Pure and Applied Functional Analysis
Volume 8, Number 4, 2023, 1235–1238

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

BANACH FIXED-POINT THEOREM YIELDS VACANT POINTS

SANJO ZLOBEC

The paper is devoted to Professor Hubertus Th. Jongen in occasion of his 75th birthday.

Abstract. Banach fixed-point theorem in the theory of complete metric spaces says that a contraction mapping T has a fixed point $x^\star$. If this point is in the interior of a compact convex set K in a Euclidean space then we give conditions when $x^\star$ can be interpreted as a vacant point. In particular, the ratio $R(x) = \|T(x) - x\|^2/\|x - x^\star\|^2$ exists on $K \setminus x^\star$ and it is undetermined (undefined) $R(x^\star) = 0/0$ at $x^\star$.

1. Introduction

We follow, in part, what an unknown author says in Wikipedia [1].

Definition 1.1. Let (X, d) be a complete metric space. Then a map $T : X \to X$ is called a contraction mapping on X if there exists q in $[0, 1)$ such that

$$d(T(x), T(y)) \leq qd(x, y)$$

for all $x, y \in X$.

Theorem 1.2. *Banach Fixed Point Theorem. Let (X, d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed-point $x^\star \in X$ i.e. $T(x^\star) = x^\star$. Furthermore, $x^\star$ can be found as follows: start with an arbitrary element $x_0 \in X$ and define a sequence x_n by $x_n = T[x_{n-1}]$ for $n \geq 1$. Then $x_n \to x^\star$.*

We refer to the above $x^\star$ as the Banach fixed point. There is a huge literature on conditions for possible convergence of iterations to $x^\star$ in Euclidean spaces. In contrast, our single result shows that $x^\star$ can be described by an undetermined ratio $0/0$. Indeed, we will show that in Euclidean space $X \equiv \mathbb{R}^n$, $x^\star$ can be formulated as a vacant point. First, we recall some elementary results from calculus on points where the derivative is zero.

2. Necessary condition for zero-derivative point

The paper is based on the result published in [6], also [7], [8].

2020 *Mathematics Subject Classification.* 46A22.
Key words and phrases. Banach fixed point, vacant point.

Theorem 2.1 (Necessary Condition for Zero-Derivative Point)**.** *Consider a compact convex set K with non-empty interior in Euclidean space $\mathbb{R}^n$. Take an interior point $x^\star \in K$. A continuously differentiable scalar function with Lipschitz derivative $f : K \to R$ has zero derivative $\nabla f(x^\star) = 0$ at $x^\star$ only if*

$$| f(x) - f(x^\star) | \leq \Lambda \, || x - x^\star ||^2$$

for some $\Lambda \geq 0$ and for every $x \in K$. Note that $\Lambda = \Lambda(x)$.

Hence it follows that convergent methods for calculating zero-derivative points of smooth, generally non-convex functions with Lipschitz derivative have a quadratic rate of convergence, [2], [6].

This implies an alternative necessary condition for optimality [8], [9]:

Theorem 2.2 (Vacant Point Formulation of Necessary Condition for Local Optimality)**.** *Consider a continuously differentiable function $f(x)$ in several variables with Lipschitz derivative on a convex compact set K with nonempty interior. If an interior point $x^\star$ of K is a local optimum of f then the ratio function*

$$r(x) = | f(x) - f(x^\star) | \, / \, || x - x^\star ||^2$$

exists on $K \setminus x^\star$ with

$$r(x^\star) = 0/0$$

undetermined (undefined) at $x^\star$.

The above condition is called the vacant point of $r(x)$ at $x^\star$. It was used in the study of L'Hospital's rule in [7] and optimality in [9].

Example 2.3. Consider $f(x) = x$ on $K = [-1, 1]$. Is the point $x^\star = 0$ a local optimum of f? The answer is negative because the ratio $r(x) = | x | \, / x^2$ is not bounded on the set $K \setminus 0$.

Example 2.4. (Properties of Fixed Points Using Theorem 2.2) Consider $f(x) = x^3$ on $K = [-2, 2]$. We obtain $r(x) = | x |^3 \, / x^2$ on $K \setminus 0$ and $r(x^\star) = 0/0$ at $x^\star = 0$. Similarly one finds properties of other fixed points $y^\star = 1$ and $z^\star = -1$.

3. Vacant point description of Banach fixed point in Euclidean space

We note that a contraction map $T : \mathbb{R}^n \to \mathbb{R}^n$ in Euclidean space is a vector function. Consider the scalar function $f(x) = || T(x) - x ||^2$ on a convex compact set K in $\mathbb{R}^n$ with an interior point $x^\star$. Assume that $x^\star$ is a Banach fixed point, i.e., $T(x^\star) = x^\star$. Then $f(x^\star) = 0$. But $f(x) \geq 0$ for every $x \in K$. This means that $x^\star$ locally optimizes f on K. If, on the other hand, we assume that $f(x)$ is continuously differentiable with Lipschitz derivative on K then $\nabla f(x^\star) = 0$. Using Theorem 2.2 this yields

Theorem 3.1 (Banach Fixed Point Theorem Yields Vacant Points)**.** *Consider a contraction map $T : \mathbb{R}^n \to \mathbb{R}^n$ in Euclidean space and its Banach fixed point $x^\star$. Assume that $x^\star$ is an interior point of a compact convex set K. We also assume that the scalar function $f(x) = || T(x) - x ||^2$ is continuously differentiable with Lipschitz derivative on K. Then the ratio function*

$$R(x) = || T(x) - x ||^2 \, / \, || x - x^\star ||^2$$

exists on $K \setminus x^\star$ *with undetermined (undefined)* $R(x^\star) = 0/0$ *at* $x^\star$*. This means that* $x^\star$ *is a vacant point of* $R(x)$.

Example 3.2. Contraction mapping can be described implicitly using iterative methods such as Newton's method for finding roots of a function $g(x)$, i.e., $x_{k+1} = x_k - g(x_k)/g'(x^k)$. Consider the cubic x^3 on $K = [-2, 2]$ starting with $x_0 = 0.1$ and its fixed point $x^\star, x_k \to x^\star = 0$. The map T is defined by a convergent sequence $\{0.1, 0.09, \ldots\}$. Here

$$\begin{aligned} R(x) &= (x^3 - x)^2/x^2 \\ &= (x^2 - 1)^2 \\ &\leq 9, \end{aligned}$$

for $x \in K \setminus 0$ and $R(x^\star) = 0/0$ confirming that $x^\star = 0$ is a vacant point. Similarly for vector functions.

4. Conclusion

Consider a contraction map T in a Euclidean space. Banach fixed point theorem says that T allows a unique fixed point $x^\star$. If $x^\star$ is an interior point of a compact convex set then we give assumptions when $x^\star$ is a vacant point. The shape of R(x) appears often in nature such as in pits and seeds of plants, where the pits correspond to $x^\star$.

Acknowledgment

This paper is motivated by "Banach fixed-point theorem" published in WIKIPEDIA by an unknown author. We partly incorporate the material from (https://planetmath.org/banachfixedpointtheorem) on PlanetMath, which is licensed under the Creative Commons Attribution/-Share-Alike License.

The author is indebted to Luka Neralić and Zrinka Lukač from University of Zagreb for organizing a special session at the conference KOI Šibenik 2020 where the results of this paper were presented in part. He also thanks Rona Valdez and Ante Pađen for interest in the subject and help with the computer.

References

[1] *Banach fixed-point theorem*, WIKIPEDIA 2020.
[2] A. Ben Israel, *Every iteration away from its fixed points is Newton*, Sept. 24, 2020. (presented at international conference KOI 2020, Šibenik, Croatia).
[3] H. Th. Jongen, J.-J. Rückmann and O. Stein, *Parametric Global Optimization: Sensitivity.* In: Floudas C. A., Pardalos P. M. (eds) *Encyclopedia of Optimization.* Springer, Boston, MA, 2001. $https://doi.org/10.1007/0-306-48332-7_382$
[4] A. N. Kolmogorov and S. V. Fomin, *Introductory real analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1970 (in Russian, translated by R. Silverman).
[5] J. F. Peters, *Amiable and almost amiable sets. Extension of the Brower fixed point theorem*, Glas. Mat. Ser. III **56** (2021), 175–194.
[6] S. Zlobec, *Characterizing zero-derivative points*, J. Global Optim. **46** (2010), 155–161.
[7] S. Zlobec, *L'Hospital's rule without derivatives*, Math. Commun. **17** (2012), 665–672.
[8] S. Zlobec, *Geometry of local optima*, Pure. Appl. Funct. Anal. **3** (2018), 743–751.
[9] S. Zlobec, *Local optima are vacant points*, Research gate, Preprint, July 2019.

Manuscript received September 6 2021
revised December 18 2021

S. Zlobec
Professor Emeritus, McGill University, Department of Mathematics and Statistics, Burnside Hall, 805 Sherbrooke Street West, Montreal, Quebec H3A 0B9, Montreal, Canada
E-mail address: sanjo.zlobec@mcgill.ca