Pure and Applied Functional Analysis
Volume 8, Number 4, 2023, 1081–1091

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

A NOTE ON MINKOWSKI–RÅDSTRÖM–HÖRMANDER SPACES

JERZY GRZYBOWSKI AND RYSZARD URBAŃSKI

Abstract. In this paper we present quasidistributive law in prelinear spaces and give a simple proof of the second distributive law in Minkowski–Rådström–Hörmander space. We also present new criteria for convex pairs of sets and for convex classes of pairs of convex sets.

1. Introduction

Minkowski–Rådström–Hörmander (MRH) space, an extension of a semigroup of compact convex sets has been investigated in a number of papers [10, 17, 18, 20]. MRH space has found an application in quasidifferential calculus of Demyanov and Rubinov [3, 4, 15, 16]. A quasidifferential is represented by a pair of compact convex sets. Since this representation is not unique, it is essential to find the inclusion-minimal one. The notion of minimal pairs was introduced in [15] and investigated in a number of papers [1, 6, 19] and many others.

In Section 2 we introduce the notion of prelinear space, which generalizes a semigroup of nonempty subsets of a vector space. We prove a quasidistributive law for convex elements of prelinear space.

In Section 3 we investigate pairs of compact convex sets with convex union. We present new characterization of convex pairs of sets and we apply properties of convex pairs to show how to convexify any pair of convex sets and how to reduce certain pairs of sets.

In Section 4 we give new criteria for convex classes, i.e. introduced in [8] quotient classes of pairs of convex compact sets in which every member has a convex union.

2. Formula for convex elements of prelinear space

We introduce a notion of a *prelinear space* for a set S with binary operation of addition $+$, neutral element 0_S and multiplication by real numbers if for any

2020 *Mathematics Subject Classification.* 20M14, 52A05, 49J52.

Key words and phrases. Second distributive law, Minkowski–Rådström–Hörmander space, convex pair of sets, convex class of pairs of sets.

$x, y, z \in S$ and $s, t \in \mathbb{R}$ the following seven conditions are satisfied.

$$(x+y)+z = x+(y+z), \tag{S1}$$

$$x+y = y+x, \tag{S2}$$

$$x+0_S = x, \tag{S3}$$

$$1x = x, \tag{S4}$$

$$0_{\mathbb{R}}x = 0_S, \tag{S5}$$

$$s(tx) = (st)x, \tag{S6}$$

$$tx+ty = t(x+y). \tag{S7}$$

An element $x \in S$ is called *convex* if $sx+tx = (s+t)x,$ for all $s, t > 0$. Notice that by (S7) the element x is convex if and only if $(-1)x$ is convex.
Let us define $t^+ := \max\{t, 0\}$ and $t^- := \max\{-t, 0\}$ for $t \in \mathbb{R}$. Then $t = t^+ - t^-$ and we observe that

$$\begin{aligned} s^+ + t^+ &= \frac{s+|s|}{2} + \frac{t+|t|}{2} = \frac{s+t+|s+t|}{2} \\ &+ \frac{|s|+|t|-|s+t|}{2} = (s+t)^+ + \frac{|s|+|t|-|s+t|}{2}. \end{aligned} \tag{2.1}$$

Since $t^- = (-t)^+$ from the formula (1) we obtain

$$s^- + t^- = (s+t)^- + \frac{|s|+|t|-|s+t|}{2}. \tag{2.2}$$

Also for prelinear S and any $x \in S$ we have

$$tx = t^+x + t^-(-1)x. \tag{2.3}$$

Proposition 2.1. (Quasidistributive law) *Let S be a prelinear space. Then an element $x \in S$ is convex if and only if for all $s, t \in \mathbb{R}$ we have*

$$sx+tx = (s+t)x + \frac{|s|+|t|-|s+t|}{2}\big(x+(-1)x\big).$$

Proof. Assuming convexity of x an applying formulas (2.1), (2.2) and (2.3) we obtain
$sx+tx = s^+x + s^-(-1)x + t^+x + t^-(-1)x = (s^+ + t^+)x + (s^- + t^-)(-1)x$

$$\begin{aligned} &= \Big((s+t)^+ + \frac{|s|+|t|-|s+t|}{2}\Big)x + \Big((s+t)^- + \frac{|s|+|t|-|s+t|}{2}\Big)(-1)x \\ &= (s+t)^+x + (s+t)^-(-1)x + \frac{|s|+|t|-|s+t|}{2}\big(x+(-1)x\big) \\ &= (s+t)x + \frac{|s|+|t|-|s+t|}{2}(x+(-1)x). \end{aligned}$$

□

Remark 2.2. Let S be a prelinear space. If all elements of S are convex then S is, by definition, a *quasilinear space* in the sense of Mayer [11, 13].

Proposition 2.3. *Let S be a quasilinear space. Then S is a linear space if and only if $x+(-1)x = 0$ for all $x \in S$.*

Proof. If $x+(-1)x = 0$ for all $x \in S$, then S with addition is a group. Also, by Proposition 2.1, we obtain the second distributive law, i.e. $sx+tx = (s+t)x$ for $x \in S, s, t \in \mathbb{R}$. □

The family $\mathcal{A}(X)$ of all nonempty subsets of a vector space X with an algebraic addition $A+B := \{a+b \mid a \in A, b \in B\}$ and multiplication $tA := \{ta \mid a \in A, t \in \mathbb{R}\}$ is an important example of prelinear space. The next corollary follows directly from Proposition 2.1.

Corollary 2.4. *Let $A \in \mathcal{A}(X)$. Then A is a convex set if and only if for all $s, t \in \mathbb{R}$ we have*

$$sA + tA = (s+t)A + \frac{|s| + |t| - |s+t|}{2}(A + (-1)A).$$

Convexity of the set A in the sense of convex analysis, and also as an element of a prelinear space $\mathcal{A}(X)$ coincide.

The formula from Corollary 2.4 allows for a simple proof of Theorem 2.5, i.e. second distributive law in the Minkowski–Rådström–Hörmander space. The formula is also useful in non-smooth analysis.

Let X be a topological vector space, $\mathcal{K}(X) \subset \mathcal{A}(X)$ be a family of all nonempty compact convex subsets of X. The relation '$\sim$' between pairs (A, B) and (C, D) belonging to $\mathcal{K}^2(X)$ is defined by condition $A + D = B + C$. Due to cancellation law, i.e. $A + B = C + B \Longrightarrow A = C$, the relation '$\sim$' is a relation of equivalence. By $[A, B]$ we denote a quotient class of the pair (A, B). We denote $\widetilde{X} := \mathcal{K}^2(X)/_{\sim}$.

The cancellation law was studied among others in [9, 18, 20] and extended to cornets as Rådström cancellation theorem in [14].

In $\widetilde{X}$ we define the addition and multiplication by real numbers:

$$\begin{aligned} \tilde{x} + \tilde{y} &:= [A + C, B + D], \\ t \cdot [A, B] &:= [t^+A + t^-B, t^+B + t^-A] \end{aligned}$$

for $\tilde{x} = [A, B], \tilde{y} = [C, D] \in \widetilde{X}$ and $t \in \mathbb{R}$. The quotient space $(\widetilde{X}, +, \cdot)$ is called the *Minkowski–Rådström–Hörmander space* over X [5].

Theorem 2.5. *Let X be a topological vector space. Then the space $(\widetilde{X}, +, \cdot)$ is the smallest up to isomorphism real vector space containing isomorphic copy of the cone $\mathcal{K}(X)$.*

Proof. The triple $(\widetilde{X}, +, \cdot)$ is a prelinear space. The proof of it was presented in [20]. Here, we are going to prove only the second distributive law. For any $s, t \geqslant 0$ and $\tilde{x} = [A, B] \in \widetilde{X}$ we have $s\tilde{x} + t\tilde{x} = [sA, sB] + [tA, tB] = [(s+t)A, (s+t)B] = (s+t)\tilde{x}$. We have just proved that all elements of prelinear space $\widetilde{X}$ are convex. Then by Proposition 2.1 for all $s, t \in \mathbb{R}$ we obtain $s\tilde{x} + t\tilde{x} = (s+t)\tilde{x} + \dfrac{|s| + |t| - |s+t|}{2}\big(\tilde{x} + (-1)\tilde{x}\big)$. Since $\tilde{x} + (-1)\tilde{x} = [A, B] + (-1)[A, B] = [A + B, B + A] = \widetilde{0}$, we have $s\tilde{x} + t\tilde{x} = (s+t)\tilde{x}$. $\square$

Remark 2.6. If in $\widetilde{X}$ we define the multiplication by real numbers $t * [A, B] := [tA, tB]$ for $\tilde{x} = [A, B] \in \widetilde{X}$ and $t \in \mathbb{R}$. Then the space $(\widetilde{X}, +, *)$ is an example of so called quasivector or q-linear space. A q-linear space (see [11], Definition 2.2, also [12]) is a commutative group with multiplication by real numbers satisfying conditions (S 6-7), having only convex elements.

An interesting example is a prelinear space $(\widetilde{X},+,*)$ given in [2], where $\widetilde{X} := \mathcal{C}(X)\times\mathcal{K}(X)/_\sim$ with $\mathcal{C}(X)$ being the family of all nonempty closed convex subsets of X. The space $\widetilde{X}$ is not a group or a quasivector space.

Notice that every element $\tilde{x}\in\widetilde{X}$ is convex and the quasidistributive law has a form

$$s*\tilde{x}+t*\tilde{x}=(s+t)*\tilde{x}+\frac{|s|+|t|-|s+t|}{2}\big(\tilde{x}+(-1)*\tilde{x}\big) \text{ for all } s,t\in\mathbb{R}.$$

Let $st<0$ then $\dfrac{|s|+|t|-|s+t|}{2}=\min\{|s|,|t|\}$ and we have

$$s*\tilde{x}+t*\tilde{x}=(s+t)*\tilde{x}+\min\{|s|,|t|\}\big(\tilde{x}+(-1)*\tilde{x}\big) \text{ for all } s,t\in\mathbb{R}.$$

If $|s|>|t|$ we get $s*\tilde{x}+t*\tilde{x}=(s+t)*\tilde{x}+t*\tilde{x}+(-1)t*\tilde{x}$. Since $(\widetilde{X},+)$ is a group, $(s+t)*\tilde{x}=s*\tilde{x}-(-1)t*\tilde{x}$. Similarly for $|t|>|s|$ we have $(s+t)*\tilde{x}=-(-1)s*\tilde{x}+t*\tilde{x}$.

3. Convex pairs of sets

A pair (A,B) of nonempty closed convex subsets of a topological vector space is a *convex pair* if the union $A\cup B$ is a convex set. By $A\vee B$ we denote a convex hull of the union $A\cup B$. We say that a set S *separates* A and B if every segment with endpoints in A and B intersects the set S.

In this section we present characterization of convex pairs in Propositions 3.1, 3.4, 3.5 and Corollaries 3.6 and 3.7. We apply properties of convex pairs to show how to convexify any pair of convex sets in Theorem 3.2 and how to reduce certain pairs of sets in Propositions 3.8 and 3.10 and in Lemma 3.12.

Proposition 3.1. (Theorem 1.1 in [7]) *Let X be a topological vector space and $A,B\subset X$ be nonempty bounded closed convex sets. The following statements are equivalent*:
(a) *The set $A\cup B$ is convex.*
(b) *The set $A\cap B$ separates the sets A and B.*
(c) *The set* $\mathrm{cl}\,(A\vee B)$ *is a summand of the set* $\mathrm{cl}\,(A+B)$.
(d) $\mathrm{cl}\,(A+B)=\mathrm{cl}\,(A\vee B+A\cap B)$.
Moreover, if one of the sets A or B is compact, the last two conditions take the following form:
(c) *The set $A\vee B$ is a summand of the sum $A+B$.*
(d) $A+B=A\vee B+A\cap B$.

The following fact shows that every pair in a given quotient class $[A,B]$ can be 'convexified' by adding to both sets the convex hull $A\vee B$. Hence each quotient class $[A,B]$ contains convex pairs. This fact is mentioned without proof in the proof of Theorem 6.3.4 in [8].

Theorem 3.2. *Let $A,B,C,D\in\mathcal{K}(X)$ and $(A,B)\sim(C,D)$. Then $(C+A\vee B, D+A\vee B)$ is a convex pair equivalent to the pair (A,B). Moreover, $(C+A\vee B)\cap(D+A\vee B)=A+D=B+C$.*

Proof. Denote $A_1:=C+A\vee B$, $B_1:=D+A\vee B$. Then $A_1\vee B_1=(C+A\vee B)\vee(D+A\vee B)=C\vee D+A\vee B$. Notice that $C+A\vee B=(A+C)\vee(B+C)=(A+C)\vee(A+D)=A+C\vee D$. On the other hand, we have $A_1+B_1=C+A\vee B+D+A\vee B=A+C\vee D+D+A\vee B=A_1\vee B_1+A+D$. Hence $A_1\vee B_1$ is a

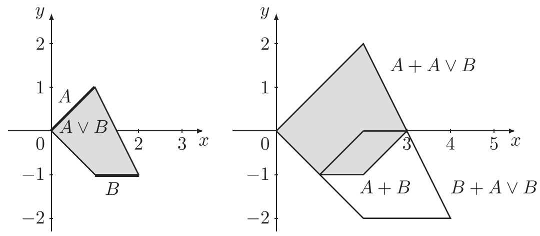

FIGURE 1. Sets described in Example 3.3

summand of $A_1 + B_1$, and by Proposition 3.1(c) the pair $(C + A \vee B, D + A \vee B)$ is convex.

Notice that $A_1+B_1 = A_1 \vee B_1 + A + D$. By Proposition 3.1(d) and the cancellation law we obtain $A_1 \cap B_1 = A + D = B + C$. $\square$

The following example illustrates Theorem 3.2 in the case of $(A, B) = (C, D)$.

Example 3.3. Let $A, B \in \mathcal{K}(\mathbb{R}^2)$, $A := (0,0) \vee (1,1)$, $B := (1,-1) \vee (2,-1)$. Then $A \vee B = (0,0) \vee (1,1) \vee (2,-1) \vee (1,-1)$. By Theorem 3.2 the pair $(A+A\vee B, B+A\vee B)$ is convex. Moreover $(A+A\vee B) \cap (B+A\vee B) = A+B$. The sets from this example are illustrated in Figure 1.

Let $A \in \mathcal{K}(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$. The *support function* h_A of the set A is a function defined by $h_A(u) := \max_{x \in A} \langle x, u \rangle$. The set $A(u) := \{x \in A \mid \langle a, u \rangle = h_A(u)\}$ is called a *support set* of A in the direction of the vector u. Let $u = (u_1, ..., u_k) \in (\mathbb{R}^n)^k$, and vectors $u_1, ..., u_k$ be pairwise orthogonal. The set $A(u) = A(u_1, ..., u_k) := \big(A(u_1, ...u_{k-1})\big)(u_k)$ is called an *iterated support set* of A in the directions $u_1, ..., u_k$. The following criterion of convexity of a pair of convex sets was given in [8].

Proposition 3.4. (Proposition 2 in [8]) *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$, $n \geqslant 2$. $A \cup B$ is convex if and only if for all $u \in \mathbb{R}^n$ we have $(A \vee B)(u) \subset A(u) \cup B(u)$.*

Criterion of convexity of a pair of convex sets can be expressed it terms of k-iterated support sets or iterated support sets that are segments contained in the boundary of $A \vee B$.

Proposition 3.5. *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$, $n \geqslant 2$. $A \cup B$ is convex if and only if for all $u = (u_1, ..., u_k) \in (\mathbb{R}^n)^k$ such that vectors $u_1, ..., u_k$ are pairwise orthogonal we have $(A \vee B)(u) \subset A(u) \cup B(u)$.*

Proof. Applying Proposition 3.4 we obtain that $A \cup B$ is convex if and only if for all $u_1 \in \mathbb{R}^n$ we have $(A \vee B)(u_1) \subset A(u_1) \cup B(u_1)$. If for a fixed u_1 we have $h_A(u_1) > h_B(u_1)$ then $(A \vee B)(u_1) = A(u_1)$. Hence for all $u_2 \in \mathbb{R}^n$ we have $(A \vee B)(u_1, u_2) = A(u_1, u_2) \subset A(u_1, u_2) \cup B(u_1, u_2)$. If for another fixed u_1 we have $h_A(u_1) = h_B(u_1)$ then $A(u_1) \vee B(u_1) = (A \vee B)(u_1) \subset A(u_1) \cup B(u_1)$. Hence a pair $(A(u_1), B(u_1))$ is convex. Thus by Proposition 3.4 we have $(A \vee B)(u_1, u_2) \subset A(u_1, u_2) \cup B(u_1, u_2)$ for all u_2 orthogonal with u_1. Therefore, the set $A \cup B$ is convex if and only if for all $u_1, u_2 \in \mathbb{R}^n$ such that u_1, u_2 are orthogonal we have $(A \vee B)(u_1, u_2) \subset A(u_1, u_2) \cup B(u_1, u_2)$. Repeating the resoning k times we obtain our proposition. $\square$

Corollary 3.6. (Criterion of convexity of a pair of convex sets) *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$, $n \geqslant 2$. The union $A \cup B$ is convex if and only if for all $u = (u_1, ..., u_{n-1}) \in (\mathbb{R}^n)^{n-1}$ such that $(A \vee B)(u)$ is a segment we have $(A \vee B)(u) \subset A(u) \cup B(u)$.*

Proof. The corollary follows from the fact that $(n-1)$-iterated support sets are segments or singletons. If $(A \vee B)(u)$ is a singleton then the inclusion $(A \vee B)(u) \subset A(u) \cup B(u)$ always hold true. □

Another corollary from Proposition 3.5 characterizes convex pairs of polytopes (convex polyhedra).

Corollary 3.7. (Criterion of convexity a pair polytopes) *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$, $n \geqslant 2$ be two polytopes. The union $A \cup B$ is convex if and only if for all $u \in (\mathbb{R}^n)^{n-1}$ such that $(A \vee B)(u)$ is a segment we have $(A \vee B)(u) \subset A(u) \cup B(u)$.*

Proof. The corollary follows from the fact that any iterated support set of a polytope is also a support set in the usual sense. □

In some important cases convexity of $A \cup B$ allows us to reduce easily a pair (A, B).

Proposition 3.8. *Let $A, A_0, B, B_0, P, I \in \mathcal{K}(X)$, $A_0 \cap P = B_0 \cap P = I$, $A = A_0 \cup P$ and $B = B_0 \cup P$. Then $(A, B) \sim (A_0, B_0)$.*

Proof. The pairs (A_0, P) and (B_0, P) are convex (see Figure 2a). Then by Proposition 3.1(d) we obtain $A_0 + P = A + I$ and $B + I = B_0 + P$. After adding sides of equations and applying the cancellation law we get $A_0 + B = A + B_0$. □

Lemma 3.9. *Let $A, B, C \in \mathcal{K}(X)$. If $A \cup B$ is convex, then $(A + C) \cap (B + C) = A \cap B + C$.*

Proof. Obviously the pair $(A+C, B+C)$ is convex. By Proposition 3.1(d) we obtain $(A + C) \cup (B + C) = A \cup B + C$. Then $(A + C) \cup (B + C) + (A + C) \cap (B + C) = A + C + B + C = (A \cup B + C) + (A \cap B + C)$. Applying the cancellation law we get $(A + C) \cap (B + C) = A \cap B + C$. □

The next proposition, illustated in Figure 2b), follows from Proposition 3.8.

Proposition 3.10. *Let $A, B, C, D, A_1, B_1, C_1, D_1, P, P_1, P_2 = P_1 + x \in \mathcal{K}(X)$, $A_1 = A \cup P, B_1 = A \cup P_1, C_1 = C \cup P, D_1 = C \cup P_2$, $B = A_1 \cup B_1, A = A_1 \cap B_1, D = C_1 \cup D_1$ and $C = C_1 \cap D_1$. Then $(A_1, B_1) \sim (C_1, D_1)$ and $(A, B) \sim (C, D)$.*

Proof. By Proposition 3.8 we obtain $(A_1, C_1) \sim (A, C)$. Similarly, $(B_1, D_1 - x) \sim (A, C - x)$. Then $(B_1, D_1) \sim (A, C)$. Hence $(A_1, C_1) \sim (B_1, D_1)$ which implies $(A_1, B_1) \sim (C_1, D_1)$.

Now, $A_1 \vee B_1 + C_1 = (A_1 + C_1) \vee (B_1 + C_1) = (A_1 + C_1) \vee (A_1 + D_1) = C_1 \vee D_1 + A_1$. Similarly, $A_1 \vee B_1 + D_1 = C_1 \vee D_1 + B_1$. Since the pairs (A_1, B_1), (C_1, D_1) are convex, by Lemma 3.9 we have $A_1 \vee B_1 + C_1 \cap D_1 = (A_1 \vee B_1 + C_1) \cap (A_1 \vee B_1 + D_1) = (C_1 \vee D_1 + A_1) \cap (C_1 \vee D_1 + B_1) = C_1 \vee D_1 + A_1 \cap B_1$. Hence $B + C = D + A$, and $(A, B) \sim (C, D)$. □

Example 3.11. Let A_1, B_1 be unions of a regular octahedron and regular tetrahedron illustrated in Figure 3. Let $A_1 \cap B_1$ be the common octahedron and $A_1 \cup B_1$ be a rhombohedron. Similarly, let C_1, D_1 be unions of an elongated octahedron and regular tetrahedron. By Proposition 3.10 we obtain equivalent pairs $(A_1, B_1) \sim (C_1, D_1)$

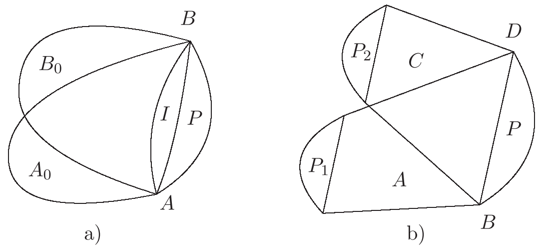

FIGURE 2. Illustrations to Propositions 3.8 and 3.10.

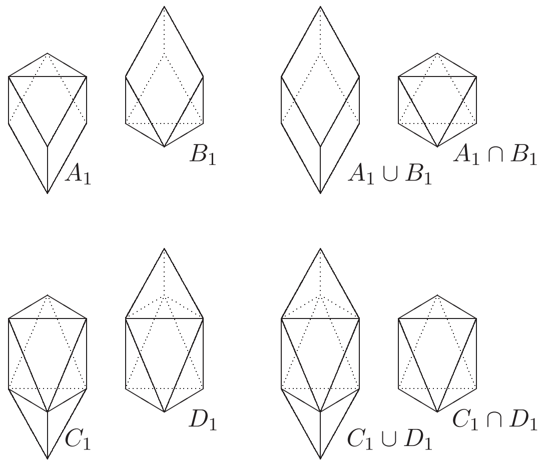

FIGURE 3. The sets described in Example 3.11

and $(A_1 \cup B_1, A_1 \cap B_1) \sim (C_1 \cup D_1, C_1 \cap D_1)$. In fact the pairs $(A_1 \cup B_1, A_1 \cap B_1)$ and $(C_1 \cup D_1, C_1 \cap D_1)$ were first example of equivalent minimal pairs not connected by translation [6].

The following lemma is used in the proof of Proposition 4.11.

Lemma 3.12. *Let $(A, B), (C, D)$ be convex pairs. Then a pair $((A + D) \vee (B + C), B + D)$ is convex and $(A + D) \vee (B + C) \vee (B + D) = (B + C \vee D) \vee (D + A \vee B)$ and $((A + D) \vee (B + C)) \cap (B + D) = (B + C \cap D) \vee (D + A \cap B)$.*

Proof. Let $E = (A+D) \vee (B+C)$ and $F = B+D$. Then $E \vee F = (A+D) \vee (B+D) \vee (B+C) = (A+D) \vee (B+D \vee C) = (A \vee B+D) \vee (B+C) = (A \vee B+D) \vee (B+D \vee C)$. Now we have $E+F = (A+D) \vee (B+C)+B+D = (A+B+2D) \vee (C+D+2B) = (D+A \vee B+A \cap B+D) \vee (B+C \vee D+C \cap D+B) \subset (B+C \vee D) \vee (D+A \vee B)+(B+C \cap D) \vee (D+A \cap B)$. Since $(B+C \cap D) \vee (D+A \cap B) \subset E \cap F$, we obtain that $E+F = E \vee F+E \cap F$ where $E \cap F = (B+C \cap D) \vee (D+A \cap B)$. $\square$

The Minkowski–Rådström–Hörmander space $\widetilde{X} = \mathcal{K}^2(X)/_\sim$ is a lattice with an ordering $[A, B] \leqslant [C, D] :\Longleftrightarrow A + D \subset B + C$. Let us notice that for any two classes $[A, B]$ and $[C, D]$ we have $\sup\{[A, B], [C, D]\} = [(A + D) \vee (B + C), B + D]$. By Lemma 3.12 convexity of pairs (A, B) and (C, D) implies convexity of the pair $((A+D) \vee (B+C), B+D)$ which is a natural representation of $\sup\{[A, B], [C, D]\}$.

Analogous result to Lemma 3.12 holds true for $\inf\{[A,B],[C,D]\} = [A+C,(A+D)\vee(B+C)]$.

4. Convex classes of pairs of sets

A *convex class* is an element of Minkowski–Rådström–Hörmander space $\widetilde{X} = \mathcal{K}^2(X)/_\sim$ such that all pairs of sets belonging to this class are convex pairs. The following sufficient condition for a convex class was given in [8].

Proposition 4.1. (Theorem 1 in [8], Theorem 6.4.4 in [16]) *Let $A,B \in \mathcal{K}(\mathbb{R}^n)$, $n \geqslant 2$. If for all $u \in \mathbb{R}^n$ we have $(A\vee B)(u) = A(u)$ or $(A\vee B)(u) = B(u)$ then the class $[A,B]$ is convex.*

Remark 4.2. Applying Proposition 4.1 to the pair (A_1,B_1) from Example 3.11 we observe that the class $[A_1,B_1]$ is convex.

The following proposition strengthens Proposition 4.1.

Proposition 4.3. *Let $A,B \in \mathcal{K}(\mathbb{R}^n)$, $n \geqslant 2$. If for all $u \in \mathbb{R}^n$ we have $(A\vee B)(u) = A(u)$ or $(A\vee B)(u) = B(u)$ or the class $[A(u),B(u)]$ is convex then the class $[A,B]$ is convex.*

Proof. Assume that the class $[A,B]$ is not convex. Then there exists a pair $(C,D) \in [A,B]$ which is not convex. By Proposition 3.4 we have $(C\vee D)(u) \not\subset C(u)\cup D(u)$ for some u. Then $(C\vee D)(u) = C(u)\vee D(u)$. We observe that the equality $A+D = B+C$ implies that $A(u)+D(u) = (A+D)(u) = (B+C)(u) = B(u)+C(u)$. Hence the pairs $\big(A(u),B(u)\big)$ and $\big(C(u),D(u)\big)$ are equivalent. Since $A\vee B+C = (A+C)\vee(B+C) = (A+C)\vee(A+D) = C\vee D+A$, we obtain that $(A\vee B, C\vee D) \sim (A,C) \sim (B,D)$. Thus $\big((A\vee B)(u),(C\vee D)(u)\big) \sim \big(A(u),C(u)\big) \sim \big(B(u),D(u)\big)$. By assumption the set $(A\vee B)(u)$ is equal to $A(u)$ or $B(u)$ or the class $[A(u),B(u)]$ is convex. First two cases imply that the set $(C\vee D)(u)$ is equal, respectively, to $C(u)$ or $D(u)$. If the class $[A(u),B(u)]$ is convex then the pair $\big(C(u),D(u)\big)$ is convex. Hence $(C\vee D)(u) = C(u)\vee D(u) \subset C(u)\cup D(u)$, a contradiction. □

Proposition 4.4. *Let $A,B \in \mathcal{K}(\mathbb{R}^n)$, $n \geqslant 2$. For all $u = (u_1,...,u_k) \in (\mathbb{R}^n)^k$, such that vectors $u_1,\dots,u_k$ are pairwise orthogonal, let the following disjunction hold true: $(A\vee B)(u) = A(u)$ or $(A\vee B)(u) = B(u)$ or the class $[A(u),B(u)]$ is convex. Then the class $[A,B]$ is convex.*

Proof. (1) The proposition follows from Proposition 4.3 by applying mathematical induction.

(2) This proposition can be also proved by applying criterion of convexity of a pair of sets from Proposition 3.5 and using a reasoning similar to the reasoning from the proof of Proposition 4.3. □

Corollary 4.5. *Let $A,B \in \mathcal{K}(\mathbb{R}^n)$, $n \geqslant 2$. For all $u = (u_1,...,u_{n-1}) \in (\mathbb{R}^n)^{n-1}$, such that vectors $u_1,\dots,u_{n-1}$ are pairwise orthogonal, let the following disjunction hold true: $(A\vee B)(u) = A(u)$ or $(A\vee B)(u) = B(u)$. Then the class $[A,B]$ is convex.*

Proof. Notice that the sets $A(u)$, $B(u)$ and $(A\vee B)(u)$ are segments or singletons. Then the class $[A(u),B(u)]$ is convex if and only if $A(u)\subset B(u)$ or $B(u)\subset A(u)$. □

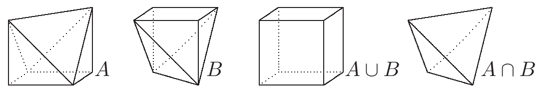

FIGURE 4. The sets from Example 4.7

Corollary 4.6. *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$, $n \geqslant 2$, be two polytopes. For all $u \in \mathbb{R}^n$, such that the set $(A \vee B)(u)$ is a segment, let the following disjunction hold true: $(A \vee B)(u) = A(u)$ or $(A \vee B)(u) = B(u)$. Then the class $[A, B]$ is convex.*

Example 4.7. Let $J := \{0,1\}^3 \subset \mathbb{R}^3$, $A := \mathrm{conv}(J \setminus \{(1,1,1),(0,0,1)\})$ and $B := \mathrm{conv}(J \setminus \{(0,1,0),(1,0,0)\})$. Notice that $A \cup B = \mathrm{conv} J$. Each edge of the union $A \cup B$ is either an edge of A or of B. By Corollary 4.6 the class $[A, B]$ is convex. Notice that a square $(A \cup B)(u)$, $u = (-1, 0, 0)$ is a union of two triangles $A(u) = \mathrm{conv}\{(0,0,0),(0,0,1),(0,1,1)\}$ and $B(u) = \mathrm{conv}\{(0,0,0),(0,1,0),(0,1,1)\}$. Hence Proposition 4.1 is not sufficient to prove that the class $[A, B]$ is convex. The sets A and B are pictured in Figure 4

Proposition 4.8. *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$, $n \geqslant 2$, be two polytopes. If the class $[A, B]$ is convex then for all $u \in \mathbb{R}^n$, such that the set $(A \vee B)(u)$ is a segment, the following disjunction holds true: $(A \vee B)(u) = A(u)$ or $(A \vee B)(u) = B(u)$.*

Proof. Assume that for some $u \in \mathbb{R}^n$ the set $(A \vee B)(u)$ is a segment, $(A \vee B)(u) \neq A(u)$ and $(A \vee B)(u) \neq B(u)$. If $(A \vee B)(u) \neq A(u) \cup B(u)$ then the pair (A, B) is not convex, and, obviously, the class $[A, B]$ is not convex. Now, assume that $(A \vee B)(u) = A(u) \cup B(u)$. Then both support sets $A(u)$ and $B(u)$ are segments contained in a straight line. Let $A(u)$ be a segment $a_1 \vee a_2$ and $B(u) := b_1 \vee b_2$. Without loosing generality we may assume that $b_2 - b_1 = t(a_2 - a_1)$ with $t \geqslant 1$ and $b_1 - a_1 = s(a_2 - a_1)$ with $s > 0$. Let I be a $(n-2)$-dimensional hypercube contained in a $(n-2)$-dimensional subspace of $\mathbb{R}^n$ perpendicular to the vectors u and $a_2 - a_1$. Let x be a vector in $\mathbb{R}^n$ such that the set $P := \mathrm{conv}(A(u) + I) \cup \{x\}$ is a pyramid with $P(u) = \{x\}$. Denote $Q := P + (b_1 - a_1) \vee (b_2 - a_2)$. We may choose x sufficiently close to the center of the set $A(u) + I$ that the pairs $(A + I, P)$ and $(B + I, Q)$ are convex. Notice that $(A + I) \cap P = (A + I)(u) = A(u) + I$ and $(B+I) \cap Q = (B+I)(u) = B(u) + I$. Then the pair $(C, D) := \big((A+I) \cup P, (B+I) \cup Q\big)$ is equivalent to (A, B). Notice that $C(u) = P(u) = \{x\}$ and $D(u) = Q(u) = x + (b_1 - a_1) \vee (b_2 - a_2)$. The equalities $b_2 - b_1 = t(a_2 - a_1)$ and $b_1 - a_1 = s(a_2 - a_1)$ imply that $b_2 - a_2 = b_1 + t(a_2 - a_1) - a_2 = (b_1 - a_1) + (t-1)(a_2 - a_1) = (s+t-1)(a_2 - a_1)$. Hence 0 does not belong to $(b_1 - a_1) \vee (b_2 - a_2)$, and the set $C(u) \cup D(u)$ is not convex. Therefore, the pair (C, D) is not convex, and the class $[A, B]$ is not convex. $\square$

Remark 4.9. If a pair of strictly convex sets A and B is convex then the class $[A, B]$ is convex.

The following theorem is a direct corollary from Corollary 4.6 and Proposition 4.8.

Theorem 4.10. *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$, $n \geqslant 2$, be two polytopes. Then the class $[A, B]$ is convex if and only if for all $u \in \mathbb{R}^n$, such that the set $(A \vee B)(u)$ is a segment, the following disjunction holds true: $(A \vee B)(u) = A(u)$ or $(A \vee B)(u) = B(u)$.*

Theorem 4.10 probably cannot be extended to all compact convex sets.

Notice that the family of all convex classes is not closed with respect to Minkowski sum. For example take subsets of one-dimensional space $\mathbb{R}$: $A := [0,1]$, $B := \{0\}$, $C := \{1\}$ and $D := [0,1]$. We obtain $A+C = [1,2]$ and $B+D = [0,1]$. The classes $[A,B]$ and $[C,D]$ are obviously convex. Yet the class $[A+C, B+D] = [\{1\},\{0\}]$ is not convex. Also, if some class $[A,B]$ is convex then $[B,A]$ is convex, too. The following Proposition shows that the family of all convex classes is closed with respect to supremum and infimum.

Proposition 4.11. *Let A,B,C,D be polytops in $X = \mathbb{R}^n$ or convex sets contained in $X = \mathbb{R}^2$. If $[A,B]$ and $[C,D]$ are convex classes, then the classes $\sup([A,B],[C,D])$ and $\inf([A,B],[C,D])$ are also convex.*

Proof. Notice that $\sup([A,B],[C,D]) = [(A+D)\vee(B+C), B+D]$. By Proposition 4.8 for all $u \in \mathbb{R}^n$ if a support set $(A\vee B)(u)$ is a segment, then $(A\vee B)(u) = A(u)$ or $(A\vee B)(u) = B(u)$ and if $(C\vee D)(u)$ is a segment, $(C\vee D)(u) = C(u)$ or $(C\vee D)(u) = D(u)$. Consider $((A+D)\vee(B+C))(u) \subset (A\vee B+C\vee D)(u)$. Denote $E = \big((A+D)\vee(B+C)\big)\vee(B+D)$ and assume that $E(u)$ is a segment.

If $h_{A+D}(u) > h_{B+C}(u)$ then $((A+D)\vee(B+C))(u) = (A+D)(u)$. Then $E(u) = \big((A+D)\vee(B+C)\vee(B+D)\big)(u) = ((A+D)\vee(B+D))(u) = ((A\vee B)+D)(u) = (A\vee B)(u)+D(u)$. Notice that the support set $(A\vee B)(u)$ is a segment or a singleton. Moreover, the class $[A,B]$ is convex. By Proposition 4.8 we have two possibilities. First, if $(A\vee B)(u) = A(u)$, then $E(u) = A(u)+D(u) = (A+D)(u) = ((A+D)\vee(B+C))(u)$. Second, if $(A\vee B)(u) = B(u)$, then $E(u) = B(u)+D(u) = (B+D)(u)$.

The case of $h_{A+D}(u) < h_{B+C}(u)$ is similar.

Let us assume that $h_{A+D}(u) = h_{B+C}(u)$. If $h_{A+D}(u) > h_{B+D}(u)$, then $E(u) = ((A+D)\vee(B+C))(u)$. If $h_{A+D}(u) = h_{B+D}(u)$, then $h_A(u) = h_B(u)$, and also $h_C(u) = h_D(u)$. Then, by Lemma 3.12, we obtain $E(u) = ((B+C\vee D)\vee(D+A\vee B))(u) = (B+C\vee D)(u)\vee(D+A\vee B)(u)$. Hence support sets $(A\vee B)(u)$ and $(C\vee D)(u)$ are segments or singletons. By convexity of classes $[A,B]$, $[C,D]$ we obtain $A(u)\subset B(u)$ or $B(u)\subset A(u)$ and $C(u)\subset D(u)$ or $D(u)\subset C(u)$. If $A(u)\subset B(u)$ and $C(u)\subset D(u)$, then $E(u) = (B+D)(u)$. In the other case $B(u)\subset A(u)$ or $D(u)\subset C(u)$, and $(B+D)(u)\subset((A+D)\vee(B+C))(u)$. Hence $E(u) = A(u)+D(u) = ((A+D)\vee(B+C))(u)$.

We have just proved the sufficient and necessary condition (Theorem 4.10) for the convexity of the class $[(A+D)\vee(B+C), B+D]$. Similarly, one can prove the convexity of the class $[A+C,(A+D)\vee(B+C)] = \inf([A,B],[C,D])$. $\square$

Let us notice that if a pair (A,B) is convex then $\min(h_A,h_B) = h_{A\cap B}$ is a convex function. If a class $[A,B]$ is convex and $f = h_A - h_B$, then for any representation $f = g - h$ as a difference of sublinear functions the function $\min(g,h)$ is convex.

References

[1] C. Bauer, *Minimal and reduced pairs of convex bodies*, Geom. Dedicata **62** (1996), 179–192.

[2] E. Caprari and J-P. Penot, *Tangentially ds functions*, Optimization **56** (1-2) (2007), 25–38.

[3] V.F. Demyanov and A.M. Rubinov, *Quasidifferential Calculus*, Optimization Software, New York, 1986.

[4] V.F. Demyanov and A.M. Rubinov, *Quasidifferentiability and Related Topics*, Nonconvex Optimization and its Applications, Kluwer, Dortrecht–Boston–London, 2000.
[5] L. Drewnowski, *Additive and countably additive correspondences*, Comment. Math. **19** (1976), 25–54.
[6] J. Grzybowski, *Minimal pairs of compact sets*, Arch. Math. **63** (1994), 173–181.
[7] J. Grzybowski, D. Pallaschke and R. Urbański, *Minimal pairs of bounded closed convex sets as minimal representation of elements of the Minkowski–Rådström–Hörmander spaces*, Banach Center Publications **84** (2009), 31–55.
[8] J. Grzybowski and R. Urbański, *On convex class of convex bodies*, Proc. Amer. Math. Soc. **125** (11) (1997), 3397–3401.
[9] J. Grzybowski and R. Urbański, *Order cancellation law in the family of bounded convex sets*, J. Global Optim. **77** (2020), 289–300.
[10] L. Hörmander, *Sur la fonction d'appui des ensembles convexes dans un espace localement convexe*, Arkiv Math. **3** (1954), 181–186.
[11] S. Markov, *On the algebraic properties of convex bodies and some applications*, J. Conv. Anal. **7** (2000), 129–166.
[12] S. Markov, *Quasilinear spaces and their relation to vector spaces*, Electron. J. Math. Comput. **2** (1) (2005), 1–21.
[13] O. Mayer, *Algebraische und metrische Strukturen in der Intervall-rechnung und einige Anwendungen*, Computing **5** (1970), 144–162.
[14] G.M. Molnár and Z. Páles, *An extension of the Rådström cancellation theorem to cornets*, Semigroup Forum **102** (2021), 765–793.
[15] D. Pallaschke, S. Scholtes and R. Urbański, *On minimal pairs of convex compact sets*, Bull. Polish Acad. Sci. Math. **39** (1991), 1–5.
[16] D. Pallaschke and R. Urbański, *Pairs of Compact Convex Sets, Fractional Arithmetic with Convex Sets*, Mathematics and Its Applications, Kluwer Academic. Publisher, Dortrecht–Boston–London, 2002.
[17] A.G. Pinsker, *The space of convex sets of a locally convex space*, Trudy Leningrad Engineering-Economic Institute **63** (1966), 13–17.
[18] H. Rådström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. **3** (1952), 165–169.
[19] S. Scholtes, *Minimal pairs of convex bodies in two dimensions,* Mathematika **39** (2) (1992), 267–273.
[20] R. Urbański, *A generalization of the Minkowski–Rådström–Hörmander Theorem*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **24** (1976), 709–715.

Manuscript received May 1 2022
revised July 10 2022

J. Grzybowski
Adam Mickiewicz University, Faculty of Mathematics and Computer Science, Uniwersytetu Poznańskiego 4, 61-614 Poznań, Poland
E-mail address: `jgrz@amu.edu.pl`

R. Urbański
Adam Mickiewicz University, Faculty of Mathematics and Computer Science, Uniwersytetu Poznańskiego 4, 61-614 Poznań, Poland
E-mail address: `rich@amu.edu.pl`