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THE ROMAN DOMINATION OF KAUTZ DIGRAPHS AND GENERALIZED KAUTZ DIGRAPHS

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ABSTRACT. A Roman domination function on a digraph $D = (V, A)$ is a function $f : V(D) \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ has an in-neighbour v with $f(v) = 2$. The weight of a Roman domination function is the value $\omega(f) = \sum_{v \in V} f(v)$. The domination number $\gamma(D)$ is the minimum cardinality of a domination set of D, and a domination set S of minimum cardinality is called a $\gamma(D)$-set of D. The Roman domination number $\gamma_R(D)$ of D is the minimum weight of a Roman domination function of D. In this paper, we investigate the Roman domination numbers of Kautz digraphs and generalized Kautz digraphs, and prove that these two classes of digraphs are Roman digraphs.

1. INTRODUCTION

In this paper we deal with digraphs which admit selfloops but no multiple arcs. Consult [13] for the notation and terminology which are not defined here. Due to the diversity of its applications in theoretical and practical problems, Roman domination has become one of the important research topics in digraph theory, more details can be found in [1, 2, 8, 10, 12]. For Roman domination one can think of any vertex representing a city in the Roman Empire. The military expenditure is too high if every city has an army stationed. Hence the best situation for the Roman Empire is to be protected by armies as few as possible, which corresponding to the Roman domination number. Nowadays, many closely related concepts on Roman domination has been investigated, for example, the Roman domination of regular graphs [8]. And this paper aims to study the Roman domination of Kautz digraphs and generalized Kautz digraphs.

Let $D = (V, A)$ be a finite directed graph with vertex set $V(D)$ and arc set $A(D)$. The order $n = n(D)$ is the number of vertices of a digraph. If uv is an arc of D, then we can also write $u \to v$, where v is an out-neighbour of u and u is an in-neighbour of v. For $v \in V(D)$, we denote the set of in-neighbourhood and out-neighbourhood

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of v by $N^-(v) = N_D^-(v)$ and $N^+(v) = N_D^+(v)$, respectively. And the closed in-neighbourhood and closed out-neighbourhood of a vertex $v \in V(D)$ are the sets $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$ and $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$, respectively. For a set $S \subseteq V$, the set of in-neighbourhood and out-neighbourhood of S by $N^-(S) = N_D^-(S) \backslash S = \cup_{v \in S} N_D^-(v) \backslash S$ and $N^+(S) = N_D^+(S) \backslash S = \cup_{v \in S} N_D^+(v) \backslash S$, respectively. And the closed in-neighbourhood and closed out-neighbourhood of S are the sets $N_D^-[S] = N^-[S] = N^-(S) \cup S$ and $N_D^+[S] = N^+[S] = N^+(S) \cup S$, respectively. The private neighbourhood $pn(v,S)$ of $v \in S$ is defined by $pn(v,S) = N^+(v) \backslash N^+(S \backslash \{v\})$. Each vertex in $pn(v,S)$ is called a private neighbour of v in S.

The in-degree and out-degree of a vertex $v \in V(D)$ are denoted by $d_D^-(v)$ and $d_D^+(v)$, respectively. The minimum in-degree, maximum in-degree, minimum out-degree and maximum out-degree among the vertices of D are represented by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively.

If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. A subset S of vertices of D is a domination set if $N^+[S] = V(D)$. The domination number $\gamma(D)$ is the minimum cardinality of a domination set of D. The domination number of D was introduced by Fu [7]. For the detail contents we refer to [4–6]. A domination set S of minimum cardinality is called a $\gamma(D)$-set of D. A Roman domination function (for short, RDF) on a digraph $D = (V, A)$ is a function $f : V(D) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ has an in-neighbour v for which $f(v) = 2$. The weight of f is $\omega(f) = \sum_{v \in V} f(v)$. In general, for a set $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$. So $\omega(f) = f(V)$. The Roman domination number $\gamma_R(D)$ of D is the minimum weight of a Roman domination function of D.

In [10], Kamaraj and Jakkammal introduced the Roman domination in digraphs. By f we can obtain the ordered partition (V_0, V_1, V_2) of V, where $V_i = \{v \in V \mid f(v) = i\}$ and let $|V_i| = n_i$, for $i = 0, 1, 2$. Note that there exists a bijection between the function $f : V(D) \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of D. So we will write $f = (V_0, V_1, V_2)$. And we say that a function $f = (V_0, V_1, V_2)$ is a γ_R-function if it is an RDF and $f(V) = \gamma_R(D)$. In this representation, the weight $\omega(f) = |V_1| + 2|V_2|$. Since $V_1 \cup V_2$ is a domination set when f is an RDF, and since placing weight 2 at the vertices of a domination set yields an RDF, we have $\gamma(D) \leq \gamma_R(D) \leq 2\gamma(D)$. In [12], Sheikholeslami and Volkmann gave a few of results on Roman domination.

If D is a digraph with $\gamma_R(D) = 2\gamma(D)$, we call D is a Roman digraph.

The de Bruijn digraph $D_B(d,t)$ $(t \geq 2, d \geq 2)$ is a directed pseudograph with the vertex set

$$V(D_B(d,t)) = \{(x_1 x_2 \cdots x_t) \,|\, x_i \in \{0, 1, \cdots, d-1\} \text{ for } i = 1, 2, \cdots, t\}$$

and the arc set

$$A(D_B(d,t)) = \{((x_1 x_2 \cdots x_t), (y_1 y_2 \cdots y_t)) \mid x_2 = y_1, x_3 = y_2, \cdots, x_t = y_{t-1}\}.$$

The digraph $D_B\,(2,3)$ is exhibited in Figure 1(a). Clearly, $D_B\,(d,1)$ is the complete digraph of order d with loop at every vertex. Since we have d choices for each of the coordinates, the order of $D_B\,(d,t)$ is $|V(D_B\,(d,t))| = d^t$.

For $t \geq 2$, the Kautz digraph $D_K\,(d,t)$ is obtained from $D_B\,(d+1,t)$ by deleting all vertices of the form $(x_1x_2\cdots x_t)$ such that $x_i = x_{i+1}$, for some $i \in \{1,2,\cdots,t-1\}$. The digraph $D_K\,(2,3)$ is exhibited in Figure 1(b). Clearly, $D_K\,(d,t)$ has no loops and is a d-regular digraph. Since we have $d+1$ choices for the first coordinate of a vertex in $D_K\,(d,t)$ and d choices for each of the other coordinates, the order of $D_K\,(d,t)$ is $|V(D_K\,(d,t))| = (d+1)d^{t-1} = d^t + d^{t-1}$. The further results of de Bruijn digraphs and Kautz digraphs were given by Araki in [3].

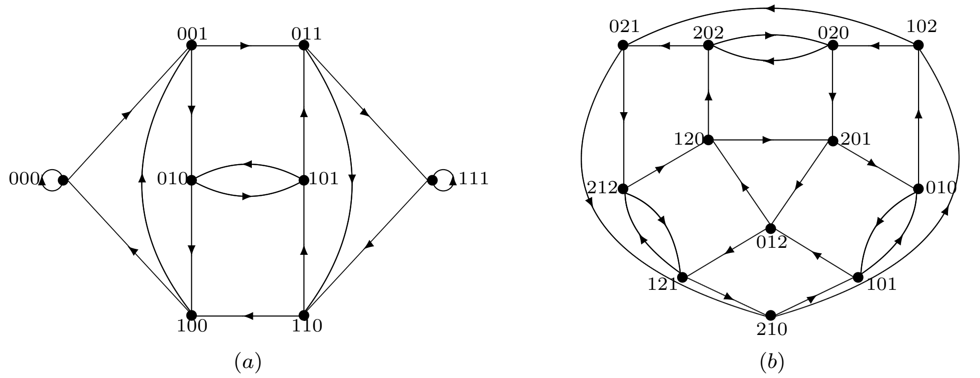

FIGURE 1. (a): The de Bruijn digraph $D_B(2,3)$; (b): The Kautz digraph $D_K(2,3)$.

As two important types of internet topologies, de Bruijn and Kautz graphs have many excellent properties such as low and constant diameter. In this paper, we investigate the Roman domination and Roman digraphs of Kautz and generalized Kautz digraphs and provide the Roman domination numbers of these two classes of digraphs, respectively. While in a new paper, which is preprinted, we discuss the Roman domination of de Bruijn graphs. To show our main results, we start with the following proposition and lemmas.

Proposition 1.1. (*Kamaraj and Jakkammal* [5], 2022).
Let $f = (V_0, V_1, V_2)$ be any $\gamma_R\,(D)$-function of a digraph D. Then

(a) $\Delta^+\,(D[V_1]) \leq 1$;
(b) *If $w \in V_1$, then $N_D^-\,(w) \cap V_2 = \emptyset$;*
(c) *If $u \in V_0$, then $\left|N_D^+\,(u) \cap V_1\right| \leq 2$;*
(d) *V_2 is a $\gamma\,(D)$-set of the induced subdigraph $D[V_0 \cup V_2]$;*
(e) *Let $H = D[V_0 \cup V_2]$. Then each vertex $v \in V_2$ with $N^-\,(v) \cap V_2 \neq \emptyset$ has at least two private neighbours relative to V_2 in the subdigraph H.*

Lemma 1.2. (*Fu, Yang and Jiang* [10], 2009). *A digraph D is a Roman digraph if and only if it has a $\gamma_R\,(D)$-function $f = (V_0, V_1, V_2)$ with $n_1 = |V_1| = 0$.*

Lemma 1.3. *Let $D_K(d,t)$ be a Kautz digraph of order n. Then $\gamma(D_k(d,t)) = d^{t-1}$.*

Proof. Let S be a minimum domination set of $D_K(d,t)$. We obtain $|S| + d|S| \geq n$ from the definition of $D_K(d,t)$. Since $n = d^t + d^{t-1}$, we have $|S| \geq \frac{n}{d+1} = d^{t-1}$. Let S_1 be a vertex subset of $D_K(d,t)$ defined as follows:

$$\begin{aligned} S_1 &= \{v_1, v_2, \cdots, v_{d^{t-1}}\} \\ &= \{(0x_2x_3\cdots x_t) \mid 0 \leq x_{j+1} \leq d,\, x_j \neq x_{j+1} \text{ for } j \in \{1,2,\cdots,t-1\}\}. \end{aligned}$$

Let $v = (x_1x_2\cdots x_t)$ be a vertex in $D_K(d,t)$. Then $N^+(v) = \{(x_2x_3\ldots x_t y) \mid 0 \leq y \leq d\}$ by the definition of $D_K(d,t)$. Hence $N^+(v_p) \cap N^+(v_q) = \emptyset$ for any $p \neq q$, $p, q \in \{1, 2, \cdots, d^{t-1}\}$. By the fact that $D_K(d,t)$ is a d-regular digraph, we have $|N^+(S_1)| = d^{t-1} \cdot d = d^t$. Hence $|N^+(S_1)| + |S_1| = d^t + d^{t-1} = |V(D_K(d,t))| = n$. This implies that S_1 is a domination set of $D_K(d,t)$ and thus $|S| \leq |S_1| = d^{t-1}$. Therefore, $\gamma(D_K(d,t)) = |S| = d^{t-1}$. The proof is completed. □

2. Roman domination number of Kautz digraphs

Let S be an arbitrary minimum domination set of D. Then for each vertex $v \in V(D)$, $N^-[v] \cap S \neq \emptyset$, and v is dominated $|N^-[v] \cap S| \geq 1$ times. We define a function rd counting the times v that is re-dominated as follows:

$$rd(v) = \left|N^-[v] \cap S\right| - 1.$$

For a vertex set $V' \subseteq V(D)$, let $rd(V') = \sum_{v \in V'} rd(v)$. Then, by Proposition 1.1, V_2 is a $\gamma(D)$-set of $D[V_0 \cup V_2]$, and this gives us the following lemma.

Lemma 2.1. *Let D be an r-regular digraph of order n. If $f = (V_0, V_1, V_2)$ is an arbitrary $\gamma_R(D)$-function of D, then*

$$rd(V(D[V_0 \cup V_2])) = (r+1)n_2 - (n - n_1),$$

where $n_1 = |V_1|$ and $n_2 = |V_2|$.

Proof. According to the definition of Roman domination function, we have $V_0 \subseteq N^+(V_2)$. Since D is an r-regular digraph, $d^+(v) = r$ for any vertex $v \in V_2$ in $D[V_0 \cup V_2]$ by Proposition 1.1(b). Then by Proposition 1.1(d), V_2 is a $\gamma(D)$-set of $D[V_0 \cup V_2]$. So

$$\begin{aligned} rd(V(D[V_0 \cup V_2])) &= \sum_{v \in V_0 \cup V_2} rd(v) = \sum_{v \in V_0 \cup V_2} \left(\left|N^-_{D[V_0 \cup V_2]}[v] \cap S\right| - 1\right) \\ &= \sum_{v \in V_0 \cup V_2} \left(\left|N^-_{D[V_0 \cup V_2]}[v] \cap V_2\right| - 1\right) \\ &= (r+1)|V_2| - (|V_0| + |V_2|) \\ &= (r+1)n_2 - (n - n_1). \end{aligned}$$

The proof is completed. □

Lemma 2.2. *Let D be an r-regular digraph of order n and $m = \left\lfloor \frac{n}{r+1} \right\rfloor$ with $n = (r+1)\,m + q$ for some $0 \leq q \leq r$. If $f = (V_0, V_1, V_2)$ is a $\gamma_R(D)$-function of D, then*

(a) $n_2 \geq \left\lceil \frac{n-n_1}{r+1} \right\rceil$;
(b) $f(V(D)) \geq 2m + \left\lceil \frac{2q+(r-1)n_1}{r+1} \right\rceil$;
(c) $f(V(D)) \geq 2m$ *for* $q = 0$;
(d) $f(V(D)) \geq 2m + 2$ *for* $q \geq 1$ *and* $(q, n_1) \neq (1,1)$.

Proof. (a) By Proposition 1.1(d), V_2 is a $\gamma(D)$-set of $D[V_0 \cup V_2]$. By Lemma 2.1 and $rd(v) = |N^-[v] \cap S| - 1 \geq 0$, we have $(r+1)\,n_2 - (n - n_1) \geq 0$. Hence $(r+1)\,n_2 \geq n - n_1$ and $n_2 \geq \left\lceil \frac{n-n_1}{r+1} \right\rceil$.

(b) Since $f(V(D)) = 2|V_2| + |V_1| = 2n_2 + n_1$, we have

$$\begin{aligned}(r+1)\,f(V(D)) &= 2(r+1)\,n_2 + (r+1)\,n_1 \\ &\geq 2n - 2n_1 + (r+1)\,n_1 \\ &= 2(r+1)\,m + 2q + (r-1)\,n_1.\end{aligned}$$

Hence $f(V(D)) \geq 2m + \left\lceil \frac{2q+(r-1)n_1}{r+1} \right\rceil$.

(c) Suppose $q = 0$. Then by (b), $f(V(D)) \geq 2m + \left\lceil \frac{2q+(r-1)n_1}{r+1} \right\rceil \geq 2m.$

(d) Suppose $q \geq 1$.

Case 1. $n_1 = 0$. By (a), $n_2 \geq \left\lceil \frac{n-n_1}{r+1} \right\rceil = \left\lceil \frac{(r+1)m+q}{r+1} \right\rceil = m+1$. Hence $f(V(D)) = 2n_2 + n_1 = 2n_2 \geq 2m + 2$.

Case 2. $n_1 = 1$, $q \geq 2$. By (b), $f(V(D)) \geq 2m + \left\lceil \frac{2q+(r-1)n_1}{r+1} \right\rceil \geq 2m + \left\lceil \frac{4+r-1}{r+1} \right\rceil = 2m + 2$.

Case 3. $n_1 \geq 2$. By (b), $f(V(D)) \geq 2m + \left\lceil \frac{2q+(r-1)n_1}{r+1} \right\rceil \geq 2m + \left\lceil \frac{2+2(r-1)}{r+1} \right\rceil = 2m + 1 + \left\lceil \frac{r-1}{r+1} \right\rceil = 2m + 2.$

The proof is completed. □

Theorem 2.3. *Let $D_K(d,t)$ be a Kautz digraph of order n. Then $\gamma_R(D_K(d,t)) = 2d^{t-1}$ and $D_K(d,t)$ is a Roman digraph.*

Proof. Let

$$V_2 = \{(0x_2 \cdots x_t) \mid 0 \leq x_{i+1} \leq d,\ x_i \neq x_{i+1} \text{ for } i \in \{1, 2, \cdots, t-1\}\},$$
$$V_1 = \emptyset, \quad V_0 = N^+(V_2).$$

By the proof of Lemma 1.3, we see that (V_1, V_2, V_3) forms a partition of $V(D_K(d,t))$. Since $V_0 = N^+(V_2)$, we further have that $f = (V_0, V_1, V_2)$ is an RDF of $D_K(d,t)$ and $f(V(D_K(d,t))) = 2d^{t-1}$. This means that $\gamma_R(D_K(d,t)) \leq 2d^{t-1}$. On the other hand, by Lemma 2.2(c), $f(V(D_K(d,t))) \geq 2\lfloor \frac{n}{d+1} \rfloor = 2d^{t-1}$ since $n = (d+1)d^{t-1}$ and

$D_K(d,t)$ is d-regular. Hence $\gamma_R(D_K(d,t)) \geq 2d^{t-1}$. Now we have $\gamma_R(D_K(d,t)) = 2d^{t-1}$.

Furthermore $f = (V_0, V_1, V_2)$ is a $\gamma_R(D_K(d,t))$-function of $D_K(d,t)$ with $|V_1| = 0$. According to Lemma 1.2, the Kautz digraph $D_K(d,t)$ is a Roman digraph.

The proof is completed. □

3. Roman domination number of generalized Kautz digraphs

In [9], Imase and Itoh studied the generalized Kautz digraph $D_I(d,n)$. In this section, the Roman domination number of generalized Kautz digraphs $D_I(d,n)$ is presented below. Let $D_I(d,n)$ be a digraph of order n with $d < n$, where $V(D_I(d,n)) = \{0, 1, \cdots, n-1\}$ and $A(D_I(d,n)) = \{(i,j) \mid j \equiv -d(i+1) + k \,(\text{mod}\, n),\ 0 \leq k \leq d-1\}$. We say that $D_I(d,n)$ is a generalized Kautz digraph. The digraph $D_I(2,8)$ is exhibited in Figure 2. It is easy to see that the generalized Kautz digraph $D_I(d,n)$ is a d-regular digraph. If $d = n-1$, then the generalized Kautz digraph $D_I(d,n)$ is isomorphic to the Kautz digraph $D_K(n-1,1)$, the Kautz digraph $D_I(n-1,n)$ is a complete digraph $\overset{\leftrightarrow}{K}_n$.

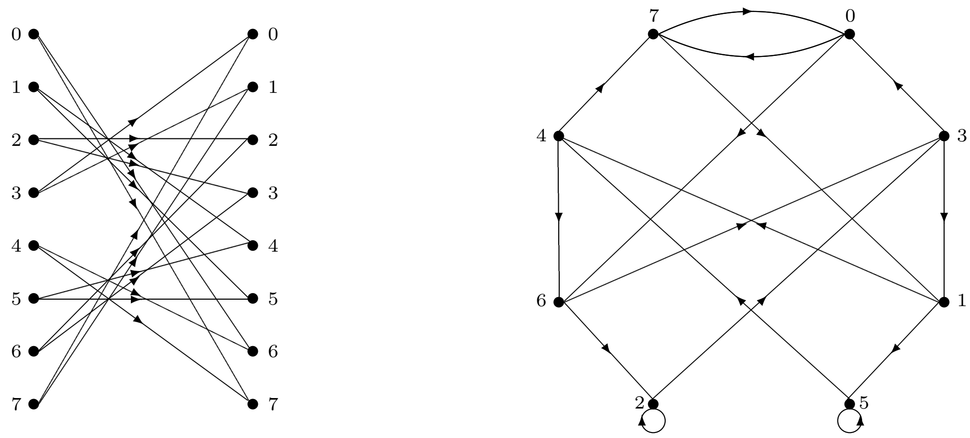

FIGURE 2. The generalized Kautz digraph $D_I(2,8)$.

In 2003, Kikuchi and Shibata [11] presented the following lemma.

Lemma 3.1. (*Kikuchi and Shibata* [13], 2003). *Let $D_I(d,n)$ be a generalized Kautz digraph of order n. If n, d are two positive integers with $d < n$, then* $\gamma(D_I(d,n)) = \left\lceil \frac{n}{d+1} \right\rceil$.

From the proof of Theorem 2.3, we see that $\gamma_R(D_K(d,t)) = 2d^{t-1}$ and $D_K(d,t)$ is a Roman digraph. The following theorem shows a method of determining the Roman domination number of $D_I(d,n)$.

Theorem 3.2. *Let $D_I(d,n)$ be a generalized Kautz digraph of order n such that n, d are two positive integers with $d < n$. If $n = m(d+1) + q$ for some $0 \le q \le d$, then*

$$\gamma_R(D_I(d,n)) = \begin{cases} 2m, & \text{if } q = 0; \\ 2m+1, & \text{if } q = 1; \\ 2m+2, & \text{if } q = 2, 3, \cdots, d. \end{cases}$$

Proof. As described in Table 1, we have the following domination structure of $D_I(d,n)$. Note that $n = (d+1)m + q$. We consider the following three cases according to the value of q.

TABLE 1. The generalized Kautz digraph $D_I(d,n)$

$i \in V(D_I(d,n))$	$N^+_{D_I(d,n)}(i)$			
0	$n-d$	$n-d+1$	$\cdots$	$n-1$
1	$n-2d$	$n-2d+1$	$\cdots$	$n-d-1$
2	$n-3d$	$n-3d+1$	$\cdots$	$n-2d-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$n-(i+1)d$	$n-(i+1)d+1$	$\cdots$	$n-id-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	$n-md$	$n-md+1$	$\cdots$	$n-(m-1)d-1$
m	$n-(m+1)d$	$n-(m+1)d+1$	$\cdots$	$n-md-1$

Case 1. $q = 0$. We have $\sum_{i=0}^{m-1} N^+(i) = \{m, m+1, \ldots, n-1\}$ (see Figure 3).

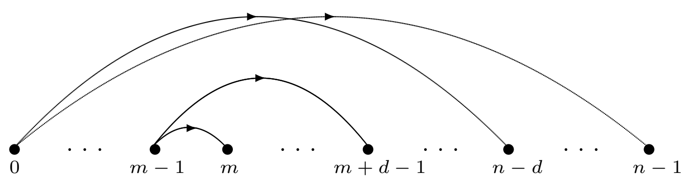

FIGURE 3. The generalized Kautz digraph $D_I(d,n)$ for $q = 0$.

Case 2. $q = 1$. We have $\sum_{i=0}^{m-1} N^+(i) = \{m+1, \ldots, n-1\}$ and $N^+(m) = \{m-d+1, m-d+2, \ldots, m\}$ (see Figure 4).

Case 3. $2 \le q \le d$. We have $\sum_{i=0}^{m-1} N^+(i) = \{m+q, \ldots, n-1\}$ and $N^+(m) = \{m-d+q, m-d+q+1, \ldots, m+q-1\}$ (see Figure 5).

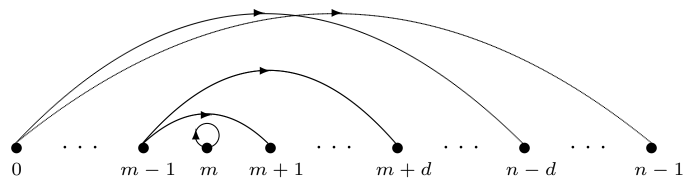

FIGURE 4. The generalized Kautz digraph $D_I(d, n)$ for $q = 1$.

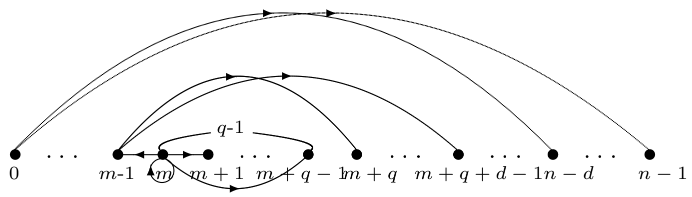

FIGURE 5. The generalized Kautz digraph $D_I(d, n)$ for $2 \le q \le r$.

Let

$$U_2 = \begin{cases} \{i : 0 \le i \le m-1\}, & \text{if } q = 0, 1; \\ \{i : 0 \le i \le m\}, & \text{if } q = 2, 3, \cdots, d. \end{cases}$$

$$U_1 = \begin{cases} \{m\}, & \text{if } q = 1; \\ \emptyset, & \text{otherwise.} \end{cases}$$

$$U_0 = N^+(U_2) \backslash U_2.$$

Thus $N^+[U_2] \cup U_1 = V(D_I(d, n))$, and $f = (V_0, V_1, V_2) = (U_0, U_1, U_2)$ is a Roman domination function of $D_I(d, n)$ with

$$f(V(D_I(d, n))) = \begin{cases} 2m, & \text{if } q = 0; \\ 2m+1, & \text{if } q = 1; \\ 2m+2, & \text{if } q = 2, 3, \cdots, d. \end{cases}$$

It follows

$$\gamma_R(D_I(d, n)) \le f(V(D_I(d, n))) = \begin{cases} 2m, & \text{if } q = 0; \\ 2m+1, & \text{if } q = 1; \\ 2m+2, & \text{if } q = 2, 3, \cdots, d. \end{cases}$$

By Lemma 2.2(b), we have $\gamma_R(D_I(d, n)) \ge 2m + \left\lceil \frac{2q+(d-1)n_1}{d+1} \right\rceil$.
Hence

$$\gamma_R(D_I(d, n)) \ge \begin{cases} 2m, & \text{if } q = 0; \\ 2m+1, & \text{if } q = 1. \end{cases}$$

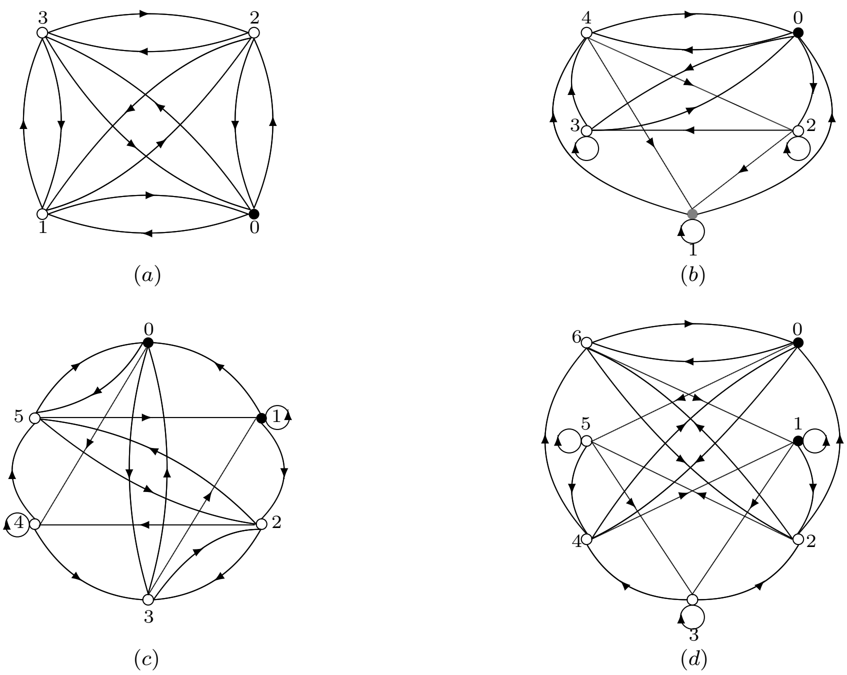

FIGURE 6. A Roman domination function on $D_I(3,n)$ for $n = 4,5,6,7$. (a): $D_I(3,4)$; (b): $D_I(3,5)$; (c): $D_I(3,6)$; (d): $D_I(3,7)$.

If $q = 2$ and $n_1 \neq 0$, we have $\gamma_R(D_I(d,n)) \geq 2m + \lceil \frac{2q+(d-1)n_1}{d+1} \rceil \geq 2m+2$. If $q = 2$ and $n_1 = 0$, then by Lemma 2.2(a) and $n = (d+1)m + q$,

$$n_2 \geq \left\lceil \frac{n-n_1}{d+1} \right\rceil = \left\lceil \frac{n}{d+1} \right\rceil = \left\lceil \frac{(d+1)m+q}{d+1} \right\rceil = m + \left\lceil \frac{q}{d+1} \right\rceil = m+1.$$

Furthermore, $\gamma_R(D_I(d,n)) = 2n_2 \geq 2m+2$, $n_2 = |V_2|$.

So

$$\gamma_R(D_I(d,n)) \geq \begin{cases} 2m, & \text{if } q = 0; \\ 2m+1, & \text{if } q = 1; \\ 2m+2, & \text{if } q = 2,3,\cdots,d. \end{cases}$$

Based on the argument above, we have

$$\gamma_R(D_I(d,n)) = \begin{cases} 2m, & \text{if } q = 0; \\ 2m+1, & \text{if } q = 1; \\ 2m+2, & \text{if } q = 2,3,\cdots,d. \end{cases}$$

The proof is completed. □

In Figure 6, we show a Roman domination function on $D_I(3,n)$ for $4 \leq n \leq 7$, where black solid dots indicate vertices in V_2, grey solid dots indicate vertices in V_1, and white hollow dots indicate vertices in V_0. It is not difficult to check that $\gamma_R(D_I(3,4)) = 2$, $\gamma_R(D_I(3,5)) = 3$, $\gamma_R(D_I(3,6)) = 4$, $\gamma_R(D_I(3,7)) = 4$.

Form Theorem 3.2, one can see the following corollary for the solution of $D_I(d,n)$.

Corollary 3.3. *The generalized Kautz digraph $D_I(d,n)$ of order n is a Roman digraph for two positive integers n,d with $d<n$, $n \not\equiv 1 \pmod{(d+1)}$.*

Proof. According to the proof of Theorem 3.2, we have $f = (V_0, V_1, V_2)$ is a $\gamma_R(D_I(d,n))$-function with $|V_1|=0$. Then by Lemma 1.2, the generalized Kautz digraph $D_I(d,n)$ of order n is a Roman digraph for two positive integers n,d with $d<n$, $n \not\equiv 1 \pmod{(d+1)}$. ☐

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