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HIERARCHICAL NASH SOLUTIONS FOR n-PERSON BARGAINING PROBLEMS

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Abstract. We characterize the collection of n-person bargaining solutions satisfying a consistency condition and all of Nash's (1950) conditions except symmetry. A bargaining solution in this collection is described by an ordered partition of the player set together with a weight vector for each element of this partition, and obtained by successively maximizing weighted Nash products.

1. Introduction

A two-person bargaining problem is a compact and convex subset S of the nonnegative orthant of $\mathbb{R}^2$, containing a (strictly) positive element, and such that for each one of its elements all nonnegative points below that element are again in S. (See Figure 1 for a typical example.) The interpretation is that the elements of S represent the possible payoff pairs to two bargainers or players; if they cannot reach an agreement on one of these pairs, then they both end up with zero. A (two-person) bargaining solution assigns a payoff pair, i.e., an element of S, to each bargaining problem S.

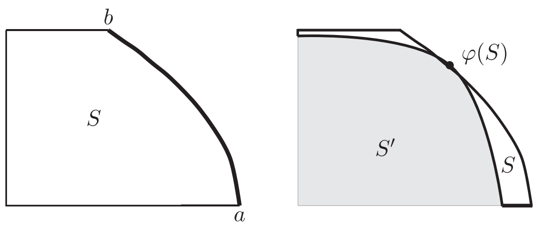

Figure 1. The left panel shows a typical two-person bargaining problem. The thick curve from a to b is the set of payoff pairs from which a Pareto optimal solution chooses. The right panel shows the same problem, plus a possible solution pair $\varphi(S)$. It also illustrates the contraction independence property: S' is a subset of S but still contains $\varphi(S)$, and contraction independence requires that $\varphi(S)$ is still the solution of S'

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In his seminal 1950 paper, Nash proposed four properties that such a bargaining solution should satisfy: Pareto optimality, scale transformation covariance, symmetry, and contraction independence.[1] Pareto optimality means that it should not be possible to give one player more without giving the other player less; hence, the bargaining solution should pick a point on the north-east boundary of the set S (see again Figure 1). Scale transformation covariance means that the solution should not depend on the units in which the payoffs are measured: if all first coordinates of the points of S are multiplied by some positive number α and all second coordinates by some positive β, then the solution payoffs should be multiplied by these numbers as well. The usual interpretation is that payoffs are von Neumann-Morgenstern expected utilities, which represent underlying preferences up to some positive scalar (e.g., Herstein and Milnor, 1953); and the solution should not depend on this particular representation. Symmetry means that if S is a symmetric set (i.e., invariant under reflection with respect to the 45-degree line), thus reflecting symmetry of the players, then these players should obtain the same payoff. The final property, contraction independence, is also the most debated one (e.g., Kalai and Smorodinsky, 1975). It says that if S shrinks to a set S' but the solution of S is still available in S', then it should not change (see Figure 1, right panel).

Nash (1950) showed that there is a unique bargaining solution satisfying these four properties: it assigns to a bargaining problem S the (unique) point where the product of the coordinates (often called the 'Nash product') is maximized. The proof of this is both simple and elegant. (See Figure 2.) By scale transformation covariance we may assume that the point where the product of the coordinates is maximized is the point $(1, 1)$, and a simple computation then shows that there is a supporting line of S at $(1, 1)$ with slope -1. In turn, this means that S is a subset of the triangle T with vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$. To T the solution assigns the point $(1, 1)$: this follows from symmetry and Pareto optimality. Finally, contraction independence then implies that $(1, 1)$ is also the point assigned to S by the solution.

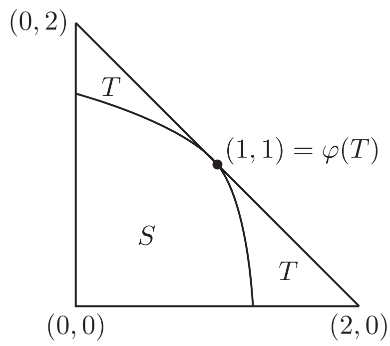

Figure 2. Illustration of the proof of Nash (1950)

De Koster et al (1983) showed that if symmetry is dropped from Nash's properties, then, for each $0 < \alpha < 1$, maximizing the product $x_1^{\alpha} x_2^{1-\alpha}$ (often called

[1]In this Introduction and later in the paper, some assumptions we make and names we use may be different from what is used in related literature, but the differences are not essential.

'nonsymmetric Nash product') also yields a bargaining solution satisfying the remaining three properties; for $\alpha = \frac{1}{2}$ the (symmetric) Nash bargaining solution is retrieved. Moreover, first maximizing the payoff for player 1 and next player 2, or the other way around, results in two more solutions with the remaining three properties (resulting in the points a and b, respectively, in the left panel of Figure 1). These are, indeed, all solutions satisfying Pareto optimality, scale transformation covariance, and contraction independence, for the case of two players.

In the present paper, which is ultimately based on preliminary versions in Peters (1983, 1992), we extend this to the case of more than two players. We impose the conditions of Pareto optimality, scale transformation covariance, and contraction independence on an n-person bargaining solution, and, additionally, require the solution to be consistent in the following sense. Suppose there are two different bargaining problems S and S', and we fix the payoffs for a subset of the players in S and in S', according to the solution. Suppose that then in both S and S', the possible payoff sets for the remaining players turn out to be identical. Then consistency requires that these players obtain the same payoffs according to the solution. For the two-person case, this condition is automatically fulfilled by Pareto optimality, but it turns out that this is not the case if there are more than two players.

In our main result (Theorem 2.4), we show that a solution satisfies Pareto optimality, scale transformation covariance, contraction independence, and consistency, if and only if it is a so-called hierarchical Nash solution. Such a solution works as follows. There is an ordered partition of the set of players, and for each element of this partition, called class, there is a positive weight vector of which the coordinates (corresponding to the players in the class) sum up to one. For a given problem S, we start with the first class and maximize the product $\prod_i x_i^{\omega_i}$ over $x \in S$, where the players i are from the first class and have weights ω_i. On the set of maximizers, we then repeat the procedure for those players in the second class who have possibly a positive payoff. And so on and so forth; we stop when there is a unique maximizer left (which happens after the last class at the latest), which is then the payoff vector assigned by the solution. In a nutshell, hierarchical Nash solutions arise by successively maximizing 'Nash products'.[2] The two-person case described earlier, is a special case. Some illustrative instances of the three-person case are given in Figure 3.

The rest of the paper is organized as follows. Section 2 presents the model and main result. Subsections deal with the two-person case, and with the requirement that every player obtains a positive payoff – both are consequences of our main result. We also show that the conditions in our main result are logically independent. Section 3 is completely devoted to the proof of the main result.

2. Model and main result

Let $n \in \mathbb{N}$ and $N = \{1, \ldots, n\}$. For $x, y \in \mathbb{R}^N$, $x \geq y$ means $x_i \geq y_i$ for all $i \in N$, and $x > y$ means $x_i > y_i$ for all $i \in N$; $x \leq y$ and $x < y$ are defined

[2]Successively maximizing Nash products also occurs in Peters and Vermeulen (2012), but there nonconvex bargaining problems are allowed.

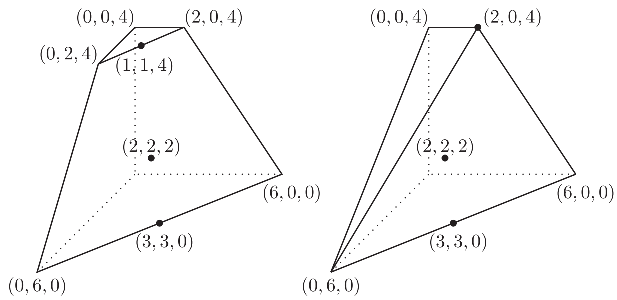

FIGURE 3. The solution with class $\{1,2,3\}$ and weights $(1/3,1/3,1/3)$ assigns to both games (left and right) the payoff vector $(2,2,2)$: at this point the product $\sqrt[3]{x_1x_2x_3}$ is maximal. The solution with classes $(\{1,2\},\{3\})$ and weights $((1/2,1/2),1)$ assigns to both games the payoff vector $(3,3,0)$. Finally, the solution with classes $(\{3\},\{1,2\})$ assigns the payoff vector $(1,1,4)$ to the left hand game, and the payoff vector $(2,0,4)$ to the right hand game

similarly. Further, $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \mid x \geq 0\}$ and $\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N \mid x > 0\}$, where $0 = (0,\ldots,0) \in \mathbb{R}^N$. A set $S \subseteq \mathbb{R}^N_+$ is *comprehensive* if for all $x \in S$ and $y \in \mathbb{R}^N_+$, $y \leq x$ implies $y \in S$.

An (n-person) *bargaining problem* is a comprehensive, convex and compact set $S \subseteq \mathbb{R}^N_+$ containing 0 and some $x \in \mathbb{R}^N$ with $x > 0$. Elements of N are also called *players*, elements of S *feasible points*, and 0 is called the *disagreement point*. By $\mathcal{B}^N$ we denote the set of all bargaining problems.

For $S \in \mathcal{B}^N$, $P(S) = \{x \in S \mid \text{for all } y \in S,\ y \geq x \text{ implies } y = x\}$ is the *Pareto optimal* subset of S. For $a \in \mathbb{R}^N_{++}$ and $x \in \mathbb{R}^N$ we write $ax = (a_1x_1,\ldots,a_nx_n)$, and for $S \in \mathcal{B}^N$ we write $aS = \{ax \mid x \in S\}$. Observe that $aS \in \mathcal{B}^N$.

An (n-person) *bargaining solution* is a map $\varphi : \mathcal{B}^N \to \mathbb{R}^N$ such that $\varphi(S) \in S$ for every $S \in \mathcal{B}^N$. We write $\varphi(S) = (\varphi_1(S),\ldots,\varphi_n(S))$ – for every $i \in N$, $\varphi_i(S)$ is the payoff to player i if φ is applied as a bargaining solution.

We are especially interested in four possible properties of φ. The first property says that it should no be possible to give some player(s) a higher payoff without lowering the payoff of some other player(s).

Pareto Optimality (PO): $\varphi(S) \in P(S)$ for every $S \in \mathcal{B}^N$.

The second property can be interpreted as saying that the payoffs should not depend on the units in which they are measured.

Scale Transformation Covariance (STC): $\varphi(aS) = a\varphi(S)$ for every $S \in \mathcal{B}^N$ and $a \in \mathbb{R}^N_{++}$.

The third property says that shrinking the set of feasible points while the assigned payoffs are still possible, should not affect those payoffs.

Contraction Independence (CI): $\varphi(S) = \varphi(T)$ for all $S, T \in \mathcal{B}^N$ such that $S \subseteq T$ and $\varphi(T) \in S$.

These three properties basically coincide with three of the four properties imposed by Nash (1950). In order to motivate the fourth property we first consider an example. Some additional notation: for a set $X \subseteq \mathbb{R}^N_+$, $\mathrm{conv}(X)$ denotes the convex hull of X, and $\mathrm{comv}(X)$ denotes the comprehensive convex hull of X, i.e., $\mathrm{comv}(X) = \{y \in \mathbb{R}^N_+ \mid y \leq x \text{ for some } x \in \mathrm{conv}(X)\}$.

Example 2.1. Let $n = 4$ and let $\varphi : \mathcal{B}^N \to \mathbb{R}^N$ be defined as follows. For $S \in \mathcal{B}^N$, let $\varphi_4(S) = \max\{x_4 \mid x \in S\}$. Let $Z = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2, x_3, \varphi_4(S)) \in S\}$. If $z > (0, 0, 0)$ for some $z \in Z$, then let $(\varphi_1 S, \varphi_2(S), \varphi_3(S))$ maximize the product $\sqrt[3]{x_1 x_2 x_3}$ on Z; otherwise, let $(\varphi_1 S, \varphi_2(S), \varphi_3(S))$ be the lexicographic maximum point of Z.[3] It is not difficult to verify that φ is well-defined and satisfies PO, STC, and CI.

Now let $S', S'' \in \mathcal{B}^N$ be the sets $S' = \mathrm{comv}\{(1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1)\}$ and $S'' = \mathrm{comv}\{(1, 0, 0, 0), (0, \frac{2}{3}, 0, 1), (0, 0, \frac{2}{3}, 1)\}$. Then $\varphi(S') = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$ and $\varphi(S'') = (0, \frac{2}{3}, 0, 1)$. If we fix player 1's payoff at $\varphi_1(S') = \frac{1}{3}$ in S', then the remaining 'slice' for the other players is the set

$$\{(x_2, x_3, x_4) \mid (\varphi_1(S'), x_2, x_3, x_4) \in S'\} = \mathrm{comv}\{(\frac{2}{3}, 0, 1), (0, \frac{2}{3}, 1)\}.$$

Similarly, fixing player 1's payoff at $\varphi_1(S'') = 0$ in S'', the remaining 'slice' for the other players is the set

$$\{(x_2, x_3, x_4) \mid (\varphi_1(S''), x_2, x_3, x_4) \in S''\} = \mathrm{comv}\{(\frac{2}{3}, 0, 1), (0, \frac{2}{3}, 1)\}.$$

Thus, given player 1's payoffs in S' and S'', the sets of feasible points for the other players are equal, but nevertheless, $(\varphi_2(S'), \varphi_3(S'), \varphi_4(S')) = (\frac{1}{3}, \frac{1}{3}, 1) \neq (\frac{2}{3}, 0, 1) = (\varphi_2(S''), \varphi_3(S''), \varphi_4(S''))$. $\triangleleft$

The fourth property excludes a situation as in Example 2.1. In order to define this property, some additional notation is required. For $x \in \mathbb{R}^N$, $X \subseteq \mathbb{R}^N$, and $\emptyset \neq M \subseteq N$, we denote $x_M = (x_i)_{i \in M}$ and $X_M = \{x_M \mid x \in X\}$. For a bargaining solution φ and $S \in \mathcal{B}^N$, we write $\varphi_M(S)$ instead of $\varphi(S)_M$.

Consistency (CONS): $\varphi_M(S) = \varphi_M(T)$ for all $S, T \in \mathcal{B}^N$ and every $\emptyset \neq M \subsetneq N$ such that $\{x_M \mid (x_M, \varphi_{N \setminus M}(S)) \in S\} = \{x_M \mid (x_M, \varphi_{N \setminus M}(T)) \in T\}$.

The consistency property is closely related to stability and consistency conditions in Lensberg (1987, 1988), Thomson and Lensberg (1989), and Chun (2002), which are usually defined for a variable population. See also, much earlier, Harsanyi (1959) and, recently, Thomson (2020).

The key result of this paper is a characterization of all bargaining solutions satisfying PO, STC, CI, and CONS. We start with the following definition.

Definition 2.2. A *weighted hierarchy* H of N is an ordered $(k+1)$-tuple of the form

$$H = \langle N^1, \ldots, N^k, \omega \rangle$$

[3] Thus, obtained by first maximizing the first coordinate, conditionally on this maximizing the second coordinate, and conditionally on this maximizing the third coordinate.

where $k \in \mathbb{N}$, $(N^1, \ldots, N^k)$ is a(n ordered) partition of N, and $\omega \in \mathbb{R}^N_{++}$ with $\sum_{i \in N^\ell} \omega_i = 1$ for every $\ell = 1, \ldots, k$. The set N^ℓ is the ℓth *class* of H. The family of all weighted hierarchies of N is denoted by $\mathcal{H}^N$.

With each weighted hierarchy of N we will associate a bargaining solution. We need again some additional notation and an observation. For a set $X \subseteq \mathbb{R}^N$ and a function $f : X \to \mathbb{R}$,

$$\arg\max\{f(x) \mid x \in X\} = \{x \in X \mid f(x) \geq f(y) \text{ for all } y \in X\}.$$

Let $\emptyset \neq M \subseteq N$, $\omega \in \mathbb{R}^N_{++}$, and let $X \subseteq \mathbb{R}^N_+$. Then we denote $M(X) = \{i \in M \mid x_i > 0 \text{ for some } x \in X\}$. If X is compact and convex, then it is not hard to verify that $\arg\max\{\prod_{i \in M(X)} x_i^{\omega_i} \mid x \in X\} \neq \emptyset$ and, moreover, that $\hat{x}_{M(X)} = \hat{y}_{M(X)}$ for all $\hat{x}, \hat{y}$ in this set.

Definition 2.3. Let $H = \langle N^1, \ldots, N^k, \omega \rangle \in \mathcal{H}^N$ and $S \in \mathcal{B}^N$. The sets $S^0, S^1, \ldots, S^k$ are defined, recursively, as follows. Let $S^0 = S$. For $\ell = 1, \ldots, k$,

$$S^\ell = \begin{cases} \arg\max\{\prod_{i \in N^\ell(S^{\ell-1})} x_i^{\omega_i^\ell} \mid x \in S^{\ell-1}\} \text{ if } N^\ell(S^{\ell-1}) \neq \emptyset, \\ S^{\ell-1} \text{ otherwise.} \end{cases}$$

By the observation preceding Definition 2.3 it follows that for every $\ell = 1, \ldots, k$ and all $x, y \in S^\ell$ we have $x_{\cup_{j=1}^\ell N^j} = y_{\cup_{j=1}^\ell N^j}$. In particular, this implies that $|S^k| = 1$. We define the bargaining solution φ^H by assigning to each $S \in \mathcal{B}^N$ the unique element of S^k, and call this a *hierarchical Nash solution.*

The definition of φ^H looks technical, but the basic idea is simple. On a given bargaining problem, we first maximize the product $\prod_{i \in N^1} x_i^{\omega_i}$. Thus, we obtain the set S^1, and by $N^2(S^1)$ we denote the subset of the agents in N^2 that can have a positive payoff on S^1. We next maximize the product $\prod_{i \in N^2(S^1)} x_i^{\omega_i}$ on S^1, and call the set of maximizers S^2; and so on and so forth, until we end up with a single point, which is then $\varphi^H(S)$. One 'extreme' case is where $k = 1$, hence $N = N^1$, in which case we only maximize once to obtain a single point. For $n = 2$ and $\omega = (\frac{1}{2}, \frac{1}{2})$, this results in the original Nash (1950) bargaining solution. Another 'extreme' case is where each class of H consists of only one player, and then $\varphi^H(S)$ is the lexicographic maximum point of S in the order given by $N^1, \ldots, N^n$.

The key result of this paper is the following theorem.

Theorem 2.4. *A bargaining solution satisfies PO, STC, CI, and CONS, if and only if it is a hierarchical Nash solution.*

The proof of Theorem 2.4 will be given in Section 3. Here, we continue with a few consequences of the theorem.

2.1. **The case** $n = 2$**.** Observe that for $N = \{1, 2\}$, consistency is implied by Pareto optimality. Thus, we obtain the following consequence of Theorem 2.4.

Corollary 2.5. *A 2-person bargaining solution satisfies PO, STC, and CI, if and only if if it is a hierarchical Nash solution.*

Corollary 2.5 was first proved in de Koster et al (1983). If $H = \langle\{1,2\},(\alpha,1-\alpha)\rangle$ for some $0 < \alpha < 1$, then for every $S \in \mathcal{B}^{\{1,2\}}$, $\varphi^H(S)$ is the feasible point where the product $x_1^\alpha x_2^{1-\alpha}$ is maximized on S. For $\alpha = \frac{1}{2}$, φ^H is the Nash bargaining solution (Nash, 1950). For $H = \langle\{1\},\{2\},1,1\rangle$, $\varphi^H(S)$ is obtained by first maximizing x_1 and on the set of maximizers maximizing x_2. Similarly, for $H = \langle\{2\},\{1\},1,1\rangle$, $\varphi^H(S)$ is obtained by first maximizing x_2 and on the set of maximizers maximizing x_1.

The bargaining solutions φ^H for $H = \langle\{1,2\},(\alpha,1-\alpha)\rangle$ for some $0 < \alpha < 1$, were first considered in Harsanyi and Selten (1972).

2.2. **Strong Individual Rationality.** Consider the following possible property of a bargaining solution φ:

Strong Individual Rationality (SIR): $\varphi(S) > 0$ for every $S \in \mathcal{B}^N$.

This condition reflects that every player should profit from reaching an agreement.

Let $H = \langle N^1,\dots,N^k,\omega\rangle$, and consider the bargaining problem $S = \{x \in \mathbb{R}^N_+ \mid \sum_{i\in N} x_i \le 1\}$. Then, clearly, $\varphi^H(S) > 0$ if and only if $k = 1$, in which case $\varphi^H(S) = \omega$. Thus, we obtain:

Lemma 2.6. *Let $H = \langle N^1,\dots,N^k,\omega\rangle \in \mathcal{H}^N$. Then φ^H satisfies SIR if and only if $k = 1$.*

The following lemma shows that adding SIR is quite powerful in our model.

Lemma 2.7. *Let bargaining solution φ satisfy SIR, STC, and CI. Then φ satisfies CONS and PO.*

Proof. In order to prove consistency, let S, T, and M be as in its definition, i.e., $S,T \in \mathcal{B}^N$ and $\emptyset \ne M \subseteq N$ such that $\{x_M \mid (x_M,\varphi_{N\setminus M}(S)) \in S\} = \{x_M \mid (x_M,\varphi_{N\setminus M}(T)) \in T\}$. Let $a \in \mathbb{R}^N_{++}$ with $a_i = 1$ for all $i \in M$ and $a_i = \varphi_i(S)/\varphi_i(T)$ for all $i \in N \setminus M$ (observe that $\varphi_i(T) > 0$ for all $i \in N$ by SIR). Then $\{x_M \mid (x_M,\varphi_{N\setminus M}(S)) \in S\} = \{x_M \mid (x_M,\varphi_{N\setminus M}(S)) \in aT\}$. Define $S' = \{s \in \mathbb{R}^N_+ \mid s \le (x_M,\varphi_{N\setminus M}(S))$ for some $(x_M,\varphi_{N\setminus M}(S)) \in S\}$ and $T' = \{t \in \mathbb{R}^N_+ \mid t \le (x_M,\varphi_{N\setminus M}(S))$ for some $(x_M,\varphi_{N\setminus M}(S)) \in aT\}$, then $S',T' \in \mathcal{B}^N$ and $S' = T'$. By CI applied twice, $\varphi(S) = \varphi(S') = \varphi(T') = \varphi(aT)$, hence by STI, $\varphi_M(S) = \varphi_M(aT) = \varphi_M(T)$. This proves consistency.

For PO, suppose that $S \in \mathcal{B}^N$ and $\varphi(S) \notin P(S)$. By SIR, $\varphi(S) > 0$. By CI, $\varphi((\alpha,\dots,\alpha)S) = \varphi(S)$ where $0 < \alpha < 1$ such that $\varphi(S) \in P((\alpha,\dots,\alpha)S)$. Hence, by STC, $\varphi(S) = (\alpha,\dots,\alpha)\varphi(S)$, a contradiction. This proves PO. □

The fact that PO is implied by SIR, STC, and CI, was already observed by Roth (1977). Theorem 2.4 and Lemmas 2.6 and 2.7 imply the following result (see also Theorem 3 in Roth, 1979).

Corollary 2.8. *A bargaining solution φ satisfies SIR, STC, and CI, if and only if there is an $H = \langle N,\omega\rangle \in \mathcal{H}^N$ such that $\varphi = \varphi^H$.*

2.3. **Independence of the properties.** We show that none of the four properties in Theorem 2.4 can be left out, by means of examples.

Example 2.9. The bargaining solution φ defined by $\varphi(S) = 0$ for every $S \in \mathcal{B}^N$ satisfies STC, CI, and CONS, but not PO. $\lhd$

Example 2.10. The lexicographic egalitarian bargaining solution φ satisfies PO, CI, and CONS, but not STC. This solution is obtained as folllows. First take the maximal point of $S \in \mathcal{B}^N$ with equal coordinates; if this is not Pareto optimal then continue with those coordinates for which equal increase is still possible and take again the maximal point of S for which these coordinates are equal; repeat this procedure until a Pareto optimal point is reached. $\lhd$

Example 2.11. Let $N = \{1, 2, 3\}$ and define the bargaining solution φ as follows. For $S \in \mathcal{B}^N$ such that $aS = \text{comv}\{(\frac{3}{2}, \frac{3}{2}, 0), (1, 1, 1)\}$ for some $a \in \mathbb{R}^N_{++}$ let $\varphi(S) = (a_1^{-1}, a_2^{-1}, a_3^{-1})(\frac{5}{4}, \frac{5}{4}, \frac{1}{2})$. For every other $S \in \mathcal{B}^N$ let $\varphi(S) = \varphi^H(S)$, where $H = \langle N, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\rangle$. Then φ satisfies PO, STC, and CONS. Take $S = \text{comv}\{(3, 0, 0), (0, 3, 0), (0, 0, 3)\}$, then $\varphi(S) = (1, 1, 1)$, but $\varphi(\text{comv}\{(\frac{3}{2}, \frac{3}{2}, 0), (1, 1, 1)\}) = (\frac{5}{4}, \frac{5}{4}, \frac{1}{2})$, which is a violation of CI. (This example is based on Lensberg (1988), p. 339.) $\lhd$

Example 2.12. Example 2.1 defines a bargaining solution which satisfies PO, STC, and CI, but not CONS. In this example, $n = 4$. An analogous example is easily defined for any $n > 4$. For $n = 2$, Corollary 2.5 implies that CONS is implied by the other three properties. For the remaining case, $n = 3$, it is still an open problem if CONS is implied by PO, STC, and CI. $\lhd$

3. Proof of Theorem 2.4

We first introduce some additional notation. Let $\emptyset \neq L \subseteq M \subseteq N$. By $e^L \in \mathbb{R}^M$ we denote the vector with $e^L_i = 1$ if $i \in L$ and $e^L_i = 0$ if $i \in M \setminus L$. Instead of $e^{\{i\}}$ we write e^i, so this is the i-th unit vector in $\mathbb{R}^M$. For $x \in \mathbb{R}^L$, we denote by $O^M(x) \in \mathbb{R}^M$ the vector with $O^M(x)_i = x_i$ if $i \in L$ and $O^M(x)_i = 0$ if $i \in M \setminus L$; and by $E^M(x) \in \mathbb{R}^M$ the vector with $E^M(x)_i = x_i$ if $i \in L$ and $E^M(x)_i = 1$ if $i \in M \setminus L$. For $S \subseteq \mathbb{R}^L$, $O^M(S) = \{O^M(x) \mid x \in S\}$ and $E^M(S) = \{E^M(x) \mid x \in S\}$. For $X \subseteq \mathbb{R}^M$ and $y \in \mathbb{R}^M$, $X + y = y + X = \{x + y \mid x \in X\}$.

We start with the if-direction of the theorem.

Proposition 3.1. *Let $H = \langle N^1, \ldots, N^k, \omega\rangle \in \mathcal{H}^N$. Then φ^H satisfies PO, STC, CI, and CONS.*

Proof. The proof of PO, STC, and CI of φ^H is left to the reader. In order to prove consistency of φ^H, let $S, T \in \mathcal{B}^N$ and $\emptyset \neq M \subsetneq N$ such that $\{x_M \mid (x_M, \varphi^H_L(S)) \in S\} = \{x_M \mid (x_M, \varphi^H_L(T)) \in T\}$, where $L = N \setminus M$. We have to show that

$$\varphi^H_M(S) = \varphi^H_M(T). \tag{3.1}$$

Let $Z = O^N(\{x_M \mid (x_M, \varphi^H_L(S)) \in S\}) = O^N(\{x_M \mid (x_M, \varphi^H_L(T)) \in T\})$, i.e., Z is the 'slice' for the players of M at the solution outcome, embedded in $\mathbb{R}^N$ by adding zeros for the players outside M. By the definition of φ^H we may assume without loss of generality that there is an $(x_M, 0_L) \in Z$ with $x_M > 0$ (otherwise restrict attention to $\{x \in S \mid x_i = 0\}$ and $\{x \in T \mid x_i = 0\}$ for those $i \in M$ with $z_i = 0$ for all $z \in Z$).

Let $L^1 = \{i \in L \mid \varphi_i^H(S) = 0\}$ and $L^2 = L \backslash L^1$. Similarly, $L^3 = \{i \in L \mid \varphi_i^H(T) = 0\}$ and $L^4 = L \setminus L^3$. By STC we may assume:

$$L^2 = \{i \in L \mid \varphi_i^H(S) = 1\} \text{ and } L^4 = \{i \in L \mid \varphi_i^H(T) = 1\}. \tag{3.2}$$

Now let $V = \text{comv}((Z \cup \{\alpha e^{L^1}\}) + e^{L^2})$ and $W = \text{comv}((Z \cup \{\beta e^{L^3}\}) + e^{L^4})$ where $e^{L^1}, \ldots, e^{L^4} \in \mathbb{R}^N$, and $\alpha, \beta > 0$ are chosen such that $\alpha e^{L^1} \in S$ and $\beta e^{L^3} \in T$. Then, by (3.2), $e^{L^2} = O^N(\varphi_{L^2}^H(S))$ and $e^{L^4} = O^N(\varphi_{L^4}^H(T))$, so that $V \subseteq S$ and $W \subseteq T$. Since $V, W \in \mathcal{B}^N$, $\varphi^H(S) \in V$, and $\varphi^H(T) \in W$, CI of φ^H implies $\varphi^H(S) = \varphi^H(V)$ and $\varphi^H(T) = \varphi^H(W)$. Since both $\varphi^H(V)$ and $\varphi^H(W)$ only depend on Z, we have that $\varphi_M^H(V) = \varphi_M^H(W)$, and hence (3.1) follows. □

In order to prove the converse of Proposition 3.1, i.e., the only-if direction of Theorem 2.4, we need some more definitions and lemmas.

For $\emptyset \neq M \subseteq N$, denote $\Delta^M = \text{comv}\{e^i \in \mathbb{R}^M \mid i \in M\}$ and $\bar{\Delta}^M = \text{comv}(E^N(\Delta^M))$. The set $\bar{\Delta}^M$ is called the *standard bargaining game* for $M \subseteq N$. As we will see, a bargaining solution with the four properties in Theorem 2.4 is completely determined by the outcomes it assigns to standard bargaining games. Before formally defining what this means, we consider a few examples.

If $n = 3$ and $\varphi(\Delta^N) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, then we will show that $\varphi = \varphi^H$ with $H = \langle N, (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})\rangle$. If $n = 3$ and $\varphi(\Delta^N) = (\frac{1}{2}, \frac{1}{2}, 0)$, then we will show that $\varphi = \varphi^H$ with $H = \langle \{1,2\}, \{3\}, (\frac{1}{2}, \frac{1}{2}, 1)\rangle$. Similarly, if $n = 4$, $\varphi(\Delta^N) = (\frac{1}{2}, 0, \frac{1}{2}, 0)$, and $\varphi(\bar{\Delta}^{\{2,4\}}) = (1, \frac{1}{4}, 1, \frac{3}{4})$, then we will show that $\varphi = \varphi^H$ with $H = \langle \{1,3\}, \{2,4\}, (\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4})\rangle$.

Formally, we have the following definition.

Definition 3.2. Let φ be a bargaining solution and let $H = \langle N^1, \ldots, N^k, \omega\rangle \in \mathcal{H}^N$ be a weighted hierarchy. Then φ *determines* H (*on standard bargaining games*) if $\varphi(\Delta^N) = O^N(\omega_{N^1})$ and for each $\ell = 2, \ldots, n$,

$$\varphi(\bar{\Delta}^{\cup_{j=\ell}^k N^j}) = e^{\cup_{j=1}^{\ell-1} N^j} + O^N(\omega_{N^\ell}).$$

The following auxiliary result is standard, but we outline the proof for completeness.

Lemma 3.3. *Let $\emptyset \neq M \subseteq N$, let S be a nonempty compact and convex subset of $\mathbb{R}_+^M$ such that $x > 0$ for some $x \in S$, and let $\omega \in \mathbb{R}_{++}^M$. Then the product $\prod_{i \in M} x_i^{\omega_i}$ has a unique maximizer on S. Moreover, $z \in S$ maximizes this product if and only if the hyperplane with equation*

$$\sum_{i \in M} \frac{\omega_i x_i}{z_i} = \sum_{i \in M} \frac{\omega_i z_i}{z_i}$$

supports S at z.

Proof. By convexity and compactness of S and strict quasiconcavity of the function $f(x) = \prod_{i \in M} x_i^{\omega_i}$, f has a unique maximum point $z > 0$ on S. By the Minkowsky separation theorem (Rockafellar, 1970, Section 11) the set S and the upper contour set $\{x \in \mathbb{R}^M \mid f(x) \geq f(z)\}$ are separated by a unique hyperplane in $\mathbb{R}^M$, the equation of which, i.e., the formula in the lemma, follows by considering the gradient of the function f at z. □

Using this result we obtain the next lemma.

Lemma 3.4. *(i) For every $H \in \mathcal{H}^N$, φ^H determines H. (ii) If a bargaining solution determines H and H' in $\mathcal{H}^N$, then $H = H'$. (iii) Every Pareto optimal bargaining solution determines some $H \in \mathcal{H}^N$.*

Proof. (i) follows from repeated application of Lemma 3.3. (ii) follows directly from Definition 3.2. To show (iii), let φ be a Pareto optimal bargaining solution. We construct H as follows. Let $N^1 = \{i \in N \mid \varphi_i(\Delta^N) > 0\}$, and, for $i \in N^1$, let $\omega_i = \varphi_i(\Delta^N)$. If $N \neq N^1$, let $N^2 = \{i \in N \setminus N^1 \mid \varphi_i(\bar{\Delta}^{N\setminus N^1}) > 0\}$, and for $i \in N^2$, let $\omega_i = \varphi_i(\bar{\Delta}^{N\setminus N^1})$; and so on and so forth. □

We now consider the two-player case. In fact, the following lemma is the only-if direction of Corollary 2.5.

Lemma 3.5. *Let $N = \{1,2\}$ and let bargaining solution φ satisfy PO, STC, and CI. Then $\varphi = \varphi^H$ for some $H \in \mathcal{H}^N$.*

Proof. Let H be the unique weighted hierarchy determined by φ (cf. Lemma 3.4, (ii) and (iii)).

If H is of the form $\langle N, \omega\rangle$ then for $S \in \mathcal{B}^N$, let $z \in S$ maximize the product $x_1^{\omega_1} x_2^{\omega_2}$ on S. Then $z > 0$ and by STC we may assume $z = (\omega_1, \omega_2)$. Then by Lemma 3.3, the fact that $\varphi(\Delta^{\{1,2\}}) = (\omega_1, \omega_2)$, and CI, it follows that $\varphi(S) = z = \varphi^H(S)$.

If $H = \langle\{1\}, \{2\}, (1,1)\rangle$, then $\varphi(\Delta^{\{1,2\}}) = (1,0)$. Let $S \in \mathcal{B}^{\{1,2\}}$ and suppose that $\varphi(S) \neq \varphi^H(S)$. We derive a contradiction. Note that $\varphi^H(S)$ is the point of S with maximal second coordinate among the points with maximal first coordinate. By PO, this implies that $\varphi_1(S) < \varphi_1^H(S)$. In view of STC, it is without loss of generality to assume that $\varphi_1^H(S) = 1$ and $\varphi_1(S)+\varphi_2(S) = 1$. Let $W = \text{comv}\{(\varphi_1^H(S), 0), \varphi(S)\} = \text{comv}\{(1,0), \varphi(S)\}$. Then $W \in \mathcal{B}$, $W \subseteq S$, and $\varphi(S) \in W$. So by CI, $\varphi(W) = \varphi(S)$. On the other hand, $W \subseteq \Delta^{\{1,2\}}$ and $\varphi(\Delta^{\{1,2\}}) = (1,0) \in W$ imply by CI that $\varphi(W) = (1,0) \neq \varphi(S)$, which is the desired contradiction.

By observing that the case $H = \langle\{2\}, \{1\}, (1,1)\rangle$ is similar, the proof is complete. □

The proof of the only-if direction of Theorem 2.4 will be based on Lemma 3.5 and the following induction hypothesis, in which we consider bargaining solutions and weighted hierarchies for subsets of N:

> For all $2 \leq m < n$, for all $M \subseteq N$ with $|M| = m$ and for all $H \in \mathcal{H}^M$, if the bargaining solution $\varphi : \mathcal{B}^M \to \mathbb{R}^M$ satisfies PO, STC, CI, and CONS, and determines $H \in \mathcal{H}^M$, then $\varphi = \varphi^H$. (*)

For an n-person bargaining solution we define solutions for subsets of the player set N, as follows.

Definition 3.6. Let $\varphi : \mathcal{B}^N \to \mathbb{R}^N$ be a bargaining solution and let $\emptyset \neq M \subseteq N$. Then $M\varphi : \mathcal{B}^M \to \mathbb{R}^M$ is the bargaining solution defined by $M\varphi(S) = \varphi_M(\text{comv}(E^N(S)))$ for every $S \in \mathcal{B}^M$.

The solutions $M\varphi$ inherit the relevant properties of φ;

Lemma 3.7. *Let the bargaining solution $\varphi : \mathcal{B}^N \to \mathbb{R}^N$ satisfy PO, STC, CI, and CONS, and let $\emptyset \neq M \subseteq N$. Then also $M\varphi$ has these properties.*

Proof. We only show that $M\varphi$ is consistent, and leave verification of the other properties to the reader. Let $S, T \in \mathcal{B}^M$ and $\emptyset \neq L \subsetneq M$ such that $\{x \in \mathbb{R}^L \mid (x, M\varphi_{M\setminus L}(S)) \in S\} = \{x \in \mathbb{R}^L \mid (x, M\varphi_{M\setminus L}(T)) \in T\}$. This implies $\{x \in \mathbb{R}^L \mid (x, \varphi_{M\setminus L}(\mathrm{comv}(E^N(S)))) \in S\} = \{x \in \mathbb{R}^L \mid (x, \varphi_{M\setminus L}(\mathrm{comv}(E^N(T)))) \in T\}$, and therefore we obtain

$$\{x \in \mathbb{R}^L \mid (x, \varphi_{N\setminus L}(\mathrm{comv}(E^N(S)))) \in \mathrm{comv}(E^N(S))\} = \{x \in \mathbb{R}^L \mid (x, \varphi_{N\setminus L}(\mathrm{comv}(E^N(T)))) \in \mathrm{comv}(E^N(T))\}.$$

Consistency of φ now implies that $\varphi_i(\mathrm{comv}(E^N(S))) = \varphi_i(\mathrm{comv}(E^N(T)))$ for all $i \in L$, and thus $M\varphi_i(S) = M\varphi_i(T)$ for all $i \in L$, as was to be proved. $\square$

The next lemma shows that the induced solution $M\varphi$ determines a corresponding weighted hierarchy.

Lemma 3.8. *Let the bargaining solution $\varphi : \mathcal{B}^N \to \mathbb{R}^N$ determine $\langle N^1, \ldots, N^k, \omega\rangle \in \mathcal{H}^N$, where $k \geq 2$. Let $M = N\setminus N^1$. Then $M\varphi$ determines $\langle N^2, \ldots, N^k, \omega_M\rangle \in \mathcal{H}^M$.*

Proof. $M\varphi(\Delta^M) = \varphi_M(\bar{\Delta}^M) = O^M(\omega_{N^2})$, $M\varphi(\mathrm{comv}(E^M(\Delta^{M\setminus N^2}))) = \varphi_M(\bar{\Delta}^{M\setminus N^2}) = \varphi_M(\bar{\Delta}^{N\setminus(N^1\cup N^2)}) = e^{N^2} + O^M(\omega_{N^3})$; and so on and so forth. $\square$

The following lemma treats the case in which the first class of a weighted hierarchy consists of exactly one player.

Lemma 3.9. *Let the bargaining solution $\varphi : \mathcal{B}^N \to \mathbb{R}^N$ satisfy PO, STC, CI, and CONS, and let φ determine $H = \langle N^1, \ldots, N^k, \omega\rangle \in \mathcal{H}^N$ with $|N^1| = 1$. Let the induction hypothesis (*) hold. Then $\varphi = \varphi^H$.*

Proof. Without loss of generality let $N^1 = \{1\}$. Let $S \in \mathcal{B}^N$.

We first prove that $\varphi_1(S) = \varphi_1^H(S)$. In order to prove this, write $u_1 = \varphi_1^H(S) = \max\{x_1 \mid x \in S\}$ and suppose to the contrary that $\varphi(S) = z \in P(S)$ with $z_1 < u_1$. Take $\beta > 0$ large enough such that $z \in V = \mathrm{comv}\{u_1 e^1, \beta e^i \in \mathbb{R}^N \mid i = 2, \ldots, n\}$, which is possible since $z_1 < u_1$. By STC and $\varphi(\Delta^N) = e^1$, $\varphi(V) = u_1 e^1$. By CI, $\varphi(V \cap S) = u_1 e^1$ as well as $\varphi(V \cap S) = z$, a contradiction. In fact, we have proved:

$$\varphi_1(T) = \varphi_1^H(T) \text{ for every } T \in \mathcal{B}^N. \tag{3.3}$$

Next, let $M = N \setminus \{1\}$ and let $L = \{i \in M \mid$ there is an $x \in \mathbb{R}^M$ such that $(\varphi_1(S), x) \in S$ and $x_i > 0\}$. Note that $\varphi_i(S) = 0$ for all $i \in M \setminus L$ and that in view of (3.3), $\{x \in \mathbb{R}^L \mid (x, \varphi_{N\setminus L}(S)) \in S\} = \{x \in \mathbb{R}^L \mid (x, \varphi_{N\setminus L}(\mathrm{comv}(e^{M\setminus L} + S))) \in \mathrm{comv}(e^{M\setminus L} + S)\}$, so that by CONS, $\varphi_L(S, d) = \varphi_L(e^{M\setminus L} + S)$. Consequently, it is without loss of generality to assume that $M = L$. In view of STC we may assume that $u_1 = 1$, so by CI: $\varphi(S) = \varphi(\mathrm{comv}(E^N(W)))$, where $W = \{x \in \mathbb{R}^M \mid (\varphi_1(S), x) \in S\}$. Hence, $\varphi_M(S) = M\varphi(W) = \varphi_M^H(S)$, where the last equality follows Lemmas 3.7 and 3.8 and the induction hypothesis (*). So we have proved that $\varphi_i(S) = \varphi_i^H(S)$ for all $i \in N$. $\square$

The next case is where the first class of a weighted hierarchy contains more than one but not all players.

Lemma 3.10. *Let the bargaining solution $\varphi : \mathcal{B}^N \to \mathbb{R}^N$ satisfy PO, STC, CI, and CONS, and let φ determine $\langle N^1, \ldots, N^k, \omega\rangle \in \mathcal{H}^N$ with $1 < |N^1| < n$. Let $S \in \mathcal{B}^N$ and $z = \varphi(S)$. Then $\prod_{i \in N^1} z_i^{\omega_i} \geq \prod_{i \in N^1} x_i^{\omega_i}$ for all $x \in S$.*

Proof. Without loss of generality let $N^1 = \{1, \ldots, s\}$ with $1 < s < n$. Let $M = N \setminus N^1$ and $q \in S$ with $q_M = 0$ and $\prod_{i=1}^s q_i^{\omega_i} \geq \prod_{i=1}^s x_i^{\omega_i}$ for all $x \in S$ with $x_M = 0$. Lemma 3.3 implies that there is a hyperplane Y in $\mathbb{R}^{N^1}$ supporting $\{x_{N^1} \in \mathbb{R}^{N^1} \mid x \in S, x_M = 0\}$ at q_{N^1} with equation $\sum_{i=1}^s \omega_i q_i^{-1} x_i = 1$. In view of STC we may assume that $q_i = \omega_i$ for $i = 1, \ldots, s$. Let $\bar{z} = O^N(z_{N^1})$. We distinguish three cases:

Case (i). $z_{N^1} = q_{N^1}$ $(= \omega_{N^1})$. Then $\prod_{i=1}^s z_i^{\omega_i} = \prod_{i=1}^s q_i^{\omega_i} \geq \prod_{i=1}^s x_i^{\omega_i}$ for all $x \in S$ with $x_M = 0$, hence $\prod_{i=1}^s z_i^{\omega_i} \geq \prod_{i=1}^s x_i^{\omega_i}$ for all $x \in S$. So in this case the conclusion of the lemma holds.

Case (ii). $z_{N^1} \notin Y$. Then z_{N^1} is in the interior of Δ^{N^1}, so $\bar{z}$ is in the relative interior of $O^N(\Delta^{N^1})$. Therefore there is $\delta > 0$ large enough such that $z \in V$, where $V = \text{comv}(O^N(\Delta^{N^1}) \cup \{\delta e^i \mid i \in M\}) \in \mathcal{B}^N$. By STC and the equalities $\varphi(\Delta^N) = O^N(\omega_{N^1}) = q$, we have $\varphi(V) = q$. Then by CI, $\varphi(V \cap S) = q$ and $\varphi(V \cap S) = z$. In particular this implies $q_{N^1} = z_{N^1}$ and $z_{N^1} \in Y$. From this contradiction we conclude that case (ii) cannot occur.

Case (iii). $z_{N^1} \in Y$, $z_{N^1} \neq q_{N^1}$. In this case, let $y \in S$ with $y_M = 0$ and $y_{N^1} = \frac{1}{2}(z_{N^1} + q_{N^1})$. Then $\prod_{i=1}^s y_i^{\omega_i} = \max\{\prod_{i=1}^s x_i^{\omega_i} \mid x \in a\Delta^{N^1}\}$ where $a \in \mathbb{R}^{N^1}_{++}$ is defined by $a_i = y_i q_i^{-1}$ for $i = 1, \ldots, s$. An elementary calculation then shows that z_{N^1} is in the interior of $a\Delta^{N^1}$, so that $\bar{z}$ is in the relative interior of $O^N(a\Delta^{N^1})$, which is a case analogous to case (ii) above. Hence, also case (iii) cannot occur, and the proof is complete. □

We can now prove the only-if direction of Theorem 2.4, that is, the following proposition.

Proposition 3.11. *Let bargaining solution $\varphi : \mathcal{B}^N \to \mathbb{R}^N$ satisfy PO, STC, CI, and CONS. Then there is a weighted hierarchy $H \in \mathcal{H}^N$ such that $\varphi = \varphi^H$.*

Proof. Let $H = \langle N^1, \ldots, N^k \rangle \in \mathcal{H}^N$ be the weighted hierarchy determined by φ, cf. Lemma 3.4. Let $S \in \mathcal{B}^N$.

If $k = 1$, then by STC we may assume without loss of generality that $\varphi^H(S) = \omega$. From Lemma 3.3 it follows that the hyperplane with equation $\sum_{i \in N} x_i = 1$ supports S at ω. Therefore, $S \subseteq \Delta^N$. Furthermore, since φ determines H, $\varphi(\Delta^N) = \omega$. So CI implies $\varphi(S) = \omega$ and, hence, $\varphi(S) = \varphi^H(S)$. The rest of the proof is based on this case and the induction hypothesis (*).

If $k > 1$ and $|N^1| = 1$, then $\varphi(S) = \varphi^H(S)$ by Lemma 3.9.

Finally, let $k > 1$ and $|N^1| > 1$. Lemma 3.10 implies that $\varphi_{N^1}(S) = \varphi^H_{N^1}(S)$. By an argument similar to the one used in the proof of Lemma 3.9, which was based on consistency, we may without loss of generality assume that there is $(x_M, \varphi_{N^1}(S)) \in S$ with $x_i > 0$ for every $i \in M = N \setminus N^1$. In view of STC, we may further assume that $\varphi_{N^1}(S) = e^{N^1} \in \mathbb{R}^{N^1}$. Hence, by CI, $\varphi(S) = \varphi(\text{comv}(E^N(\{x \in \mathbb{R}^M \mid (x_M, \varphi_{N^1}(S)) \in S\})))$, so $\varphi_M(S) = M\varphi(\{x \in \mathbb{R}^M \mid (x_M, \varphi(S)) \in S\}) = \varphi^H_{N^1}(S)_M$, where the last equality follows from Lemmas 3.7 and 3.8 and the induction hypothesis (*). We conclude that $\varphi(S) = \varphi^H(S)$. $\square$

Theorem 2.4 now follows from Propositions 3.1 and 3.11.

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