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SUPERIORIZATION TECHNIQUE WITH A PROJECTED SUBGRADIENT METHOD FOR GAMES

ALEXANDER J. ZASLAVSKI

Abstract. In our recent paper we studied a constrained minimization problem with a convex objective function and with a feasible region, which is the intersection of finitely many closed convex constraint sets. We used a projected subgradient method combined with a dynamic string-averaging projection method, with variable strings and variable weights, as a feasibility-seeking algorithm. It was shown that any sequence, generated by the superiorized version of a dynamic string-averaging projection algorithm, not only converges to a feasible point but, additionally, also either its limit point solves the constrained minimization problem or the sequence is strictly Fejér monotone with respect to the solution set. In the present paper we use the superiorization technique in order to obtain an analogous result for a game with a finite number of players.

1. Introduction

During more than sixty years now, there has been a lot of activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [3, 5, 12, 14, 15, 18–20, 22–25, 28, 29] and the references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also covers the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility and common fixed point problems, which find important applications in engineering and medical sciences [9, 13, 16, 26–29].

Let (X, ρ) be a complete metric space. In [5] it was studied the influence of errors on the convergence of orbits of nonexpansive mappings in metric spaces and it was obtained the following result (see also Theorem 2.72 of [25]).

Theorem 1.1. *Let $A : X \to X$ satisfy*

$$\rho(Ax, Ay) \le \rho(x, y) \text{ for all } x, y \in X, \tag{1.1}$$

let $F(A)$ be the set of all fixed points of A and let for each $x \in X$, the sequence $\{A^n x\}_{n=1}^{\infty}$ converges in (X, ρ).

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Assume that $\{x_n\}_{n=0}^{\infty} \subset X$, $\{r_n\}_{n=0}^{\infty} \subset (0, \infty)$ *satisfies*

$$\sum_{n=0}^{\infty} r_n < \infty$$

and that

$$\rho(x_{n+1}, Ax_n) \le r_n,\ n = 0, 1, \dots.$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges to a fixed point of* A *in* (X, ρ).

Theorem 1.1 found interesting applications and is an important ingredient in superiorization and perturbation resilience of algorithms. See [2,4,6,7,10,11,17,21] and the references mentioned therein. The superiorization methodology works by taking an iterative algorithm, investigating its perturbation resilience, and then using proactively such perturbations in order to "force" the perturbed algorithm to do in addition to its original task something useful. This methodology can be explained by the following result on convergence of inexact iterates.

Assume that $(X, \|\cdot\|)$ is a Banach space, $\rho(x, y) = \|x - y\|$ for all $x, y \in X$, a mapping $A : X \to X$ satisfies (1.1) and that for each $x \in X$, the sequence $\{A^n x\}_{n=1}^{\infty}$ converges in the norm topology. $x_0 \in X$, $\{\beta_k\}_{k=0}^{\infty}$ is a sequence of positive numbers satisfying

$$\sum_{k=0}^{\infty} \beta_k < \infty, \tag{1.2}$$

$\{v_k\}_{k=0}^{\infty} \subset X$ is a norm bounded sequence and that for any integer $k \ge 0$,

$$x_{k+1} = A(x_k + \beta_k v_k). \tag{1.3}$$

Then it follows from Theorem 1.1 that the sequence $\{x_k\}_{k=0}^{\infty}$ converges in the norm topology of X and its limit is a fixed point of A. In this case the mapping A is called bounded perturbations resilient (see [6] and Definition 10 of [10]). In other words, if exact iterates of a nonexpansive mapping converge, then its inexact iterates with bounded summable perturbations converge too.

Now assume that $x_0 \in X$ and the sequence $\{\beta_k\}_{k=0}^{\infty}$ satisfying (1.2) are given and we need to find an approximate fixed point of A. In order to meet this goal we construct a sequence $\{x_k\}_{k=1}^{\infty}$ defined by (1.3). Under an appropriate choice of the bounded sequence $\{v_k\}_{k=0}^{\infty}$, the sequence $\{x_k\}_{k=1}^{\infty}$ possesses some useful property. For example, the sequence $\{f(x_k)\}_{k=1}^{\infty}$ can be decreasing, where f is a given function. This superiorization methodology was used in [11] in order to study a constrained minimization problem with a convex objective function and with a feasible region, which is the intersection of finitely many closed convex constraint sets. In [11] it was used a projected normalized subgradient method combined with a dynamic string-averaging projection method, with variable strings and variable weights, as a feasibility-seeking algorithm, which was introduced in [8]. It is shown that any sequence, generated by the superiorized version of a dynamic string-averaging projection algorithm, not only converges to a feasible point but, additionally, also either its limit point solves the constrained minimization problem or the sequence is strictly Fejér monotone with respect to a subset of the solution set. It should be mentioned that in [11] it was used a projected normalized subgradient method. It means that for any iteration a subgradient should be normalized. In [30] we showed that the main result of [11] is also true without normalization of subgradients if the objective function is Lipschitz. Of course, this makes our computations much easier. In the present paper we use the superiorization technique in order to obtain an analogous result for a game with a finite number of players.

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with a inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$.

For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{\|x - y\| : \ y \in E\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \leq r\}.$$

In the sequel we use the following well-known proposition [10].

Proposition 1.2. *Let D be a nonempty closed convex subset of X. Then for each $x \in X$ there is a unique point $P_D(x) \in D$ satisfying*

$$\|x - P_D(x)\| = \inf\{\|x - y\| : \ y \in D\}.$$

Moreover,

$$\|P_D(x) - P_D(y)\| \leq \|x - y\| \text{ for all } x, y \in X$$

and for each $x \in X$ and each $z \in D$,

$$\langle z - P_D(x), x - P_D(x) \rangle \leq 0$$

and

$$\|z - P_D(x)\|^2 + \|x - P_D(x)\|^2 \leq \|z - x\|^2.$$

Suppose that $C_1, \dots, C_m$ are nonempty closed convex subsets of X where m is a natural number. Set

$$C = \cap_{i=1}^m C_i.$$

We suppose that $C \neq \emptyset$. For $i = 1, \dots, m$ set

$$P_i = P_{C_i}.$$

By an index vector, we a mean a vector $t = (t_1, \dots, t_q)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, q$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q, \ P[t] = P_{t_q} \cdots P_{t_1}.$$

A finite set Ω of index vectors is called fit if for each $i \in \{1, \dots, m\}$, there exists $t = (t_1, \dots, t_q) \in \Omega$ such that $t_s = i$ for some $s \in \{1, \dots, q\}$.

It is easy to see that for each index vector t

$$\|P[t](x) - P[t](y)\| \leq \|x - y\| \text{ for all } x, y \in X,$$

$$P[t](x) = x \text{ for all } x \in C.$$

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a fit finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ be such that } \sum_{t \in \Omega} w(t) = 1.$$

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{t \in \Omega} w(t)P[t](x),\ x \in X.$$

It is easy to see that

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| \le \|x - y\| \text{ for all } x, y \in X,$$

$$P_{\Omega,w}(x) = x \text{ for all } x \in C.$$

We use the following assumption.

(A) For each $\epsilon > 0$ and each $M > 0$ there exists $\delta = \delta(\epsilon, M) > 0$ such that for each $x \in B(0, M)$ satisfying $d(x, C_i) \le \delta$, $i = 1, \dots, m$ the inequality $d(x, C) \le \epsilon$ holds.

Note that if the space X is finite-dimensional, then assumption (A) always holds [10]. It means that if a point is closed to any set C_i, $i = 1, \dots, m$, then it is close to their intersection C.

In the sequel we assume that Assumption (A) holds. Fix a number

$$\Delta \in (0, m^{-1})$$

and an integer

$$\bar{q} \ge m.$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \le \bar{q} \text{ for all } t \in \Omega,$$

$$w(t) \ge \Delta \text{ for all } t \in \Omega.$$

The following result was obtained in [10].

Theorem 1.3. *Let $\{\beta_k\}_{k=0}^{\infty}$ be a sequence of nonnegative numbers such that $\sum_{k=0}^{\infty} \beta_k < \infty$, $\{v_k\}_{k=0}^{\infty} \subset X$ be a norm bounded sequence, $\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$, $x_0 \in X$, and let for any integer $k \ge 0$,*

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k + \beta_k v_k).$$

Then the sequence $\{x_k\}_{k=0}^{\infty}$ converges in the norm topology of X and its limit belongs to C.

In the proof of this result assumption (A) is used.

2. The main result

Assume that m is a natural number and that for each $i = 1, \dots, m$, m_i is a natural number and $(X_i, \langle \cdot, \cdot \rangle)$ is a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm. Assume that for each $i \in \{1, \dots, m\}$,

$$U_i \subset X_i$$

is an open convex set,

$$C_{i,j} \subset U_i,\ j = 1, \dots, m_i$$

are nonempty, closed, convex sets,

$$C_i = \cap_{j=1}^{m_i} C_{i,j} \ne \emptyset$$

and that

$$f_i : \prod_{j=1}^{m} U_j \to R^1$$

is a continuous function which represents the criterion of i-th player. For each $i \in \{1, \dots, m\}$, each $x = (x_1, \dots, x_m) \in \prod_{j=1}^m X_j$ and each $\xi \in X_i$ set

$$(x^{-i}, \xi) = y = (y_1, \dots, y_m) \in \prod_{j=1}^{m} X_j$$

such that

$$y_j = x_j,\ j \in \{1, \dots, m\} \setminus \{i\},\ y_i = \xi.$$

Assume that $L \geq 1$ and that for each $i \in \{1, \dots, m\}$ and each $x = (x_1, \dots, x_m) \in \prod_{j=1}^m U_j$ the function

$$f_i(x^{-i}, \cdot) : U_i \to R^1$$

is convex, satisfies for each $y_1, y_2 \in U_i$

$$|f_i(x^{-i}, y_1) - f_i(x^{-i}, y_2)| \leq L\|y_1 - y_2\| \tag{2.1}$$

and has a point of minimum on C_i. For each $i \in \{1, \dots, m\}$ and each $j \in \{1, \dots, m_i\}$ set

$$P_{i,j} = P_{C_{i,j}} : X_i \to C_{i,j}. \tag{2.2}$$

Let $i \in \{1, \dots, m\}$. By an (i)-index vector, we a mean a vector $t = (t_1, \dots, t_q)$ such that $t_j \in \{1, \dots, m_i\}$ for all $j = 1, \dots, q$.

For an (i)-index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q,\ P[t] = P_{i,t_q} \cdots P_{i,t_1}. \tag{2.3}$$

A finite set Ω of (i)-index vectors is called (i)-fit if for each $j \in \{1, \dots, m_i\}$, there exists $t = (t_1, \dots, t_q) \in \Omega$ such that $t_s = j$ for some $s \in \{1, \dots, q\}$. Denote by $\mathcal{M}^{(i)}$ the collection of all pairs (Ω, w), where Ω is an (i)-fit finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ be such that } \sum_{t \in \Omega} w(t) = 1. \tag{2.4}$$

Let $(\Omega, w) \in \mathcal{M}^{(i)}$. Define

$$P_{\Omega,w}(x) = \sum_{t \in \Omega} w(t) P[t](x),\ x \in X_i. \tag{2.5}$$

It is easy to see that

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| \leq \|x - y\| \text{ for all } x, y \in X_i, \tag{2.6}$$

$$P_{\Omega,w}(x) = x \text{ for all } x \in C_i. \tag{2.7}$$

We assume that for each $i \in \{1, \dots, m\}$ the following assumption holds.

(A) For each $\epsilon > 0$ and each $M > 0$ there exists $\delta = \delta(\epsilon, M) > 0$ such that for each $x \in X_i$ satisfying $\|x\| \leq M$ and $d(x, C_{i,j}) \leq \delta,\ j = 1, \dots, m_i$ the inequality $d(x, C_i) \leq \epsilon$ holds.

Note that if the space X is finite-dimensional, then assumption (A) always holds [10]. It means that if a point is closed to any set $C_{i,j}$, $j = 1, \dots, m_i$, then it is close to their intersection C_i.

Let $i \in \{1, \dots, m\}$. Fix a number

$$\Delta_i \in (0, m_i^{-1}) \tag{2.8}$$

and an integer

$$\bar{q}_i \geq m_i. \tag{2.9}$$

Denote by $\mathcal{M}_*^{(i)}$ the set of all $(\Omega, w) \in \mathcal{M}^{(i)}$ such that

$$p(t) \leq \bar{q}_i \text{ for all } t \in \Omega, \tag{2.10}$$

$$w(t) \geq \Delta_i \text{ for all } t \in \Omega. \tag{2.11}$$

For each $x = (x_1, \dots, x_m) \in \prod_{j=1}^m U_j$ define

$$(\partial f_i / \partial x_i)(x) = \{l \in X_i :$$

$$f_i(x^{-i}, \xi) - f_i(x) \geq \langle l, \xi - x_i \rangle \text{ for each } \xi \in U_i\}. \tag{2.12}$$

Now we describe our algorithm.

Suppose that for each $i \in \{1, \dots, m\}$,

$$\{(\Omega_{i,k}, w_{i,k})\}_{k=1}^\infty \subset \mathcal{M}_*^{(i)},$$

N is a natural number, $N_k \in \{1, \dots, N\}$ for each integer $k \geq 0$,

$$\alpha_{k,j}^{(i)} \in (0, 1] \text{ for all integers, } k \geq 0 \text{ and } j \in \{1, \dots, N_k\}, \tag{2.13}$$

$$\sum_{k=0}^{\infty} \sum_{j=1}^{N_k} \alpha_{k,j}^{(i)} < \infty, \tag{2.14}$$

$$x_0 = (x_0^{(1)}, \dots, x_0^{(m)}),$$
$$x_0^{(i)} \in U_i, \ i = 1, \dots, m.$$

Define

$$\{x_k^{(i)}\}_{k=0}^\infty \subset U_i, \ i = 1, \dots, m,$$
$$\{l_{k,j}^{(i)} : \ k = 0, 1, \dots, \ j \in \{1, \dots, N_k\}\} \subset X_i, \ i =, \dots, m,$$
$$\{x_{k,j}^{(i)} : \ k = 0, 1, \dots, \ j \in \{1, \dots, N_k\}\} \subset U_i, \ i = 1, \dots, m$$

and

$$x_{k,j-1} = (x_{k,j-1}^{(1)}, \dots, x_{k,j-1}^{(m)}), \ k = 0, 1, \dots, \ j = 1, \dots, N_k$$

as follows: for each integer $k \geq 1$ and each $i \in \{1, \dots, m\}$,

$$x_{k-1,0}^{(i)} = x_{k-1}^{(i)}, \tag{2.15}$$

and for each integer $j \in \{1, \dots, N_k\}$,

$$l_{k-1,j}^{(i)} \in (\partial f_i / \partial x_i)(x_{k-1,j-1}), \tag{2.16}$$

$$x_{k-1,j}^{(i)} = x_{k-1,j-1}^{(i)} - \alpha_{k-1,j}^{(i)} e_{k-1,j}^{(i)}, \tag{2.17}$$

$$x_k^{(i)} = x_{k,N_k}^{(i)} = P_{\Omega_{i,k},w_{i,k}}(x_{k-1}^{(i)}, N_k). \tag{2.18}$$

We prove the following result.

Theorem 2.1. *For each $i \in \{1,\dots,m\}$, the sequence $\{x_k^{(i)}\}_{k=0}^{\infty}$ converges in the norm topology of X_i to $x_*^{(i)} \in C_i$,*

$$x_* = (x_*^{(1)},\dots,x_*^{(m)})$$

and at least one of the following assertions holds:

(a)

$$f_i(x_*^{(1)},\dots,x_*^{(m)}) \le f_i(x_*^{-i},\xi) \text{ for each } \xi \in C_i;$$

(b) if

$$f_i(x_*^{(1)},\dots,x_*^{(m)}) > \inf\{f_i(x_*^{-i},\xi) :\ \xi \in C_i\}$$

and

$$C_{min}^i = \{\xi \in C_i :\ f_i(x_*^{-i},\xi) \le f_i(x_*^{-i},\eta) \text{ for each } \eta \in C_i\},$$

then there exist a natural number k_0 and $\Delta_0 \in (0,1)$, such that for each $\xi \in C_{min}^i$ and each integer $k \ge k_0$,

$$\|x_k^{(i)} - \xi\|^2 \le \|x_{k-1}^{(i)} - \xi\|^2 - \Delta_0 \sum_{p=1}^{N_k} \alpha_{k-1,p}^{(i)}.$$

3. Auxiliary result

Lemma 3.1. *Let $i \in \{1,\dots,m\}$, $x_*^{(j)} \in U_j$, $j = 1,\dots,m$, $x_* = (x_*^{(1)},\dots,x_*^{(m)})$,*

$$\bar{x} \in C_i,\ \alpha \in (0,1],\ \Delta > 0,\ \widehat{x} \in C_i, \tag{3.1}$$

$$f_i(x_*^{-i},\widehat{x}) \ge f_i(x_*^{-i},\bar{x}) + \Delta, \tag{3.2}$$

$$v \in (\partial f_i/\partial x_i)(x_*^{-i},\widehat{x}), \tag{3.3}$$

$$(\Omega, w) \in \mathcal{M}_*^{(i)}, \tag{3.4}$$

$$y - P_{\Omega,w}(\widehat{x} - \alpha v). \tag{3.5}$$

Then

$$\|\widehat{x} - \alpha v - \bar{x}\|^2 \le \|\widehat{x} - \bar{x}\|^2 + \alpha^2 L^2 - 2\alpha\Delta$$

and

$$\|y - \bar{x}\|^2 \le \|\widehat{x} - \bar{x}\|^2 + \alpha^2 L^2 - 2\alpha\Delta.$$

Proof. By (2.1) and (3.3),

$$\|v\| \le L, \tag{3.6}$$

$$\langle v, \bar{x} - \widehat{x}\rangle \le f_i(x_*^{-i},\bar{x}) - f_i(x_*^{-i},\widehat{x}\rangle. \tag{3.7}$$

By (2.7), (3.4)-(3.7) and the inclusion $\bar{x} \in C_i$,

$$\begin{aligned}\|y - \bar{x}\|^2 &= \|P_{\Omega,w}(\widehat{x} - \alpha x) - \bar{x}\|^2 \\ &\le \|\widehat{x} - \alpha v - \bar{x}\|^2 \\ &= \|\widehat{x} - \bar{x}\|^2 + \alpha^2\|v\|^2 - 2\alpha\langle v, \widehat{x} - \bar{x}\rangle\end{aligned}$$

$$\leq \|\widehat{x} - \bar{x}|^2 + \alpha^2 L^2 + 2\alpha(f_i(x_*^{-i}, \bar{x}) - f_i(x_*^{-i}, \widehat{x}))$$
$$\leq \|x - \widehat{x}\|^2 + \alpha^2 L^2 - 2\alpha\Delta.$$

The relation above implies

$$\|\widehat{x} - \alpha v - \bar{x}\|^2 \leq \|x - \bar{x}\|^2 + \alpha^2 L^2 - 2\alpha\Delta.$$

Lemma 3.1 is proved. □

4. Proof of Theorem 2.1

Let $i \in \{1, \dots, m\}$. By (2.18), for each integer $k \geq 1$,

$$x_k^{(i)} = P_{\Omega_{i,k}, w_{i,k}}(x_{k-1}^{(i)}, N_k). \tag{4.1}$$

By (2.15) and (2.17), for each integer $k \geq 1$ and each $p \in \{1, \dots, N_k\}$,

$$\begin{aligned} \|x_{k-1,p}^{(i)} - x_{k-1}^{(i)}\| &= \|\sum_{j=1}^{p}(x_{k-1,j}^{(i)} - x_{k-1,j-1}^{(i)})\| \\ &\leq \sum_{j=1}^{N_k} \|x_{k-1,j}^{(i)} - x_{k-1,j-1}^{(i)}\| \\ &\leq \sum_{j=1}^{N_k} \alpha_{k-1,j}^{(i)} \|l_{k-1,j}^{(i)}\|. \end{aligned} \tag{4.2}$$

By (2.1), (2.16) and (4.2), for each integer $k \geq 1$ and each $p \in \{1, \dots, N_k\}$,

$$\|x_{k-1,p}^{(i)} - x_{k-1}^{(i)}\| \leq L \sum_{j=1}^{N_k} \alpha_{k-1,j}^{(i)} \tag{4.3}$$

and in particular

$$\|x_{k-1,N_k}^{(i)} - x_{k-1}^{(i)}\| \leq L \sum_{j=1}^{N_k} \alpha_{k-1,j}^{(i)}. \tag{4.4}$$

In view of (2.14) and (4.3),

$$\sum_{k=1}^{\infty} \|\|x_{k-1,N_k}^{(i)} - x_{k-1}^{(i)}\| \leq L \sum_{k=1}^{\infty} \sum_{j=1}^{N_k} \alpha_{k-1,j}^{(i)}. \tag{4.5}$$

Theorem 1.3 and equations (4.1) and (4.5) imply that there exists

$$x_*^{(i)} = \lim_{k \to \infty} x_k^{(i)} \in C_i \tag{4.6}$$

in the norm topology.

Let $x_* = (x_*^{(1)}, \dots, x_*^{(m)})$. Assume that $i \in \{1, \dots, m\}$ and that (a) does not hold. Let

$$\widehat{x} \in C_{min}^i. \tag{4.7}$$

There is $\Delta_0 > 0$ such that

$$f_i(x_*^{(1)}, \dots, x_*^{(m)}) > f_i(x_*^{-i}, \widehat{x}) + 4\Delta_0. \tag{4.8}$$

By the continuity of f_i and (2.14), (4.3), (4.6)-(4.8), there exists a natural number k_1 such that for each integer $k \ge k_1$ and each $j \in \{0, \dots, N_k\}$,

$$f_i(x_{k,j}) > f_i(x^{-i}_{k,j}, \widehat{x}) + 3\Delta_0 \tag{4.9}$$

and

$$\alpha^{(i)}_{k-1,j} L^2 \le \Delta_0. \tag{4.10}$$

Let an integer $k \ge k_1 + 1$ and $p \in \{1, \dots, N_k\}$. Equations (2.16)-(2.18), (4.9), (4.10) and Lemma 3.1 applied with

$$\bar{x} = \widehat{x},\ x_* = x_{k-1,p},\ v = l^{(i)}_{k-1,p},$$

$$\widehat{x} = x^{(i)}_{k-1,p},\ \bar{x} = \widehat{x},\ \Delta = 3\Delta_0,\ \alpha = \alpha^{(i)}_{k-1,p},\ y = x^{(i)}_{k,p}$$

imply that

$$\|x^{(i)}_{k-1,p} - \widehat{x}\|^2 \le \|x^{(i)}_{k-1,p-1} - \widehat{x}\|^2 - 6\alpha^{(i)}_{k-1,p}\Delta_0 + (\alpha^{(i)}_{k-1,p})^2 L^2$$

$$\le \|x^{(i)}_{k-1,p-1} - \widehat{x}\|^2 - \alpha^{(i)}_{k-1,p}\Delta_0. \tag{4.11}$$

By (2.15), (2.18) and (4.11),

$$\begin{aligned}\|x^{(i)}_{k-1} - \widehat{x}\|^2 - \|x^{(i)}_{k} - \widehat{x}\|^2 &\ge \|x^{(i)}_{k-1,0} - \widehat{x}\|^2 - \|x^{(i)}_{k-1,N_k} - \widehat{x}\|^2 \\ &= \sum_{p=1}^{N_k} (\|x^{(i)}_{k-1,p-1} - \widehat{x}\|^2 - \|\|x^{(i)}_{k-1,p} - \widehat{x}\|^2) \\ &\ge \Delta_0 \sum_{p=1}^{N_k} \alpha^{(i)}_{k-1,p}\end{aligned}$$

and

$$\|x^{(i)}_{k} - \widehat{x}\|^2 \le \|x^{(i)}_{k-1} - \widehat{x}\|^2 - \Delta_0 \sum_{p=1}^{N_k} \alpha^{(i)}_{k-1,p}.$$

Theorem 2.1 is proved.

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A. J. Zaslavski
Department of Mathematics, The Technion – Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: ajzasl@technion.ac.il