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MINIMUM QUASI$_1$-PANCYCLIC MULTIPARTITE TOURNAMENTS

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Abstract. In 1994, Guo and Volkmann showed that every partite set of a strong c-partite tournament, where $c \geq 3$, contains at least one vertex that lies on an ℓ-cycle for all $\ell \in \{3, \dots, c\}$, a generalization of Moon's theorem concerning tournaments. Since then, the question whether this bound is sharp has been an open problem.

In his master's thesis, Auclair answered this question in the affirmative by constructing suitable examples. In this paper, we extend his results by giving an infinite family of multipartite tournaments with exactly c vertices that lie on an ℓ-cycle for all $\ell \in \{3, \dots, c\}$. We call such digraphs minimum quasi$_1$-pancyclic. In particular, our family contains examples with arbitrarily large partite sets and minimum semi-degree.

Furthermore, we characterize all minimum quasi$_1$-pancyclic 3-partite tournaments.

Additionally, we notice a new generalization of Moon's theorem in semicomplete multipartite digraphs, namely: Every vertex of a strong semicomplete c-partite digraph D, where $c \geq 3$, belongs to $c-2$ cycles whose lengths are at least 3 and are pairwise distinct.

1. Introduction and terminology

For terminology and notation not explicitly introduced here, we refer to Bang-Jensen and Gutin [2]. For a *digraph* (short for directed graph) D, we denote its vertex set by $V(D)$ and its arc set by $A(D)$. Instead of $(x, y) \in A(G)$, for convenience, we mostly write $xy \in A(G)$ or $x \to y$. For sets of vertices $X, Y \subseteq V(D)$, by $X \to Y$ ($X \Rightarrow Y$, respectively) we denote the fact that $x \to y$ ($y \nrightarrow x$, respectively) for all $x \in X$ and $y \in Y$. A sequence $P = x_0 \dots x_\ell$ of vertices such that $x_i x_{i+1} \in A(D)$, for all $i \in \{0, \dots, \ell-1\}$, is called an (x_0, x_ℓ)-*walk* of length ℓ in D. P is a *closed* walk, if $x_0 = x_\ell$, it is a *path*, if all of its vertices are pairwise distinct, and it is a *cycle*, if all of its vertices are pairwise distinct except for $x_0 = x_\ell$. A digraph is *strong*, if it contains an (x, y)-path for every pair $\{x, y\}$ of its vertices. Let $X \subseteq V(D)$. Then $D[X]$ is the subdigraph of D induced by X, i.e., $V(D[X]) = X$ and $A(D[X]) = \{xy \mid x, y \in X, xy \in A(D)\}$. $D - X$ denotes the subdigraph $D[V(D) \setminus X]$. For a single vertex x, we also write $D - x$. The *in-degree* of a vertex x, denoted by $d^-(x)$, is the number of arcs that end in x. The *out-degree* $d^+(x)$ is defined analogously. We then call $\delta^-(D) = \min_{x \in V(D)} d^-(x)$

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the *minimum in-degree*, $\delta^+(D) = \min_{x\in V(D)} d^+(x)$ the *minimum out-degree*, and $\delta^0(D) = \min\{\delta^-(D), \delta^+(D)\}$ the *minimum semi-degree* of D.

For a positive integer c, a *c-partite tournament* D is an orientation of a complete c-partite graph, which is to say $V(D)$ can be decomposed into c pairwise disjoint, non-empty, independent subsets $V_1, \dots, V_c$ – called *partite sets* of D – such that two vertices of distinct partite sets are connected by exactly one arc. If there is *at least* one arc between vertices of distinct partite sets, we call the considered digraph *semicomplete c-partite*. We then define the function

$$\mathrm{p} = \mathrm{p}_D : V(D) \to \{1, \dots, c\},\ x \mapsto i,\ \text{if and only if } x \in V_i$$

that assigns each vertex the index of its partite set. The class of *multipartite tournaments* contains all c-partite tournaments for all positive integers c. Semicomplete multipartite digraphs are defined analogously. We call a semicomplete multipartite digraph *proper* (*rich*, respectively), if at least one (each, respectively) of its partite sets contains at least two vertices. A *tournament* is a multipartite tournament whose every partite set contains exactly one vertex.

As a first result, we consider the following decomposition of the vertex set of multipartite tournaments, due to Tewes and Volkmann, which will play an important role in the proofs to come.

Theorem 1.1 (Tewes and Volkmann [12], 1999). *Let D be a non-strong, c-partite tournament with partite sets $V_1, \dots, V_c$. Then there exists a unique decomposition of $V(D)$ into pairwise disjoint subsets $X_1, \dots, X_r$, where X_i is the vertex set of a strong component of D or $X_i \subseteq V_\ell$ for some $\ell \in \{1, \dots, c\}$ such that $X_i \Rightarrow X_j$ for $1 \le i < j \le r$ and there are $x_i \in X_i$ and $y_i \in X_{i+1}$ such that $x_i \to y_i$ for $1 \le i < r$.*

We call $X_1, \dots, X_r$ the *multipartite decomposition* of the multipartite tournament D.

Moon's theorem, arguably one of the most central results on tournaments, states that every vertex of a strong tournament on n vertices is contained in an ℓ-cycle of length ℓ for all $\ell \in \{3, \dots, n\}$, a property which we call vertex-pancyclicity.

Theorem 1.2 (Moon [11], 1966). *Every strong tournament is vertex-pancyclic.*

However, Moon's theorem does not hold for strong multipartite tournaments, at least not in the specific formulation given in Theorem 1.2. This is known since at least 1976, when Bondy [3] showed the existence of strong c-partite tournaments on $n > c$ vertices without ℓ-cycles for all $\ell > c$ (cf. Figure 1). Challenged by this obstacle, researchers considered competing concepts to capture the essence of Moon's theorem in a result that would extend to multipartite tournaments.

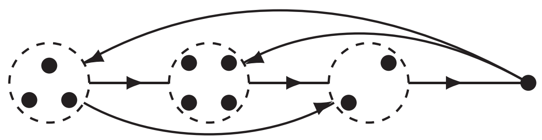

FIGURE 1. A strong 4-partite tournament without ℓ-cycles for all $\ell > 4$.

2. Generalizations of Moon's theorem

In this section, we give a short overview of some known generalizations of Moon's theorem (Theorem 1.2) for multipartite tournaments and introduce a new one. In the literature the term *quasi-pancyclicity* has been used in this context to express varying concepts. To avoid confusion, we therefore introduce the terms *quasi*$_{\mathrm{x}}$-*pancyclicity*, where $\mathrm{x} \in \{\mathrm{p}, \mathrm{o}, \mathrm{nl}, \mathrm{l}\}$, whose meanings will become apparent in the following paragraphs.

We begin with an approach by Goddard and Oellermann. Instead of counting each vertex of a cycle, they would count the number of partite sets represented in said path or cycle. A *quasi*$_{\mathrm{p}}$-k-*cycle* in a semicomplete multipartite digraph is then a cycle which contains vertices from exactly k different partite sets. Since every partite set of a tournament contains exactly one vertex, in a tournament both concepts are equal. Now, a vertex in a semicomplete c-partite digraph is called *quasi*$_{\mathrm{p}}$-*pancyclic*, if it belongs to a quasi$_{\mathrm{p}}$-k-cycle for each $k \in \{3, \ldots, c\}$. Again, in tournaments, quasi$_{\mathrm{p}}$-pancyclicity equals the common concept of pancyclicity. Their efforts resulted in the following generalization of Moon's theorem. Note that it is stated only for strong multipartite tournaments, but, since every strong semicomplete multipartite digraph contains a strong multipartite tournament on the same vertex set, the result also holds for the larger class.

Theorem 2.1 (Goddard and Oellermann [4], 1991)**.** *Every vertex of a strong c-partite tournament, where $c \geq 3$, is quasi$_{\mathrm{p}}$-pancyclic.*

Furthermore, Guo and Volkmann found that the cycles considered in Theorem 2.1 can be particularly chosen.

Theorem 2.2 (Guo and Volkmann [9], 2004)**.** *Every vertex of a strong semicomplete c-partite digraph D, where $c \geq 3$, belongs to a quasi$_{\mathrm{p}}$-k-cycle C^k for each $k \in \{3, \ldots, c\}$ such that $V(C^3) \subset \ldots \subset V(C^c)$.*

In 1999, Guo introduced the concept of an *outpath* of a vertex x in a digraph as a path starting at x such that x dominates the endvertex of the path only if the endvertex also dominates x. In tournaments, the start- and endvertex of every path are connected by exactly one arc and thus, the outpaths of length $\ell - 1$ of a tournament are exactly its cycles of length ℓ. We therefore call a vertex in a semicomplete c-partite digraph *quasi*$_{\mathrm{o}}$-*pancyclic*, if it has an outpath of length $\ell - 1$ for each $\ell \in \{3, \ldots, c\}$. Consequently, the following result also includes Moon's theorem.

Theorem 2.3 (Guo [5], 1999)**.** *Every vertex of a strong c-partite tournament, where $c \geq 3$, is quasi$_{\mathrm{o}}$-pancyclic.*

We also notice that Moon's theorem may be equivalently expressed as follows: For every strong tournament D of order c, we have

$$\mathrm{nl}(D) \;:=\; \min_{v \in V(D)} |\{\ell \geq 3 \mid v \text{ lies on an } \ell\text{-cycle in } D\}| \;=\; c - 2.$$

Attempts to extend this concept to multipartite tournaments have been stifled in part by the following result.

Theorem 2.4 (Guo, Pinkernell, and Volkmann [6], 1997)**.** *If D is a strong c-partite tournament, where $c \geq 3$, and v an arbitrary vertex of D, then v is contained in an ℓ- or $(\ell+1)$-cycle for all $\ell \in \{3, \ldots, c\}$.*

Since it is best possible, at first glance, Theorem 2.4 suggests that $\mathrm{nl}(D)$ would be approximately of size $c/2 - 1$ for general strong c-partite tournaments. In order to disprove this notion, we consider the following result due to Tewes and Volkmann, which, as Theorem 2.1, also holds for strong semicomplete c-partite digraphs.

Theorem 2.5 (Tewes and Volkmann [12], 1999)**.** *If D is a strong c-partite tournament, where $c \geq 4$, then there are two distinct vertices $u_1, u_2 \in V(D)$ such that $D - u_i$ is strong for $i \in \{1, 2\}$.*

We now state a new generalization of Moon's theorem for semicomplete multipartite digraphs, which is meant to provide a new point of view on the original result and to inspire further work on pancyclicity in semicomplete multipartite digraphs in the same vein.

Theorem 2.6. *Every vertex of a strong semicomplete c-partite digraph D, where $c \geq 3$, belongs to $c - 2$ cycles whose lengths are at least 3 and are pairwise distinct.*

Proof. Let v be an arbitrary vertex of D. From Theorem 2.1, we see that v belongs to a quasi_{p}-c-cycle C^c.

Let $k \in \{3, \ldots, c\}$. Suppose that we have found quasi_{p}-i-cycles C^i that contain v for all $i \in \{k, \ldots, c\}$ such that $V(C^k) \subsetneq \ldots \subsetneq V(C^c)$. If $k = 3$, then $C^3, \ldots, C^c$ are obviously $c - 2$ cycles with the desired properties. Otherwise, $D[V(C^k)]$ is a strong semicomplete k-partite digraph, where $k \geq 4$, and therefore contains a vertex $u \neq v$ such that $D[V(C^k)] - u$ is strong, by Theorem 2.5. $D[V(C^k)] - u$ has at least $k - 1 \geq 3$ partite sets and thus, by Theorem 2.1, there is a quasi_{p}-$(k-1)$-cycle C^{k-1} in $D[V(C^k)] - u$ that contains v. $V(C^{k-1}) \subsetneq V(C^k) \subsetneq \ldots \subsetneq V(C^c)$ is an obvious consequence and the result follows by induction. □

Since Theorem 2.6 is aleady implied by Theorem 2.2, we note two things. First, the short independent proof of Theorem 2.6 is included here, since the proof of Theorem 2.2 was omitted in [9]. Second, we mainly state Theorem 2.6 not for the result itself, but as motivation for the following definition.

We call a vertex in a semicomplete c-partite digraph *quasi_{nl}-pancyclic* when it belongs to $c - 2$ cycles whose lengths are at least 3 and are pairwise distinct. We may then express Theorem 2.6 as: Every vertex of a strong semicomplete c-partite digraph, where $c \geq 3$, is quasi_{nl}-pancyclic.

Since quasi_{nl}-pancyclicity does not require the inclusion of vertices from a certain number of partite sets, it is a new and independent view on pancyclicity in semicomplete multipartite digraphs. Therefore, other results on pancyclicity in tournaments that are proven not to hold for semicomplete multipartite digraphs using quasi_{p}-pancyclicity, might very well hold for quasi_{nl}-pancyclicity.

In 1994, Guo and Volkmann went yet another route to generalize Moon's theorem. In their terminology, a vertex of a semicomplete c-partite digraph is *quasi_{l}-pancyclic*, if it belongs to a cycle of length ℓ for all $\ell \in \{3, \ldots, c\}$, which, by Bondy [3], is the best one can hope for in a general strong semicomplete multipartite digraph.

Still, not every vertex of a strong multipartite tournament can be guaranteed to be quasi l-pancyclic. However, Guo and Volkmann could prove that every partite set of a strong multipartite tournament contains at least one quasi l-pancyclic vertex, which, again, also holds for strong semicomplete multipartite digraphs and generalizes Moon's theorem.

Theorem 2.7 (Guo and Volkmann [7], 1994)**.** *Every partite set of a strong c-partite tournament, where $c \geq 3$, contains a quasi*l*-pancyclic vertex.*

The question whether this bound is sharp – i.e., whether there exist proper c-partite tournaments with exactly c quasi l-pancyclic vertices which we call *minimum quasi*l*-pancyclic* – remained an open problem until 2013, when Auclair [1] was able to construct some examples of such digraphs in his master's thesis. Figure 2 depicts one example he found, where the vertices of the upper row are quasi l-pancyclic while those of the lower row are not. In the following section, we extend his results by giving an infinite family of minimum quasi l-pancyclic multipartite tournaments and characterize all minimum quasi l-pancyclic 3-partite tournaments.

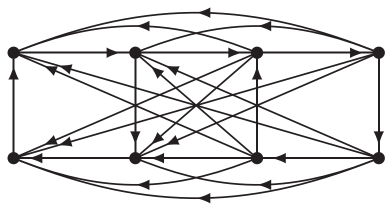

Figure 2. A minimum quasi l-pancyclic 4-partite tournament.

3. Minimum quasi l-pancyclic multipartite tournaments

We begin this section with the definition of a particular digraph, denoted by Q_c, that Las Vergnas [10] proved to be the unique strong tournament on c vertices with exactly two vertices whose removal leaves the tournament still strong (cf. Theorem 2.5). Q_c has been shown to play an important role in previous results on cycles in multipartite tournaments – e.g., in the characterization of all rich strong c-partite tournaments without a $(c+1)$-cycle (see [8]) – and will appear as a subdigraph in the subsequent definitions of minimum quasi l-pancyclic multipartite tournaments.

Definition 3.1. Let $c \geq 3$ be an integer. We define the strong tournament $Q_c = (V, A)$ through $V := \{x_1, \ldots, x_c\}$ and

$$A := \{x_i x_{i+1} \mid i \in \{1, \ldots, c-1\}\} \cup \{x_j x_i \mid c \geq j \geq i+2 \geq 3\}.$$

We now define an infinite family of multipartite tournaments which we will prove to be minimum quasi l-pancyclic.

Definition 3.2. A c-partite tournament D, where $c \geq 3$, is contained in the class $\mathcal{R}_3^c$, if and only if the following conditions are met. D is rich and $V(D)$ can be decomposed into disjoint vertex sets:

$$X \;\; = \{x_1, \ldots, x_c\} \text{ such that } D[X] = Q_c, \text{ as given in Definition 3.1,}$$
$$Y^+ = \{y \in V_{\mathrm{p}(x_c)} \cup V_{\mathrm{p}(x_{c-1})} \mid y \Rightarrow X\},$$

$Y^- = \{y \in V_{\mathrm{p}(x_1)} \cup V_{\mathrm{p}(x_2)} \mid X \Rightarrow y\}$, and
$Y_j^\pm = \{y \in V_{\mathrm{p}(x_j)} \mid \{x_{j+1}, \dots, x_c\} \to y \to \{x_1, \dots, x_{j-1}\}\}$
for $j \in \{2, \dots, c-1\}$.

Furthermore, $D-X$ can be decomposed into $c-1$ bipartite tournaments $D_1, \dots, D_{c-1}$ (where we allow $D_j = (\emptyset, \emptyset)$ for some of them) such that the following holds:

(a) $Y^+ \subseteq V(D_{c-1}) \subseteq Y^+ \cup Y_{c-1}^\pm$.
The initial component of the multipartite decomposition of D_{c-1} is either a subset of $Y_{c-1}^\pm$ or said component is strong and contains at least one vertex of $Y_{c-1}^\pm$.

(b) $Y^- \subseteq V(D_1) \subseteq Y^- \cup Y_2^\pm$.
The terminal component of the multipartite decomposition of D_1 is either a subset of $Y_2^\pm$ or said component is strong and contains at least one vertex of $Y_2^\pm$.

(c) $V(D_j) \subseteq Y_j^\pm \cup Y_{j+1}^\pm$ for all $j \in \{2, \dots, c-2\}$.

(d) $V(D_j) \Rightarrow V(D_i)$ in D for all $1 \le i < j \le c-1$.

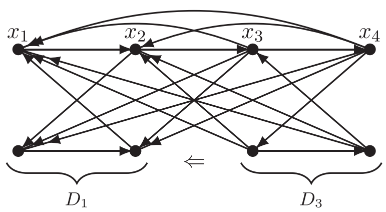

FIGURE 3. A digraph contained in $\mathcal{R}_3^4$.

In fact, we can prove a stronger result.

Theorem 3.3. *Let $c \ge 3$ be an integer and $D \in \mathcal{R}_3^c$. Then D is a rich strong c-partite tournament with exactly c vertices that lie on a 3-cycle.*

Proof. D is rich by definition. Since $D[X] = Q_c$ is a strong tournament of order c, each of its vertices is contained in a 3-cycle, by Theorem 1.2. All that remains to be shown is that no vertex of $V(D-X)$ lies on a 3-cycle. The decomposition of $V(D)$ into the vertex sets X, Y^+, Y^-, and $Y_j^\pm$ obviously prevents 3-cycles that contain exactly one vertex of $V(D-X)$ and two vertices of X.

Suppose that there is a 3-cycle $yy'xy$ in D with $y, y' \in V(D-X)$ and $x \in X$. $y \to y'$ combined with Definition 3.2 (d) implies that $y \in V(D_j)$ and $y' \in V(D_i)$ for some $1 \le i \le j \le c-1$. If $i < j$, then $y \in Y_{j'}^\pm$ and $y' \in Y_{i'}^\pm$ with $i' < j'$. If $i = j$, then $y, y' \in Y_i^\pm \cup Y_{i+1}^\pm$. In both cases, $y \Rightarrow x$ for all $x \in X$ such that $y' \to x$, a contradiction.

Finally, Definition 3.2 (a) and (d) guarantee, that there is a path from X to every vertex of $V(D-X)$ and Definition 3.2 (b) and (d) guarantee the converse. Since $D[X] = Q_c$ is strong, so is D. □

Corollary 3.4. *Let $c \ge 3$ be an integer and $D \in \mathcal{R}_3^c$. Then D is minimum quasi$_1$-pancyclic.*

As an easy implication, we obtain the existence of minimum quasi$_1$-pancyclic multipartite tournaments with partite sets of all sizes.

Corollary 3.5. *Let $s_1, \ldots, s_c \geq 2$ be a sequence of integers, where $c \geq 4$. Then there exists a c-partite tournament $D \in \mathcal{R}_3^c$ with partite sets $V_1, \ldots, V_c$ such that $|V_i| = s_i$ for all $i \in \{1, \ldots, c\}$.*

Proof. We construct a multipartite tournament $D \in \mathcal{R}_3^c$ with the desired properties. Let $V(D) := X \cup Y_1 \cup \ldots \cup Y_c$, where $X := \{x_1, \ldots, x_c\}$ and $Y_i = \{y_i^1, \ldots, y_i^{s_i - 1}\}$ such that $V_{\mathrm{p}(x_i)} = \{x_i\} \cup Y_i$ for all $i \in \{1, \ldots, c\}$. Let $D[X] := Q_c$, $Y^+ := Y_c$, $Y^- := Y_1$, and $Y^\pm := Y_i$ for all $i \in \{2, \ldots, c-1\}$ in accordance with the conditions of Definition 3.2. Furthermore, let $D_1 := D[Y^- \cup Y_2]$ and $D_{c-1} := D[Y_{c-1} \cup Y^+]$ be bipartite tournaments such that $Y^- \to Y_2$ and $Y_{c-1} \to Y^+$, let $D_2 = D_{c-2} := (\emptyset, \emptyset)$, and let $D_i := D[Y_i]$ consist of isolated vertices for all $i \in \{3, \ldots, c-3\}$. We complete D to a multipartite tournament in $\mathcal{R}_3^c$ by adding arcs such that $V(D_j) \to V(D_i)$ for all $1 \leq i < j \leq c-1$, which concludes the proof. $\square$

Corollary 3.6 (Auclair [1], 2013)**.** *Let $s_1, \ldots, s_c$ be a sequence of positive integers, where $c \geq 4$. Then there exists a minimum quasi$_1$-pancyclic c-partite tournament with partite sets $V_1, \ldots, V_c$ such that $|V_i| = s_i$ for all $i \in \{1, \ldots, c\}$.*

Proof. We replace all elements $s_i = 1$ by 2 and construct a c-partite tournament $D \in \mathcal{R}_3^c$ as in the proof of Corollary 3.5. Suppose that $s_i = 1$ for some $i \in \{1, \ldots, c\}$. If we delete Y_i from the digraph D to produce a c-partite tournament such that $|V_i| = s_i = 1$, then the vertices of X remain quasi$_1$-pancyclic, since $D[X] = Q_c$ is a strong tournament and the vertices of $V(D) \setminus X$ are still not contained in a 3-cycle. Thus, $D - Y_i$ is still minimum quasi$_1$-pancyclic, as long as it is strong. It is easy to see that this is the case, unless we delete Y_2 while $Y_1 = Y^- \neq \emptyset$ or we delete Y_{c-1} while $Y_c = Y^+ \neq \emptyset$. Therefore, to guarantee that $D - Y_i$ remains strong, before we begin the construction of D, we just have to rearrange the sequence $s_1, \ldots, s_c$ in such a way that s_2 and s_{c-1} are its largest elements. The result follows by Corollary 3.4. $\square$

To realize that there exist minimum quasi$_1$-pancyclic multipartite tournaments with arbitrarily large minimum semi-degree δ^0, let us consider the bipartite tournament Bi_k (cf. Figure 4).

Definition 3.7. Let $k \geq 2$ be an integer. We define the bipartite tournament $Bi_k = (V, A)$ through $V := \{x_0, \ldots, x_{2^k - 1}\} \cup \{y_0, \ldots, y_{2^k - 1}\}$ and

$$A := \{x_i y_{i+j \mod 2^k},\ y_i x_{i+j+1 \mod 2^k} \mid 0 \leq i \leq 2^k - 1, 0 \leq j \leq 2^{k-1} - 1\}.$$

Obviously, for every vertex $x \in V(Bi_k)$, we have $\delta^+(x) = \delta^-(x) = 2^{k-1}$ and thus, $\delta^0(Bi_k) = 2^{k-1}$.

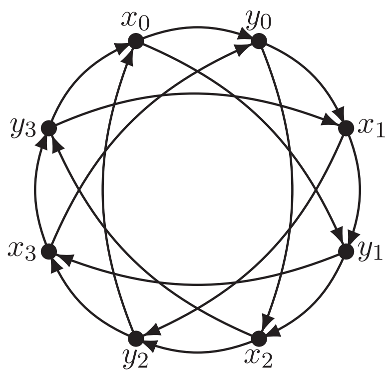

FIGURE 4. The bipartite tournament Bi_2.

Corollary 3.8. *Let $c, d \geq 3$ be integers. Then there exists a $D \in \mathcal{R}_3^c$ with $\delta^0(D) \geq d$.*

Proof. Let k be an integer with $2^{k-1} \geq d$. We construct a c-partite tournament D on the vertex set $V := X \cup \{x'_i, y'_i, x^*_i, y^*_i \mid i \in \{0, \ldots, 2^k - 1\}\}$, where $X := \{x_1, \ldots, x_c\}$ such that $V_{\mathrm{p}(x_1)} := \{x_1, x'_i \mid 0 \leq i \leq 2^k - 1\}$, $V_{\mathrm{p}(x_2)} := \{x_2, y'_i \mid 0 \leq i \leq 2^k - 1\}$, $V_{\mathrm{p}(x_{c-1})} := \{x_{c-1}, x^*_i \mid 0 \leq i \leq 2^k - 1\}$, and $V_{\mathrm{p}(x_c)} := \{x_c, y^*_i \mid 0 \leq i \leq 2^k - 1\}$. Furthermore, let $D_1 := D[\{x'_i, y'_i \mid 0 \leq i \leq 2^k - 1\}]$ and $D_{c-1} := D[\{x^*_i, y^*_i \mid 0 \leq i \leq 2^k - 1\}]$ equal two disjoint copies of Bi_k with $V[D_{c-1}] \Rightarrow V[D_1]$ and $D[X] = Q_c$. To complete D to a digraph adhering to the conditions of Definition 3.2, we set $Y^+ := V(D_{c-1}) \setminus \{x^*_0\}$, $Y^- := V(D_1) \setminus \{y'_0\}$, $Y_2^{\pm} := \{y'_0\}$, and $Y_{c-1}^{\pm} := \{x^*_0\}$. Thus, $D \in \mathcal{R}_3^c$. Since $\delta^0(D) \geq 2^{k-1} \geq d$ by Definition 3.7, the result follows. □

Corollary 3.9 (Auclair [1], 2013)**.** *Let $c, d \geq 3$ be integers. Then there exists a minimum quasi$_1$-pancyclic c-partite tournament D with $\delta^0(D) \geq d$.*

Since the structure of general minimum quasi$_1$-pancyclic multipartite tournaments seems too complex to be characterized in one fell swoop, we restrict our remaining considerations to 3-partite tournaments. We start with those that are not rich and define the following class of digraphs.

Definition 3.10. A 3-partite tournament D is contained in the class $\mathcal{B}_3$, if and only if the following conditions are met. $V(D)$ can be decomposed into five disjoint vertex sets:

$X = \{x_1, x_2, x_3\}$ such that $x_1x_2x_3x_1$ is a 3-cycle in D,
$Y^+ = \{y \in V_{\mathrm{p}(x_1)} \cup V_{\mathrm{p}(x_2)} \mid y \Rightarrow X\}$,
$Y^- = \{y \in V_{\mathrm{p}(x_1)} \cup V_{\mathrm{p}(x_2)} \mid X \Rightarrow y\}$,
$Y_1^{\pm} = \{y \in V_{\mathrm{p}(x_1)} \mid x_2 \to y \to x_3\}$, and
$Y_2^{\pm} = \{y \in V_{\mathrm{p}(x_2)} \mid x_3 \to y \to x_1\}$.

Note that, except for X, we allow for some or all of the sets to be empty.

Furthermore, $D - X$ can be decomposed into two bipartite tournaments D' and D'' (where we allow $D' = (\emptyset, \emptyset)$ and/or $D'' = (\emptyset, \emptyset)$) such that the following holds:

(a) $V(D') = Y^+ \cup Y_1^{\pm}$.

If $V(D') \neq \emptyset$, then the initial component of the multipartite decomposition of D' is either a subset of $Y_1^{\pm}$ or it is strong and contains at least one vertex of $Y_1^{\pm}$.

(b) $V(D'') = Y^- \cup Y_2^{\pm}$.
If $V(D'') \neq \emptyset$, then the terminal component of the multipartite decomposition of D'' is either a subset of $Y_2^{\pm}$ or it is strong and contains at least one vertex of $Y_2^{\pm}$.

(c) $V(D') \Rightarrow V(D'')$ in D.

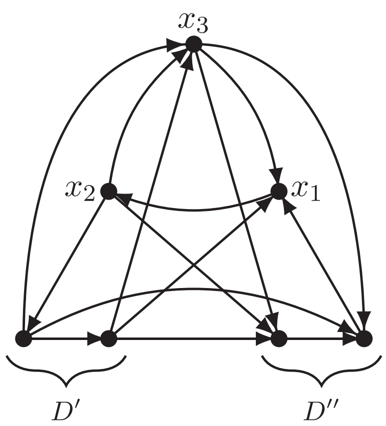

FIGURE 5. A 3-partite tournament contained in $\mathcal{B}_3$.

The following lemma shows that $\mathcal{B}_3$ contains exactly the non-rich minimum quasi$_1$-pancyclic 3-partite tournaments.

Lemma 3.11. *A non-rich* 3*-partite tournament D is minimum quasi*$_1$*-pancyclic, if and only if $D \in \mathcal{B}_3$.*

Proof. A non-rich minimum quasi$_1$-pancyclic 3-partite tournament D contains a 3-cycle $x_1x_2x_3x_1$ such that $V_{p(x_3)} = \{x_3\}$. $D \in \mathcal{B}_3$ contains such a cycle by definition. Let $X = \{x_1, x_2, x_3\}$. The decomposition of $V(D - X)$ into vertex sets Y^+, Y^-, $Y_1^{\pm}$, and $Y_2^{\pm}$, as given in Definition 3.10, is a necessary and sufficient condition to prevent 3-cycles of the form $x_3x_1yx_3$ and $x_3yx_2x_3$ with $y \in V(D-X)$. Furthermore, $Y^+ \cup Y_1^{\pm} \Rightarrow Y^- \cup Y_2^{\pm}$ is necessary and sufficient to prevent 3-cycles of the form $x_3yy'x_3$ in D with $y, y' \in V(D - X)$. Since $D - x_3$ is bipartite, there are no other possible 3-cycles in D containing a vertex of $V(D - X)$. We now consider $D' = D[Y^+ \cup Y_1^{\pm}]$ and $D'' = D[Y^- \cup Y_2^{\pm}]$. Since $Y^+ \Rightarrow V(D) \setminus Y_1^{\pm}$, condition 3.10 (a) is necessary to guarantee that D is strong, as is condition 3.10 (b), by symmetry. Together, conditions 3.10 (a) and (b) are sufficient. □

We complete our characterization, by proving that $\mathcal{R}_3^3$ actually contains all rich minimum quasi$_1$-pancyclic 3-partite tournaments.

Lemma 3.12. *A rich* 3*-partite tournament D is minimum quasi*$_1$*-pancyclic, if and only if $D \in \mathcal{R}_3^3$.*

Proof. Let D be a rich minimum quasi$_1$-pancyclic 3-partite tournament. By Corollary 3.4, we only need to prove $D \in \mathcal{R}_3^3$.

Let $X = \{x_1, x_2, x_3\}$ be the set of quasi$_1$-pancyclic vertices that form the unique 3-cycle $x_1x_2x_3x_1$ in D. Furthermore, let $Y_i := V(D) \cap V_{\mathrm{p}(x_i)} \setminus \{x_i\}$ for $i \in \{1, 2, 3\}$. Note, that $Y_i \neq \emptyset$ for all $i \in \{1, 2, 3\}$, since D is rich.

(1) Suppose that two of the sets Y_1, Y_2, Y_3 each contain a vertex that is dominated by a vertex in X and dominates another vertex in X.

Without loss of generality, we may assume that there are such vertices $y_1 \in Y_1$ and $y_2 \in Y_2$ – i.e., $x_2 \to y_1 \to x_3$ and $x_3 \to y_2 \to x_1$, since neither y_1 nor y_2 is contained in a 3-cycle. $y_1 \to y_2$ follows for the same reason. Particularly, we find the 4-cycle $C = y_1y_2x_1x_2y_1$. We now consider a vertex $y_3 \in Y_3$. If $D[\{y_3\} \cup V(C)]$ were strong, y_3 would be contained in a 3-cycle, by Theorem 2.7. Thus, either $y_3 \Rightarrow V(C)$ or $V(C) \Rightarrow y_3$ holds.

Suppose that $V(C) \Rightarrow y_3$. Since D is strong, there is a shortest path P from y_3 to a vertex in $\{x_3\} \cup V(C)$. If that vertex is not x_3, again, we find at least one vertex from Y_3 that is contained in a 3-cycle in $D[\{y_3\} \cup V(P) \cup V(C)]$, a contradiction. Thus, P ends in x_3. Now $C' = y_3Px_3y_2x_1y_3$ is a cycle that contains vertices from three partite sets but not x_2 and hence, there is at least one vertex from Y_2 that lies on a 3-cycle in the induced subdigraph $D[V(C')]$, another contradiction.

Suppose now that $y_3 \Rightarrow V(C)$ holds. By the same arguments as above, we find that a shortest path from $\{x_3\} \cup V(C)$ to y_3 must start at x_3. We obtain another cycle $C' = x_3Py_3x_2y_1x_3$ that contains vertices from three partite sets but not x_1, which implies the existence of a vertex from Y_1 on a 3-cycle in the induced subdigraph $D[V(C')]$. Consequently, all in all, there are no two partite sets with the assumed properties.

(2) Suppose that there are three vertices $y_1 \in Y_1, y_2 \in Y_2, y_3 \in Y_3$ such that $X \Rightarrow \{y_1, y_2, y_3\}$ or $\{y_1, y_2, y_3\} \Rightarrow X$.

Without loss of generality, we may assume that $X \Rightarrow \{y_1, y_2, y_3\}$. Otherwise, we consider D^{-1}. Since $D[\{y_1, y_2, y_3\}]$ cannot be a 3-cycle, we may further assume that $\{y_1, y_2\} \to y_3$. Since D is strong, there is a shortest path P from y_3 to a vertex $x_i \in X$. If $i \in \{1, 2\}$, then the vertex set of the cycle $y_3Px_iy_{3-i}y_3$ induces a strong 3-partite subtournament of $D - x_3$ and thus implies a vertex from Y_3 on a 3-cycle, a contradiction. Therefore, P ends in x_3. If P contains vertices of all partite sets, then $y_3Px_3x_1y_3$ analogously implies a vertex from Y_2 on a 3-cycle. If P does not contain any vertex in Y_i for some $i \in \{1, 2\}$, then the cycle $y_3Px_3y_iy_3$ produces the same result. Thus, in any case, we reach the contradiction that some $y \in Y$ lies on a 3-cycle.

(3) Suppose that two of the sets Y_1, Y_2, Y_3 are dominated by X or two of them dominate X.

Without loss of generality, we may assume that Y_1 and Y_2 are dominated by X. Otherwise, we consider D^{-1} and/or rename the respective vertex sets. Let $y_1 \in Y_1$. Since D is strong, there is a shortest path P from y_1 to a vertex $y_3 \in Y_3$. By (2), we know that $X \not\Rightarrow y_3$. If $y_3 \not\to x_2$, then y_3 is contained in the 3-cycle $y_3x_1x_2y_3$. Hence, $y_3 \to x_2$. Since P does not contain any vertices from X, the vertex set of the cycle

$y_1Py_3x_2y_1$ induces a strong 3-partite tournament without x_1, in contradiction to the fact that no vertex from Y_1 is contained in a 3-cycle in D.

(4) Suppose that $y \Rightarrow X$ or $X \Rightarrow y$ for all $y \in Y := Y_1 \cup Y_2 \cup Y_3$.

Since D is strong, there are vertices $y, y' \in Y$ and $x \in X$ from pairwise distinct partite sets such that $X \Rightarrow y$, $y \to y'$, and $y' \Rightarrow X$. Thus, the 3-cycle $xyy'x$ implies a contradiction.

(5) Suppose that there are three vertices $y_1 \in Y_1, y_2 \in Y_2, y_3 \in Y_3$ such that $y_1 \Rightarrow X$, $x_3 \to y_2 \to x_1$, and $X \Rightarrow y_3$.

Let P be a shortest path from a vertex $x_i \in X$ to y_1. If $i = 3$ (which is particularly the case, if $y_2 \to y_1$), then the vertex set of the cycle $y_1x_2x_3Py_1$ induces a 3-partite subtournament of $D - x_1$. If $i = 2$, then $D[\{x_1\} \cup V(P) \cup \{y_2\}]$ is strong. Let P' be a shortest path from y_3 to a vertex in $X \cup V(P) \cup \{y_2\}$. If the endvertex of P' is not x_3, then $D[\{x_1\} \cup V(P) \cup \{y_2\} \cup V(P')]$ is a strong 3-partite tournament without x_3. Otherwise, the vertex set of the closed walk $y_3P'x_3y_2x_1y_3$ induces a strong 3-partite subtournament of $D - x_2$. Finally, if $i = 1$, then the vertex set of the closed walk $y_1x_3y_2x_1Py_1$ induces a strong 3-partite tournament without x_2 in D. In all cases, some vertex $y \in Y$ lies on a 3-cycle in some strong 3-partite subtournament, a contradiction.

(6) By (1-5), we may assume that there are three vertices $y_1 \in Y_1, y_2 \in Y_2, y_3 \in Y_3$ such that $X \Rightarrow y_1$, $x_3 \to y_2 \to x_1$, and $y_3 \Rightarrow X$. Suppose that a vertex $y_1' \in Y_1$ exists such that $y_1' \Rightarrow X$ or that a vertex $y_3' \in Y_3$ exists such that $X \Rightarrow y_3'$.

Without loss of generality, we may assume the existence of $y_1' \in Y_1$ with $y_1' \Rightarrow X$. Otherwise, we consider D^{-1}. By (1) and (5), we know then that $Y_3 \Rightarrow X$. Furthermore, for all $y \in Y_1$ either $y \Rightarrow X$ or $X \Rightarrow y$ holds, by (1). Thus, every vertex in $V(D - X)$ that is dominated by a vertex in X, is also dominated by x_3. Consequently, there is a shortest path P from X to y_1' that starts in x_3. Now, the vertex set of the cycle $x_3Py_1'x_2x_3$ induces a strong 3-partite subtournament of $D - x_1$, which implies the existence of a vertex from Y_1 on a 3-cycle, a contradiction.

Therefore, all in all, except for rotation of the cycle $x_1x_2x_3x_1$, we have $X \Rightarrow Y_1$, $Y_3 \Rightarrow X$, $Y_3 \Rightarrow Y_1$, and $x_3 \to y_2 \to x_1$ for at least one vertex $y_2 \in Y_2$, i.e., $V(D)$ can be decomposed into X, Y^+, Y^-, and $Y_2^{\pm}$ as in Definition 3.2.

Now, consider the multipartite decomposition $Z_1, \ldots, Z_r$ of $D - X$. Obviously, any strong component must be bipartite to prevent 3-cycles in $D - X$. Let $i_0 \in \{1, \ldots, r\}$ be maximal such that $Z_{i_0} \cap Y^+ \cap Y_3 \neq \emptyset$. Let $Z_{i_0+1}^+ := Z_{i_0+1} \cap Y^+$, if $Z_{i_0+1} \subseteq Y_2$, and $Z_{i_0+1}^+ := \emptyset$, otherwise. Let $Z_{i_0+1}^- := Z_{i_0+1} \setminus Z_{i_0+1}^+$. We now define $D_2 := D[\bigcup_{i=1}^{i_0} Z_i \cup Z_{i_0+1}^+]$ and $D_1 := D[Z_{i_0+1}^- \cup \bigcup_{i=i_0+2}^{r} Z_i]$. Then Definition 3.2 (d) holds by the definition of a multipartite decomposition. Furthermore, $Y^+ \cap Y_3 \subseteq V(D_2)$ by the choice of i_0.

Suppose that there is a vertex $y \in Y^+ \cap V(D_1)$. We then have $y \in Y_2 \cap \bigcup_{i=i_0+2}^{r} Z_i$ or $D[Z_{i_0}]$ is strong and $y \in Y_2 \cap Z_{i_0}$. Either case implies the existence of a vertex $y' \in V(D_1) \cap Y_1$ such that $y' \to y$ and thus, the 3-cycle $y'yx_3y'$ poses a contradiction. Therefore, $Y^+ \subseteq V(D_2)$.

Suppose now that there is a vertex $y \in V(D_2) \cap Y^-$. If $y \in Z_i$ for an $i < i_0$, then there is a vertex $y' \in Z_{i_0} \cap Y^+ \cap Y_3$ by the choice of i_0 and $y \to y'$ follows by the structure of the multipartite decomposition. Hence, $yy'x_2y$ is a 3-cycle, if $y \in Y_1$, and $yy'x_1y$ one, if $y \in Y_2$. If $y \in Z_{i_0}$, then $y \to y'$ for some $y' \in Z_{i_0} \cap Y^+ \cap Y_3$, since $D[Z_{i_0}]$ is a strong bipartite subtournament and we reach the same contradiction. Therefore, our assumption is wrong and conditions (a) and (b) from Definition 3.2 hold, where the statements on the initial and terminal components of D_2 and D_1, respectively, are a direct consequence of the fact that D is strong. □

We combine our previous partial results to the following characterization of minimum quasi$_1$-pancyclic 3-partite tournaments.

Theorem 3.13. *A 3-partite tournament D is minimum quasi$_1$-pancyclic, if and only if $D \in \mathcal{B}_3 \cup \mathcal{R}_3^3$.*

4. Open problems

Obviously, the ultimate goal of further research on minimum quasi$_1$-pancyclic multipartite tournaments would be a complete characterization.

Problem 4.1. Characterize all minimum quasi$_1$-pancyclic multipartite tournaments.

But the following problem would be a more manageable natural next step in this direction.

Problem 4.2. Let $c \geq 4$ be an integer. Characterize all minimum quasi$_1$-pancyclic c-partite tournaments.

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