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DENSITY AND GENERICITY OF WELL-POSED VECTOR OPTIMIZATION PROBLEMS

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Abstract. In this paper we consider well-posedness properties of vector optimization problems with objective function $f : X \to Y$ where X and Y are Banach spaces and Y is partially ordered by a closed convex pointed cone with nonempty interior. The vector well-posedness notion considered in this paper is the one due to Dentcheva and Helbig [5], which is a natural extension of Tykhonov well-posedness for scalar optimization problems. When a scalar optimization problem is considered it is possible to prove (see e.g. [21], [28]) that under some assumptions the set of functions for which the related optimization problem is Tykhonov well-posed is dense or even more is "big" i.e. contains a dense G_δ set (these results are called genericity results). The aim of this paper is to extend these genericity results to vector optimization problems.

1. Introduction

Well-posedness properties are important qualitative characterizations for scalar and vector optimization problems. In particular, the notion of well-posedness plays a central role in stability theory for scalar optimization (see e.g. [6]). The well-posedness notion for scalar functions dates back to Hadamard [11] and to Tykhonov [26]. Extensions to vector and set-valued cases are presented in several papers and are still a topic of research (see e.g. [1], [2], [4], [5], [6], [10], [14], [15], [19], [21], [23]). In [23] a classification of vector well-posedness notions into two groups is given: pointwise and global notions. The definitions of the first group consider a fixed efficient point (or the image of an efficient point) and deal with well-posedness of the vector optimization problem at this point. This approach imposes that the minimizing sequences related to the considered point are well-behaved. Since in the vector case the solution set is typically not a singleton, there is also a class of definitions, called global notions, that involve the efficient frontier as a whole.
In scalar optimization, a crucial point is the identification of classes of objective functions for which the related optimization problem enjoys well-posedness properties. It is known (see e.g. [6]) that, when X is finite-dimensional, scalar optimization problems with convex objective function $f : X \to \mathbb{R}$ enjoy well-posedness properties. Similarly, it is known that vector optimization problems with cone-convex objective

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function $f : X \to Y$ with X and Y finite-dimensional, enjoy well posedness properties (see. eg. [23]).
When functions $f : X \to \mathbb{R}$ with X infinite-dimensional are considered, it is known that convexity does not guarantee well-posedness (see e.g. [6]). In this case it is interesting to investigate density properties of well-posed optimization problems. A stronger version of these results leads to find classes of functions for which the subset of well-posed optimization problems is "big" in the sense of Baire category, i.e. contains a dense G_δ set (see e.g. [16], [21] and references therein).
The aim of this paper is to extend these results, called genericity results, to vector optimization problems with objective function $f : X \to Y$ where X and Y are Banach spaces. In our investigation we will focus on the pointwise well-posedness notion for vector functions due to Dentcheva and Helbig [5].
The outline of the paper is the following. In Section 2 we introduce the notations and we recall some preliminary notions. In Section 3 we recall some scalar and vector well-posedness notions. In Section 4 we give results concerning density of well-posed vector optimization problems, without convexity assumptions. Section 5 is devoted to genericity results under cone-convexity assumptions.

2. Preliminaries

In the following X and Y are Banach spaces. We consider a function $f : X \to Y$ (results in this paper hold true also when $f : A \subseteq X \to Y$ where A is closed). Let $C \subseteq Y$ a closed, convex, pointed cone with nonempty interior, endowing Y with a partial order in the following way

$$\begin{aligned} y_1 \leq_C y_2 &\iff y_2 - y_1 \in C \\ y_1 <_C y_2 &\iff y_2 - y_1 \in \operatorname{int} C \end{aligned} \tag{2.1}$$

For a set $A \subseteq X$ we denote by $\operatorname{diam} A$ the diameter of A, i.e.

$$\operatorname{diam} A = \sup\{\|x - y\| : \ x, y \in A\}$$

We denote by B the closed unit ball both in X and Y (from the context it will be clear to which space we refer), by Y^* the topological dual space of Y and by C^* the positive polar cone of C, i.e.

$$C^* = \{\xi \in Y^* : \langle \xi, c\rangle \geq 0, \ \forall c \in C\}$$

Consider the vector optimization problem

$$(X, f) \qquad \min f(x), \quad x \in X.$$

A point $\bar{x} \in X$ is called an efficient solution for problem (X, f) when

$$(f(X) - f(\bar{x})) \cap (-C) = \{0\}$$

We denote by $\operatorname{Eff}(X, f)$ the set of all efficient solutions for problem (X, f). A point $\bar{x} \in X$ is called a weakly efficient solution for problem (X, f) when

$$(f(X) - f(\bar{x})) \cap (-\operatorname{int} C) = \emptyset.$$

We denote by WEff(X, f) the set of weakly efficient solutions for problem (X, f). We recall (see e.g. [3]) that a point $\bar{x} \in X$ is said to be a strictly efficient solution for problem (X, f) when, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(f(X) - f(\bar{x})) \cap (\delta B - C) \subseteq \varepsilon B \tag{2.2}$$

We denote by StEff(X, f) the set of strictly efficient solutions for problem (X, f). Clearly StEff$(X, f) \subseteq$ Eff$(X, f) \subseteq$ WEff(X, f).

Definition 2.1 ([20])**.** A function $f : X \to Y$, is said to be $C-$convex if $\forall x, z \in X$ and $t \in [0, 1]$ it holds

$$f(tx + (1 - t)z) \in tf(x) + (1 - t)f(z) - C$$

Proposition 2.2 ([20])**.** *$f : X \to Y$ is C-convex if and only if functions $g_\xi(x) = \langle \xi, f(x)\rangle$ are convex for every $\xi \in C^*$.*

We recall also that a function $f : X \to Y$ is said to be $*$-quasiconvex when functions $g_\xi(x) = \langle \xi, f(x)\rangle$ are quasiconvex for every $\xi \in C^*$ (see e.g. [18]).
For $y \in Y$ we denote by $L_f^C(y) := \{x \in X : f(x) \in y - C\}$ the corresponding sublevel set. We say that $f : X \to Y$ is C-lower semicontinuous (C-lsc for short) when $L_f^C(y)$ is closed for every $y \in Y$ [20].
Now, we recall, the notion of oriented distance between a point $y \in Y$ and a set $A \subseteq Y$, denoted by $D_A(y)$.

Definition 2.3. For a set $A \subseteq Y$ the oriented distance is the function $D_A : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined as

$$D_A(y) = d_A(y) - d_{Y \setminus A}(y) \tag{2.3}$$

with $d_\emptyset(y) = +\infty$.

Function D_A was introduced in [12], [13] to analyze the geometry of nonsmooth optimization problems and obtain necessary optimality conditions. The next result summarizes some basic properties of function D_A.

Proposition 2.4 ([8], [27])**.** *If the set A is nonempty and $A \neq Y$, then*

1. *D_A is real valued;*
2. *D_A is 1-Lipschitzian;*
3. *$D_A(y) < 0$ for every $y \in \operatorname{int} A$, $D_A(y) = 0$ for every $y \in \partial A$ and $D_A(y) > 0$ for every $y \in \operatorname{int}(Y \setminus A)$ (∂A denotes the boundary of the set A);*
4. *if A is closed, then it holds $A = \{y : D_A(y) \leq 0\}$;*
5. *if A is convex, then D_A is convex;*
6. *if A is a cone, then D_A is positively homogeneouos;*
7. *if A is a closed convex cone, then D_A is nonincreasing with respect to the ordering relation induced on Y, i.e. the following is true: if $y_1, y_2 \in Y$ then $y_1 - y_2 \in A \Rightarrow D_A(y_1) \leq D_A(y_2)$; if A has nonempty interior, then $y_1 - y_2 \in \operatorname{int} A \Rightarrow D_A(y_1) < D_A(y_2)$:*
8. *It holds*

$$D_A(y) = \max_{\xi \in C^* \cap \partial B} \langle \xi, y\rangle \tag{2.4}$$

Theorem 2.5 ([23]). *If $f : X \to Y$ is C-convex, then for every $y \in Y$, function $D_{-C}(f(x) - y)$ is convex.*

Let $\bar{x} \in X$. We associate to problem (X, f) the scalar problem

$$(X, D_{-C}) \qquad\qquad \min D_{-C}(f(x) - f(\bar{x})) \ , \ x \in X$$

The relations of the solutions of this problem with those of problem (X, f) are investigated in [8], [23], [27]. For the convenience of the reader, we quote the characterization of efficient points and weakly efficient points.

Theorem 2.6 ([8], [23], [27]). *Let $f : X \to Y$.*

1. *$\bar{x} \in \mathrm{WEff}\,(X, f)$ if and only if $\bar{x}$ is a solution of problem (X, D_{-C}).*
2. *If $\bar{x}$ is the unique solution of problem (X, D_{-C}), then $\bar{x} \in \mathrm{Eff}\,(X, f)$.*

3. Well-posedness for scalar and vector optimization problems

3.1. **Well-posedness for scalar optimization problems.** In this section we recall the notion of well-posednsess for functions $f : X \to \mathbb{R}$ introduced by Tykhonov [26]. For a complete treatment of this notion and of its generalizations one can refer to [6], [21]. Clearly in this case problem (X, f) reduces to a scalar minimization problem.

Definition 3.1. Let $f : X \to \mathbb{R}$. Problem (X, f) is said to be Tykhonov well-posed (T-wp for short) if:

1. there exists a unique $\bar{x} \in X$ such that $f(\bar{x}) \leq f(x)$, $\forall x \in X$;
2. every sequence x_n such that $f(x_n) \to \inf_X f$ is such that $x_n \to \bar{x}$.

Next proposition provides a useful characterization of Tykhonov well-posedness. It is called the Furi-Vignoli criterion [7].

Proposition 3.2. *Let $f : X \to \mathbb{R}$ be lower semicontinuous (lsc). Then problem (X, f) is T-wp if and only if $\inf_{a > \inf_X f} \operatorname{diam} L_f(a) = 0$, where $L_f(a) = \{x \in X : f(x) \leq a\}$.*

The following result concerning well-posedness of convex functions defined on a finite-dimensional space is well-known.

Theorem 3.3 (see e.g. [6]). *Let X be finite-dimensional and $f : X \to \mathbb{R}$ be a convex function with a unique minimizer. Then problem (X, f) is T-wp.*

Theorem 3.3 does not hold when X is infinite-dimensional as the following example shows (see e.g. [6]).

Example 3.4. Let X be a separable Hilbert space with orthonormal basis $\{e_n, n \in \mathbb{N}\}$. Let $f(x) = \sum_{n=1}^{+\infty} \frac{\langle x, e_n \rangle^2}{n^2}$. Then f is continuous, convex and has $\bar{x} = 0$ as unique minimizer, but problem (X, f) is not T-wp. Indeed the sequence $\sqrt{n} e_n$ is an unbounded minimizing sequence.

Consider now the space

$$\Gamma := \{f : X \to \mathbb{R} : f \text{ is convex and lsc}\}.$$

We endow Γ with a distance compatible with the uniform convergenge on bounded sets (see e.g. [21]). Let $\theta \in X$ be fixed and for any two functions $f, g \in \Gamma$ and for every $i \in \mathbb{N}$ set

$$\|f - g\|_i = \sup_{\|x-\theta\| \le i} |f(x) - g(x)|.$$

If $\|f - g\|_i = \infty$ for some i, then set $d(f, g) = 1$, otherwise

$$d(f, g) = \sum_{i=1}^{\infty} 2^{-i} \frac{\|f - g\|_i}{1 + \|f - g\|_i}.$$

When X is infinite-dimensional, it can be shown that the set of functions $f \in \Gamma$ such that problem (X, f) is T-wp is "big" in the sense that contains a dense G_δ set (see e.g. [21])

Theorem 3.5 ([21])**.** *Let X be a Banach space and consider the set Γ, equipped with the topology of uniform convergence. Then the set of functions $f \in \Gamma$ such that problem (X, f) is T-wp contains a dense G_δ set .*

If the convexity assumption in Γ is dropped, weaker variants of Theorem 3.5 hold, in which density of the class of functions $f \in \Gamma$ such that problem (X, f) is T-wp is proven. We recall e.g. the following result [21].

Theorem 3.6. *Consider the set*

$$\tilde{\Gamma} := \{f : X \to \mathbb{R} : f \text{ is positive and lsc}\}.$$

equipped with the topology of uniform convergence. Then the set of functions $f \in \tilde{\Gamma}$ such that problem (X, f) is T-wp is dense.

Next results (see e.g. [21]) will be useful in the following.

Proposition 3.7. *Let $f : X \to \mathbb{R}$, assume f has a minimum point $\bar{x} \in X$ and let $g(x) = f(x) + a\|x - \bar{x}\|$ with $a > 0$. Then problem (X, g) is T-wp.*

Theorem 3.8. (Ekeland's Variational Principle) *Let $f : X \to \mathbb{R}$ be a lsc, lower bounded function. Let $\varepsilon > 0$, $r > 0$ and $\bar{x} \in X$ be such that $f(\bar{x}) < \inf_X f + r\varepsilon$. Then, there exists $\hat{x} \in X$ enjoying the following properties:*

1. $\|\hat{x} - \bar{x}\| < r$;
2. $f(\hat{x}) < f(\bar{x}) - \varepsilon\|\bar{x} - \hat{x}\|$;
3. *Problem* (X, g) *with* $g(x) = f(x) + \varepsilon\|x - \hat{x}\|$ *is T-wp.*

3.2. **Well-posedness for vector optimization problems.** Several generalizations of the well-posedness notion to vector optimization problems have been proposed. We refer to [23] for a survey on the topic and a study of the relations among different well-posedness concepts. In that paper vector well-posedness notions have been divided in two classes: pointwise and global notions. Notions in the first class define the well-posedness of a vector problem with respect to a fixed efficient solution, while in the global notions the set of efficient solutions or weakly efficient solutions is considered as a whole.
In this paper we focus on the notion of well-posedness due to Dentcheva and Helbig [5] (DH-well-posedness) which is a pointwise notion according to [23].

Definition 3.9. Let $f : X \to \mathbb{R}$. Problem (X, f) is said to be DH-well-posed (DH-wp for short) at $\bar{x} \in \mathrm{Eff}\ (X, f)$ if

$$\inf_{\alpha>0} \operatorname{diam} L_f^C(f(\bar{x}) + \alpha c) = 0, \quad \forall c \in C,$$

where $L_f^C(f(\bar{x}) + \alpha c) = \{x \in X : f(x) \in f(\bar{x}) + \alpha c - C\}$.

In [23] it has been proven that DH- well-posedness is the strongest among the pointwise well-posedness notions, that is if problem (X, f) is DH-wp at $\bar{x} \in X$ then it is well-posed at $\bar{x}$ according to the other pointwise well-posedness notions known in the literature. The next result gives a useful characterization of DH–well-posedness.

Theorem 3.10 ([10], [23])**.** *Problem (X, f) is DH-wp at $\bar{x} \in \mathrm{Eff}\,(X, f)$ if and only if problem (X, D_{-C}) is T-wp.*

The following theorem (see [23]) gives a generalization of Theorem 3.3.

Theorem 3.11. *Let X and Y be finite-dimensional. Assume $f : X \to Y$ is a C-convex function, $\bar{x} \in \mathrm{Eff}\,(X, f)$ and $f^{-1}(f(\bar{x})) = \{\bar{x}\}$. Then problem (X, f) is DH-wp at $\bar{x}$.*

DH-well-posedness imposes some restrictions on the set $\mathrm{Eff}\,(X, f)$. Indeed, if problem (X, f) is DH-well-posed at $\bar{x} \in \mathrm{Eff}\ (X, f)$ then $\bar{x} \in \mathrm{StEff}\,(X, f)$. This property is typical of the vector case and shows that most of the vector well-posedness notions require implicitly stronger properties than the simple good behavior of minimizing sequences.

Theorem 3.12 ([23])**.** *If $f : X \to Y$ is continuous and problem (X, f) is DH-wp at $\bar{x} \in \mathrm{Eff}\,(X, f)$, then $\bar{x} \in \mathrm{StMin}\,(X, f)$.*

4. Density of DH-well-posed functions

The first result in this section shows that if the set of functions

$$\mathcal{H} = \{f : X \to Y : \mathrm{Eff}\,(X, f) \neq \emptyset\}$$

is endowed with the topology of uniform convergence on bounded sets, then the set of functions $g \in \mathcal{H}$ enjoying DH-well-posedness properties is dense in $\mathcal{H}$.

Theorem 4.1. *Let $f \in \mathcal{H}$. Then, for every $\bar{x} \in \mathrm{Eff}\,(X, f)$, there exists a sequence of functions $f_n : X \to Y$ such that $f_n \to f$ in the uniform convergence on bounded sets, $\bar{x} \in \mathrm{Eff}\,(X, f_n)$ for every n and problem (X, f_n) is DH-wp at $\bar{x}$. Further, if f is continuous then $\bar{x} \in \mathrm{StEff}\,(X, f_n)$ for every n.*

Proof. Let $k^0 \in \operatorname{int} C$ be fixed and consider the sequence of functions

$$f_n(x) = f(x) + \frac{1}{n}\|x - \bar{x}\|k^0$$

Since $\bar{x} \in \mathrm{Eff}\,(X, f)$, it holds

$$f(x) - f(\bar{x}) \notin -C, \ \forall x \in X, \ x \neq \bar{x} \tag{4.1}$$

Hence

$$f(x) - f_n(\bar{x}) = f(x) - f(\bar{x}) + \frac{1}{n}\|x - \bar{x}\|k^0 \notin -C, \ \forall x \in X, \ x \neq \bar{x}$$

since (4.1) holds. This entails $\bar{x} \in \mathrm{Eff}\,(X, f_n)$ $\forall n$. Since $\mathrm{Eff}\,(X, f_n) \subseteq \mathrm{WEff}\,(X, f_n)$, Theorem 2.6 implies

$$D_{-C}(f_n(x) - f_n(\bar{x})) \geq D_{-C}(f_n(\bar{x}) - f_n(\bar{x})) = 0$$

for every $x \in X$. Now we prove problem (X, f_n) is DH-wp at $\bar{x}$ for every n. From Theorem 3.10 we know that problem (X, f) is DH-wp at $\bar{x} \in X$ if and only if the scalar problem (X, D_{-C}) is T-wp. Since $\mathrm{int}\, C \neq \emptyset$, C^* has a closed convex weak*-compact base

$$G = \{\xi \in C^* : \langle \xi, k^0 \rangle = 1\} \tag{4.2}$$

(see e.g. [17]). According to [23] there exists a constant $\alpha > 0$ such that

$$D_{-C}(f_n(x) - f(\bar{x})) \geq \alpha \max_{\xi \in G} \langle \xi, f_n(x) - f_n(\bar{x}) \rangle$$

$$= \alpha \max_{\xi \in G} \langle \xi, f(x) - f(\bar{x}) + \frac{1}{n} \|x - \bar{x}\| k^0 \rangle$$

$$= \alpha \max_{\xi \in G} \langle \xi, f(x) - f(\bar{x}) \rangle + \frac{1}{n} \|x - \bar{x}\|$$

For a fixed n, let x_k be a minimizing sequence for $D_{-C}(f_n(x) - f_n(\bar{x}))$, that is $D_{-C}(f_n(x_k) - f_n(\bar{x})) \to 0$. If $x_k \not\to \bar{x}$ we get

$$D_{-C}(f_n(x_k) - f_n(\bar{x})) \geq \alpha \max_{\xi \in G} \langle \xi, f(x_k) - f(\bar{x}) \rangle + \frac{1}{n} \|x_k - \bar{x}\|$$

$$\geq \inf_{k \in \mathbb{N}} \frac{1}{n} \|x_k - \bar{x}\| > 0$$

which contradicts to x_k minimizing sequence for $D_{-C}(f_n(x) - f_n(\bar{x}))$ (the last inequality follows since $\bar{x} \in \mathrm{Eff}\,(X, f)$ implies $\max_{\xi \in G} \langle \xi, f(x_k) - f(\bar{x}) \rangle \geq 0$ $\forall x \in X$). Hence $x_k \to \bar{x}$ and problem (X, f_n) is DH-wp at $\bar{x}$. Finally, we get the desired result observing that $f_n \to f$ in the uniform convergence on bounded sets. If f is continuous, f_n is continuous and apply Theorem 3.12 to conclude the proof. □

To prove the second density result in this section we need the following definition and the next lemma.

Definition 4.2 ([10])**.** We say that $f : X \to Y$ is C-bounded from below by $\xi \in C^* \backslash \{0\}$ when $\inf_{x \in X} \langle \xi, f(x) \rangle > -\infty$.

Let $\bar{x} \in \mathrm{Eff}\,(X, f)$, consider function

$$h_{\bar{\xi}}(x) = \langle \bar{\xi}, f(x) \rangle$$

and the associated scalar minimization problem

$$(X, h_{\bar{\xi}}) \qquad \min h_{\bar{\xi}}(x) \ , \ x \in X$$

Lemma 4.3. *Assume there exists $\bar{\xi} \in C^* \backslash \{0\}$ such that problem $(X, h_{\bar{\xi}})$ is T-wp. Then there exists a point $\bar{x} \in X$ such that problem (X, f) is DH-wp at $\bar{x}$.*

Proof. Without loss of generality let $\bar{\xi} \in C^* \cap \partial B$. Since problem $(X, h_{\bar{\xi}})$ is T-wp it follows the exixtence of a point $\bar{x} \in X$ such that $\bar{x}$ is the unique minimum point for $h_{\bar{\xi}}$ over X and hence

$$h_{\bar{\xi}}(x) - h_{\bar{\xi}}(\bar{x}) = \langle \bar{\xi}, f(x) - f(\bar{x}) \rangle > 0 \ \forall x \in X \backslash \{\bar{x}\}$$

Since, by Proposition 2.4

$$D_{-C}(f(x) - f(\bar{x})) = \max_{\xi \in C^* \cap \partial B} \langle \xi, f(x) - f(\bar{x}) \rangle$$

it follows

$$D_{-C}(f(x) - f(\bar{x})) \geq \langle \bar{\xi}, f(x) - f(\bar{x}) \rangle = h_{\bar{\xi}}(x) - h_{\bar{\xi}}(\bar{x}) > 0 \ \forall x \in X \backslash \{\bar{x}\} \tag{4.3}$$

By Theorem 2.6, $\bar{x} \in \mathrm{Eff}\,(X, f)$. Assume problem (X, f) is not DH-wp at $\bar{x}$. Since $\bar{x} \in \mathrm{Eff}\,(X, f) \subseteq \mathrm{WEff}(X, f)$, by Theorem 2.6 it holds

$$D_{-C}(f(x) - f(\bar{x})) \geq 0, \ \forall x \in X$$

and by Theorem 3.10 problem (X, D_{-C}) is not T-wp. Then there exists a sequence $x_n \in X$ such that $D_{-C}(f(x_n) - f(\bar{x})) \to 0$ but $x_n \not\to \bar{x}$. From $D_{-C}(f(x_n) - f(\bar{x})) \to 0$, by (4.3) it follows $h_{\bar{\xi}}(x_n) \to h_{\bar{\xi}}(\bar{x})$ which contradicts problem $(X, h_{\bar{\xi}})$ is T-wp since $x_n \not\to \bar{x}$. □

In the next result we drop the asumption $\mathrm{Eff}\,(x, f) \neq \emptyset$ of Theorem 4.1 and we show that if the set of functions

$$\mathcal{H}' = \{f : X \to Y : \exists\ \xi \in C^* \backslash \{0\} \text{ such that } f \text{ is } C-\text{bounded from below by } \xi\}$$

is endowed with the topology of uniform convergence on bounded sets, then the set of functions $g \in \mathcal{H}'$ enjoying DH-wp properties is dense in $\mathcal{H}'$. We endow $\mathcal{H}'$ with a distance compatible with the uniform convergenge on bounded sets (see e.g. [21]). Fix $\theta \in X$ and for any two functions $f, g \in \mathcal{H}'$ and $i \in \mathbb{N}$, set

$$\|f - g\|_i = \sup_{\|x - \theta\| \leq i} \|f(x) - g(x)\| \,.$$

If $\|f - g\|_i = \infty$ for some i, then set $d(f, g) = 1$, otherwise

$$d(f, g) = \sum_{i=1}^{\infty} 2^{-i} \frac{\|f - g\|_i}{1 + \|f - g\|_i}. \tag{4.4}$$

Theorem 4.4. *Assume there exists $\bar{\xi} \in C^* \backslash \{0\}$ such that $f : X \to Y$ is C-bounded from below by $\bar{\xi}$ and $\langle \bar{\xi}, f(x) \rangle$ is lsc with respect to $x \in X$. Then, there exists a sequence of functions $f_n : X \to Y$ uniformly converging to f on the bounded sets, such that* $\mathrm{Eff}\,(X, f_n) \neq \emptyset$ *for every n and problem (X, f_n) is DH-wp at some* $\bar{x}_n \in \mathrm{Eff}\,(X, f_n)$.

Proof. Let $k^0 \in \mathrm{int}\, C$ be such that $\langle \bar{\xi}, k^0 \rangle = 1$. Fix $\sigma > 0$ and take j so large that setting

$$g(x) = f(x) + \frac{1}{j} \|x - \theta\| k^0 \tag{4.5}$$

it holds $d(f,g) < \frac{\sigma}{2}$. Now set

$$g_{\bar{\xi}}(x) = \langle \bar{\xi}, g(x) \rangle = \langle \bar{\xi}, f(x) \rangle + \frac{1}{j}\|x - \theta\| \tag{4.6}$$

and observe that $\langle \bar{\xi}, f(x) \rangle$ is lower bounded by Definition 4.2. Hence, for any $\delta > 0$ we can find $M > 0$ such that

$$\{x \in X : g_{\bar{\xi}}(x) \leq \inf_{x \in X} g_{\bar{\xi}}(x) + \delta\} \subseteq B(\theta, M)$$

where $B(\theta, M)$ is the ball centered at θ with radius M. Let $s = \sum_{k=0}^{+\infty} \frac{1}{2^k}(k+M)\|k^0\|$ and apply Theorem 3.8 with $\varepsilon = \frac{\sigma}{2s}$ and arbitrary r to find a point $\bar{x} = \bar{x}_\sigma \in X$ such that $\|\bar{x} - \theta\| \leq M$, $\bar{x}$ is the unique minimizer of

$$h_{\bar{\xi}}(x) = \langle \bar{\xi}, g(x) \rangle + \varepsilon\|x - \bar{x}\|$$

and problem $(X, h_{\bar{\xi}})$ is T-wp. Let

$$h(x) = g(x) + \varepsilon\|x - \bar{x}\|k^0$$

and observe that since $\bar{x}$ minimizes $h_{\bar{\xi}}(x)$, it holds

$$h_{\bar{\xi}}(x) - h_{\bar{\xi}}(\bar{x}) = \langle \bar{\xi}, h(x) - h(\bar{x}) \rangle > 0, \ \forall x \in X \backslash \{\bar{x}\}$$

which implies

$$\begin{aligned} D_{-C}(h(x) - h(\bar{x})) &= \max_{\xi \in C^* \cap \partial B} \langle \xi, h(x) - h(\bar{x}) \rangle \\ &\geq \langle \bar{\xi}, h(x) - h(\bar{x}) \rangle > 0, \ \forall x \in X \backslash \{\bar{x}\} \end{aligned}$$

Hence, Theorem 2.6 implies $\bar{x} \in \mathrm{Eff}\,(X, h)$. Combining Theorem 2.6 and Lemma 4.3, we obtain that problem

$$(X,h) \qquad \min h(x)\ ,\ x \in X$$

is DH-wp at $\bar{x}$. Now observe that

$$\|h(x) - g(x)\|_i \leq \varepsilon\|k^0\|(i + M)$$

It follows $d(h,g) \leq \varepsilon s = \frac{\sigma}{2}$ and then $d(f,h) < \sigma$. Take now $\sigma = \frac{1}{n}, n = 1, 2, \ldots$ and set $\bar{x}_n = \bar{x}_\sigma$ to complete the proof. ☐

Remark 4.5. The lower semicontinuity hypothesis on $\langle \bar{\xi}, f(x) \rangle$ is satisfied when functions in $\mathcal{H}'$ are assumed to be $C-lsc$. Indeed, in this case all functions $\langle \xi, f(x) \rangle$ with $\xi \in C^*$ are lsc.

The next result shows that under some hypotheses, the assumptions in Theorem 4.4 are weaker than those in Theorem 4.1. We recall the following result.

Theorem 4.6 (Sion's Minimax Theorem [24], [25]). *Let Z be a compact convex subset of a linear topological space and W a convex subset of a linear topological space. Let g be a real-valued function on $Z \times W$ such that*

i) *$g(\cdot, w)$ is upper semicontinuous and quasi-concave on Z $\forall w \in W$;*
ii) *$g(z, \cdot)$ is lower semicontinuous and quasi-convex on W $\forall z \in Z$.*

Then

$$\sup_{z \in Z} \inf_{w \in W} g(z,w) = \inf_{w \in W} \sup_{z \in Z} g(z,w)$$

Proposition 4.7. *Let $f : X \to Y$ be $*$-quasiconvex and C-lsc. Then, if* $\mathrm{Eff}\,(X, f) \neq \emptyset$*, there exists $\bar{\xi} \in C^*\backslash\{0\}$ such that f is C-bounded from below by $\bar{\xi}$.*

Proof. Assume $\mathrm{Eff}\,(X, f) \neq \emptyset$ and let $\bar{x} \in \mathrm{Eff}\,(X, f)$. Ab absurdo assume that for every $\xi \in C^*\backslash\{0\}$ it holds

$$\inf_{x\in X}\langle \xi, f(x)\rangle = \inf_{x\in X}\langle \xi, f(x) - f(\bar{x})\rangle = -\infty$$

Since $\mathrm{int}\, C \neq \emptyset$, C^* has a weak*-compact base G. Function $g(\xi, x) = \langle \xi, f(x)-f(\bar{x})\rangle$, $\xi \in G$, $x \in X$, is linear and continuous with respect to ξ and quasiconvex with respect to x. Further, since f is C-lsc with respect to $x \in X$, $g(\xi, x)$ is lsc with respect to $x \in X$. Since $\bar{x} \in \mathrm{Eff}\,(X, f)$, it holds $\max_{\xi\in G}\langle \xi, f(x)-f(\bar{x})\rangle \geq 0$ for every $x \in X$. Apply Sion's Minimax Theorem to get the following chain of equalities

$$-\infty = \sup_{\xi\in G}\inf_{x\in X}\langle \xi, f(x) - f(\bar{x})\rangle = \inf_{x\in X}\sup_{\xi\in G}\langle \xi, f(x) - f(\bar{x})\rangle$$

which implies there exists $\tilde{x} \in X$ such that $\sup_{\xi\in G}\langle \xi, f(\tilde{x}) - f(\bar{x})\rangle < 0$. A contradiction to $\bar{x} \in \mathrm{Eff}\,(X, f)$. □

Generalized convexity assumptions in the previous reult cannot be removed as the following example shows.

Example 4.8. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = C^* = \mathbb{R}^2_+$, $f : X \to Y$ defined as $f(x) = (x, -x^3)$ is not $*$-quasiconvex. We have $\mathrm{Eff}(X, f) = \mathbb{R} \neq \emptyset$ but for any $\xi \in C^*\backslash\{0\}$ we have $\inf_{x\in X}\langle \xi, f(x)\rangle = -\infty$. Hence does not exist $\xi \in C^*\backslash\{0\}$ such that $\langle \xi, f(x)\rangle$ is bounded from below.

5. Genericity of DH-well-posedness for C-convex functions

In this section we show that the set of C-convex and C-lsc functions $f : X \to Y$ enjoying DH-well-posedness properties contains a dense G_δ set. To prove the main theorem in this section we need some preliminary results.

Proposition 5.1. *Let $f : X \to \mathbb{R}$ a convex and lsc function, $\bar{x} \in X$ and set*

$$g(x) = f(x) + a\|x - \bar{x}\|^\alpha,\ a > 0, \alpha \geq 1 \tag{5.1}$$

Then $\lim_{\|x\|\to+\infty} g(x) = +\infty$. Furthermore $g(x)$ is lower bounded.

Proof. We prove that for every sequence $x_n \in X$ with $\|x_n\| \to +\infty$ it holds $\lim_{n\to+\infty} g(x_n) = +\infty$. Denote by X^* the topological dual space of X. Since $f(x)$ is convex, the set $\partial f(\bar{x}) \subseteq X^*$ of all subgradients of f at $\bar{x}$ is nonempty and by definition of subgradient [9], for every continuous linear functional $v \in \partial f(\bar{x})$ it holds $f(x) \geq f(\bar{x}) + v(x - \bar{x}), \forall x \in X$. Hence,

$$\begin{aligned}\lim_{n\to+\infty} g(x_n) &= \lim_{n\to+\infty}\left[f(x_n) + a\|x_n - \bar{x}\|^\alpha\right]\\ &\geq \lim_{n\to+\infty}\left(f(\bar{x}) + v(x_n - \bar{x}) + a\|x_n - \bar{x}\|^\alpha\right)\end{aligned}$$

$$= \lim_{n\to+\infty}\left[f(\bar{x}) + \|x_n - \bar{x}\|^\alpha\left(v\left(\frac{x_n - \bar{x}}{\|x_n - \bar{x}\|}\right)\|x_n - \bar{x}\|^{1-\alpha} + a\right)\right] = +\infty$$

(the last equality follows since a continuous linear functional is bounded). To prove that $g(x)$ is lower bounded observe that for every $M \in \mathbb{R}$, there exists $k > 0$ such that $g(x) > M$ for $\|x\| > k$. If we take $A = \{x \in X : \|x\| \leq k\}$, $g(x)$ is lower bounded on the bounded set A (see e.g. [21]), which concludes the proof. □

Corollary 5.2. *Let $f : X \to Y$ be a C-convex, C-lsc function and for $\xi \in C^*\backslash\{0\}$ and $a > 0, \alpha \geq 1$ set*

$$g_\xi(x) = \langle \xi, f(x)\rangle + a\|x - x^0\|^\alpha \tag{5.2}$$

Then, $\lim_{\|x\|\to+\infty} g_\xi(x) = +\infty$ *and g is lower bounded.*

Proof. The proof follows from Proposition 5.1 since f C-convex and C-lsc implies g convex and lsc for every $\xi \in C^*\backslash\{0\}$. □

Let $\mathcal{F}$ be the set of C-convex and C-lsc functions $f : X \to Y$. We endow $\mathcal{F}$ with the distance defined by (4.4), compatible with the topology of uniform convergence on bounded sets.

Theorem 5.3. *Let $\mathcal{F}$ be the set of C-convex and C-lsc functions endowed with the topology of uniform convergence on bounded sets and let $\tilde{\mathcal{F}}$ be the set of functions $f \in \mathcal{F}$ such that* $\operatorname{Eff}(X, f) \neq \emptyset$ *and problem (X, f) is DH-wp at some point $\bar{x} \in$* $\operatorname{Eff}(X, f)$*. Then $\tilde{\mathcal{F}}$ contains a dense G_δ set .*

Proof. The initial argument of the proof is inspired to that of Theorem 2.1 in [22]. If we fix $k^0 \in \operatorname{int} C$, we can find $\bar{\xi} \in C^*$ such that $\langle \bar{\xi}, k^0\rangle = 1$. Consider the set

$$\mathcal{Z} = \{z : X \to \mathbb{R} \text{ such that } z(x) = \langle \bar{\xi}, f(x)\rangle, f \in \mathcal{F}\}$$

Since f is C-lsc, z is lsc. Endow $\mathcal{Z}$ with the topology of uniform convergence on bounded sets and let $S : \mathcal{F} \to \mathcal{Z}$ be the map $S(f) = z$, with z defined as before. Then S is a continuous map. Let

$$\mathcal{A}_n = \{z \in \mathcal{Z} : \exists a > \inf_{x\in X} z,\ \operatorname{diam} L_z(a) < \frac{1}{n}\} \tag{5.3}$$

where $L_z(a) = \{x \in X : z(x) \leq a\}$. Observe that $L_z(a)$ are closed convex sets since z is convex and lsc. It is known (see e.g. [21]) that if $z_n \to z$ in the uniform convergence, then $\operatorname{diam} L_{z_n}(a) \to \operatorname{diam} L_z(a)$, which gives continuity of the diam function. Hence $\mathcal{A}_n$ is an open set for all n and this implies $S^{-1}(\mathcal{A}_n)$ is an open set for all n. We claim that the set $\mathcal{W}$ of those functions $h \in \mathcal{F}$ such that problem $(X, S(h))$ is T-wp is dense in $\mathcal{F}$. Since

$$\mathcal{W} = \bigcap_{n=1}^{+\infty} S^{-1}(\mathcal{A}_n)$$

(see Proposition 3.2) then it is a G_δ set i.e. the countable intersection of open sets. Let $f \in \mathcal{F}$, $\sigma > 0$ and take j so large that setting

$$g(x) = f(x) + \frac{1}{j}\|x - \theta\|k^0 \tag{5.4}$$

it holds $d(f, g) < \frac{\sigma}{2}$. Setting

$$g_{\bar{\xi}}(x) = \langle \bar{\xi}, g(x)\rangle \tag{5.5}$$

we have $\lim_{\|x\|\to+\infty} g_{\bar{\xi}}(x) = +\infty$ and $g_{\bar{\xi}}$ is lower bounded by Corollary 5.2. The proof now follows along the lines of Theorem 4.4. For any $\delta > 0$ we can find $M > 0$ such that

$$\{x \in X : g(x) \leq \inf_{x\in X} g(x) + \delta\} \subseteq B(\theta, M)$$

Let $h : X \to Y$ be defined as

$$h(x) = g(x) + \varepsilon\|x - \bar{x}\|k^0$$

and let $s = \sum_{k=0}^{+\infty} \frac{1}{2^k}(k + M)\|k^0\|$. Apply Theorem 3.8 with $\varepsilon = \frac{\sigma}{2s}$ and arbitrary r to find a point $\bar{x} = \bar{x}_\sigma \in X$ such that $\|\bar{x} - \theta\| \leq M$, $\bar{x}$ is the unique minimizer of

$$S(h)(x) = \langle\bar{\xi}, g(x)\rangle + \varepsilon\|x - \bar{x}\|$$

and problem $(X, S(h))$ is T-wp. Hence $h \in \mathcal{W}$. This implies that problem (X, h) is DH-wp at $\bar{x}$ by Lemma 4.3. Now observe that

$$\|h(x) - g(x)\|_i \leq \varepsilon\|k^0\|(i + M)$$

It follows $d(h, g) \leq \varepsilon s = \frac{\sigma}{2}$ and then $d(f, h) < \sigma$. Hence $\mathcal{F}$ contains a dense G_δ set, which concludes the proof. □

6. Concluding remarks

We conclude this paper recalling that in Chapter 9 of [28] the author considers vector minimization problems with objective function mapping from a complete metric space X to $\mathbb{R}^n$. The space $\mathbb{R}^n$ is endowed with the Pareto order induced by the cone $\mathbb{R}^n_+$, i.e. for $y = (y_1, \dots, y_n), z = (z_1, \dots, z_n) \in \mathbb{R}^n$, $y \leq z$ means $y_i \leq z_i$, $i = 1, \dots, n$. Denote by $\mathcal{A}$ the set of all functions $f = (f_1, \dots, f_n)$ with $f_i : X \to \mathbb{R}$, $i = 1, \dots, n$, which are continuous and bounded from below, i.e. there exists $a \in \mathbb{R}^n$ such that $f(x) \geq a$, $\forall x \in X$. In [28] it is shown that when $\mathcal{A}$ is endowed with a natural complete metric, then there exists a dense G_δ subset of $\mathcal{A}$ such that for any element of this subset the set of efficient points of the corresponding vector optimization problem is nonempty and compact. Similar generic properties are obtained in [28] under lower semicontinuity assumptions.

Furthermore, in the above mentioned fremework, the author of [28] considers sequences which are minimizing according to the following definition. We denote by $f(X)$ the image of X through fiunction f and by $\operatorname{cl} D$ the closure of the set $D \subseteq \mathbb{R}^n$ and by $\operatorname{Min}(D)$ the set of Pareto minimal points of D, with respect to to the order $\mathbb{R}^n_+$, i.e. $y \in \operatorname{Min}(D)$ when does not exist any $x \in D\backslash\{y\}$ such that $x \leq y$.

Definition 6.1. A sequence $x_n \in X$ is called minimizing when there exists a sequence $y_n \in \operatorname{Min}(\operatorname{cl} f(X))$ and a sequence $\alpha_i \in (0, +\infty)$ with $\alpha_n \to 0$ and

$$f(x_n) \leq y_n + \alpha_n e \tag{6.1}$$

where $e = (1, \dots, 1) \in \mathbb{R}^n$.

Theorem 9.14 in [28] shows that there exists a dense G_δ subset $\mathcal{F}$ of $\mathcal{A}$ such that for $f \in \mathcal{F}$ every minimizing sequence admits a convergent subsequence. Minimizing sequences of the type considered in Definition 6.1 are related to global-well-posedness (see e.g. [1], [2], [14], [15]).

The results in this paper (particularly Theorem 5.3) can be viewed as counterparts

of the results in [28] under different settings and assumptions.
Indeed, in this paper we consider functions with values in a Banach space Y ordered by a closed convex pointed cone C with nonepty interior and in Theorem 5.3 we consider functions enjoing cone-convexity properties. Under these assumptions we show that both existence of efficient points and pointwise well-posedness are generic properties.
The mentioned results in [28] suggest that further research could investigate similar properties considering global well-posedness notions.

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