3)}_{n,2},\dots,r^{(3)}_{n,p})\|_* \\
&+ \|(r^{(2)}_{n,1},r^{(2)}_{n,2},\dots,r^{(2)}_{n,p})\|_* \\
\le {}& \dots \\
\le {}& \big(1-\alpha_{n,2}+\alpha_{n,2}(1-\alpha_{n,3})\theta+\alpha_{n,2}\alpha_{n,3}(1-\alpha_{n,4})\theta^2+\dots \\
&+ \prod_{j=2}^{q-2}\alpha_{n,j}(1-\alpha_{n,q-1})\theta^{q-3}\big)\|(x_{n,1},x_{n,2},\dots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_* \\
&+ \prod_{j=2}^{q-1}\alpha_{n,j}\theta^{q-2}\|(z^{(q-1)}_{n,1},z^{(q-1)}_{n,2},\dots,z^{(q-1)}_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_* \\
&+ \prod_{j=2}^{q-1}\alpha_{n,j}\theta^{q-3}\sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z^{(q-1)}_{n,1},z^{(q-1)}_{n,2},\dots,z^{(q-1)}_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*)
\end{aligned}
$$

$$
\begin{aligned}
&+ \prod_{j=2}^{q-2} \alpha_{n,j} \theta^{q-4} \sum_{i=1}^{p} a_{n,i} \phi(\theta \|(z_{n,1}^{(q-2)}, z_{n,2}^{(q-2)}, \dots, z_{n,p}^{(q-2)}) - (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)\|_*) \\
&+ \dots + \alpha_{n,2}\alpha_{n,3}\theta \sum_{i=1}^{p} a_{n,i} \phi(\theta \|(z_{n,1}^{(3)}, z_{n,2}^{(3)}, \dots, z_{n,p}^{(3)}) - (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)\|_*) \\
&+ \alpha_{n,2} \sum_{i=1}^{p} a_{n,i} \phi(\theta \|(z_{n,1}^{(2)}, z_{n,2}^{(2)}, \dots, z_{n,p}^{(2)}) - (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)\|_*) \\
&+ \big(\prod_{j=2}^{q-1} \alpha_{n,j} \theta^{q-3} + \prod_{j=2}^{q-2} \alpha_{n,j} \theta^{q-4} + \dots + \alpha_{n,2}\alpha_{n,3}\theta + \alpha_{n,2} \big) \sum_{i=1}^{p} b_{n,i} \\
&+ \prod_{j=2}^{q-1} \alpha_{n,j} \theta^{q-3} \|(e_{n,1}'^{(q-1)}, e_{n,2}'^{(q-1)}, \dots, e_{n,p}'^{(q-1)})\|_* \\
&+ \prod_{j=2}^{q-2} \alpha_{n,j} \theta^{q-4} \|(e_{n,1}'^{(q-2)}, e_{n,2}'^{(q-2)}, \dots, e_{n,p}'^{(q-2)})\|_* \\
&+ \dots + \alpha_{n,2}\alpha_{n,3}\theta \|(e_{n,1}'^{(3)}, e_{n,2}'^{(3)}, \dots, e_{n,p}'^{(3)})\|_* + \alpha_{n,2} \|(e_{n,1}'^{(2)}, e_{n,2}'^{(2)}, \dots, e_{n,p}'^{(2)})\|_* \\
&+ \prod_{j=2}^{q-2} \alpha_{n,j} \theta^{q-3} \|(e_{n,1}''^{(q-1)}, e_{n,2}''^{(q-1)}, \dots, e_{n,p}''^{(q-1)})\|_* \\
&+ \prod_{j=2}^{q-3} \alpha_{n,j} \theta^{q-4} \|(e_{n,1}''^{(q-2)}, e_{n,2}''^{(q-2)}, \dots, e_{n,p}''^{(q-2)})\|_* \\
&+ \dots + \alpha_{n,2}\theta \|(e_{n,1}''^{(3)}, e_{n,2}''^{(3)}, \dots, e_{n,p}''^{(3)})\|_* + \|(e_{n,1}''^{(2)}, e_{n,2}''^{(2)}, \dots, e_{n,p}''^{(2)})\|_* \\
&+ \prod_{j=2}^{q-2} \alpha_{n,j} \theta^{q-3} \|(r_{n,1}^{(q-1)}, r_{n,2}^{(q-1)}, \dots, r_{n,p}^{(q-1)})\|_* \\
&+ \prod_{j=2}^{q-3} \alpha_{n,j} \theta^{q-4} \|(r_{n,1}^{(q-2)}, r_{n,2}^{(q-2)}, \dots, r_{n,p}^{(q-2)})\|_* \\
&+ \dots + \alpha_{n,2}\theta \|(r_{n,1}^{(3)}, r_{n,2}^{(3)}, \dots, r_{n,p}^{(3)})\|_* + \|(r_{n,1}^{(2)}, r_{n,2}^{(2)}, \dots, r_{n,p}^{(2)})\|_* \\
\le& \big(1 - \alpha_{n,2} + \alpha_{n,2}(1-\alpha_{n,3})\theta + \alpha_{n,2}\alpha_{n,3}(1-\alpha_{n,4})\theta^2 + \dots \\
&+ \prod_{j=2}^{q-2} \alpha_{n,j}(1-\alpha_{n,q-1})\theta^{q-3}\big) \|(x_{n,1}, x_{n,2}, \dots, x_{n,p}) - (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)\|_* \\
&+ \prod_{j=2}^{q-1} \alpha_{n,j} \theta^{q-2} [(1-\alpha_{n,q}) \|(x_{n,1}, x_{n,2}, \dots, x_{n,p}) - (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)\|_* \\
&+ \alpha_{n,q}\theta \|(x_{n,1}, x_{n,2}, \dots, x_{n,p}) - (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)\|_*
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha_{n,q}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(x_{n,1},x_{n,2},\dots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*)\\
&+\alpha_{n,q}\sum_{i=1}^{p}b_{n,i}+\alpha_{n,q}\|(e'^{(q)}_{n,1},e'^{(q)}_{n,2},\dots,e'^{(q)}_{n,p})\|_*\\
&+\|(e''^{(q)}_{n,1},e''^{(q)}_{n,2},\dots,e''^{(q)}_{n,p})\|_*+\|(r^{(q)}_{n,1},r^{(q)}_{n,2},\dots,r^{(q)}_{n,p})\|_*]\\
&+\prod_{j=2}^{q-1}\alpha_{n,j}\theta^{q-3}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z^{(q-1)}_{n,1},z^{(q-1)}_{n,2},\dots,z^{(q-1)}_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*)\\
&+\prod_{j=2}^{q-2}\alpha_{n,j}\theta^{q-4}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z^{(q-2)}_{n,1},z^{(q-2)}_{n,2},\dots,z^{(q-2)}_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*)\\
&+\dots+\alpha_{n,2}\alpha_{n,3}\theta\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z^{(3)}_{n,1},z^{(3)}_{n,2},\dots,z^{(3)}_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*)\\
&+\alpha_{n,2}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z^{(2)}_{n,1},z^{(2)}_{n,2},\dots,z^{(2)}_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*)\\
&+\big(\prod_{j=2}^{q-1}\alpha_{n,j}\theta^{q-3}+\prod_{j=2}^{q-2}\alpha_{n,j}\theta^{q-4}+\dots+\alpha_{n,2}\alpha_{n,3}\theta+\alpha_{n,2}\big)\sum_{i=1}^{p}b_{n,i}\\
&+\prod_{j=2}^{q-1}\alpha_{n,j}\theta^{q-3}\|(e'^{(q-1)}_{n,1},e'^{(q-1)}_{n,2},\dots,e'^{(q-1)}_{n,p})\|_*\\
&+\prod_{j=2}^{q-2}\alpha_{n,j}\theta^{q-4}\|(e'^{(q-2)}_{n,1},e'^{(q-2)}_{n,2},\dots,e'^{(q-2)}_{n,p})\|_*\\
&+\dots+\alpha_{n,2}\alpha_{n,3}\theta\|(e'^{(3)}_{n,1},e'^{(3)}_{n,2},\dots,e'^{(3)}_{n,p})\|_*+\alpha_{n,2}\|(e'^{(2)}_{n,1},e'^{(2)}_{n,2},\dots,e'^{(2)}_{n,p})\|_*\\
&+\prod_{j=2}^{q-2}\alpha_{n,j}\theta^{q-3}\|(e''^{(q-1)}_{n,1},e''^{(q-1)}_{n,2},\dots,e''^{(q-1)}_{n,p})\|_*\\
&+\prod_{j=2}^{q-3}\alpha_{n,j}\theta^{q-4}\|(e''^{(q-2)}_{n,1},e''^{(q-2)}_{n,2},\dots,e''^{(q-2)}_{n,p})\|_*\\
&+\dots+\alpha_{n,2}\theta\|(e''^{(3)}_{n,1},e''^{(3)}_{n,2},\dots,e''^{(3)}_{n,p})\|_*+\|(e''^{(2)}_{n,1},e''^{(2)}_{n,2},\dots,e''^{(2)}_{n,p})\|_*\\
&+\prod_{j=2}^{q-2}\alpha_{n,j}\theta^{q-3}\|(r^{(q-1)}_{n,1},r^{(q-1)}_{n,2},\dots,r^{(q-1)}_{n,p})\|_*\\
&+\prod_{j=2}^{q-3}\alpha_{n,j}\theta^{q-4}\|(r^{(q-2)}_{n,1},r^{(q-2)}_{n,2},\dots,r^{(q-2)}_{n,p})\|_*\\
&+\dots+\alpha_{n,2}\theta\|(r^{(3)}_{n,1},r^{(3)}_{n,2},\dots,r^{(3)}_{n,p})\|_*+\|(r^{(2)}_{n,1},r^{(2)}_{n,2},\dots,r^{(2)}_{n,p})\|_*
\end{aligned}
$$

$$
\begin{aligned}
&= \big(1-\alpha_{n,2}+\alpha_{n,2}(1-\alpha_{n,3})\theta+\alpha_{n,2}\alpha_{n,3}(1-\alpha_{n,4})\theta^2 \\
&\quad +\cdots+\prod_{j=2}^{q-1}\alpha_{n,j}(1-\alpha_{n,q})\theta^{q-2}+\prod_{j=2}^{q}\alpha_{n,j}\theta^{q-1}\big)\|(x_{n,1},x_{n,2},\ldots,x_{n,p}) \\
&\quad -(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_* \\
&\quad +\prod_{j=2}^{q}\alpha_{n,j}\theta^{q-2}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(x_{n,1},x_{n,2},\ldots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*) \\
&\quad +\prod_{j=2}^{q-1}\alpha_{n,j}\theta^{q-3}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z_{n,1}^{(q-1)},z_{n,2}^{(q-1)},\ldots,z_{n,p}^{(q-1)})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*) \\
&\quad +\prod_{j=2}^{q-2}\alpha_{n,j}\theta^{q-4}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z_{n,1}^{(q-2)},z_{n,2}^{(q-2)},\ldots,z_{n,p}^{(q-2)})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*) \\
&\quad +\cdots+\alpha_{n,2}\alpha_{n,3}\theta\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z_{n,1}^{(3)},z_{n,2}^{(3)},\ldots,z_{n,p}^{(3)})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*) \\
&\quad +\alpha_{n,2}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z_{n,1}^{(2)},z_{n,2}^{(2)},\ldots,z_{n,p}^{(2)})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*) \\
&\quad +\big(\prod_{j=2}^{q}\alpha_{n,j}\theta^{q-2}+\prod_{j=2}^{q-1}\alpha_{n,j}\theta^{q-3}+\cdots+\alpha_{n,2}\alpha_{n,3}\theta+\alpha_{n,2}\big)\sum_{i=1}^{p}b_{n,i} \\
&\quad +\prod_{j=2}^{q}\alpha_{n,j}\theta^{q-2}\|(e_{n,1}^{\prime(q)},e_{n,2}^{\prime(q)},\ldots,e_{n,p}^{\prime(q)})\|_* \\
&\quad +\prod_{j=2}^{q-1}\alpha_{n,j}\theta^{q-3}\|(e_{n,1}^{\prime(q-1)},e_{n,2}^{\prime(q-1)},\ldots,e_{n,p}^{\prime(q-1)})\|_* \\
&\quad +\prod_{j=2}^{q-2}\alpha_{n,j}\theta^{q-4}\|(e_{n,1}^{\prime(q-2)},e_{n,2}^{\prime(q-2)},\ldots,e_{n,p}^{\prime(q-2)})\|_* \\
&\quad +\cdots+\alpha_{n,2}\alpha_{n,3}\theta\|(e_{n,1}^{\prime(3)},e_{n,2}^{\prime(3)},\ldots,e_{n,p}^{\prime(3)})\|_*+\alpha_{n,2}\|(e_{n,1}^{\prime(2)},e_{n,2}^{\prime(2)},\ldots,e_{n,p}^{\prime(2)})\|_* \\
&\quad +\prod_{j=2}^{q-1}\alpha_{n,j}\theta^{q-2}\|(e_{n,1}^{\prime\prime(q)},e_{n,2}^{\prime\prime(q)},\ldots,e_{n,p}^{\prime\prime(q)})\|_* \\
&\quad +\prod_{j=2}^{q-2}\alpha_{n,j}\theta^{q-3}\|(e_{n,1}^{\prime\prime(q-1)},e_{n,2}^{\prime\prime(q-1)},\ldots,e_{n,p}^{\prime\prime(q-1)})\|_* \\
&\quad +\prod_{j=2}^{q-3}\alpha_{n,j}\theta^{q-4}\|(e_{n,1}^{\prime\prime(q-2)},e_{n,2}^{\prime\prime(q-2)},\ldots,e_{n,p}^{\prime\prime(q-2)})\|_*+\cdots \\
&\quad +\alpha_{n,2}\theta\|(e_{n,1}^{\prime\prime(3)},e_{n,2}^{\prime\prime(3)},\ldots,e_{n,p}^{\prime\prime(3)})\|_*+\|(e_{n,1}^{\prime\prime(2)},e_{n,2}^{\prime\prime(2)},\ldots,e_{n,p}^{\prime\prime(2)})\|_*
\end{aligned} \tag{4.11}
$$

$$+\prod_{j=2}^{q-1}\alpha_{n,j}\theta^{q-2}\|(r_{n,1}^{(q)},r_{n,2}^{(q)},\ldots,r_{n,p}^{(q)})\|_*$$
$$+\prod_{j=2}^{q-2}\alpha_{n,j}\theta^{q-3}\|(r_{n,1}^{(q-1)},r_{n,2}^{(q-1)},\ldots,r_{n,p}^{(q-1)})\|_*$$
$$+\prod_{j=2}^{q-3}\alpha_{n,j}\theta^{q-4}\|(r_{n,1}^{(q-2)},r_{n,2}^{(q-2)},\ldots,r_{n,p}^{(q-2)})\|_*$$
$$+\cdots+\alpha_{n,2}\theta\|(r_{n,1}^{(3)},r_{n,2}^{(3)},\ldots,r_{n,p}^{(3)})\|_*+\|(r_{n,1}^{(2)},r_{n,2}^{(2)},\ldots,r_{n,p}^{(2)})\|_*.$$

Using (4.8), (4.11) and in virtue of the fact that $0<\alpha\leq\prod_{j=1}^{q}\alpha_{n,j}$, for all $n\in\mathbb{N}\cup\{0\}$, we obtain

$$\|(x_{n+1,1},x_{n+1,2},\ldots,x_{n+1,p})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*$$
$$\leq(1-\alpha_{n,1})(x_{n,1},x_{n,2},\ldots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*$$
$$+\alpha_{n,1}\theta[1-\alpha_{n,2}+\alpha_{n,2}(1-\alpha_{n,3})\theta+\alpha_{n,2}\alpha_{n,3}(1-\alpha_{n,4})\theta^2+\ldots$$
$$+\prod_{j=2}^{q-1}\alpha_{n,j}(1-\alpha_{n,q})\theta^{q-2}+\prod_{j=2}^{q}\alpha_{n,j}\theta^{q-1})\|(x_{n,1},x_{n,2},\ldots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*$$
$$+\prod_{j=2}^{q}\alpha_{n,j}\theta^{q-2}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(x_{n,1},x_{n,2},\ldots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*)$$
$$+\prod_{j=2}^{q-1}\alpha_{n,j}\theta^{q-3}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z_{n,1}^{(q-1)},z_{n,2}^{(q-1)},\ldots,z_{n,p}^{(q-1)})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*)$$
$$+\prod_{j=2}^{q-2}\alpha_{n,j}\theta^{q-4}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z_{n,1}^{(q-2)},z_{n,2}^{(q-2)},\ldots,z_{n,p}^{(q-2)})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*)$$
$$+\cdots+\alpha_{n,2}\alpha_{n,3}\theta\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z_{n,1}^{(3)},z_{n,2}^{(3)},\ldots,z_{n,p}^{(3)})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*)$$
$$+\alpha_{n,2}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z_{n,1}^{(2)},z_{n,2}^{(2)},\ldots,z_{n,p}^{(2)})-(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p)\|_*)$$
$$+\big(\prod_{j=2}^{q}\alpha_{n,j}\theta^{q-2}+\prod_{j=2}^{q-1}\alpha_{n,j}\theta^{q-3}+\prod_{j=2}^{q-2}\alpha_{n,j}\theta^{q-4}+\cdots+\alpha_{n,2}\alpha_{n,3}\theta+\alpha_{n,2}\big)\sum_{i=1}^{p}b_{n,i}$$
$$+\prod_{j=2}^{q}\alpha_{n,j}\theta^{q-2}\|(e_{n,1}^{\prime(q)},e_{n,2}^{\prime(q)},\ldots,e_{n,p}^{\prime(q)})\|_*+\prod_{j=2}^{q-1}\alpha_{n,j}\theta^{q-3}\|(e_{n,1}^{\prime(q-1)},e_{n,2}^{\prime(q-1)},\ldots,e_{n,p}^{\prime(q-1)})\|_*$$
$$+\prod_{j=2}^{q-2}\alpha_{n,j}\theta^{q-4}\|(e_{n,1}^{\prime(q-2)},e_{n,2}^{\prime(q-2)},\ldots,e_{n,p}^{\prime(q-2)})\|_*+\cdots+\alpha_{n,2}\alpha_{n,3}\theta\|(e_{n,1}^{\prime(3)},e_{n,2}^{\prime(3)},\ldots,e_{n,p}^{\prime(3)})\|_*$$

$$
\begin{aligned}
&+ \alpha_{n,2}\|(e'^{(2)}_{n,1}, e'^{(2)}_{n,2}, \ldots, e'^{(2)}_{n,p})\|_* + \prod_{j=2}^{q-1} \alpha_{n,j}\theta^{q-2}\|(e''^{(q)}_{n,1}, e''^{(q)}_{n,2}, \ldots, e''^{(q)}_{n,p})\|_* \\
&+ \prod_{j=2}^{q-2} \alpha_{n,j}\theta^{q-3}\|(e''^{(q-1)}_{n,1}, e''^{(q-1)}_{n,2}, \ldots, e''^{(q-1)}_{n,p})\|_* \\
&+ \prod_{j=2}^{q-3} \alpha_{n,j}\theta^{q-4}\|(e''^{(q-2)}_{n,1}, e''^{(q-2)}_{n,2}, \ldots, e''^{(q-2)}_{n,p})\|_* \\
&+ \cdots + \alpha_{n,2}\theta\|(e''^{(3)}_{n,1}, e''^{(3)}_{n,2}, \ldots, e''^{(3)}_{n,p})\|_* + \|(e''^{(2)}_{n,1}, e''^{(2)}_{n,2}, \ldots, e''^{(2)}_{n,p})\|_* \\
&+ \prod_{j=2}^{q-1} \alpha_{n,j}\theta^{q-1}\|(r^{(q)}_{n,1}, r^{(q)}_{n,2}, \ldots, r^{(q)}_{n,p})\|_* + \prod_{j=2}^{q-2} \alpha_{n,j}\theta^{q-3}\|(r^{(q-1)}_{n,1}, r^{(q-1)}_{n,2}, \ldots, r^{(q-1)}_{n,p})\|_* \\
&+ \prod_{j=2}^{q-3} \alpha_{n,j}\theta^{q-4}\|(r^{(q-2)}_{n,1}, r^{(q-2)}_{n,2}, \ldots, r^{(q-2)}_{n,p})\|_* + \cdots + \alpha_{n,2}\theta\|(r^{(3)}_{n,1}, r^{(3)}_{n,2}, \ldots, r^{(3)}_{n,p})\|_* \\
&+ \|(r^{(2)}_{n,1}, r^{(2)}_{n,2}, \ldots, r^{(2)}_{n,p})\|_*] + \alpha_{n,1}\sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z^{(1)}_{n,1}, z^{(1)}_{n,2}, \ldots, z^{(1)}_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*) \\
&+ \alpha_{n,1}\sum_{i=1}^{b} b_{n,i} + \alpha_{n,1}\|(e'^{(1)}_{n,1}, e'^{(1)}_{n,2}, \ldots, e'^{(1)}_{n,p})\|_* + \|(e''^{(1)}_{n,1}, e''^{(1)}_{n,2}, \ldots, e''^{(1)}_{n,p})\|_* \\
&+ \|(r^{(1)}_{n,1}, r^{(1)}_{n,2}, \ldots, r^{(1)}_{n,p})\|_* \\
&= \big(1 - \alpha_{n,1} + \alpha_{n,1}(1-\alpha_{n,2})\theta + \alpha_{n,1}\alpha_{n,2}(1-\alpha_{n,3})\theta^2 + \alpha_{n,1}\alpha_{n,2}\alpha_{n,3}(1-\alpha_{n,4})\theta^3 + \ldots \\
&+ \prod_{j=1}^{q-1} \alpha_{n,j}(1-\alpha_{n,q})\theta^{q-1} + \prod_{j=1}^{q} \alpha_{n,j}\theta^{q}\big)\|(x_{n,1}, x_{n,2}, \ldots, x_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_* \\
&+ \prod_{j=1}^{q} \alpha_{n,j}\theta^{q-1}\sum_{i=1}^{p} a_{n,i}\phi(\theta\|(x_{n,1}, x_{n,2}, \ldots, x_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*) \\
&+ \prod_{j=1}^{q-1} \alpha_{n,j}\theta^{q-2}\sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z^{(q-1)}_{n,1}, z^{(q-1)}_{n,2}, \ldots, z^{(q-1)}_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*) \\
&+ \prod_{j=1}^{q-2} \alpha_{n,j}\theta^{q-3}\sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z^{(q-2)}_{n,1}, z^{(q-2)}_{n,2}, \ldots, z^{(q-2)}_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*) \\
&+ \cdots + \alpha_{n,1}\alpha_{n,2}\alpha_{n,3}\theta^2\sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z^{(3)}_{n,1}, z^{(3)}_{n,2}, \ldots, z^{(3)}_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*) \\
&+ \alpha_{n,1}\alpha_{n,2}\theta\sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z^{(2)}_{n,1}, z^{(2)}_{n,2}, \ldots, z^{(2)}_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*)
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha_{n,1}\sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z_{n,1}^{(1)}, z_{n,2}^{(1)}, \dots, z_{n,p}^{(1)}) - (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)\|_*) \\
&+ \big(\prod_{j=1}^{q}\alpha_{n,j}\theta^{q-1} + \prod_{j=1}^{q-1}\alpha_{n,j}\theta^{q-2} + \prod_{j=1}^{q-2}\alpha_{n,j}\theta^{q-3} + \dots + \alpha_{n,1}\alpha_{n,2}\alpha_{n,3}\theta^2 \\
&+ \alpha_{n,1}\alpha_{n,2}\theta + \alpha_{n,1}\big)\sum_{i=1}^{p} b_{n,i} + \prod_{j=1}^{q}\alpha_{n,j}\theta^{q-1}\|(e_{n,1}'^{(q)}, e_{n,2}'^{(q)}, \dots, e_{n,p}'^{(q)})\|_* \\
&+ \prod_{j=1}^{q-1}\alpha_{n,j}\theta^{q-2}\|(e_{n,1}'^{(q-1)}, e_{n,2}'^{(q-1)}, \dots, e_{n,p}'^{(q-1)})\|_* \\
&+ \prod_{j=2}^{q-2}\alpha_{n,j}\theta^{q-3}\|(e_{n,1}'^{(q-2)}, e_{n,2}'^{(q-2)}, \dots, e_{n,p}'^{(q-2)})\|_* \\
&+ \dots + \alpha_{n,1}\alpha_{n,2}\alpha_{n,3}\theta^2\|(e_{n,1}'^{(3)}, e_{n,2}'^{(3)}, \dots, e_{n,p}'^{(3)})\|_* \\
&+ \alpha_{n,1}\alpha_{n,2}\theta\|(e_{n,1}'^{(2)}, e_{n,2}'^{(2)}, \dots, e_{n,p}'^{(2)})\|_* + \alpha_{n,1}\|(e_{n,1}'^{(1)}, e_{n,2}'^{(1)}, \dots, e_{n,p}'^{(1)})\|_* \\
&+ \prod_{j=1}^{q-1}\alpha_{n,j}\theta^{q-1}\|(e_{n,1}''^{(q)}, e_{n,2}''^{(q)}, \dots, e_{n,p}''^{(q)})\|_* \\
&+ \prod_{j=1}^{q-2}\alpha_{n,j}\theta^{q-2}\|(e_{n,1}''^{(q-1)}, e_{n,2}''^{(q-1)}, \dots, e_{n,p}''^{(q-1)})\|_* \\
&+ \prod_{j=1}^{q-3}\alpha_{n,j}\theta^{q-3}\|(e_{n,1}''^{(q-2)}, e_{n,2}''^{(q-2)}, \dots, e_{n,p}''^{(q-2)})\|_* \\
&+ \dots + \alpha_{n,1}\alpha_{n,2}\theta^2\|(e_{n,1}''^{(3)}, e_{n,2}''^{(3)}, \dots, e_{n,p}''^{(3)})\|_* + \alpha_{n,1}\theta\|(e_{n,1}''^{(2)}, e_{n,2}''^{(2)}, \dots, e_{n,p}''^{(2)})\|_* \\
&+ \|(e_{n,1}''^{(1)}, e_{n,2}''^{(1)}, \dots, e_{n,p}''^{(1)})\|_* + \prod_{j=1}^{q-1}\alpha_{n,j}\theta^{q-1}\|(r_{n,1}^{(q)}, r_{n,2}^{(q)}, \dots, r_{n,p}^{(q)})\|_* \\
&+ \prod_{j=1}^{q-2}\alpha_{n,j}\theta^{q-2}\|(r_{n,1}^{(q-1)}, r_{n,2}^{(q-1)}, \dots, r_{n,p}^{(q-1)})\|_* \\
&+ \prod_{j=1}^{q-3}\alpha_{n,j}\theta^{q-3}\|(r_{n,1}^{(q-2)}, r_{n,2}^{(q-2)}, \dots, r_{n,p}^{(q-2)})\|_* + \dots \\
&+ \alpha_{n,1}\alpha_{n,2}\theta^2\|(r_{n,1}^{(3)}, r_{n,2}^{(3)}, \dots, r_{n,p}^{(3)})\|_* + \alpha_{n,1}\theta\|(r_{n,1}^{(2)}, r_{n,2}^{(2)}, \dots, r_{n,p}^{(2)})\|_* \\
&+ \|(r_{n,1}^{(1)}, r_{n,2}^{(1)}, \dots, r_{n,p}^{(1)})\|_* \\
&= \big(1 - \alpha_{n,1} + \alpha_{n,1}\theta - \alpha_{n,1}\alpha_{n,2}\theta + \alpha_{n,1}\alpha_{n,2}\theta^2 - \alpha_{n,1}\alpha_{n,2}\alpha_{n,3}\theta^2 + \alpha_{n,1}\alpha_{n,2}\alpha_{n,3}\theta^3 \\
&- \alpha_{n,1}\alpha_{n,2}\alpha_{n,3}\alpha_{n,4}\theta^3 + \dots - \prod_{j=1}^{q-1}\alpha_{n,j}\theta^{q-2} + \prod_{j=1}^{q-1}\alpha_{n,j}\theta^{q-1} - \prod_{j=1}^{q}\alpha_{n,j}\theta^{q-1}
\end{aligned}
$$

$$
\begin{aligned}
&+ \prod_{j=1}^{q} \alpha_{n,j}\theta^{q}\big)\|(x_{n,1}, x_{n,2}, \ldots, x_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_* \\
&+ \prod_{j=1}^{q} \alpha_{n,j}\theta^{q-1} \sum_{i=1}^{p} a_{n,i}\phi(\theta\|(x_{n,1}, x_{n,2}, \ldots, x_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*) \\
&+ \sum_{k=1}^{q-1} \prod_{j=1}^{k} \alpha_{n,j}\theta^{k-1} \sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z_{n,1}^{(k)}, z_{n,2}^{(k)}, \ldots, z_{n,p}^{(k)}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*) \\
&+ \sum_{t=1}^{q} \prod_{j=1}^{t} \alpha_{n,j}\theta^{t-1} \sum_{i=1}^{p} b_{n,i} + \sum_{i=1}^{q} \prod_{j=1}^{i} \alpha_{n,j}\theta^{i-1}\|(e_{n,1}'^{(i)}, e_{n,2}'^{(i)}, \ldots, e_{n,p}'^{(i)})\|_* \\
&+ \sum_{i=2}^{q} \prod_{j=1}^{i-1} \alpha_{n,j}\theta^{i-1}\|(e_{n,1}''^{(i)}, e_{n,2}''^{(i)}, \ldots, e_{n,p}''^{(i)})\|_* \\
&+ \sum_{i=2}^{q} \prod_{j=1}^{i-1} \alpha_{n,j}\theta^{i-1}\|(r_{n,1}^{(i)}, r_{n,2}^{(i)}, \ldots, r_{n,p}^{(i)})\|_* \\
&+ \|(e_{n,1}''^{(1)}, e_{n,2}''^{(1)}, \ldots, e_{n,p}''^{(1)})\|_* + \|(r_{n,1}^{(1)}, r_{n,2}^{(1)}, \ldots, r_{n,p}^{(1)})\|_* \\
&\le \big(1 - \alpha_{n,1} + \alpha_{n,1} - \alpha_{n,1}\alpha_{n,2}\theta + \alpha_{n,1}\alpha_{n,2}\theta - \alpha_{n,1}\alpha_{n,2}\alpha_{n,3}\theta^2 + \alpha_{n,1}\alpha_{n,2}\alpha_{n,3}\theta^2 \\
&- \alpha_{n,1}\alpha_{n,2}\alpha_{n,3}\alpha_{n,4}\theta^3 + \cdots - \prod_{j=1}^{q-1} \alpha_{n,j}\theta^{q-2} + \prod_{j=1}^{q-1} \alpha_{n,j}\theta^{q-2} - \prod_{j=1}^{q} \alpha_{n,j}\theta^{q-1} \\
&+ \prod_{j=1}^{q} \alpha_{n,j}\theta^{q}\big)\|(x_{n,1}, x_{n,2}, \ldots, x_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_* \\
&+ \prod_{j=1}^{q} \alpha_{n,j}\theta^{q-1} \sum_{i=1}^{p} a_{n,i}\phi(\theta\|(x_{n,1}, x_{n,2}, \ldots, x_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*) \\
&+ \sum_{k=1}^{q-1} \prod_{j=1}^{k} \alpha_{n,j}\theta^{k-1} \sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z_{n,1}^{(k)}, z_{n,2}^{(k)}, \ldots, z_{n,p}^{(k)}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*) \\
&+ \sum_{t=1}^{q} \prod_{j=1}^{t} \alpha_{n,j}\theta^{t-1} \sum_{i=1}^{p} b_{n,i} + \sum_{i=1}^{q} \prod_{j=1}^{i} \alpha_{n,j}\theta^{i-1}\|(e_{n,1}'^{(i)}, e_{n,2}'^{(i)}, \ldots, e_{n,p}'^{(i)})\|_* \\
&+ \sum_{i=2}^{q} \prod_{j=1}^{i-1} \alpha_{n,j}\theta^{i-1}\|(e_{n,1}''^{(i)}, e_{n,2}''^{(i)}, \ldots, e_{n,p}''^{(i)})\|_* \\
&+ \sum_{i=2}^{q} \prod_{j=1}^{i-1} \alpha_{n,j}\theta^{i-1}\|(r_{n,1}^{(i)}, r_{n,2}^{(i)}, \ldots, r_{n,p}^{(i)})\|_* \\
&+ \|(e_{n,1}''^{(1)}, e_{n,2}''^{(1)}, \ldots, e_{n,p}''^{(1)})\|_* + \|(r_{n,1}^{(1)}, r_{n,2}^{(1)}, \ldots, r_{n,p}^{(1)})\|_*
\end{aligned}
$$

$$= \big(1 - \prod_{j=1}^{q} \alpha_{n,j}\theta^{q-1} + \prod_{j=1}^{q} \alpha_{n,j}\theta^{q}\big)\|(x_{n,1}, x_{n,2}, \ldots, x_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*$$

$$+ \prod_{j=1}^{q} \alpha_{n,j}\theta^{q-1} \sum_{i=1}^{p} a_{n,i}\phi(\theta\|(x_{n,1}, x_{n,2}, \ldots, x_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*)$$

$$+ \sum_{k=1}^{q-1} \prod_{j=1}^{k} \alpha_{n,j}\theta^{k-1} \sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z_{n,1}^{(k)}, z_{n,2}^{(k)}, \ldots, z_{n,p}^{(k)}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*)$$

$$+ \sum_{t=1}^{q} \prod_{j=1}^{t} \alpha_{n,j}\theta^{t-1} \sum_{i=1}^{p} b_{n,i} + \sum_{i=1}^{q} \prod_{j=1}^{i} \alpha_{n,j}\theta^{i-1}\|(e_{n,1}'^{(i)}, e_{n,2}'^{(i)}, \ldots, e_{n,p}'^{(i)})\|_*$$

$$+ \sum_{i=2}^{q} \prod_{j=1}^{i-1} \alpha_{n,j}\theta^{i-1}\|(e_{n,1}''^{(i)}, e_{n,2}''^{(i)}, \ldots, e_{n,p}''^{(i)})\|_*$$

$$+ \sum_{i=2}^{q} \prod_{j=1}^{i-1} \alpha_{n,j}\theta^{i-1}\|(r_{n,1}^{(i)}, r_{n,2}^{(i)}, \ldots, r_{n,p}^{(i)})\|_*$$

$$+ \|(e_{n,1}''^{(1)}, e_{n,2}''^{(1)}, \ldots, e_{n,p}''^{(1)})\|_* + \|(r_{n,1}^{(1)}, r_{n,2}^{(1)}, \ldots, r_{n,p}^{(1)})\|_*$$

$$= \big(1 - \theta^{q-1}(1-\theta) \prod_{j=1}^{q} \alpha_{n,j}\big)\|(x_{n,1}, x_{n,2}, \ldots, x_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*$$

$$+ \prod_{j=1}^{q} \alpha_{n,j}\theta^{q-1} \sum_{i=1}^{p} a_{n,i}\phi(\theta\|(x_{n,1}, x_{n,2}, \ldots, x_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*)$$

$$+ \sum_{k=1}^{q-1} \prod_{j=1}^{k} \alpha_{n,j}\theta^{k-1} \sum_{i=1}^{p} a_{n,i}\phi(\theta\|(z_{n,1}^{(k)}, z_{n,2}^{(k)}, \ldots, z_{n,p}^{(k)}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*)$$

$$+ \sum_{t=1}^{q} \prod_{j=1}^{t} \alpha_{n,j}\theta^{t-1} \sum_{i=1}^{p} b_{n,i} + \sum_{i=1}^{q} \prod_{j=1}^{i} \alpha_{n,j}\theta^{i-1}\|(e_{n,1}'^{(i)}, e_{n,2}'^{(i)}, \ldots, e_{n,p}'^{(i)})\|_*$$

$$+ \sum_{i=2}^{q} \prod_{j=1}^{i-1} \alpha_{n,j}\theta^{i-1}\|(e_{n,1}''^{(i)}, e_{n,2}''^{(i)}, \ldots, e_{n,p}''^{(i)})\|_*$$

$$+ \sum_{i=2}^{q} \prod_{j=1}^{i-1} \alpha_{n,j}\theta^{i-1}\|(r_{n,1}^{(i)}, r_{n,2}^{(i)}, \ldots, r_{n,p}^{(i)})\|_*$$

$$+ \|(e_{n,1}''^{(1)}, e_{n,2}''^{(1)}, \ldots, e_{n,p}''^{(1)})\|_* + \|(r_{n,1}^{(1)}, r_{n,2}^{(1)}, \ldots, r_{n,p}^{(1)})\|_*$$

$$\leq \big(1 - \theta^{q-1}(1-\theta) \prod_{j=1}^{q} \alpha_{n,j}\big)\|(x_{n,1}, x_{n,2}, \ldots, x_{n,p}) - (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)\|_*$$

$$+ \theta^{q-1}(1-\theta) \prod_{j=1}^{q} \alpha_{n,j} \frac{\Delta_n}{\theta^{q-1}(1-\theta)\alpha}$$

$$+\sum_{i=2}^{q}\prod_{j=1}^{i-1}\alpha_{n,j}\theta^{i-1}\|(e''^{(i)}_{n,1},e''^{(i)}_{n,2},\dots,e''^{(i)}_{n,p})\|_*$$
$$+\sum_{i=2}^{q}\prod_{j=1}^{i-1}\alpha_{n,j}\theta^{i-1}\|(r^{(i)}_{n,1},r^{(i)}_{n,2},\dots,r^{(i)}_{n,p})\|_*$$
$$+\|(e''^{(1)}_{n,1},e''^{(1)}_{n,2},\dots,e''^{(1)}_{n,p})\|_*$$
$$+\|(r^{(1)}_{n,1},r^{(1)}_{n,2},\dots,r^{(1i)}_{n,p})\|_*, \tag{4.12}$$

where for each $n\in\mathbb{N}\cup\{0\}$,

$$\begin{aligned}\Delta_n=&\prod_{j=1}^{q}\alpha_{n,j}\theta^{q-1}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(x_{n,1},x_{n,2},\dots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*)\\&+\sum_{k=1}^{q-1}\prod_{j=1}^{k}\alpha_{n,j}\theta^{k-1}\sum_{i=1}^{p}a_{n,i}\phi(\theta\|(z^{(k)}_{n,1},z^{(k)}_{n,2},\dots,z^{(k)}_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*)\\&+\sum_{t=1}^{q}\prod_{j=1}^{t}\alpha_{n,j}\theta^{t-1}\sum_{i=1}^{p}b_{n,i}+\sum_{i=1}^{q}\prod_{j=1}^{i}\alpha_{n,j}\theta^{i-1}\|(e'^{(i)}_{n,1},e'^{(i)}_{n,2},\dots,e'^{(i)}_{n,p})\|_*.\end{aligned}$$

Obviously, (4.3) ensures that for each $i\in\Gamma$,

$$\lim_{n\to\infty}\|(e''^{(j)}_{n,1},e''^{(j)}_{n,2},\dots,e''^{(j)}_{n,p})\|_*=\lim_{n\to\infty}\|(r^{(j)}_{n,1},r^{(j)}_{n,2},\dots,r^{(j)}_{n,p})\|_*=0.$$

Let us now take for each $n\geq 0$,

$$\begin{aligned}a_n&=\|(x_{n,1},x_{n,2},\dots,x_{n,p})-(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)\|_*,\\ \mu_n&=\theta^{q-1}(1-\theta)\prod_{j=1}^{q}\alpha_{n,j},\ b_n=\frac{\Delta_n}{\theta^{q-1}(1-\theta)\alpha},\\ c_n&=\sum_{i=2}^{q}\prod_{j=1}^{i-1}\alpha_{n,j}\theta^{i-1}\|(e''^{(i)}_{n,1},e''^{(i)}_{n,2},\dots,e''^{(i)}_{n,p})\|_*\\&\quad+\sum_{i=2}^{q}\prod_{j=1}^{i-1}\alpha_{n,j}\theta^{i-1}\|(r^{(i)}_{n,1},r^{(i)}_{n,2},\dots,r^{(i)}_{n,p})\|_*\\&\quad+\|(e''^{(1)}_{n,1},e''^{(1)}_{n,2},\dots,e''^{(1)}_{n,p})\|_*+\|(r^{(1)}_{n,1},r^{(1)}_{n,2},\dots,r^{(1)}_{n,p})\|_*.\end{aligned}$$

Since $\lim_{n\to\infty}\|(e'^{(i)}_{n,1},e'^{(i)}_{n,2},\dots,e'^{(i)}_{n,p})\|_*=0$ and $\lim_{n\to\infty}a_{n,i}=\lim_{n\to\infty}b_{n,i}=0$ for $i=1,2,\dots,p$, we observe that all the conditions of Lemma 4.3 are satisfied and so Lemma 4.3 and (4.12) imply that $\lim_{n\to\infty}(x_{n,1},x_{n,2},\dots,x_{n,p})=(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)$. Thus, the iterative sequence $\{(x_{n,1},x_{n,2},\dots,x_{n,p})\}_{n=0}^{\infty}$ generated by Algorithm 4.1 converges strongly to the only element $(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_p)$ of $\mathrm{Fix}(Q)\cap\Phi_{\mathrm{SGNVLI}}$. The proof is finished. □

We obtain the following corollary as a direct consequence of the above theorem immediately.

Corollary 4.5. *Suppose that $X_i, F_i, T_i, P_i, \varphi_i, \eta_i, g_i, h_i$ $(i = 1, 2, \dots, p)$ are the same as in Theorem 3.7 and all the conditions of Theorem 3.7 hold. Then, the iterative sequence $\{(x_{n,1}, x_{n,2}, \dots, x_{n,p})\}_{n=0}^{\infty}$ generated by Algorithm 4.2 converges strongly to the unique solution $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)$ of the* SGNVLI (3.1).

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