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PROPORTIONATE CHANGE OF OUTPUTS AND OF INPUTS IN DEA

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The paper is devoted to Professor Hubertus Th. Jongen on the occasion of his 75th birthday.

Abstract. In this review paper, we consider sensitivity analysis for the proportionate change of outputs and of inputs in Data Envelopment Analysis (DEA). Sensitivity analysis is studied in the additive model for the proportionate changes of outputs, of inputs, and of simultaneous proportionate changes of outputs and of inputs. Cases of proportionate changes of outputs and inputs with different coefficients of proportionality and of changes of discretionary outputs and inputs are also studied for the additive model. Sensitivity analysis in the CCR model is considered for the simultaneous proportionate changes of outputs and of inputs, for the proportionate changes of a subset of inputs and/or of outputs and for the proportionate changes of outputs and inputs with different coefficients of proportionality.

1. Introduction

Sensitivity analysis of the proportionate change of outputs and/or of inputs in Data Envelopment Analysis (DEA) for the additive model was studied in Charnes and Neralić [12], Neralić and Sexton [23], Neralić [17], and for the Charnes-Cooper-Rhodes (CCR) ratio model in Charnes and Neralić [9], Guddat et al. [16], Neralić and Sexton [23], Charnes and Neralić [11]. Results are based on the sensitivity and stability analysis for different cases of perturbations of outputs and of inputs in the additive model and in the CCR model in Charnes and Neralić [8], Charnes and Neralić [6], Charnes and Neralić [7], Charnes and Neralić [10], Charnes and Neralić [12], Neralić [20]. For a recent review paper on the algorithmic approach to sensitivity analysis in DEA, see Neralić and Wendell [24]. New results on enlarging the radius of stability and stability regions in DEA are included in Neralić and Wendell [25]. Some recent results on sensitivity and stability analysis in DEA can be found in Cooper et al. [13].

To the best of my knowledge, the first result in sensitivity analysis of the additive model in DEA for the case of the decrease (increase) of outputs of Pareto-Koopmans efficient (inefficient) Decision Making Units (DMUs) and the increase (decrease) of inputs of Pareto-Koopmans efficient (inefficient) DMUs respectively for which the

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efficiency of an efficient DMU_0 is preserved, appeared in Neralić [18]. More details on sensitivity in data envelopment analysis for simultaneous perturbations of all data for the additive model can be found in Neralić [22] and for the CCR model in Neralić [21]. Another approach to sensitivity analysis of DEA models for the simultaneous changes in all the data for the percentage change case and for the absolute change case was studied in Seiford and Zhu [26].

Pareto-Koopmans efficiency in Data Envelopment Analysis can be defined in the following way: "A DMU is fully efficient if and only if it is not possible to improve any input or output without worsening some other input or output". (See Definition 3.3 in the book [14], p. 45, which recognizes the contributions of the economists Vilfredo Paretto and Tjalling Koopmans.) "Improving" in the definition means decrease for inputs and increase for outputs, and "worsening" means increase for inputs and decrease for outputs.

There is also the other definition of a Pareto-efficient (minimum) point for a finite set of functions. Namely, a Pareto-efficient (or Pareto-Koopmans efficient) point of functions $g_1(x), g_2(x), \ldots, g_k(x)$ is a point $x^\star$ such that there is no other point x in the domain of these functions such that

$$g_k(x) \leq g_k(x^\star), k = 1, 2, \ldots, p$$

with at least one strict inequality. Charnes and Cooper [2], Chapter IX, showed that $x^\star$ is a Pareto-efficient if and only if $x^\star$ is an optimal solution to the mathematical (goal) program

$$\min_x \sum_{k=1}^{p} g_k(x), g_k(x) \leq g_k(x^\star), k = 1, 2, \ldots, p.$$

The major DEA models are really the Charnes-Cooper test for vector optimality of a multiple-objective program. For efficient production we wish to maximize on outputs while minimizing on inputs. Details on testing an empirical input-output point X_0, Y_0 for Pareto-Koopmans efficiency in the case of the additive model can be found in Charnes et al. [4], pp. 96-97.

Efficient points are very important in multi-objective optimization (see, for example, Wendell and Lee [27]). Namely, for a subset X of a finite dimensional Euclidean space and p functions $g_k : X \to \mathbb{R}$, $k = 1, 2, \ldots, p$, we consider vector minimization problem

$$\min g(x), x \in X$$

with $g(x) = [g_1(x)\ g_2(x)\ \ldots\ g_p(x)]^T$. Let an auxiliary problem $P(\bar{x})$ be

$$h(\bar{x}) \equiv \inf \sum_{k=1}^{p} g_k(x), g(x) \leq g(\bar{x}), x \in X$$

where $\bar{x}$ is any point in X. It can be shown that $x^0 \in X$ is efficient if and only if $h(x^0) = \sum_{k=1}^{p} g_k(x^0)$. Let $x^\star \in X$ be an optimal solution to auxiliary problem $P(\bar{x})$. Then $x^\star$ is efficient and $g(x^\star) \leq g(\bar{x})$, with at least one strong inequality sign $<$. For details, see Theorem 1 and Theorem 2 with proofs in Wendell and Lee [27], pp. 406-407.

The paper is organized as follows. Sensitivity analysis of the proportionate change of outputs and/or of inputs in the additive model is studied in Section 2. Subsection 2.1 contains some preliminaries for the additive model. The proportionate change of outputs, of inputs and of outputs and inputs simultaneously is considered in Subsections 2.2, 2.3 and 2.4 respectively.

The case with different coefficients of proportionality for outputs and inputs is studied in Subsection 2.5 and the case of discretionary outputs and/or inputs is considered in Subsection 2.6. Similar cases for the CCR model are studied in Section 3. Some preliminaries for the CCR model are contained in Subsection 3.1. Simultaneous proportionate changes of inputs and of outputs are considered for the CCR model in Subsection 3.2. Subsection 3.3 contains a sensitivity analysis of the proportionate change of a subset of inputs and/or of outputs. Conclusions and some suggestions for further research are presented in the last section.

2. Sensitivity analysis of the proportionate change of outputs and/or of inputs in the Additive model

2.1. **Preliminaries.** Let us suppose that there are n Decision Making Units (DMUs) with m inputs and s outputs. Let x_{ij} be the observed amount of the ith type of input of the jth DMU ($x_{ij} > 0$, $i = 1, 2, \dots, m, j = 1, 2, \dots, n$) and let y_{rj} be the observed amount of output of the rth type for the jth DMU ($y_{rj} > 0$, $r = 1, 2, \dots, s, j = 1, 2, \dots, n$). Let Y_j, X_j be the observed vectors of outputs and inputs of the DMU$_j$, respectively, $j = 1, 2, \dots, n$. Let e be the column vector of ones and let T as a superscript denote the transpose. In order to see if the DMU$_{j0}$ = DMU$_0$ is Pareto-Koopmans efficient according to the additive model, the following linear programming problem should be solved:

$$\min 0\lambda_1 + \cdots + 0\lambda_0 + \cdots + 0\lambda_n - e^T s^+ - e^T s^-$$

subject to

$$\begin{array}{rrrcr} Y_1\lambda_1 + \cdots + Y_0\lambda_0 + \cdots + Y_n\lambda_n & - \; s^+ & & = & Y_0 \\ -X_1\lambda_1 - \cdots - X_0\lambda_0 - \cdots - X_n\lambda_n & & - \; s^- & = & -X_0 \\ \lambda_1 + \cdots + \lambda_0 + \cdots + \lambda_n & & & = & 1 \\ \lambda_1, \dots, \lambda_0, \dots, \lambda_n, & s^+, & s^- & \geq & 0, \end{array} \tag{2.1}$$

where $Y_0 = Y_{j_0}, X_0 = X_{j_0}, \lambda_0 = \lambda_{j_0}$. DMU$_0$ is Pareto - Koopmans efficient if and only if for the optimal solution $(\lambda^*, s^{+*}, s^{-*})$ of the linear programming problem (2.1) $\min(-e^T s^+ - e^T s^-) = -e^T s^{+*} - e^T s^{-*} = 0$ holds (for details, see Charnes and Cooper [3]).

We are interested in variations of all outputs and/or inputs of a Pareto-Koopmans efficient DMU$_0$ preserving its efficiency. An increase of any output cannot worsen an already achieved efficiency rating. Upward variations of outputs are not possible in the efficiency rating for an efficient DMU$_0$. Hence we can pay attention to downward variations of outputs which can be written as

$$\widehat{y}_{r0} = y_{r0} - \alpha_r > 0, \quad \alpha_r \geq 0, \quad r = 1, 2, \dots, s. \tag{2.2}$$

Similarly, a decrease of any input cannot worsen an already achieved efficiency rating. Downward variations of inputs are not possible in the efficiency rating for an efficient DMU_0. Hence we can restrict attention to upward variations of inputs of an efficient DMU_0 which can be written as

$$\widehat{x}_{i0} = x_{i0} + \beta_i, \quad \beta_i \geq 0, \qquad i = 1, 2, \ldots, m. \tag{2.3}$$

For a Pareto-Koopmans efficient DMU_0 there is a basic optimal solution $(\lambda^*, s^{+*}, s^{-*})$ of the linear programming problem (2.1) with $\lambda_0^* = 1,\ \lambda_j^* = 0,\ j \neq j_0,\ j = 1, 2, \ldots, n,\ s^{+*} = s^{-*} = 0$ and the corresponding optimal basis matrix

$$B = \begin{bmatrix} Y_B & -I_B^+ & 0 \\ -X_B & 0 & -I_B^- \\ e^T & 0 & 0 \end{bmatrix}.$$

Let the inverse of the matrix B be

$$B^{-1} = \left[b_{ij}^{-1} \right], \quad i,\ j = 1, 2, \ldots, s + m.$$

Let $P_j,\ j = 1, 2, \ldots, n + s + m + 1$ be the columns of the matrix and let P_0 be the right-hand side of the linear programming problem (2.1). We will use the following notations:

$$\begin{aligned} \Gamma_j &= B^{-1} P_j,\ j = 0, 1, \ldots, n + s + m + 1, \\ \omega^T &= c_B^T B^{-1}, \\ z_j &= c_B^T B^{-1} P_j \\ &= \omega^T P_j,\ j = 0, 1, \ldots, n + s + m + 1. \end{aligned}$$

The simultaneous change of outputs (2.2) and change of inputs (2.3) leads to the following change of the optimal basis matrix B

$$\widehat{B} = B + D \tag{2.4}$$

with

$$D = \begin{array}{c} \begin{array}{ccccccc} & & & k & & & \\ & & & \downarrow & & & \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & -\alpha_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\alpha_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\alpha_s & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\beta_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\beta_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\beta_m & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array}, \tag{2.5}$$

and the following change of the right-hand side vector

$$\widehat{P}_0 = P_0 + \left[-\alpha_1\ -\alpha_2\ \ldots -\alpha_s\ -\beta_1\ -\beta_2\ \ldots -\beta_m\ 0 \right]^T, \tag{2.6}$$

where index k corresponds to the optimal basic variable $\lambda_0^* = \lambda_k^* = 1$. It is easy to show that for matrices B^{-1} and D the following holds:

$$B^{-1} D B^{-1} D = p B^{-1} D, \tag{2.7}$$

where

$$p = -\left(\sum_{t=1}^{s} b_{k,t}^{-1}\alpha_t + \sum_{t=1}^{m} b_{k,s+t}^{-1}\beta_t\right). \tag{2.8}$$

Theorem 2.1. *Let us suppose that DMU_0 is Pareto-Koopmans efficient and that for p in (2.8) $1+p \neq 0$ holds. Then conditions*

$$\frac{1}{1+p}\omega^T D\Gamma_j \geq z_j - c_j, \quad j \text{ an index of nonbasic variables,} \tag{2.9}$$

are sufficient for DMU_0 to preserve efficiency after the simultaneous change of outputs (2.2) and change of inputs (2.3). If $1+p>0$, conditions (2.9) can be written as the following system of inequalities

$$\sum_{t=1}^{s}(b_{k,t}^{-1}\bar{c}_j - \omega_t\Gamma_{kj})\alpha_t + \sum_{t=1}^{m}(b_{k,s+t}^{-1}\bar{c}_j - \omega_{s+t}\Gamma_{kj})\beta_t \geq \bar{c}_j, \tag{2.10}$$
$$j \text{ an index of nonbasic variables,}$$

with $\bar{c}_j = z_j - c_j$. If $1+p<0$ holds, the inequality sign $\geq$ in (2.10) should be changed to $\leq$.

For the proof and details, see Charnes and Neralić [10], Theorem 2 and Remark 3.

Using Theorem 2.1 we can get the following two corollaries:

Corollary 2.2. *Let us suppose that we are interested only in the change (2.2) of outputs of a Pareto-Koopmans efficient DMU_0, which means that we have to put $\beta_1 = \beta_2 = \ldots = \beta_m = 0$ in (2.3), (2.5), (2.6) and (2.8). In that case, if*

$$1+p_1 = 1 - \sum_{t=1}^{s} b_{k,t}^{-1}\alpha_t > 0, \tag{2.11}$$

conditions

$$\sum_{t=1}^{s}(b_{k,t}^{-1}\bar{c}_j - \omega_t\Gamma_{kj})\alpha_t \geq \bar{c}_j, \quad j \text{ an index of nonbasic variables,} \tag{2.12}$$

are sufficient for DMU_0 to preserve efficiency after the change (2.2) of outputs. If $1+p_1<0$ in (2.11) holds, the inequality sign $\geq$ in (2.12) should be changed into $\leq$.

Corollary 2.3. *Let us suppose that we are interested only in the change (2.3) of inputs of a Pareto-Koopmans efficient DMU_0, which means that we have to put $\alpha_1 = \alpha_2 = \ldots = \alpha_s = 0$ in (2.2), (2.5), (2.6) and (2.8). In that case, if*

$$1+p_2 = 1 - \sum_{t=1}^{m} b_{k,s+t}^{-1}\beta_t > 0, \tag{2.13}$$

conditions

$$\sum_{t=1}^{m}(b_{k,s+t}^{-1}\bar{c}_j - \omega_{s+t}\Gamma_{kj})\beta_t \geq \bar{c}_j, \quad j \text{ an index of nonbasic variables,} \tag{2.14}$$

are sufficient for DMU_0 to preserve efficiency after the change (2.3) of inputs. If $1+p_2<0$ in (2.13) holds, the inequality sign $\geq$ in (2.14) should be changed into $\leq$.

2.2. **Proportionate change of outputs.** For fixed inputs, we are interested in the proportionate change (decrease) of all outputs of a Pareto-Koopmans efficient DMU_0

$$\widehat{y}_{r0} = \widehat{\alpha} y_{r0}, \quad 0 < \widehat{\alpha} \leq 1, \quad r = 1, 2, \ldots, s, \tag{2.15}$$

preserving efficiency. We want to find the sufficient conditions for DMU_0 to preserve efficiency and the minimal value $\widehat{\alpha}^\star$ of $\widehat{\alpha}$ for which the efficiency of DMU_0 is preserved after the proportionate change (2.15) of outputs.

Theorem 2.4. *Let us suppose that DMU_0 is Pareto-Koopmans efficient and let (2.11) hold with $\alpha_t = (1-\widehat{\alpha}) y_{t0}, t = 1, 2, \ldots, s$. Let*

$$a_1 = \sum_{t=1}^{s} b_{kt}^{-1} y_{t0}, \quad a_2 = \sum_{t=1}^{s} \omega_t y_{t0}, \tag{2.16}$$

$$d_j = -a_2 \Gamma_{kj} + a_1 \bar{c}_j, \quad j = 1, 2, \ldots, n+s+m+1. \tag{2.17}$$

Conditions

$$\widehat{\alpha} d_j \leq d_j - \bar{c}_j, \quad j \text{ an index of nonbasic variables}, \tag{2.18}$$

are sufficient for DMU_0 to preserve efficiency after the proportionate change (2.15) of outputs.

Proof. If we put

$$\widehat{\alpha} = 1 - \alpha, \quad 0 \leq \alpha < 1, \tag{2.19}$$

and

$$\alpha_r = \alpha y_{r0}, \quad \alpha_r \geq 0, \quad r = 1, 2, \ldots, s, \tag{2.20}$$

we can write (2.15) as

$$\begin{aligned} \widehat{y}_{r0} &= y_{r0} - \alpha y_{r0} \\ &= y_{r0} - \alpha_r > 0, \quad \alpha_r \geq 0, \quad r = 1, 2, \ldots, s. \end{aligned} \tag{2.21}$$

This means that the proportionate change (2.15) of outputs is the special case of the change (2.2) of outputs, with $\alpha_r, \ r = 1, 2, \ldots, s$ in (2.20) and α in (2.19).

Let us suppose that conditions (2.18) are satisfied. Then, using (2.16), (2.17) and (2.19) it is easy to show that conditions (2.18) are equivalent to conditions (2.12). Therefore, in accordance to Corollary 2.2, it follows that conditions (2.12) and also (2.18) are sufficient for DMU_0 to continue to be efficient after the proportionate change (2.15) of outputs, which completes the proof. $\square$

Remark 2.5. For

$$J_1 = \{ j \mid d_j < 0, \ j \text{ an index of nonbasic variables} \}$$

using (2.15) and (2.18) it follows

$$1 - \min\{ \frac{\bar{c}_j}{d_j} \mid j \varepsilon J_1 \} \leq \widehat{\alpha} \leq 1. \tag{2.22}$$

The minimal value $\widehat{\alpha}^\star$ of $\widehat{\alpha}$ for which the efficiency of DMU_0 is preserved after the change (2.15) according to (2.22) is

$$\widehat{\alpha}^\star = 1 - \min\{\frac{\bar{c}_j}{d_j} \mid j \varepsilon J_1\}. \tag{2.23}$$

The maximal percentage of decrease of all outputs preserving the efficiency of DMU_0 after the change (2.15) is $\alpha^\star 100\% = (1-\widehat{\alpha}^\star)100\%$.

Remark 2.6. The value $1-\widehat{\alpha}^\star$ is a measure of the stability of efficiency for an efficient DMU_0. If for an efficient DMU_1 and DMU_2 $1-\widehat{\alpha}_1^\star > 1-\widehat{\alpha}_2^\star$ holds, it can be said that DMU_1 is more stable than DMU_2. In other words, DMU_1 is less sensitive to the proportionate change of outputs preserving efficiency than DMU_2.

2.3. **Proportionate change of inputs.** For fixed outputs let us consider the simultaneous proportionate change (increase) of all inputs

$$\widehat{x}_{i0} = \widehat{\beta} x_{i0}, \quad \widehat{\beta} \geq 1, \quad i = 1,2,\ldots,m, \tag{2.24}$$

of a Pareto-Koopmans efficient DMU_0 preserving efficiency. We are interested in sufficient conditions for DMU_0 to preserve efficiency after the proportionate change (2.24) of inputs. We also want to find the maximal value $\widehat{\beta}^\star$ of $\widehat{\beta}$ for which the efficiency of DMU_0 is preserved after the proportionate change (2.24) of inputs.

Theorem 2.7. *Let us suppose that DMU_0 is Pareto-Koopmans efficient and let (2.13) hold with $\beta_t = (\widehat{\beta}-1)x_{t0}$, $t = 1,2,\ldots,m$. Let*

$$b_1 = \sum_{t=1}^{m} b^{-1}_{k,s+t} x_{t0}, \quad b_2 = \sum_{t=1}^{m} \omega_{s+t} x_{t0}, \tag{2.25}$$

$$e_j = b_1 \bar{c}_j - b_2 \Gamma_{kj}, \quad j = 1,2,\ldots,n+s+m+1. \tag{2.26}$$

Conditions

$$\widehat{\beta} e_j \geq e_j + \widehat{c}_j, \quad j \text{ an index of nonbasic variables}, \tag{2.27}$$

are sufficient for DMU_0 to continue to be efficient after the proportionate change (2.24) of inputs.

Proof. Using the substitutions

$$\widehat{\beta} = 1 + \beta, \quad \beta \geq 0, \tag{2.28}$$

and

$$\beta_i = \beta x_{i0}, \quad \beta_i \geq 0, \quad i = 1,2,\ldots,m, \tag{2.29}$$

we can write (2.24) as

$$\begin{aligned} \widehat{x}_{i0} &= x_{i0} + \beta x_{i0} \\ &= x_{i0} + \beta_i, \quad \beta_i \geq 0, \quad i = 1,2,\ldots,m. \end{aligned} \tag{2.30}$$

This means that the proportionate change (2.24) of inputs is the special case of the change (2.3) of inputs with β_i, $i = 1,2,\ldots,m$ in (2.29) and β in (2.28).

Let us suppose that conditions (2.27) are satisfied. Then using (2.25), (2.26) and (2.28) it follows that conditions (2.27) are equivalent to conditions (2.14). This means that according to Corollary 2.3, conditions (2.14) and also (2.27) are sufficient for DMU_0 to preserve efficiency after the proportionate change (2.24) of inputs which completes the proof. □

Remark 2.8. For

$$J_2 = \{j \mid e_j < 0, \quad j \text{ an index of nonbasic variables}\},$$

it follows from (2.24) and (2.27) that

$$1 \leq \widehat{\beta} \leq 1 + \min\{\frac{\bar{c}_j}{e_j} \mid j \varepsilon J_2\}. \tag{2.31}$$

This means that the maximal value $\widehat{\beta}^\star$ of $\widehat{\beta}$ for which the efficiency of DMU_0 is preserved after the change (2.24) according to (2.31) is

$$\widehat{\beta}^\star = 1 + \min\{\frac{\bar{c}_j}{e_j} \mid j \varepsilon J_2\}. \tag{2.32}$$

The maximal percentage of increase of all inputs preserving the efficiency of DMU_0 after the change (2.24) is $\beta^\star$ 100% = $(\widehat{\beta}^\star - 1)100\%$.

Remark 2.9. The value $\widehat{\beta}^\star - 1$ is a measure of stability of efficiency for an efficient DMU_0. If for efficient DMU_1 and DMU_2 $\widehat{\beta}_1^\star - 1 > \widehat{\beta}_2^\star - 1$ holds, it can be said that DMU_1 is more stable than DMU_2. In other words, DMU_1 is less sensitive to the proportionate change of outputs preserving efficiency than DMU_2.

2.4. **Simultaneous proportionate change of outputs and of inputs.** Let us consider the simultaneous proportionate change (decrease) (2.15)

$$\widehat{y}_{r0} = \widehat{\alpha} y_{r0}, \quad 0 < \widehat{\alpha} \leq 1, \quad r = 1, 2, \ldots, s,$$

of outputs and proportionate change (increase) (2.24)

$$\widehat{x}_{i0} = \widehat{\beta} x_{i0}, \quad \widehat{\beta} \geq 1, \quad i = 1, 2, \ldots, m,$$

of inputs. We are interested in sufficient conditions for DMU_0 to preserve efficiency after the simultaneous changes (2.15) and (2.24). We are also interested in the region $\widehat{A}_0$ which is the solution set of the corresponding system of inequalities in $\widehat{\alpha}$ and $\widehat{\beta}$ in the coordinate system $\widehat{\alpha}\widehat{O}\widehat{\beta}$. The area of that region is a measure of stability of efficiency for DMU_0.

Theorem 2.10. *Let us suppose that DMU_0 is Pareto-Koopmans efficient. Let for p in (2.8) with $\alpha_t = (1 - \widehat{\alpha}) y_{t0}$, $t = 1, 2, \ldots, s$ and $\beta_t = (\widehat{\beta} - 1) x_{t0}$, $t = 1, 2, \ldots, m$ holds*

$$1 + p = 1 - \Big[(1 - \widehat{\alpha}) a_1 + (\widehat{\beta} - 1) b_1\Big] > 0, \tag{2.33}$$

with a_1 in (2.16) and b_1 in (2.25). Then conditions

$$(1 - \widehat{\alpha}) d_j + (\widehat{\beta} - 1) e_j \geq \bar{c}_j, \quad j \text{ an index of nonbasic variables} \tag{2.34}$$

with d_j in (2.17), e_j in (2.26), $\bar{c}_j = z_j - c_j$, are sufficient for DMU_0 to continue to be efficient after the simultaneous proportionate change (2.15) of outputs and proportionate change (2.24) of inputs.

Proof. Using substitutions and arguments in the proof of Theorem 3.1 and in the proof of Theorem 3.5, it is easy to show that for the special case of the simultaneous proportionate change (2.15) of outputs and the proportionate change (2.24) of inputs, conditions (2.34) are equivalent to conditions (2.10) in Theorem 2.1. This means that conditions (2.34) are sufficient for DMU_0 to preserve efficiency after changes (2.15) and (2.24) and completes the proof. $\square$

Remark 2.11. The system of inequalities (2.34) together with conditions (2.15), (2.24) and (2.33) for $\widehat{\alpha}$ and $\widehat{\beta}$ gives the region $\widehat{A}_0$ in the plane with the coordinate system $\widehat{\alpha}\widehat{O}\widehat{\beta}$. For each point $(\widehat{\alpha}, \widehat{\beta})$ in the region $\widehat{A}_0$, the efficiency of DMU_0 will be preserved after the simultaneous proportionate change (2.15) of outputs and the proportionate change (2.24) of inputs.

Remark 2.12. The area of the region $\widehat{A}_0$ is a measure of stability of efficiency for an efficient DMU_0. If for an efficient DMU_1 and DMU_2 $\widehat{A}_1 > \widehat{A}_2$ holds, it can be said that DMU_1 is more stable than DMU_2. In other words, DMU_1 is less sensitive to the simultaneous proportionate change of outputs and the proportionate change of inputs preserving efficiency than DMU_2. The measure of stability of efficiency among efficient DMUs can also be based on the proportionate change of inputs (or outputs) (see Remark 2.6 and Remark 2.9).

2.5. **Proportionate change of inputs and outputs with different coefficients of proportionality.** Let us consider the simultaneous proportionate change (increase) of all inputs

$$\widehat{x}_{i0} = \widehat{\beta}_i x_{i0}, \quad \widehat{\beta}_i \geq 1, \quad i = 1, 2, \ldots, m, \tag{2.35}$$

and the proportionate change (decrease) of all outputs

$$\widehat{y}_{r0} = \widehat{\alpha}_r y_{r0}, \quad 0 < \widehat{\alpha}_r \leq 1, \quad r = 1, 2, \ldots, s, \tag{2.36}$$

of an efficient DMU_0 preserving efficiency. We are interested in the sufficient conditions for DMU_0 to preserve efficiency after the simultaneous changes (2.35) and (2.36).

Theorem 2.13. *Let us suppose that DMU_0 is Pareto-Koopmans efficient according to the additive model (2.1). Let*

$$1 + p = 1 - \sum_{t=1}^{s} b_{kt}^{-1} y_{t0}(1 - \widehat{\alpha}_t) - \sum_{t=1}^{m} b_{k,s+t}^{-1} x_{t0}(\widehat{\beta}_t - 1) > 0. \tag{2.37}$$

Then the conditions

$$\left[-\sum_{t=1}^{s} \omega_t y_{t0}(1 - \widehat{\alpha}_t) - \sum_{t=1}^{m} \omega_{s+t} x_{t0}(\widehat{\beta}_t - 1) \right] \Gamma_{kj} \quad + \tag{2.38}$$

$$+ \left[\sum_{t=1}^{s} b_{kt}^{-1} y_{t0}(1 - \widehat{\alpha}_t) + \sum_{t=1}^{m} b_{k,s+t}^{-1} x_{t0}(\widehat{\beta}_t - 1) \right] \bar{c}_j \quad \geq \quad \bar{c}_j, \tag{2.39}$$

j an index of nonbasic variables,

are sufficient for DMU_0 to preserve efficiency after the changes (2.35) and (2.36).

Proof. The proof, based on Theorem 2.1, is omitted because it is similar to the proof of Theorem 2 in Charnes and Neralić [11]. □

2.6. **Simultaneous proportionate change of discretionary outputs and inputs in the additive model.** Let us consider the simultaneous proportionate change (decrease) of discretionary outputs

$$\widehat{y}_{r0} = \widehat{\alpha} y_{r0}, \quad 0 < \widehat{\alpha} \leq 1, \quad r = 1, 2, \ldots, \bar{s}, \tag{2.40}$$

with no change of non-discretionary outputs

$$\widehat{y}_{r0} = y_{r0}, \quad r = \bar{s}+1, \bar{s}+2, \ldots, s, \tag{2.41}$$

and proportionate change (increase) of discretionary inputs

$$\widehat{x}_{i0} = \widehat{\beta} x_{i0}, \quad \widehat{\beta} \geq 1, \quad i = 1, 2, \ldots, \overline{m}, \tag{2.42}$$

with no change of non-discretionary inputs

$$\widehat{x}_{i0} = x_{i0}, \quad i = \overline{m}+1, \overline{m}+2, \ldots, m, \tag{2.43}$$

of Pareto-Koopmans efficient DMU_0 preserving efficiency. We are interested in sufficient conditions for DMU_0 to preserve efficiency after the simultaneous proportionate change (2.40) of discretionary outputs and proportionate change (2.42) of discretionary inputs.

Theorem 2.14. *Let us suppose that DMU_0 is Pareto-Koopmans efficient. Let*

$$a_1 = \sum_{t=1}^{\bar{s}} b_{kt}^{-1} y_{t0}, \quad a_2 = \sum_{t=1}^{\bar{s}} \omega_t y_{t0}, \tag{2.44}$$

$$b_1 = \sum_{t=1}^{\overline{m}} b_{k,s+t}^{-1} x_{t0}, \quad b_2 = \sum_{t=1}^{\overline{m}} \omega_{s+t} x_{t0}, \tag{2.45}$$

$$d_j = -a_2 \Gamma_{kj} + a_1 \bar{c}_j, \quad e_j = b_1 \bar{c}_j - b_2 \Gamma_{kj}, \quad j = 1, 2, \ldots, n+s+m+1, \tag{2.46}$$

with $\bar{c}_j = z_j - c_j$. Let for p in (2.9) with $\alpha_t = (1-\widehat{\alpha}) y_{t0}, t = 1, 2, \ldots, \bar{s}$ and $\beta_t = (\widehat{\beta}-1) x_{t0},\ t = 1, 2, \ldots, \overline{m}$ holds

$$1 + p = 1 - \left[(1-\widehat{\alpha}) a_1 + (\widehat{\beta}-1) b_1\right] > 0. \tag{2.47}$$

Then, conditions

$$(1-\widehat{\alpha}) d_j + (\widehat{\beta}-1) e_j \geq \bar{c}_j, \quad j \text{ an index of nonbasic variables}, \tag{2.48}$$

are sufficient for DMU_0 to continue to be efficient after the simultaneous proportionate change (2.40) of discretionary outputs and the proportionate change (2.42) of discretionary inputs.

Proof. The proof is omitted because it is similar to the proof of Theorem 2 in Charnes and Neralić [11] for the CCR model. □

Remark 2.15. The system of inequalities (2.48) together with conditions (2.40), (2.42) and (2.47) for $\widehat{\alpha}$ and $\widehat{\beta}$ gives the region $\widehat{A}_0$ in the plane with the coordinate system $\widehat{\alpha}\widehat{O}\widehat{\beta}$. For each point $(\widehat{\alpha}, \widehat{\beta})$ in the region $\widehat{A}_0$, the efficiency of DMU_0 will be preserved after the simultaneous proportionate change (2.40) of discretionary outputs and the proportionate change (2.42) of discretionary inputs. The area of the corresponding region about the efficient point DMU_0 within which perturbations (2.40) and (2.42) keep it efficient is a measure of the stability of efficiency at DMU_0.

Remark 2.16. If we are interested in the proportionate change (2.40) of discretionary outputs of DMU_0 only preserving its efficiency, we can get from Theorem 2.14 sufficient conditions for that case. Using

$$J_1 = \{j \mid d_j < 0,\ j \textit{ an index of nonbasic variables}\}$$

it is easy to show that the minimal value $\widehat{\alpha}^\star$ of $\widehat{\alpha}$ for which the efficiency of DMU_0 is preserved after the change (2.40) is

$$\widehat{\alpha}^\star = 1 - \min\{\frac{\bar{c}_j}{d_j} \mid j \varepsilon J_1\}.$$

The maximal percentage of decrease of all discretionary outputs preserving efficiency of DMU_0 after the change (2.40) is $\alpha^\star 100\% = (1 - \widehat{\alpha}^\star)100\%$. For each $\widehat{\alpha}$ such that $\widehat{\alpha}^\star \le \widehat{\alpha} \le 1$, the efficiency of DMU_0 will be preserved after the proportionate change (2.40) of discretionary outputs. For the corresponding outputs $\widehat{y}^\star_{r0} \le \widehat{y}_{r0} \le y_{r0}, r = 1, 2, \ldots, \bar{s}$ holds. The volume of the region

$$\prod_{r=1}^{\bar{s}} [\widehat{y}^\star_{r0}, y_{r0}]$$

is a measure of stability of efficiency of DMU_0 at the proportionate change (2.40) of discretionary outputs.

A similar result holds for the case of the proportionate change (2.42) of discretionary inputs only.

3. Sensitivity analysis of the proportionate change of outputs and/or of inputs in the CCR model

3.1. **Preliminaries.** Let us suppose that there are n Decision Making Units (DMUs) with m inputs and s outputs. Let x_{ij} be the observed amount of the ith type of input of the jth DMU ($x_{ij} > 0$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$) and let y_{rj} be the observed amount of output of the rth type for the jth DMU ($y_{rj} > 0$, $r = 1, 2, \ldots, s, j = 1, 2, \ldots, n$). Let Y_j, X_j be the observed vectors of outputs and inputs of the DMU_j, respectively, $j = 1, 2, \ldots, n$. Let e be the column vector of ones and let T as a superscript denote the transpose. In order to see if the $\mathrm{DMU}_{j0} = \mathrm{DMU}_0$ is efficient according to the CCR ratio model, the following linear programming problem should be solved:

$$\min 0\lambda_1 + \cdots + 0\lambda_0 + \cdots + 0\lambda_n - \varepsilon e^T s^+ - \varepsilon e^T s^- + \theta$$

subject to

$$
(3.1)\qquad
\begin{array}{rcccrcl}
Y_1\lambda_1+\cdots+Y_0\lambda_0+\cdots+Y_n\lambda_n & - & s^+ & & & = & Y_0\\
-X_1\lambda_1-\cdots-X_0\lambda_0-\cdots-X_n\lambda_n & - & s^- & + & X_0\theta & = & 0\\
\lambda_1,\ldots,\lambda_0,\ldots,\lambda_n, & s^+, & s^- & & & \geq & 0,
\end{array}
$$

with $Y_0=Y_{j_0}, X_0=X_{j_0}, \lambda_0=\lambda_{j_0}$ and θ unconstrained. The symbol ε represents the infinitesimal we use to generate the non-Archimedean ordered extension field we shall use. In this extension field ε is less than every positive number in our base field, but greater than zero. DMU_0 is DEA efficient if and only if for the optimal solution $(\lambda^*, s^{+*}, s^{-*}, \theta^*)$ of the linear programming problem (3.1) both of the following are satisfied (for details, see Charnes and Cooper [3]):

$$
(3.2)\qquad
\begin{array}{ccccl}
\min\ \theta & = & \theta^* & = & 1\\
s^{+*} & = & s^{-*} & = & 0, \text{ in all alternative optima.}
\end{array}
$$

We are interested in variations of all inputs and all outputs of an efficient DMU_0 preserving efficiency. A decrease of any input cannot worsen an already achieved efficiency rating. Downward variations of inputs are not possible in the efficiency rating for an efficient DMU_0 . Hence we can restrict attention to upward variations of inputs of an efficient DMU_0 which can be written as

$$
(3.3)\qquad \widehat{x}_{i0}=x_{i0}+\beta_i,\quad \beta_i\geq 0,\qquad i=1,2,\ldots,m.
$$

Similarly, an increase of any output cannot worsen an already achieved efficiency rating. Upward variations of outputs are not possible in the efficiency rating for an efficient DMU_0. Hence we can restrict attention to downward variations of outputs which can be written as

$$
(3.4)\qquad \widehat{y}_{r0}=y_{r0}-\alpha_r>0,\quad \alpha_r\geq 0,\quad r=1,2,\ldots,s.
$$

For an efficient DMU_0 because of (3.2), vectors $[\ Y_0\quad -X_0\ ]^T$ and $[\ 0\quad X_0\ ]^T$ must occur in some optimal basis, which means that there is a basic optimal solution to (3.1) with $\lambda_0^*=1$ and $\theta^*=1$. Changes (3.3) and (3.4) are then accompanied by alterations in the inverse B^{-1} of the optimal basis matrix

$$
(3.5)\qquad B=\begin{bmatrix} Y_B & -I_B^+ & 0 & 0\\ -X_B & 0 & -I_B^- & X_0\end{bmatrix},
$$

which corresponds to the optimal solution $(\lambda^*, s^{+*}, s^{-*}, \theta^*)$ of (2.1) with $\lambda_0^*=1$ and $\theta^*=1$. Let

$$
B^{-1}=\left[b_{ij}^{-1}\right],\quad i,\ j=1,2,\ldots,s+m,
$$

be the inverse of the optimal basis B in (3.5). Let $P_j,\ j=1,2,\ldots,n{+}s{+}m{+}1$ be the columns of the matrix and let P_0 be the right-hand side of the linear programming problem (3.1). The simultaneous change of outputs (3.3) and inputs (3.4) leads to the following change of the optimal basis matrix B

$$
(3.6)\qquad \widehat{B}=B+\triangle B
$$

with

$$\begin{array}{cc} & \begin{array}{ccccccc} & & & k & & & s+m \\ & & & \downarrow & & & \downarrow \end{array} \\ \triangle B = & \begin{pmatrix} 0 & \cdots & 0 & -\alpha_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\alpha_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\alpha_s & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\beta_1 & 0 & \cdots & \beta_1 \\ 0 & \cdots & 0 & -\beta_2 & 0 & \cdots & \beta_2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\beta_m & 0 & \cdots & \beta_m \end{pmatrix} \end{array} \tag{3.7}$$

and the following change of the right-hand side vector

$$\widehat{P}_0 = P_0 + [-\alpha_1 \; - \alpha_2 \; \ldots - \alpha_s \; 0 \; \ldots 0 \;]^T , \tag{3.8}$$

where indexes k and $s+m$ correspond to the optimal basic variables $\lambda_0^* = 1$ and $\theta^* = 1$ respectively. Using matrices

$$U_{(s+m)\times 2} = \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 \\ \vdots & \vdots \\ \alpha_s & \alpha_s \\ \beta_1 & 0 \\ \beta_2 & 0 \\ \vdots & \vdots \\ \beta_m & 0 \end{bmatrix} \tag{3.9}$$

and

$$\begin{array}{cc} & \begin{array}{ccccccccc} & & & k & & & & & s+m \\ & & & \downarrow & & & & & \downarrow \end{array} \\ V^T_{2\times(s+m)} = & \begin{pmatrix} 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \end{array} \tag{3.10}$$

we can write the perturbation matrix (3.7) as $\triangle B = UV^T$. Let us use the abbreviation

$$M = I + V^T B^{-1} U,$$

where matrix M is nonsingular, with

$$\begin{aligned} det\ M \;\; &= \;\; 1 - \sum_{t=1}^{s} b^{-1}_{k,t}\alpha_t + \sum_{t=1}^{m}(-b^{-1}_{k,s+t} + b^{-1}_{s+m,s+t})\beta_t + \\ &+ \;\; (\sum_{t=1}^{s} b^{-1}_{s+m,t}\alpha_t)(\sum_{t=1}^{m} b^{-1}_{k,s+t}\beta_t) - (\sum_{t=1}^{s} b^{-1}_{k,t}\alpha_t)(\sum_{t=1}^{m} b^{-1}_{s+m,s+t}\beta_t), \end{aligned} \tag{3.11}$$

and

$$D = UM^{-1}V^T. \tag{3.12}$$

Theorem 3.1. *Conditions*

$$\omega^T D\Gamma_j \geq z_j - c_j, \quad j \text{ an index of nonbasic variables,} \tag{3.13}$$

are sufficient for DMU$_0$ to be efficient after the simultaneous changes of inputs (3.3) and of outputs (3.4). If $detM > 0$*, conditions (3.13) can be written in the following way*

$$\gamma_k \Gamma_{kj} + \gamma_{s+m}\Gamma_{s+m,j} \geq (z_j - c_j) det\ M, \tag{3.14}$$

with

$$\gamma_k = -(1 + \sum_{t=1}^{m} b^{-1}_{s+m,s+t}\beta_t)(\sum_{t=1}^{s}\omega_t\alpha_t) + (-1 + \sum_{t=1}^{s} b^{-1}_{s+m,t}\alpha_t)(\sum_{t=1}^{m}\omega_{s+t}\beta_t),$$

and

$$\gamma_{s+m} = (\sum_{t=1}^{m} b^{-1}_{k,s+t}\beta_t)(\sum_{t=1}^{s}\omega_t\alpha_t) + (1 - \sum_{t=1}^{s} b^{-1}_{k,t}\alpha_t)(\sum_{t=1}^{m}\omega_{s+t}\beta_t).$$

For the proof and details see, Charnes and Neralić [8], Theorem 1 and Remark 1.

3.2. **Simultaneous proportionate change of inputs and outputs in the CCR model.** Let us consider the simultaneous proportionate change (increase) of all inputs

$$\widehat{x}_{i0} = \widehat{\beta} x_{i0}, \quad \widehat{\beta} \geq 1, \quad i = 1, 2, \ldots, m, \tag{3.15}$$

and the proportionate change (decrease) of all outputs

$$\widehat{y}_{r0} = \widehat{\alpha} y_{r0}, \quad 0 < \widehat{\alpha} \leq 1, \quad r = 1, 2, \ldots, s, \tag{3.16}$$

of an efficient DMU$_0$ preserving efficiency. We are interested in sufficient conditions for DMU$_0$ to preserve efficiency after the simultaneous changes (3.15) and (3.16).

Theorem 3.2. *Let us suppose that DMU$_0$ is efficient and let*

$$det\ M = 1 - a_1(1-\widehat{\alpha}) + (-b_1 + b_2)(\widehat{\beta} - 1) + (a_2 b_1 - a_1 b_2)(1-\widehat{\alpha})(\widehat{\beta} - 1) > 0, \tag{3.17}$$

with

$$a_1 = \sum_{t=1}^{s} b^{-1}_{kt} y_{t0}, \quad a_2 = \sum_{t=1}^{s} b^{-1}_{s+m,t} y_{t0}, \quad b_1 = \sum_{t=1}^{m} b^{-1}_{k,s+t} x_{t0}, \quad b_2 = \sum_{t=1}^{m} b^{-1}_{s+m,s+t} x_{t0}. \tag{3.18}$$

Let

$$a_3 = \sum_{t=1}^{s} \omega_t y_{t0}, \quad b_3 = \sum_{t=1}^{m} \omega_{s+t} x_{t0}, \tag{3.19}$$

$$d_j = -a_3 \Gamma_{kj} + a_1 \bar{c}_j, \quad e_j = -b_3(\Gamma_{kj} - \Gamma_{s+m,j}) - (-b_1 + b_2)\bar{c}_j, \tag{3.20}$$

$$f_j = (a_2 b_3 - a_3 b_2)\Gamma_{kj} + (a_3 b_1 - a_1 b_3)\Gamma_{s+m,j} - (a_2 b_1 - a_1 b_2)\bar{c}_j, \tag{3.21}$$

$$j = 1, 2, \ldots, n + s + m + 1,$$

with $\bar{c}_j = z_j - c_j$*. Then, the conditions*

$$d_j(1-\widehat{\alpha}) + e_j(\widehat{\beta} - 1) + f_j(1-\widehat{\alpha})(\widehat{\beta} - 1) \geq \bar{c}_j, \tag{3.22}$$

j an index of nonbasic variables,

are sufficient for DMU_0 *to preserve efficiency after the simultaneous proportionate changes of inputs (3.15) and of outputs (3.16).*

Proof. For the proof and details, see Charnes and Neralić [11]. □

Remark 3.3. The system of inequalities (3.22) together with conditions (3.15), (3.16) and (3.17) for $\widehat{\alpha}$ and $\widehat{\beta}$ gives the region $\widehat{A}_0$ in the plane with the coordinate system $\widehat{\alpha}\widehat{O}\widehat{\beta}$. For each point $(\widehat{\alpha}, \widehat{\beta})$ in the region $\widehat{A}_0$, the efficiency of DMU_0 will be preserved after the simultaneous proportionate changes of inputs (3.15) and outputs (3.16).

Remark 3.4. The area of the region $\widehat{A}_0$ is a measure of stability of efficiency for an efficient DMU_0. For example, if for efficient DMU_1 and DMU_2 $\widehat{A}_1 > \widehat{A}_2$ holds, it can be said that "DMU_1 is relatively more stable than DMU_2" because DMU_1 is less sensitive to the simultaneous proportionate change of inputs and outputs preserving efficiency than DMU_2. The measure of stability of efficiency among efficient DMUs can also be based on the proportionate change of inputs (or outputs) as it was suggested in Banker and Gifford [1] and used in Charnes and Neralić [9].

3.3. **Sensitivity analysis of the proportionate change of a subset of inputs and/or of outputs.** We are interested in the proportionate change of a subset of inputs and/or of a subset of outputs of an efficient DMU_0 preserving efficiency. An increase of any output cannot worsen an already achieved efficiency rating. Upward variations of outputs are not possible in the efficiency rating for an efficient DMU_0. Hence, without loss of generality, we can focus attention on the proportionate decrease of the subset of the first $\bar{s}$ ($\bar{s} < s$) outputs which can be written as

$$\widehat{y}_{r0} = \widehat{\alpha} y_{r0}, \quad 0 < \widehat{\alpha} \leq 1, \quad r = 1, 2, \ldots, \bar{s}, \tag{3.23}$$

with the last $s - \bar{s}$ outputs fixed

$$\widehat{y}_{r0} = y_{r0}, \quad r = \bar{s}+1, \bar{s}+2, \ldots, s. \tag{3.24}$$

Similarly, a decrease of any input cannot worsen an already achieved efficiency rating. Downward variations of inputs are not possible in the efficiency rating for an efficient DMU_0 . Hence we can focus attention on the upward variations of inputs of an efficient DMU_0. Without loss of generality, let us consider the proportionate increase of the subset of the first $\overline{m}$ ($\overline{m} < m$) inputs, with the last $m - \overline{m}$ inputs fixed. It can be written as

$$\widehat{x}_{i0} = \widehat{\beta} x_{i0}, \quad \widehat{\beta} \geq 1, \quad i = 1, 2, \ldots, \overline{m}, \tag{3.25}$$

and

$$\widehat{x}_{i0} = x_{i0}, \quad i = \overline{m}+1, \overline{m}+2, \ldots, m. \tag{3.26}$$

Let us introduce the following substitution

$$\widehat{\alpha} = 1 - \alpha, \quad 0 \leq \alpha < 1. \tag{3.27}$$

Using (3.27) in (3.23), we have

$$\widehat{y}_{r0} = y_{r0} - \alpha_r > 0, \quad r = 1, 2, \ldots, \bar{s}, \tag{3.28}$$

with

$$\alpha_r = \alpha y_{r0}, \quad \alpha_r \geq 0, \quad r = 1, 2, \ldots, \bar{s}. \tag{3.29}$$

Because of (3.24), we have

$$\alpha_r = 0, \quad r = \bar{s}+1, \bar{s}+2, \ldots, s. \tag{3.30}$$

Let us also introduce the substitution

$$\widehat{\beta} = 1 + \beta, \quad \beta \geq 0. \tag{3.31}$$

Using (3.31) in (3.25) we have

$$\widehat{x}_{i0} = x_{i0} + \beta_i, \quad i = 1, 2, \ldots, \overline{m}, \tag{3.32}$$

with

$$\beta_i = \beta x_{i0}, \quad \beta_i \geq 0, \quad i = 1, 2, \ldots, \overline{m}. \tag{3.33}$$

Because of (3.26), we have

$$\beta_i = 0, \quad i = \overline{m}+1, \overline{m}+2, \ldots, m. \tag{3.34}$$

This means that the proportionate change of a subset of outputs (3.23), with the other outputs fixed (3.24), can be considered as the additive change (3.28), with α_r in (3.29) and (3.30). Similarly, the proportionate change of a subset of inputs (3.25), with the other inputs fixed (3.26), can be considered as the additive change (3.32) with β_i in (3.33) and (3.34). Therefore, we will consider the additive changes (3.28) of outputs together with (3.30) and/or additive changes (3.32) of inputs together with (3.34).

For an efficient DMU_0 because of (3.2), vectors $[\ Y_0 \quad -X_0\ ]^T$ and $[\ 0 \quad X_0\ ]^T$ must occur in some optimal basis, which means that there is a basic optimal solution to (3.1) with $\lambda_0^* = 1$ and $\theta^* = 1$. Similarly as in Charnes and Neralić [8], simultaneous changes (3.28), together with (3.30), and changes (3.32), together with (3.34), are then accompanied by alterations in the inverse B^{-1} of the optimal basis matrix

$$B = \begin{bmatrix} Y_B & -I_B^+ & 0 & 0 \\ -X_B & 0 & -I_B^- & X_0 \end{bmatrix}, \tag{3.35}$$

which corresponds to the optimal solution $(\lambda^*, s^{+*}, s^{-*}, \theta^*)$ of (3.1) with $\lambda_0^* = 1$ and $\theta^* = 1$. Let

$$B^{-1} = \left[b_{ij}^{-1}\right], \quad i,\ j = 1, 2, \ldots, s+m,$$

be the inverse of the optimal basis B in (3.35). Let $P_j,\ j = 1, 2, \ldots, n+s+m+1$ be the columns of the matrix and let P_0 be the right-hand side of the linear programming problem (3.1).

The simultaneous change of outputs (3.28) together with (3.30) and of inputs (3.32) together with (3.34) leads to the following change of the optimal basis matrix B

$$\widehat{B} = B + \triangle B \tag{3.36}$$

with

$$
\begin{array}{c}
\begin{array}{ccccccc} & & & k & & & s+m \\ & & & \downarrow & & & \downarrow \end{array} \\
\triangle B = \begin{pmatrix}
0 & \cdots & 0 & -\alpha_1 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & -\alpha_{\bar{s}} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & -\beta_1 & 0 & \cdots & \beta_1 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & -\beta_{\overline{m}} & 0 & \cdots & \beta_{\overline{m}} \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\tag{3.37}
$$

and the following change of the right-hand side vector

$$\widehat{P}_0 = P_0 + [-\alpha_1 \ -\alpha_2 \ \ldots - \alpha_{\bar{s}} \ 0 \ \ldots 0 \ ]^T , \tag{3.38}$$

where indexes k and $s+m$ correspond to the optimal basic variables $\lambda_0^* = 1$ and $\theta^* = 1$ respectively. Similarly as in subsection 3.1 above we can use the corresponding matrices $U_{(s+m)\times 2}$ and $V^T_{2\times(s+m)}$ in order to write the perturbation matrix (3.37) as

$$\triangle B = UV^T. \tag{3.39}$$

As in Charnes and Neralić [8], because of (3.36) and (3.39) we can use the Sherman-Morrison-Woodbury formula (see, for example, Golub and Loan [15], p. 3) to get the following perturbed basis inverse

$$\begin{aligned} (\widehat{B})^{-1} &= (B + UV^T)^{-1} \\ &= B^{-1} - B^{-1}U(I + V^T B^{-1}U)^{-1}V^T B^{-1}. \end{aligned} \tag{3.40}$$

Using the abbreviation

$$D = U(I + V^T B^{-1}U)^{-1}V^T \tag{3.41}$$

we can write (3.40) as

$$\begin{aligned} (\widehat{B})^{-1} &= B^{-1} - B^{-1}DB^{-1} \\ &= B^{-1}(I - DB^{-1}) \\ &= (I - B^{-1}D)B^{-1}. \end{aligned} \tag{3.42}$$

Also, using matrices U and V^T we can get $V^T B^{-1}U$ and

$$M = I + V^T B^{-1}U \tag{3.43}$$

with $\det M$, M^{-1} and matrix

$$D = UM^{-1}V^T \tag{3.44}$$

with elements

$$d_{t,k} = -\frac{1}{\det M}(1+\sum_{t=1}^{\overline{m}} b^{-1}_{s+m,s+t}\beta_t)\alpha_t,\ t=1,2,\ldots,\bar{s}, \tag{3.45}$$

$$d_{s+t,k} = \frac{1}{\det M}(-1+\sum_{t=1}^{\bar{s}} b^{-1}_{s+m,t}\alpha_t)\beta_t,\ t=1,2,\ldots,\overline{m}, \tag{3.46}$$

$$d_{t,s+m} = \frac{1}{\det M}(\sum_{t=1}^{\overline{m}} b^{-1}_{k,s+t}\beta_t)\alpha_t,\ t=1,2,\ldots,\bar{s}, \tag{3.47}$$

$$d_{s+t,s+m} = \frac{1}{\det M}(1-\sum_{t=1}^{\bar{s}} b^{-1}_{k,t}\alpha_t)\beta_t,\ t=1,2,\ldots,\overline{m}. \tag{3.48}$$

Now we can prove the following

Theorem 3.5. *Let us suppose that DMU_0 is efficient. Conditions*

$$\omega^T D\Gamma_j \geq z_j - c_j, \quad j \text{ an index of nonbasic variables,} \tag{3.49}$$

are sufficient for DMU_0 to continue to be efficient after the simultaneous proportionate changes of a subset of outputs (3.23) and of a subset of inputs (3.25).

Proof. The proof is omitted because it is similar to the proof of Theorem 1 in Charnes and Neralić [8]. (See also the proof of Theorem 2.1 in Charnes and Neralić [7].) □

Remark 3.6. In conditions (3.49) there is $\det M$ in the denominator of elements of matrix D. Consequently in general, there are two possibilities, one for the case of $\det M > 0$ and the other for the case $\det M < 0$. In either case, the condition on $\det M$, together with constraints (3.28) - (3.30) and (3.32) - (3.34), should be added to conditions (3.49). The solution set of the corresponding system of inequalities S_{j0} will be a set of points $(\alpha_1,\ldots,\alpha_{\bar{s}},0,\ldots,0,\beta_1,\ldots,\beta_{\overline{m}},0,\ldots,0)$ in $\mathbb{R}^{s\times m}$. Because of (3.29) and (3.33), we can get the corresponding system of inequalities in α,β and the solution set $\bar{S}_{j0}$. Using substitutions (3.27) and (3.31) we can also get the corresponding system of inequalities in $\widehat{\alpha},\widehat{\beta}$ and the solution set S^*_{j0}. For all points $(\widehat{\alpha},\widehat{\beta})$ in the solution set S^*_{j0} after the changes of outputs according to (3.23), (3.24) and the changes of inputs according to (3.25) and (3.26), the efficiency of DMU_0 will be preserved. The solution set S^*_{j0} gives the region A^*_{j0} in the plane with the coordinate system $\widehat{\alpha}\widehat{O}\widehat{\beta}$.

Remark 3.7. Using points $(\widehat{\alpha},\widehat{\beta})$ from the set S^*_{j0} in (3.23) and (3.25) with (3.24), (3.26) we can get the corresponding *region of efficiency* R_{j0} around DMU_{j0}. The volume of the region of efficiency around the efficient point, within which perturbations (3.23) and (3.25) keep it efficient, is an important property of the (empirical) efficient production function at this point. It is a measure of stability of efficiency at that point. If $R_1 > R_2$ holds for an efficient DMU_1 and DMU_2, this means that DMU_1 is more stable that DMU_2 in preserving efficiency at the simultaneous proportionate changes (3.23)and (3.25).

For the case with $\det M > 0$ (see also Theorem 2 in Charnes and Neralić [11]), it is easy to get from Theorem 3.2 the following

Corollary 3.8. *Let us suppose that DMU_0 is efficient and let*

$$\det\ M = 1 - a_1(1-\widehat{\alpha}) + (-b_1+b_2)(\widehat{\beta}-1) + (a_2b_1 - a_1b_2)(1-\widehat{\alpha})(\widehat{\beta}-1) > 0, \tag{3.50}$$

with

$$a_1 = \sum_{t=1}^{\bar{s}} b_{kt}^{-1} y_{t0}, \quad a_2 = \sum_{t=1}^{\bar{s}} b_{s+m,t}^{-1} y_{t0}, \quad b_1 = \sum_{t=1}^{\overline{m}} b_{k,s+t}^{-1} x_{t0}, \quad b_2 = \sum_{t=1}^{\overline{m}} b_{s+m,s+t}^{-1} x_{t0}. \tag{3.51}$$

Let

$$a_3 = \sum_{t=1}^{\bar{s}} \omega_t y_{t0}, \quad b_3 = \sum_{t=1}^{\overline{m}} \omega_{s+t} x_{t0}, \tag{3.52}$$

$$d_j = -a_3\Gamma_{kj} + a_1\bar{c}_j, \quad e_j = -b_3(\Gamma_{kj} - \Gamma_{s+m,j}) - (-b_1+b_2)\bar{c}_j, \tag{3.53}$$

$$f_j = (a_2b_3 - a_3b_2)\Gamma_{kj} + (a_3b_1 - a_1b_3)\Gamma_{s+m,j} - (a_2b_1 - a_1b_2)\bar{c}_j, \tag{3.54}$$
$$j = 1, 2, \ldots, n+s+m+1,$$

with $\bar{c}_j = z_j - c_j$. Then, the conditions

$$d_j(1-\widehat{\alpha}) + e_j(\widehat{\beta}-1) + f_j(1-\widehat{\alpha})(\widehat{\beta}-1) \geq \bar{c}_j, \tag{3.55}$$
$$j \text{ an index of nonbasic variables},$$

are sufficient for DMU_0 to preserve efficiency after the simultaneous proportionate changes of a subset of outputs (3.23) and of a subset of inputs (3.25).

For fixed inputs, we can consider the proportionate change (decrease) of a subset of outputs (3.23) with the other outputs fixed (3.24). In that case, because of $\beta_1 = \beta_2 = \ldots = \beta_{\overline{m}} = 0$ in (3.32) and of (3.34), it is easy to get the corresponding matrix D_1 from the matrix D with elements in (3.45) - (3.48). With the matrix D_1 instead of matrix D from Theorem 3.2, we have the following

Corollary 3.9. *Conditions*

$$\omega^T D_1 \Gamma_j \geq z_j - c_j, \quad j \text{ an index of nonbasic variables}, \tag{3.56}$$

are sufficient for DMU_0 to be efficient after the proportionate changes of a subset of outputs (3.23) with the other outputs and all inputs fixed.

For fixed outputs, we can consider the proportionate change (increase) of a subset of inputs (3.25) with the other inputs fixed (3.26). In that case, because of $\alpha_1 = \alpha_2 = \ldots = \alpha_{\bar{s}} = 0$ in (3.28) and of (3.30), it is easy to get the corresponding matrix D_2 from the matrix D with elements in (3.45) - (3.48). With the matrix D_2 instead of matrix D from Theorem 3.2, we have the following

Corollary 3.10. *Conditions*

$$\omega^T D_2 \Gamma_j \geq z_j - c_j, \quad j \text{ an index of nonbasic variables}, \tag{3.57}$$

are sufficient for DMU_0 to be efficient after the proportionate change of a subset of inputs (3.25) with the other inputs and all outputs fixed.

3.4. **Sensitivity analysis of the proportionate change of inputs and outputs in the CCR model with different coefficients of proportionality.** Let us consider the simultaneous proportionate change (increase) of all inputs

$$\widehat{x}_{i0} = \widehat{\beta}_i x_{i0}, \quad \widehat{\beta}_i \geq 1, \quad i = 1, 2, \ldots, m, \tag{3.58}$$

and the proportionate change (decrease) of all outputs

$$\widehat{y}_{r0} = \widehat{\alpha}_r y_{r0}, \quad 0 < \widehat{\alpha}_r \leq 1, \quad r = 1, 2, \ldots, s, \tag{3.59}$$

of an efficient DMU_0 preserving efficiency. We are interested in sufficient conditions for DMU_0 to preserve efficiency after the simultaneous changes (3.58) and (3.59).

Let us also introduce the following notations:

$$A_1 = \sum_{t=1}^{s} b_{kt}^{-1} y_{t0}(1 - \widehat{\alpha}_t), A_2 = \sum_{t=1}^{s} b_{s+m,t}^{-1} y_{t0}(1 - \widehat{\alpha}_t), A_3 = \sum_{t=1}^{s} \omega_t y_{t0}(1 - \widehat{\alpha}_t), \tag{3.60}$$

(3.61)

$$B_1 = \sum_{t=1}^{m} b_{k,s+t}^{-1} x_{t0}(\widehat{\beta}_t - 1), B_2 = \sum_{t=1}^{m} b_{s+m,s+t}^{-1} x_{t0}(\widehat{\beta}_t - 1), B_3 = \sum_{t=1}^{m} \omega_{s+t} x_{t0}(\widehat{\beta}_t - 1),$$

Theorem 3.11. *Let us suppose that DMU_0 is efficient and let*

$$\det\ M = 1 - A_1 - B_1 + B_2 + A_2 B_1 - A_1 B_2 > 0. \tag{3.62}$$

Then the conditions

$$\begin{aligned} & (A_3 - B_3 + A_2 B_3 - A_3 B_2)\Gamma_{kj} + (B_3 + A_3 B_1 - A_1 B_3)\Gamma_{s+m,j} + \\ + \quad & (A_1 + B_1 - B_2 - A_2 B_1 + A_1 B_2)\bar{c}_j \geq \bar{c}_j, \\ & \qquad j \text{ an index of nonbasic variables}, \end{aligned} \tag{3.63}$$

with $\bar{c}_j = z_j - c_j$, are sufficient for DMU_0 to preserve efficiency after the simultaneous proportionate changes of inputs (3.58) and of outputs (3.59).

Proof. It is easy to see that the proportionate change of inputs (3.58) is the special case of the change

$$\widehat{x}_{i0} = x_{i0} + \beta_i, \ \beta_i \geq 0, i = 1, 2, \ldots, m \tag{3.64}$$

with

$$\beta_i = (\widehat{\beta}_i - 1)x_{i0}, \quad \widehat{\beta}_i \geq 1, \quad i = 1, 2, \ldots, m. \tag{3.65}$$

A similar result holds for the proportionate change of outputs (3.59) which is the special case of the change

$$\widehat{y}_{r0} = y_{r0} - \alpha_r > 0, \quad 0 \leq \alpha_r < y_{r0}, \quad r = 1, 2, \ldots, s, \tag{3.66}$$

with

$$\alpha_r = (1 - \widehat{\alpha}_r)y_{r0}, \quad 0 < \widehat{\alpha}_r \leq 1, \quad r = 1, 2, \ldots, s. \tag{3.67}$$

Let us suppose that conditions (3.63) are satisfied. Then using (3.64), (3.65) and (3.62) it is easy to show that conditions (3.62) are equivalent to conditions (34) in Guddat et al. [16], p. 196, for the case with $\beta_i, i = 1, 2, \ldots, m$ in (2.21) and $\alpha_r, r = 1, 2, \ldots, s$ in (3.66). But conditions (34) in Guddat et al. [16] are sufficient for DMU_0 to preserve efficiency after the simultaneous changes (3.63) and (3.65). Thus, the statement of the theorem follows. $\square$

Corollary 3.12. *Let us suppose that DMU_0 is efficient. Let $\widehat{\beta}_i = \widehat{\beta},\ i = 1, 2, \ldots, m,$ in (3.58) and let $\widehat{\alpha}_r = \widehat{\alpha},\ r = 1, 2, \ldots, s$ in (3.59). Let*

$$d_j = -a_3\Gamma_{kj} + a_1\bar{c}_j, e_j = -b_3(\Gamma_{kj} - \Gamma_{s+m,j}) - (-b_1 + b_2)\bar{c}_j, \tag{3.68}$$

$$f_j = (a_2b_3 - a_3b_2)\Gamma_{kj} + (a_3b_1 - a_1b_3)\Gamma_{s+m,j} - (a_2b_1 - a_1b_2)\bar{c}_j, \tag{3.69}$$

$$j = 1, 2, \ldots, n + s + m + 1,$$

with $\bar{c}_j = z_j - c_j$. Let

$$\det\ M = 1 - a_1(1 - \widehat{\alpha}) + (-b_1 + b_2)(\widehat{\beta} - 1) + (a_2b_1 - a_1b_2)(1 - \widehat{\alpha})(\widehat{\beta} - 1) > 0. \tag{3.70}$$

Then the conditions

$$d_j(1 - \widehat{\alpha}) + e_j(\widehat{\beta} - 1) + f_j(1 - \widehat{\alpha})(\widehat{\beta} - 1) \geq \bar{c}_j, \tag{3.71}$$

j an index of nonbasic variables,

are sufficient for DMU_0 to preserve efficiency after the proportionate change of inputs (3.58) with the same coefficient of proportionality $\widehat{\beta}$ for all inputs and the proportionate change of outputs (3.59) with the same coefficient of proportionality $\widehat{\alpha}$ for all outputs.

Remark 3.13. For details about the case in Corollary 3.12, see Theorem 2 in Neralić [21].

Corollary 3.14. *Let us suppose that DMU_0 is efficient and let $\widehat{\beta}_i = \widehat{\beta}, i = 1, 2, \ldots, m$ in (3.58) and $\widehat{\alpha}_r = 1, r = 1, 2, \ldots, s$ in (3.59). Let*

$$\det M = 1 + (-b_1 + b_2)(\widehat{\beta} - 1) > 0. \tag{3.72}$$

Then the conditions

$$(-b_3(\Gamma_{kj} - \Gamma_{s+m,j}) - (-b_1 + b_2)\bar{c}_j)(\widehat{\beta} - 1) \geq \bar{c}_j), \tag{3.73}$$

j an index of nonbasic variables,

are sufficient for DMU_0 to preserve efficiency after the proportionate change (3.58) of inputs only with the same coefficient of proportionality $\widehat{\beta}$ for all inputs.

Remark 3.15. For details about the case in Corollary 3.14, see Theorem 3.1 in Charnes and Neralić [9].

Corollary 3.16. *Let us suppose that DMU_0 is efficient and let $\widehat{\beta}_i = 1, i = 1, 2, \ldots, m$ in (3.58) and $\widehat{\alpha}_r = \widehat{\alpha}, r = 1, 2, \ldots, s$ in (3.59). Let*

$$\det M = 1 - a_1(1 - \widehat{\alpha}) > 0. \tag{3.74}$$

Then the conditions

$$(-a_3\Gamma_{kj} + a_1\bar{c}_j)(1 - \widehat{\alpha}) \geq \bar{c}_j, \tag{3.75}$$

j an index of nonbasic variables,

are sufficient for DMU_0 to preserve efficiency after the proportionate change (3.59) of outputs only with the same coefficient of proportionality $\widehat{\alpha}$ for all outputs.

Remark 3.17. For details about the case in Corollary 3.16, see Theorem 4.1 in Charnes and Neralić [9].

4. Summary and conclusions

In this paper, sensitivity analysis of the proportionate change of outputs and/or of inputs is considered for different cases in the additive model and in the CCR model of DEA. Sufficient conditions for an efficient DMU_0 to preserve efficiency are reviewed for several cases of proportionate change of outputs and/or of inputs.

Sensitivity analysis of the proportionate change of outputs and/or of inputs in the additive model is studied in Section 2. Subsection 2.1 contains some preliminaries for the additive model. The proportionate change of outputs, of inputs and of outputs and inputs simultaneously is considered in Subsections 2.2, 2.3 and 2.4 respectively. The case with different coefficients of proportionality for outputs and inputs is studied in Subsection 2.5 and the case of discretionary outputs and/or inputs is considered in Subsection 2.6. Similar cases for the CCR model are studied in Section 3. Subsection 3.1 includes some preliminaries for the CCR model. Simultaneous proportionate changes of inputs and of outputs are considered for the CCR model in Subsection 3.2. Subsection 3.3 contains a sensitivity analysis of the proportionate change of a subset of inputs and/or of outputs. In the case with two coefficients of proportionality, one for outputs and the other for inputs, sufficiency conditions give for each efficient DMU_0 a region of efficiency and an area the size of which is a measure of stability of efficiency at the proportionate change of a subset of inputs and/or of a subset of outputs.

Because the conditions obtained are only sufficient, an open question remains related to the form of necessary and sufficient conditions for an efficient DMU_0 to preserve efficiency after the proportionate changes of outputs and/or of inputs in the additive, CCR and BCC models of DEA. The question remains about the efficiency preservation of all efficient DMUs according to the CCR and the BCC model of DEA under the proportionate changes of all data. The case of the proportionate change of all data in the CCR model with preservation of the efficiency of all efficient and the inefficiency of all inefficient DMUs seems to be interesting. An application of the results based on the proportionate change of outputs and/or inputs using data for a real world problem seems to be a challenge.

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