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RECENT ADVANCES IN MULTIOBJECTIVE CONVEX SEMI-INFINITE OPTIMIZATION

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Abstract. This paper reviews the existing literature on multiobjective (or vector) semi-infinite optimization problems, which are defined by finitely many convex objective functions of finitely many variables whose feasible sets are described by infinitely many convex constraints. The paper shows several applications of this type of optimization problems and presents a state-of-the-art review of its methods and theoretical developments (in particular, optimality, duality, and stability).

1. Introduction

The adjective "semi-infinite" was coined in 1964 by K.O. Kortanek [57] for those optimization problems whose feasible sets can be expressed as

$$X = \{x \in \mathbb{R}^n : g(x,t) \leq 0, t \in T\},$$

where the index set T is infinite and $g_t := g(\cdot, t)$ is a scalar function for all $t \in T$. These semi-infinite programming (SIP in short) problems are called *continuous* if T is compact and $(t,x) \longmapsto g_t(x)$ is continuous on $T \times \mathbb{R}^n$, and *scalar* (respectively, *multiobjective* or *vector*) when they have a real-valued (a vector-valued, resp.) objective function. The adjective "vector" is also referred in the title of some works to optimization problems where the objective function takes values in (possibly infinite dimensional) linear spaces or even in more general algebraic structures (e.g., ordered groups, as in [41]). Along this paper we denote the objective function by $f : \mathbb{R}^n \to \mathbb{R}^p$, with $p = 1$ when SIP is scalar and $p \geq 2$ otherwise, in which case the $j-$th component of f is denoted by f_j. A SIP problem is said to be *linear* (LSIP) when f_j is linear for all j and g_t is affine for all t, *convex* (CSIP) when all these functions are convex, and *nonconvex* (NCSIP) when at least one of these functions is nonconvex. An important extension of SIP, called *generalized* (GSIP), deals with those optimization problems whose feasible set can be expressed as $X = \{x \in \mathbb{R}^n : g(x,t) \leq 0, t \in T(x)\}$, meaning that the index set $T(x)$ depends on the decision variable x.

According to MathSciNet, more than 800 published works containing one of the phrases "semi-infinite programming" or "semi-infinite optimization" in their titles have been published since 1964 (the term "programming" being approximately twice

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more frequent than "optimization"). Kortanek was also one of the authors of the first seven papers on SIP (more precisely on LSIP) published between 1965 and 1973, co-authored by Charnes, Cooper or Gustafson. Regarding nonlinear SIP (NLSIP), the first two papers were published in 1976 and 1978 by Hoffmann and Klostermair and by Hettich and Jongen [42], respectively. The latter paper has been very influential in numerical SIP as it provided the theoretical tools for the (finite) local reduction approach based on suitable optimality conditions involving finitely many constraints. Kortanek, Hettich and Jongen, all of them authors of up to 20 publications including the term "semi-infinite" in their titles, are generally considered the founding fathers of SIP.

All SIP problems considered in the literature have been of scalar nature until the publication by Todorov of 4 papers on multiobjective LSIP at the end of the 1980s ([86], [87], [88], [89]). It is worth observing that the adjective "multiobjective" for these SIP problems became progressively more frequent than "vector": to be precise, "multiobjective" is twice more frequent in the titles than "vector" in the more than 90 papers published up to now on nonscalar SIP. In contrast with the literature on scalar SIP, where most papers deal with numerical methods, most works published on multiobjective SIP are focused on theoretical aspects as optimality conditions (the majority), duality, and stability of the different sets of solutions, while they seldom deal with numerical methods, probably because these problems are usually solved in practice via scalarization, that is, reducing them to a parametric family of scalar SIP problems providing different types of optimal solutions. This could be the reason why, to the best of our knowledge, this is the first survey paper dealing with multiobjective SIP, while scalar SIP has been reviewed many times, the first one by Hettich and Kortanek in a famous survey on all types of scalar SIP problems published in 1993 [43]. The surveys published after [66] (2007) deal with particular types of scalar SIP problems: generalized SIP ([85], 2012), LSIP ([31], 2018), and NCSIP ([20], 2021).

The paper is organized as follows. In Section 2 we show some motivating examples of multiobjective CSIP and discuss the linearization of multiobjective CSIP problems. Section 3 analyzes the scalarization of multiobjective CSIP problems and reviews contributions to scalar CSIP numerical methods dated after 2007. Section 4 compares the main constraint qualifications (CQs in short) in CSIP allowing to characterize, in Section 5, different types of solutions. Finally, Sections 6 and 7 discuss duality and stability issues.

2. Preliminaries

We consider in this paper multiobjective optimization problems of the form

$$(2.1) \qquad (CSIP) \quad \text{“}\min_{x\in\mathbb{R}^n}\text{”}\ f(x) := (f_1(x), \ldots, f_p(x)) \text{ s.t. } g(t,x) \leq 0,\ t \in T,$$

where T is infinite, $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a real-valued convex function for all $i \in I := \{1, \ldots, p\}$ and $g_t : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a convex lower semicontinuous (lsc in brief) proper function for all $t \in T$. One can aggregate to $(CSIP)$ a set constraint $x \in C$, where $C \subset \mathbb{R}^n$ is a given closed and convex set, by adding to (2.1) the constraint $\delta_C(x) \leq 0$, with $\delta_C : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ being the indicator function of C (i.e., $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ otherwise). We also assume along the paper that the (closed and convex) feasible set of $(CSIP)$ is non-empty, that is,

$$X := \{x \in \mathbb{R}^n : g_t(x) \leq 0, \forall t \in T\} \neq \emptyset.$$

We also consider the subfamily of multiobjective linear SIP problems, that can be formulated as

$$(2.2) \qquad (LSIP) \quad \text{“}\min_{x\in\mathbb{R}^n}\text{”}\ f(x) := \left(c_1'x, \ldots, c_p'x\right) \text{ s.t. } a_t'x \leq b_t,\ t \in T,$$

whose data are the vectors c_i, $i \in I$, and a_t, $t \in T$, and the scalars b_t, $t \in T$, and where $c_i'x$ represents the scalar product of c_i by x.

The task “min” in $(CSIP)$ and $(LSIP)$ consists in finding solutions whose definitions involve three partial orderings on $\mathbb{R}^p$. Given $x, y \in \mathbb{R}^p$, we write $x \leqq y$ (resp. $x < y$) when $x_i \leq y_i$ (resp. $x_i < y_i$) for all $i \in I$. Moreover, we write $x \leq y$ if $x \leqq y$ and $x \neq y$. We only consider here four types of solutions for $(CSIP)$:

- $\widehat{x} \in X$ is a *weak efficient solution* of $(CSIP)$ if it does not exist $x \in X$ such that $f(x) < f(\widehat{x})$.
- $\widehat{x} \in X$ is an *efficient solution* (or *Pareto minimizer*) of $(CSIP)$ if it does not exist $x \in X$ such that $f(x) \leq f(\widehat{x})$.
- $\widehat{x} \in X$ is a (*Geoffrion*) *proper efficient solution* of $(CSIP)$ if there exists $M > 0$ such that for all $i \in I$ and $x \in S$ with $f_i(x) < f_i(\widehat{x})$, there exists $j \in \{1, ..., p\}$ such that

$$f_j(x) > f_j(\widehat{x}) \text{ and } \frac{f_i(\widehat{x}) - f_i(x)}{f_j(x) - f_j(\widehat{x})} < M.$$

  Observe that there are many notions of proper efficiency in the literature, as those of Benson, Borwein and Henig, but all of them are equivalent to the above one thanks to the convexity assumption (see, e.g., [21]).
- $\widehat{x} \in X$ is a *sharp efficient solution* (or *isolated efficient solution*) of $(CSIP)$ if there exists $\nu > 0$ such that

$$\max_{i\in I}\{f_i(x) - f_i(\widehat{x})\} \geq \nu \left\|x - \widehat{x}\right\|,\ \forall x \in X,$$

  where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

The corresponding sets of solutions are denoted by X_W, X_E, X_P, and X_S, respectively. One has $X_S \subset X_P \subset X_E \subset X_W$. Moreover, $X = X_W$ whenever one component of f is identically zero, and $X = X_P$ in the trivial case that f is the null function. It is also known that $f(X_P)$ is dense in $f(X_E)$ ([40]; see also [21, Theorem 3.17]). Due to the convexity of $(CSIP)$, X_W is connected [21, Theorem 3.38]. If X is bounded, then X_E is connected, too [21, Theorem 3.40].

Roughly speaking, X_W and X_P are easily computable via scalarization, X_E (the so-called Pareto frontier of $(CSIP)$) is preferred by most users in most applications, and X_S is stable under Lipschitzian perturbations of the objective functions in the following sense: if $\widehat{x} \in X_S$ for (SIP), the same is true when $f(x)$ is replaced by $f(x) + Cx$, when the $p \times n$ matrix C is sufficiently small [29]. We do not consider local solutions in this survey because, due to the convexity assumptions, local and global concepts coincide in CSIP. When $p = 1$ (scalar SIP), $X_W = X_E = X_P$ is the optimal set while X_S is the set of sharp minima (also called strongly unique solutions).

2.1. **Some examples.** Let us now present some examples of multiobjective CSIP problems.

Example 1 (Simultaneous Chebyshev best approximation)**.** ([7], [37]) Let $\left\{\psi_j^{(i)}\right\}_{j=1}^{n}$, $i = 1, \ldots, p$ $(p > 1)$, be p families of n real-valued continuous functions on the interval $[a, b]$. Let $\psi_0^{(i)}$, $i \in I := \{1, \ldots, p\}$, be p given real valued continuous functions on an interval $[a, b]$. For $i \in I$ define the function

$$f_i(x) := \left\| \psi_0^{(i)} - \sum_{j=1}^{n} x_j \psi_j^{(i)} \right\|_\infty = \max_{t\in[a,b]} \left| \psi_0^{(i)}(t) - \sum_{j=1}^{n} x_j \psi_j^{(i)}(t) \right|,$$

where $\|\cdot\|_\infty$ denotes the Chebyshev (or uniform) norm on the linear space $C([a, b])$ of real-valued continuous functions on $[a, b]$. The simultaneous Chebyshev best approximation problem consists in finding a solution $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ of the multiobjective non-differentiable optimization problem

$$\text{“}\min\text{”}\,(f_1(x), \ldots, f_p(x)) \ \text{ s.t. } \ x \in C \subset \mathbb{R}^n, \tag{2.3}$$

where C stands for some closed convex set. The epigraphical reformulation of (2.3) yields the equivalent problem

$$\text{“}\min_{(x,y)\in C\times\mathbb{R}^p}\text{”} \ \ (y_1, \ldots, y_p) \ \text{ s.t. } \left\| \psi_0^{(i)} - \sum_{j=1}^{n} x_j \psi_j^{(i)} \right\|_\infty \le y_i, \ i \in I,$$

which can be reformulated as

$$\begin{array}{lll} (CSIP) & \text{“}\min_{(x,y)\in C\times\mathbb{R}^p}\text{”} & (y_1, \ldots, y_p) \\ & \text{s.t.} & \psi_0^{(i)}(t) - \sum_{j=1}^{n} x_j \psi_j^{(i)}(t) \le y_i, i \in I, t \in [a, b], \\ & & -\psi_0^{(i)}(t) + \sum_{j=1}^{n} x_j \psi_j^{(i)}(t) \le y_i, i \in I, t \in [a, b], \end{array}$$

where all the involved functions are linear in the variables x and y, except the hidden constraint function, δ_C. So, $(CSIP)$ is a multiobjective nonlinear CSIP problem.

Example 2 (Multinorm one-sided best approximation)**.** [26] Let $\psi_0, \psi_1, ..., \psi_n$ be continuous functions on an interval $[a, b]$ and let $\{\|\cdot\|_i, i = 1, ..., p\}$ be a given family of norms on $C([a, b])$. Consider the problem consisting of computing a “good” approximation to ψ_0 from above on $[a, b]$ by means of linear combinations of $\psi_1, ..., \psi_n$, when it is not obvious how to measure the approximation error based on the norms $\|\cdot\|_i$, $i \in I$. For each $i \in I$, we denote by $f_i(x) := \left\| \psi_0 - \sum_{j=1}^{n} x_j \psi_j \right\|_i$ the error, measured with $\|\cdot\|_i$, corresponding to a decision x. We thus have the multiobjective linearly constrained convex problem

$$\begin{array}{lll} (CSIP) & \text{“}\min_{x\in\mathbb{R}^n}\text{”} & (f_1(x), \ldots, f_p(x)) \\ & \text{s.t.} & \psi_0(t) \le \sum_{i=1}^{n} \psi_i(t) x_i, t \in [a, b]. \end{array}$$

Consider now the particular case where the family of norms is limited to the Chebyshev and the Manhattan norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$. Since $\psi_0(t) \le \sum_{i=1}^{n} \psi_i(t) x_i$, for all

$t \in [a, b]$ and for all feasible x,

$$\left\| \psi_0 - \sum_{i=1}^{n} x_i \psi_i \right\|_\infty = \max_{t \in [a,b]} \left\{ \sum_{i=1}^{n} x_i \psi_i(t) - \psi_0(t) \right\}$$

and

$$\begin{aligned} \|\psi_0 - \textstyle\sum_{i=1}^n x_i \psi_i\|_1 &= \int_a^b \left[\textstyle\sum_{i=1}^n \psi_i(t) x_i - \psi_0(t)\right] dt \\ &= \textstyle\sum_{i=1}^n \left(\int_a^b \psi_i(t) dt \right) x_i - \int_a^b \psi_0(t) dt. \end{aligned}$$

So, one has to solve the continuous biobjective LSIP problem

$$\begin{array}{lll} (LSIP) & \text{“}\min_{(x,x_{n+1}) \in \mathbb{R}^{n+1}}\text{”} & \left(x_{n+1}, \sum_{i=1}^n \left(\int_a^b \psi_i(t) dt \right) x_i \right) \\ & \text{s.t.} & \sum_{i=1}^n \psi_i(t) x_i \geq \psi_0(t), t \in [a, b], \\ & & x_{n+1} \geq \psi_0(t) - \sum_{i=1}^n \psi_i(t) x_i \geq -x_{n+1},\ t \in [a, b]. \end{array}$$

Example 3 (Robust convex optimization)**.** ([19], [44], [10], [47]) Consider an uncertain optimization problem

$$\begin{array}{lll} (P) & \text{”}\min_{x \in \mathbb{R}^n}\text{”} & f(x) = (f_1(x), \ldots, f_p(x)) \\ & \text{s.t.} & g_j(x) \leq 0,\ j = 1, ..., q, \end{array}$$

whose data f_i and g_j, are the result of perturbing those of some deterministic convex nominal problem

$$\begin{array}{lll} (\overline{P}) & \text{“}\min_{x \in \mathbb{R}^n}\text{”} & \overline{f}(x) = \left(\overline{f}_1(x), \ldots, \overline{f}_p(x)\right) \\ & & \text{s.t. } \overline{g}_j(x) \leq 0,\ j = 1, ..., q, \end{array}$$

such that $\overline{f}_i,\ i \in I := \{1, ..., p\}$, and $\overline{g}_j,\ j \in J = \{1, ..., q\}$, are convex functions. For the sake of simplicity, we make the assumption that the data of (P) depend on parameters ranging on independent uncertainty sets: $\mathcal{U}_i$, for f_i, $i \in I$, and $\mathcal{V}_j$, for g_j, $j \in J$ (singletons for the deterministic functions). The uncertainty set of (P) is $\mathcal{U} \times \mathcal{V}$ with $\mathcal{U} := \prod_{i=1}^p \mathcal{U}_i$ and $\mathcal{V} := \prod_{j=1}^q \mathcal{V}_j$. Each couple $(u, v) \in \mathcal{U} \times \mathcal{V}$ defines a scenario problem

$$\begin{array}{lll} (P_{u,v})_{u \in \mathcal{U}, v \in \mathcal{V}} & \text{“}\min_{x \in \mathbb{R}^n}\text{”} & f(x, u) = (f_i(x, u_i), ..., f_p(x, u_p)) \\ & \text{s.t.} & g_j(x, v_j) \leq 0,\ j \in J, \end{array}$$

with $(P_{\overline{u},\overline{v}}) \equiv \left(\overline{P}\right)$ for some nominal scenario $(\overline{u}, \overline{v}) \in \mathcal{U} \times \mathcal{V}$. In most practical applications each nominal function $\overline{h} \in \left\{\overline{f}_i, i \in I\right\} \cup \left\{\overline{g}_j, j \in J\right\}$ has a compact convex uncertainty set $\mathcal{W} \subset \mathbb{R}^{n+1}$ such that $0_{n+1} \in \operatorname{int} \mathcal{W}$ and is subject to affine perturbations, that is,

$$h(x, w) = \overline{h}(x) + w'x + z, (w, z) \in \mathcal{W}.$$

This uncertainty set is $\mathcal{W} = \mathcal{U}_i$ for $\overline{h} = \overline{f}_i$ and $\mathcal{W} = \mathcal{V}_j$ for $\overline{h} = \overline{g}_j$, so that we are assuming that $\mathcal{U}_i$ and $\mathcal{V}_j$ are compact convex subsets of $\mathbb{R}^{n+1}$ whose interiors contain the origin 0_{n+1}. The objective functions of the robust (or pessimistic) counterpart of (P) are the worst case functions $\varphi_i(\cdot) := \sup_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i)$ while the constraints are all conceivable perturbations of the constraints. In conclusion, since all these functions are convex, the robust counterpart of (P) is the multiobjective CSIP problem

$$\begin{array}{lll}(RCSIP) & \text{“}\min_{x\in\mathbb{R}^n}\text{”} & \varphi(x) := \big(\varphi_1(x), \ldots, \varphi_p(x)\big) \\ & \text{s.t.} & g_j(x, v_j) \leq 0,\ v_j \in \mathcal{V}_j,, j \in J.\end{array}$$

2.2. **Linearization of CSIP problems.** It is worth mentioning that any multiobjective CSIP problem as the one in (2.1) can be reformulated as an LSIP (that is, at least theoretically, any multiobjective CSIP problem is an LSIP one in desguise). To do this (as in [26]), we must recall some convex analysis concepts. Given a set $\emptyset \neq A \subseteq \mathbb{R}^n$, the closure of A, the interior of A, the relative interior of A, the linear span of A, the convex hull of A, and the convex cone (containing the origin) generated by A are respectively denoted by $\overline{A}$, $\operatorname{int} A$, $\operatorname{ri} A$, $\operatorname{span} A$, $\operatorname{conv} A$, and $\operatorname{cone} A$, respectively. The negative polar cone of A is $A^0 := \{x \in \mathbb{R}^n : a'x \leq 0, \forall a \in A\}$ and its strictly negative polar cone $A^- := \{x \in \mathbb{R}^n : a'x < 0, \forall a \in A\}$.

Given a function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$, the effective domain of h, the graph of h, the epigraph of h and the convex subdifferential of h are respectively denoted by $\operatorname{dom} h$, $\operatorname{gph} h$, $\operatorname{epi} h$ and ∂h. The Fenchel conjugate of h is the function $h^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that

$$h^*(v) = \sup\{v'x - h(x) : x \in \operatorname{dom} h\}.$$

It is well-known that, if h is a proper lower semicontinuous convex function, then h^* enjoys the same properties and its conjugate

$$h^{**}(x) = \sup\{v'x - h^*(v) : v \in \operatorname{dom} h^*\}$$

coincides with h.

Since all functions involved in $(CSIP)$ are proper, lsc and convex, we have

$$h_i(x) = h_i^{**}(x) = \sup_{u\in\operatorname{dom}\ h_i^*} \{x'u - h_i^*(u)\}, \forall i \in I. \tag{2.4}$$

Analogously, given $t \in T$,

$$\begin{aligned} g_t(x) \leq 0 &\iff g_t^{**}(x) \leq 0 \\ &\iff x'u - g_t^*(u) \leq 0, \forall u \in \operatorname{dom} g_t^* \\ &\iff u'x \leq g_t^*(u), \forall u \in \operatorname{dom} g_t^*. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5) we get that $(CSIP)$ is equivalent to the multiobjective LSIP problem

$$\begin{array}{lll}(LSIP) & \text{“}\min_{(z,x)\in\mathbb{R}^{n+p}}\text{”} & f(z, x) = (z_1, ..., z_p) \\ & \text{s.t.} & -z_i + u'x \leq h_i^*(u),\ u \in \operatorname{dom}\ h_i^*, i \in I, \\ & & u'x \leq g_t^*(u),\ (u, t) \in \operatorname{dom} g_t^* \times T.\end{array}$$

The drawback with this reformulation (linearization) of $(CSIP)$ is that some of its relevant properties (e.g., the compactness of the index set or desirable constraint qualifications) are not inherited by $(LSIP)$. However, direct linearizations may be occasionally useful.

Example 4. [26] Suppose that we can invest a capital M in n shares. For $j \in \{1, ..., n\}$, we denote by x_j the amount to be invested in the $j-$th share, and by r_j its rate of return. Obvious constraints are $\sum_{j=1}^{n} x_j = M$ and $x_j \geq 0$, $j = 1, ..., n$.

We express these, and possibly other linear constraints, in a compact way as $a_j' x \geq b_j, j = 1, ..., q$. In the (unrealistic) absence of uncertainty, the problem to be solved is the linear optimization one

$$(P) \quad \max_{x\in\mathbb{R}^n} r'x \ \text{ s.t. } \ a_j'x \geq b_j, j = 1, ..., q.$$

Unfortunately, r is in practice an uncertain vector. The uncertain problem (P) can be modeled in a variety of ways, taking into account that the decision maker intends to maximize its return at a minimum risk. If the probability distribution of r is unknown, the first objective for a pessimistic decision maker consists of maximizing $\inf_{r\in R} r'x$ (or, equivalently, minimizing $\sup_{r\in R}(-r'x)$), where $R \subset \mathbb{R}^n$ denotes the set of conceivable values of r. Concerning the risk, it is usually identified with the variance of the portfolio x, i.e., the uncertain number $x'Vx$, where V denotes the (positive definite) matrix of variances-covariances of r. So, the second objective consists of minimizing the quadratic convex function $h(x) := x'Vx$. Consequently, we have a biobjective LSIP problem. Indeed, since $h^*(u) = \frac{1}{4}u'V^{-1}u$ for all $u \in \mathbb{R}^n$, the equivalent problem is

$$\begin{array}{lll}(LSIP) & \text{“}\min_{(x,z)\in\mathbb{R}^{n+2}}\text{”} & f(z,x) = (z_1, z_2) \\ & \text{s.t.} & r'x + z_1 \geq 0, r \in R, \\ & & -u'x + z_2 \geq -\frac{1}{4}u'V^{-1}u, u \in \mathbb{R}^n, \\ & & a_j'x \geq b_j, j = 1, ..., q.\end{array}$$

We have mentioned at the beginning of this section two desirable properties of X : non-emptyness (assumed along the paper) and boundedness. The well-known characterizations of both properties in terms of the data for linear systems [30, Theorems 4.4 and 9.3] allow to obtain convex counterparts via the linear representation of X in (2.5):

$$X \neq \emptyset \iff (0_n, -1) \notin \overline{\text{cone}\left(\bigcup_{t\in T} \text{epi}\, g_t^* \cup \{(0_n, 1)\}\right)},$$

in which case,

$$\begin{array}{rl}X \text{ is bounded} & \iff (0_n, 1) \notin \text{int cone}\left(\bigcup_{t\in T} \text{epi}\, g_t^* \cup \{(0_n, 1)\}\right) \\ & \iff \text{cone}\left(\bigcup_{t\in T} \text{dom}\, g_t^*\right) = \mathbb{R}^n.\end{array}$$

3. Numerical issues

Multiobjective CSIP problems are usually solved via scalarization, which consists in obtaining solutions of the $(CSIP)$ in (2.1) by solving instances of some parametric scalar $(CSIP)$ problem. We now briefly describe the application to the multiobjective $(CSIP)$ in (2.1) of four of the many available scalarization methods (see, e.g., [1], [50], [54], [65], [106], etc.).

In the *linear weighted sum method* the parametric set is the standard simplex in $\mathbb{R}^p$,

$$\Delta_+^p := \{w \in \mathbb{R}^p : w_1 + ... + w_p = 1, w \geqq 0_p\}, \tag{3.1}$$

whose relative interior we denote by

$$\Delta_{++}^p = \{w \in \mathbb{R}^p : w_1 + ... + w_p = 1, w > 0_p\}.$$

The parametric (weighted) problem, depending on the parameter $w \in \Delta_{+}^{p}$, is the scalar CSIP problem

$$(3.2) \qquad (CSIPw) \quad \min_{x\in\mathbb{R}^n} \sum_{i=1}^{p} w_i f_i(x) \text{ s.t. } g(t,x) \leq 0,\ t \in T.$$

Let $\overline{x} \in X$. Then, according to [21, Propositions 3.9 and 3.10, and Theorems 3.11 and 3.15],

1. $\overline{x} \in X_P$ if and only if there exists $w \in \Delta_{++}^{p}$ such that $\overline{x}$ is a minimizer of $(CSIPw)$.
2. $\overline{x} \in X_W$ if and only if there exists $w \in \Delta_{+}^{p}$ such that $\overline{x}$ is a minimizer of $(CSIPw)$.

So, thanks to the convexity of $(CSIP)$, optimality for the weighted sum scalarization with $w \in \Delta_{++}^{p}$ (resp. $w \in \Delta_{+}^{p}$) is a necessary and sufficient condition for proper (weak, resp.) efficiency. Hence, this approach allows to compute all proper and weak efficient solutions.

In the *ε-constraint method* the parametric set is $\mathbb{R}^p \times I$ and the parametric problem corresponding to $(\varepsilon, j) \in \mathbb{R}^p \times I$ is the scalar CSIP problem

$$(3.3) \qquad \begin{array}{lll} (CSIP(\varepsilon,j)) & \min_{x\in\mathbb{R}^n} & f_j(x) \\ & \text{s.t.} & f_i(x) \leq \varepsilon_i, i \in I \diagdown \{j\}, \\ & & g(t,x) \leq 0,\ t \in T. \end{array}$$

Let $\overline{x} \in X$. Then, according to [21, Proposition 4.3 and Theorem 4.5] (based on [8]),

1. If $\overline{x}$ is a minimizer of $(CSIP(\varepsilon,j))$ for some $(\varepsilon,j) \in \mathbb{R}^p \times I$, then $\overline{x} \in X_W$.
2. $\overline{x} \in X_E$ if and only if there exists a vector $\varepsilon \in \mathbb{R}^p$ such that $\overline{x}$ is a minimizer of $(CSIP(\varepsilon,j))$ for all $j \in I$.

So, this approach allows to obtain all efficient solutions (the most important solutions from the practical point of view), but not all the weakly efficient solutions of $(CSIP)$.

In the so-called *hybrid method* the parametric set is again Δ_{+}^{p}. This method, which is inspired in the previous ones, requires the previous computation of a feasible solution $x^0 \in X$ of $(CSIP)$. Then, the parametric problem corresponding to $w \in \Delta_{+}^{p}$ is the scalar CSIP problem

$$(3.4) \qquad \begin{array}{lll} (CSIPw) & \min_{x\in\mathbb{R}^n} & \sum\limits_{i=1}^{p} w_i f_i(x) \\ & \text{s.t.} & f_i(x) \leq f_i\left(x^0\right), i \in I, \\ & & g(t,x) \leq 0,\ t \in T. \end{array}$$

Given $\overline{x} \in X$ and $w \in \Delta_{++}^{p}$, by [21, Theorem 4.7] (based on [36]), $\overline{x} \in X_E$ if and only if $\overline{x}$ is a minimizer of $(CSIPw)$. Observing the independence of the chosen vector $w \in \Delta_{++}^{p}$, Charnes and Cooper proposed to scalarize with the vector of all ones 1_n, idea that has inspired an algorithm [54], where the authors sum up to the objective function $\sum\limits_{i=1}^{p} f_i(x)$ in $(CSIP1_n)$ a regularization term $\frac{1}{2}\left\|x-\overline{x}\right\|^2$ (as in the proximal algorithm for convex NLP) for a suitable $\overline{x}$, so that the objective function $\sum\limits_{i=1}^{p} f_i(\cdot) + \frac{1}{2}\left\|\cdot-\overline{x}\right\|^2$ becomes strongly convex instead of just convex.

Other scalarization methods with CSIP parametric problems are the elastic constraint method [21, Section 4.3] and weighted Chebyshev scalarization [21, Section 4.5], whereas Benson's scalarization method does not [21, Section 4.4], except in the case that all objective functions are linear (as it happens in multiobjective LSIP, e.g., in Example 4). In all these methods, the parametric problems aggregate finitely many convex constraints to the initial constraint system representing X, so that the continuity of $(CSIP)$ is inherited by the substitute problems.

Since the last survey paper covering scalar CSIP [66] was published in 2007, we now briefly review subsequent contributions to the numerical treatment of $(CSIP)$ in (2.1), with $p = 1$. Two main difficulties for the numerical treatment of $(CSIP)$ are:

1. The fact that checking the feasibility of a given $x \in \mathbb{R}^n$ requires to compute the optimal value of the so-called *lower level problem* at x,

$$Q(x): \quad \max_{t \in T} g(\overline{x}, t),$$

which is a global optimization problem.

2. The fact that possibly all constraints are inactive at any feasible solution, i.e., the *set of active constraints* at $\overline{x}$, i.e. $T_0(\overline{x}) = \{t \in T : g(\overline{x}, t) = 0\}$, could be empty for all $\overline{x} \in X$ (even on the boundary of X). To avoid this undesirable situation constraint qualifications are needed.

- **Penalty and smoothing methods**
  [3] proposes a unified framework concerning old and new Remez-type algorithms and integral methods coupled with penalty and smoothing methods for continuous CSIP that converge assuming that f has bounded lower level sets and the Slater constraint qualification defined in Section 4 holds; [4] compares several implementations of one of the algorithmic schemes proposed in [3], RPSALG, tackling the lower level problem $Q(x)$ with a variant of the cutting angle method called ECAM, a global optimization procedure for solving Lipschitz programming problems. These variants are compared with the unique publicly available SIP solver, NSIPS, on a battery of test problems.
  The penalty method in [102] assumes that $(CSIP)$ is continuous and all data functions are continuously differentiable. Numerical tests are also provided.
- **Exchange methods**
  Most exchange method solve sequences of finitely relaxed subproblems, that is, replace T by a finite subset of T at each step. This is the approach of [107], which proposes an exchange method for continuous CSIP problems such that all data are continuously differentiable, f has bounded lower level sets, and the Slater constraint qualification holds. The method has finite termination whenever f is strictly convex. The main features of this method are that only those active constraints with positive Lagrange multipliers are kept and that, instead of solving $Q(x)$ at each iteration to detect the (almost) most violated constraint, the algorithm works with indexes whose associated constraints are only slightly violated. The computational efficiency of this

method is compared with that of central cutting plane algorithm and with the semi-infinite solver *fseminf* in MATLAB toolbox. It is also applied to the FIR filter design problem. A sequel of this paper is [25], whose main idea consists in perturbing the non-strictly convex objective function f making it strictly convex. It is shown that, under certain assumptions, a good approximate solution of the original problem can be found in a finite number of steps. The method is illustrated with an application to the so-called sparse broadband beamformed design problem.

The exchange method in [72] assumes that $(CSIP)$ is continuous, T is a polyhedral convex set, and both f and g are continuously differentiable. In contrast with the traditional exchange methods, this one generates a sequence of auxiliary semi-ínfinite problems indexed by T whose constraints are quadratic approximations of the initial constraints (the so-called refined problems, that are in practice replaced by NLP problems. Some numerical tests are provided.

- **Cutting surface methods**

In 1993 Kortanek and No [58] extended to CSIP the classical Gribik's linear cutting plane method, which is based on an efficient grid management scheme to generate cuts.

The central cutting plane method in [108] applies to CSIP problems of the form

$$(CSIP) \qquad \min_{x\in C} f(x) \text{ s.t. } g(t,x) \leq 0, \ t \in T, \tag{3.5}$$

where $C \subset \mathbb{R}^n$ is a given compact convex set and g is continuously differentiable. Observe that the scalarized problems (3.3) and (3.4) have this form, with C being lower level sets of f_i. Some numerical examples are provided. The cutting surface method of [68] applies to CSIP problems of the form

$$(CSIP) \qquad \min_{x\in C} f(x) = x_1 \text{ s.t. } g(t,x) \leq 0, \ t \in T, \tag{3.6}$$

such that $C \subset \mathbb{R}^n$ is a given compact convex set, the strong Slater condition defined in Section 4 holds and all constraint functions g_t are subdifferentiable and uniformly bounded. The method is inspired in the mentioned cutting plane one of Kortanek and No, just replacing planes by nonlinear surfaces, more precisely convex cuts generated directly from the constraints. The model (3.6) corresponds to the CSIP formulation of distributionally robust optimization problems in which the uncertainty set consists of probability distributions with given bounds on their moments. The paper contains numerical tests and applications to moment robust optimization and portfolio optimization.

- **Bundle methods**

The bundle method of [73] applies to CSIP problems formulated as in (3.5) and such that $T \subset \mathbb{R}^m$ is compact and $g(\cdot,x)$ is upper semicontinuous. The idea of the method consists in replacing $(CSIP)$ by an unconstrained problem by using the so-called improvement function

$$H_y(x) := \max\{f(x) - f(y), \max_{t\in T} g(t,x)\},$$

for which, $\overline{x}$ is a minimizer of $(CSIP)$ if and only if it is a global minimum of $H_{\overline{x}}$. The paper provides computational results for non-smooth convex semi-infinite problems and moment robust optimization, and comparisons with the cutting plane method of Kortanek and No [58] and the exchange method of Zhang, Wu, and López [107].

The discretization method proposed by [74] considers CSIP problems as in $(CSIP)$ in (2.1) satisfying the Slater constraint qualification such that $T \subset \mathbb{R}^m$ is compact and g is locally Lipschitz. It is inspired in the classical discretization methods and the bundle methods of Sagastizábal and Solodov [82]. The authors present some numerical tests collected.

- **Stochastic approximation methods**

  Wei, Haskell, and Zhao have recently proposed two stochastic approximation methods.

  The method in [100] applies to continuous CSIP problems of the form (3.5) where C is a compact and convex subset of $\mathbb{R}^n$, T is a convex body in some low dimensional space, and all data functions are Lipschitz continuous and subdifferentiable. The two CoMirror algorithms proposed in this paper use random sampling to approximately solve the cut generation problem and their convergence analysis is based on general error bounds. These methods are compared with those of Calafiore and Campi [6] on a parametric test problem taken from the latter paper.

  The inexact first-order primal-dual algorithm method based on Monte Carlo sampling proposed in [101] applies to a class of SIP problems slightly larger than CSIP as the constraints are required to be Lipschitz continuous (but not necessarily convex), the rest of assumptions being as the ones in the previous method. The paper contains some numerical tests.

4. Data qualifications

The central theme of the CSIP theory is the characterization of the sets of solutions in terms of convex cones contained in the decision space $\mathbb{R}^n$ or in terms of the existence of Karush-Kuhn-Tucker (KKT in short) multipliers popularized in all branches of optimization after the publication of [59]. As a general rule, to obtain checkable necessary optimality conditions and duality theorems for a given constrained optimization problem, one needs to assume some property of the data called *data qualification.* In particular, those data qualifications only involving the constraint functions $\{g_t, t \in T\}$ are called *constraint qualifications* (CQs), those only involving the objective functions $\{f_i, i \in I\}$ are called *objective qualifications* (OQs), and those data qualifications which involve both constraint and objective functions are called *mixed qualifications* (MQs). The oldest CQ, published in 1950, was introduced by M. Slater in a seminal work on scalar NLP, and was later adapted to almost any optimization field, e.g., to scalar LSIP by Charnes, Cooper and Kortanek in the 1960s. Many of the data qualifications below also have a long history. For instance, the locally Farkas-Minkowski MQ was first defined in [80] for scalar LSIP, and then extended to convex scalar CSIP in [30] and to scalar convex infinite programming (dealing with optimization problems with infinite dimensional decision space and infinitely many convex constraints) in [18]; weaker mixed qualifications have been introduced in [63] and [62] in scalar CSIP and convex infinite programming, respectively. For the sake of brevity we reduce to a minimum the historical notes in the following list of the most common data qualifications encountered in the CSIP literature. More historical details can be found in many papers dealing with optimality conditions as [60], [14], [15], [26], [27], [37], [62], etc.

Main constraint qualifications

- The *Slater constraint qualification* (SCQ in brief, introduced in [26] and [27] in the frameworks of multiobjective LSIP and CSIP, respectively) holds when there exists $\overline{x} \in \mathbb{R}^n$ (called *Slater point*) such that $g_t(\overline{x}) < 0$, $\forall t \in T$, i.e., $T_0(\overline{x}) = \emptyset$.
- The *Mangasarian-Fromovitz constraint qualification* (MFCQ, [48] in NLSIP) holds at $\widehat{x} \in X$ if

$$G(\widehat{x}) := \bigcup_{t \in T(\widehat{x})} \partial g_t(\widehat{x}) \neq \emptyset$$

and (its strictly negative polar cone)

$$G^{-}(\widehat{x}) \neq \emptyset.$$

- The *perturbed Mangasarian-Fromovitz constraint qualification* (PMFCQ, [71] in NLSIP) holds at $\widehat{x} \in X$ if there exists $w \in \mathbb{R}^n$ such that

$$\inf_{\varepsilon > 0} \sup \left\{ \xi' w : \xi \in \bigcup_{t \in T_\varepsilon(\widehat{x})} \partial g_t(\widehat{x}) \right\} < 0,$$

where $T_\varepsilon(\widehat{x}) := \{t \in T : \varepsilon \leq -g_t(\widehat{x})\}$ is the set of $\varepsilon-$active indices at $\widehat{x}$.
- The *local Farkas-Minkowski constraint qualification* (LFMCQ, [30] in scalar CSIP) holds at $\widehat{x} \in X$ when

$$\operatorname{cone} G(\widehat{x}) = D^0(X, \widehat{x}),$$

where $D(X; \widehat{x})$ is the cone of feasible directions at $\widehat{x}$.
- The *Abadie constraint qualification* (ACQ, [61] in scalar CSIP; [29] in multiobjective CSIP) holds at $\widehat{x} \in X$ when $G(\widehat{x}) \neq \emptyset$ and

$$G^0(\widehat{x}) \subseteq \overline{D(X, \widehat{x})},$$

where $\overline{D(X, \widehat{x})}$ is, in the convex setting, the *tangent cone* at $\widehat{x}$, i.e.,

$$\{v \in \mathbb{R}^n : \exists t_r \downarrow 0, \ \exists v_r \to v \text{ such that } \widehat{x} + t_r v_r \in X, \ \forall r \in \mathbb{N}\}.$$

Main objective qualification

- The *Maeda objective qualification* (MOQ, [67] in multiobjective NLP; [29] in multiobjective CSIP) holds at $\widehat{x} \in X$ when

$$F^0(\widehat{x}) \subseteq \{0_n\} \cup \bigcup_{i=i}^{p} \partial f_i(\widehat{x})^{-},$$

where

$$F(\widehat{x}) := \bigcup_{i \in I} \partial f_i(\widehat{x}).$$

Main mixed qualifications

- The *weak Abadie mixed qualification* (WAMQ, [30] in scalar CSIP) holds at $\widehat{x} \in X$ when $G(\widehat{x}) \neq \emptyset$ and

$$F^{-}(\widehat{x}) \cap G^{0}(\widehat{x}) \subseteq \overline{D(X, \widehat{x})}.$$

- The *extended Abadie mixed qualification* (EAMQ, MOQ, [67] in multiobjective NLP; [29] in multiobjective CSIP) holds at $\widehat{x} \in X$ when $G(\widehat{x}) \neq \emptyset$ and

$$F^{0}(\widehat{x}) \cap G^{0}(\widehat{x}) \subseteq \bigcap_{i \in I} \overline{D(Q^{i}(\widehat{x}), \widehat{x})},$$

where, for each $i \in I$,

$$Q^{i}(\widehat{x}) := \{x \in X : f_k(x) \leq f_k(\widehat{x}),\ \forall k \in I \setminus \{i\}\}.$$

- The *Local Farkas-Minkowski mixed qualification* (LFMMQ, [64] in scalar LSIP; [34] in multiobjective CSIP) holds at $\widehat{x} \in X$ when

$$[-\operatorname{conv} F(\widehat{x})] \cap D^{0}(X; \widehat{x}) \subset \operatorname{cone} G(\widehat{x}).$$

Diagram 1 summarizes the main relationships between the above data qualifications [29, Theorem 1]. Some implications are true under additional assumptions: [1] $(CSIP)$ is continuous; [2] $G(\widehat{x}) \neq \emptyset$; [3] $p = 1$.

$$\begin{array}{ccccc}
\text{PMFCQ} & ^{[2]}\rightarrow & \text{MFCQ} & \leftarrow^{[2]} & \text{SCQ} \\
 & & \downarrow^{[1]} & & \\
 & & \text{LFMCQ} & \rightarrow & \text{LFMMQ} \\
 & & \downarrow & & \\
\text{MOQ} & & \text{ACQ} & ^{[3]}\rightarrow & \text{EAMQ} \\
 & & \downarrow & \swarrow & \\
 & & \text{WAMQ} & &
\end{array}$$

Diagram 1

Observe that MOQ is the unique DQ not connected in the above graph gathering the existing relationships.

5. Optimality

This section presents characterizations of the sets X_W, X_E, X_P, and X_S by means of Karush-Kuhn-Tucker (KKT in short) type optimality conditions and the attainment of the zero-value by the *gap function*

$$\vartheta : \bigcup\nolimits_{x \in \mathbb{R}^n} \left(\{x\} \times \prod_{i=1}^{p} \partial f_i(x) \times \Delta_{+}^{p} \right) \longrightarrow \overline{\mathbb{R}},$$

where Δ_+^p was defined in (3.1), such that

$$\vartheta(x,\xi,\lambda) := \sup_{y\in X}\left(\sum_{i=1}^{p}\lambda_i\xi_i'(x-y)\right).$$

Obviously,

$$\vartheta(x,\xi,\lambda)\geq 0, \forall x\in X, \forall \xi\in\prod_{i=1}^{p}\partial f_i\left(x\right), \forall\lambda\in\Delta_+^p.$$

We say that $\vartheta(x,\cdot,\cdot)$ *attains zero-value* on a set Δ such that $\emptyset\neq\Delta\subset\mathbb{R}^p$ at $x\in X$ if there exist $\xi\in\prod_{i=1}^{p}\partial f_i(x)$ and $\lambda\in\Delta$ such that $\vartheta(x,\xi,\lambda)=0$.

By $\mathbb{R}_+^{(T(\widehat{x}))}$ we denote the positive cone in the vector space $\mathbb{R}^{(T(\widehat{x}))}$ of real-valued functions on $T(\widehat{x})$ with finite support, that is, a function $\beta: T(\widehat{x})\longrightarrow\mathbb{R}_+$ belongs to $\mathbb{R}_+^{(T(\widehat{x}))}$ if $\beta_t=\beta\left(t\right)=0$ for all $t\in T(\widehat{x})$ except for finitely many indices.

- **Weak efficiency**

  The *weak KKT condition* holds at $\widehat{x}\in X$ if there exist $\alpha\in\Delta_+^p$ and $\beta\in\mathbb{R}_+^{(T(\widehat{x}))}$ such that

$$0_n\in\sum_{i=1}^{p}\alpha_i\partial f_i(\widehat{x})+\sum_{t\in T(\widehat{x})}\beta_t\partial g_t(\widehat{x}),$$

  or, equivalently, if $0_n\in\operatorname{conv}F(\widehat{x})+\operatorname{cone}G(\widehat{x})$.
  Regarding the weak KKT condition, by [29, Theorem 4].

$$\text{weak KKT holds at } \widehat{x}\in X\Longrightarrow\widehat{x}\in X_W,$$

  by [29, Theorem 2(i)], under WAMQ,

$$\widehat{x}\in X_W\Longrightarrow 0_n\in\operatorname{conv}F(\widehat{x})+\overline{\operatorname{cone}G(\widehat{x})},$$

  and, by [34, Theorem 4.1], under the stronger LFMMQ,

$$\widehat{x}\in X_W\Longleftrightarrow\text{weak KKT holds at } \widehat{x}\in X.$$

  Regarding the gap function,

$$\vartheta(\widehat{x},\cdot,\cdot)\text{ attains zero-value on }\Delta_+^p\Longrightarrow\widehat{x}\in X_W$$

  and by [34, Theorem 4.2], under LFMMQ,

$$\widehat{x}\in X_W\Longleftrightarrow\vartheta(\widehat{x},\cdot,\cdot)\text{ attains zero-value on }\Delta_+^p.$$

- **Efficiency**

  The *strong KKT condition* holds at $\widehat{x}\in X$ when there exist $\alpha\in\Delta_{++}^p$ and $\beta\in\mathbb{R}_+^{(T(\widehat{x}))}$ such that

$$0_n\in\sum_{i=1}^{p}\alpha_i\partial f_i(\widehat{x})+\sum_{t\in T(\widehat{x})}\beta_t\partial g_t(\widehat{x}).$$

One has

$$0_n \in \operatorname{ri}\operatorname{conv} F(\widehat{x}) + \operatorname{cone} G(\widehat{x}) \Longrightarrow \text{strong KKT holds at } \widehat{x} \Longrightarrow \widehat{x} \in X_E.$$

Moreover, if all objective functions are differentiable at $\widehat{x}$,

$$\text{strong KKT holds at } \widehat{x} \Longleftrightarrow 0_n \in \operatorname{ri}\operatorname{conv} F(\widehat{x}) + \operatorname{cone} G(\widehat{x}).$$

In general, by [29, Theorem 7],

$$\text{strong KKT holds at } \widehat{x} \Longrightarrow \widehat{x} \in X_E,$$

and, according to [29, Theorem 6], the converse statement holds whenever EAMQ and MOQ hold at $\widehat{x}$, i.e.

$$\widehat{x} \in X_E \Longleftrightarrow \text{strong KKT holds at } \widehat{x}.$$

Regarding the gap function, by [29, Theorem 8(i)],

$$\vartheta(\widehat{x}, \cdot, \cdot) \text{ attains zero-value on } \Delta^p_{++} \Longrightarrow \widehat{x} \in X_E$$

and the converse holds, by [29, Theorem 8(ii)], under EAMQ and MOQ, that is,

$$\widehat{x} \in X_E \Longleftrightarrow \vartheta(\widehat{x}, \cdot, \cdot) \text{ attains zero-value on } \Delta^p_{++}.$$

- **Proper efficiency**

  In general,

$$\text{strong KKT holds at } \widehat{x} \Longrightarrow \widehat{x} \in X_P$$

and, according to [34, Theorem 5.1], if LFMMQ holds at $\widehat{x}$, one has

$$\widehat{x} \in X_P \Longleftrightarrow \text{strong KKT holds at } \widehat{x} \in X.$$

Similarly, it is always true that

$$\vartheta(\widehat{x}, \cdot, \cdot) \text{ attains zero-value on } \Delta^p_{++} \Longrightarrow \widehat{x} \in X_P,$$

and, according to [34, Theorem 5.2], the converse statement holds whenever LFMMQ holds at $\widehat{x}$, i.e.,

$$\widehat{x} \in X_P \Longleftrightarrow \vartheta(\widehat{x}, \cdot, \cdot) \text{ attains zero-value on } \Delta^p_{++}.$$

- **Sharp efficiency**

  The *perturbed KKT condition* holds at $\widehat{x} \in X$ when there exist $\nu > 0$ such that, $\forall w \in \nu\mathbb{B}_n$, there exist $\alpha \in \Delta^p_+$ and $\beta \in \mathbb{R}^{(T(\widehat{x}))}_+$ such that

$$w \in \sum_{i=1}^{p} \alpha_i \partial f_i(\widehat{x}) + \sum_{t \in T(\widehat{x})} \beta_t \partial g_t(\widehat{x}).$$

By [29, Theorem 10], we always have that

$$\text{perturbed KKT holds at } \widehat{x} \Longleftrightarrow 0_n \in \operatorname{int}\left(\operatorname{conv} F(\widehat{x}) + \operatorname{cone} G(\widehat{x})\right) \Longrightarrow \widehat{x} \in X_S.$$

If $(CSIP)$ is continuous and the PMFCQ holds at $\widehat{x}$, by [29, Theorem 9(ii)],

$$\widehat{x} \in X_S \Longleftrightarrow \text{perturbed KKT holds at } \widehat{x}.$$

Finally, assume that $(CSIP)$ is continuous, that all constraints are continuously differentiable at $\widehat{x} \in X$ and that PMFCQ holds at $\widehat{x}$. Then, by [29, Theorem 9(i)],

$$\widehat{x} \in X_S \Longleftrightarrow \left\{ \begin{array}{c} \exists \nu > 0 : \forall w \in \nu \mathbb{B}_n, \exists \xi \in \prod_{i=1}^p \partial f_i(\widehat{x}) \text{ and } \exists \lambda \in \mathbb{R}^p_{++} \\ \text{such that } \vartheta(\widehat{x}, \xi - w, \lambda) = 0 \end{array} \right\}.$$

Optimality conditions for multiobjective LSIP were provided in [26] and [49], not all of them being straightforward consequence of the above ones, but the bulk of papers providing optimality conditions for multiobjective SIP deal with NCSIP problems involving locally Lipschitz continuous data. Recall that if $h : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is a proper convex function, then it is locally Lipschitz on the interior of its domain [105, Theorem 2.1.5], so that our $(CSIP)$ in (2.1) has locally Lipschitz data whenever all the involved functions are real-valued, in which case extended subdifferentials as those of Clark and Mordukhovich can be used. The abundant literature on optimality conditions for multiobjective SIP problems with locally Lipschitz data includes [16], [17], [46], [55], [71], [78], [79], [94], etc., while [98] provides optimality conditions for multiobjective SIP problems under geodesic convexity assumptions. Moreover, [83] provides optimality conditions for approximate Pareto solutions of multiobjective programming problems posed in Banach spaces with locally Lipschitz data in terms of Clarke subdifferentials, while [15] provides new KKT type conditions for $\widehat{x} \in X_W$ in terms of limiting subdifferentials.

6. Duality

Given two scalar optimization problems

$$(P) \ \ \min\nolimits_{x \in X} f(x) \ \text{ and } \ (D) \ \ \max\nolimits_{z \in Z} h(z),$$

one says that the pair $(P) - (D)$ satisfies weak duality if $h(z) \leq f(x)$ for all $x \in X$ and $z \in Z$, i.e, if $\sup_Z h \leq \inf_X f$. If $\sup_Z h = \inf_X f$ and the supremum (respectively, infimum) is attained then strong duality (reverse strong duality, resp.) holds. Typically, these duality theorems require the satisfaction of data qualifications. In the case of the scalar CSIP problem in (2.1) with $p = 1$, different dual problems have been defined, among them the Lagrange-Haar dual

$$(LCSID) \quad \max\nolimits_{\lambda \in \mathbb{R}_+^{(T)}} \quad h(\lambda),$$

with objective function $h(\lambda) := \inf_{y \in \mathbb{R}^n} \left\{ f(y) + \sum_{t \in T} \lambda_t g_t(y) \right\}$ and feasible set $Z = \mathbb{R}_+^{(T)}$, and variants where $Z \subsetneq \mathbb{R}_+^{(T)}$. Strong duality theorems and reverse strong duality theorems for the pair $(CSIP) - (LCSID)$ and some variants can be found in [28], [32], [33], and references therein (observe that most duality theorems for CSIP in these papers are also valid for convex infinite optimization problems, where the decision space is an arbitrary locally convex space instead of $\mathbb{R}^n$, but the SIP case is also discussed in the three mentioned papers). The most common data qualifications for the strong duality theorems are SCQ (and relaxed CQs) or closed cone CQs involving the set $\bigcup_{t\in T} \operatorname{epi} g_t^*$, while the assumptions are of topological nature for the reverse strong duality theorems. Other dual problems can be associated with $(CSIP)$ under additional assumptions. For instance, if all data are differentiable, the Wolfe dual of $(CSIP)$ is

$$\begin{array}{lll}(WCSID) & \max_{(y,\lambda)\in\mathbb{R}^n\times\mathbb{R}_+^{(T)}} & f(y)+\sum_{t\in T}\lambda_t g_t(y)\\ & \text{s.t.} & \nabla f(y)+\sum_{t\in T}\lambda_t \nabla g_t(y)=0_n,\end{array}$$

with feasible set $Z=\mathbb{R}^n\times\mathbb{R}_+^{(T)}$, and, if $(CSIP)$ is a continuous LSIP problem, as $(LSIP)$ in (2.2) with $p=1$, its continuous dual is

$$\begin{array}{lll}(CLSID) & \max_{\mu\in\mathcal{C}'_+(T)} & \int_T b_t\, d\mu(t)\\ & \text{s.t.} & \int_T a_t d\mu(t)=c,\end{array}$$

whose feasible set $Z=\mathcal{C}'_+(T)$ is the cone of non-negative regular Borel measures on T. Duality theorems for $(CLSID)$ can be found in the classical book on continuous linear problems [2] and references therein, while $(WCSID)$ is still to be investigated.

Let us now consider the multiobjective counterpart. Given two multiobjective optimization problems

$$(P)\ \text{“}\min_{x\in X}\text{”} f(x) \ \text{ and } \ (D)\ \text{“}\max_{z\in Z}\text{”} h(z),$$

we say that the pair $(P)-(D)$ satisfies weak duality if $h(z)\leqq f(x)$ for all $x\in X$ and $z\in Z$, and the duality theorems guarantee the equality $h(z)=f(x)$ with attainment (i.e., existence of some type of solution) of either (D) (in the strong duality theorems) or (P) (in the reverse strong duality theorems). The types of solutions are not necessarily the same for (P) and (D).

The natural dual problem for $(CSIP)$ is the Lagrange-Haar dual

$$(LCSID)\quad \text{“}\max_{\lambda\in\mathbb{R}_+^{(T)}}\text{”}\quad h(\lambda)=(h_1(\lambda),...,h_p(\lambda)),$$

where $h_i(\lambda):=\inf_{y\in\mathbb{R}^n}\left\{f_i(y)+\sum_{t\in T}\lambda_t g_t(y)\right\}$, $i=1,...,p$. The inequality $h(z)\leqq f(x)$ for all $x\in X$ and $z\in Z$ holds for free from the scalar case with objective functions f_i and h_i, $i=1,...,p$. However, to the authors' knowledge, the unique published paper dealing with duality in multiobjective CSIP is [38], for continuous $(CSIP)$ problems. The authors reformulate their primal problem by rewriting X as solution set of a conic system as follows: $X=\left\{x\in\mathbb{R}^n : g(x,\cdot)\in-\mathcal{C}(T)_+\right\}$, where $\mathcal{C}(T)_+$ is the positive cone in the space $\mathcal{C}(T)$ of real-valued continuous functions on T. Then, following [45], they associate with $(CSIP)$ the continuous dual problem

$$(CCSID)\quad \text{“}\max_{(\lambda,\mu,y)\in\mathbb{Z}}\text{”}\quad h(\lambda,\mu,y)=y,$$

where

$$Z:=\left\{(\lambda,\mu,y)\in\mathbb{A}\times\mathcal{C}'_+(T)\times\mathbb{R}^p : \lambda' y\leq\inf_{y\in\mathbb{R}^n}\left\{\lambda' f(y)+\int_T g(y,t)\,d\mu(t)\right\}\right\},$$

$\mathbb{A}$ being a suitable subset of $\mathbb{R}^p_+$. In the strong duality theorem [38, Theorem 2.3], under SCQ, and taking $\mathbb{A} = \mathbb{R}^p_{++}$, it is proved, that if $\widehat{x} \in X_P$, there exists $\left(\widehat{\lambda}, \widehat{\mu}, \widehat{y}\right) \in Z_E$ such that $f(\widehat{x}) = h\left(\widehat{\lambda}, \widehat{\mu}, \widehat{y}\right)$. Moreover, it is shown that reverse strong duality holds assuming the compactness of Z, more precisely, that if $\left(\widehat{\lambda}, \widehat{\mu}, \widehat{y}\right) \in Z_E$, then there exists $\widehat{x} \in X_P$ such that $f(\widehat{x}) = h\left(\widehat{\lambda}, \widehat{\mu}, \widehat{y}\right)$. Analogously, in the strong duality theorem [38, Theorem 2.5], also under SCQ, and taking $\mathbb{A} = \mathbb{R}^p_+ \setminus \{0_p\}$, the same is proved just replacing $\widehat{x} \in X_P$ by $\widehat{x} \in X_W$ and Z_E by Z_W in the first statement and Z_E by Z_W and X_P by X_W in the second one.

In the particular case of multiobjective LSIP, the following Wolfe-type dual problem is associated in [49] with the problem $(LSIP)$ in (2.2):

$$\begin{array}{lll} (LSID) & \text{“}\max_{(y,\lambda,\alpha)\in\mathbb{R}^n\times\mathbb{R}^{(T)}_+\times\mathbb{R}^p_{++}}\text{”} & h(y,\lambda,\alpha) \\ & \text{s.t.} & \sum_{i=1}^p \alpha_i c_i + \sum_{t\in T}\lambda_t a_t = 0_n, \\ & & \sum_{i=1}^p \alpha_i = 1, \end{array}$$

where

$$h(y,\lambda,\alpha) = \left(c_1'y + \sum_{t\in T}\lambda_t\left(a_t'y - b_t\right), ..., c_p'y + \sum_{t\in T}\lambda_t\left(a_t'y - b_t\right)\right).$$

The authors provide a strong duality theorem [49, Theorem 4.2] asserting under EAMQ that, if span $\{c_1, ..., c_p\} = \mathbb{R}^n$ and $\widehat{x} \in X_E$, then there exists $\left(\widehat{y}, \widehat{\lambda}, \widehat{\alpha}\right) \in Z_W$ such that $f(\widehat{x}) = h\left(\widehat{y}, \widehat{\lambda}, \widehat{\alpha}\right)$. A similar result is valid for a Mond-Weir-type dual problem.

Duality theorems for multiobjective NCSIP can be found in [81] and [104], for multiobjective SIP problems with convex constraints and DC objective functions, in [93], for multiobjective SIP problems with convex constraints and multiple interval-valued objective functions, in [56] [78] and [79], for similar problems with with vanishing constraints, in [99], for multiobjective SIP problems with locally Lipschitz data, in [69] and [70], for multiobjective SIP problems with generalized convex data (e,g., pseudoconvex and quasiconvex), in [98] for multiobjective SIP problems under geodesic convexity assumptions, etc.

7. Stability

Almost all the existing literature on stability of multiobjective SIP problems concerns the effect of small perturbations of the nominal data on either the set of weak efficient solutions or the set of efficient solutions (the Pareto frontier). The model consists of embedding the nominal problem $(CSIP)$ into a family of parameterized CSIP problems of the form

$$(CSIP_\theta) \quad \text{“}\min_{x\in\mathbb{R}^n}\text{”}\ f(x,\theta) := (f_1(x,\theta), \dots, f_p(x,\theta)) \ \text{s.t. } g(t,x,\theta) \le 0,\ t \in T,$$

with $\theta \in \Theta$, the so-called *space of parameters*, with $(CSIP_{\overline{\theta}}) = (CSIP)$ for certain $\overline{\theta} \in \Theta$, so that the perturbations are required to preserve the number n of decision variables and the index set T. The topology on Θ usually corresponds to some measure (e.g., a pseudometric) of the size of the admissible perturbations. There is a consensus about the convenience of measuring the distance between two parameters as the supremum of some distance between functions for all the data. In the case of linear functions, $\langle p, \cdot\rangle$ and $\langle q, \cdot\rangle$, this distance uses to be some norm of the difference of their gradients, i.e., $d(p,q) = \|p-q\|$, while in the case of extended real-valued functions it is customary to consider an expansive family of compact convex sets $\{B_k, k \in \mathbb{N}\}$ covering $\mathbb{R}^n$ (as the integer multiples of the unit closed ball $\mathbb{B}_n$), and define the pseudo-distance between g and h, $g, h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, as

$$d(g,h) := \sum_{k=1}^{+\infty} 2^{-k} \min\{1, d_k(g,h)\},$$

where

$$d_k(g,h) := \sup_{x\in B_k} |f(x) - h(x)|, \forall k \in \mathbb{N},$$

with the convention that $(+\infty) - (+\infty) = 0$ and $|-\infty| = +\infty, |+\infty| = +\infty$.

Let us recall the definitions of the stability concepts favorites of the researchers in this area. Consider a set-valued mapping $\mathcal{M} : \Theta \rightrightarrows \mathbb{R}^n$, and let $\overline{\theta} \in \Theta$ be such that $\mathcal{M}\left(\overline{\theta}\right) \neq \emptyset$.

- $\mathcal{M}$ is *(Berge-Kuratowski) lower semicontinuous* (lsc, in brief) at $\overline{\theta}$ if for each open set $U \subset \mathbb{R}^n$ verifying $\mathcal{M}\left(\overline{\theta}\right) \cap U \neq \emptyset$, there exists a neighborhood V of $\overline{\theta}$ such that $\mathcal{M}(\theta) \cap U \neq \emptyset$, for all $\theta \in V$.
- $\mathcal{M}$ is *(Berge-Kuratowski) upper semicontinuous* (usc) at $\overline{\theta}$ if for each open set $U \subset \mathbb{R}^n$ verifying $\mathcal{M}\left(\overline{\theta}\right) \subset U$, there exists a neighborhood V of $\overline{\theta}$ such that $\mathcal{M}(\theta) \subset U$, for all $\theta \in V$.
- $\mathcal{M}$ is *Painlevé-Kuratowski convergent* (pkc) at $\overline{\theta}$ when for each sequence $(\theta_r)_{r=1}^{\infty} \subset \Theta$ such that $\lim_{r\to\infty} \theta_r = \overline{\theta}$, there exists $p \in \mathbb{N}$ such that $\mathcal{M}(\theta_r) \neq \emptyset$ for all $r \geq p$ and $\lim_{r\to\infty} \mathcal{M}(\theta_r) = \mathcal{M}\left(\overline{\theta}\right)$ in the Painlevé-Kuratowski sense, i.e.
$$\liminf_{\theta_r\to\overline{\theta},\ r\geq p} \mathcal{M}(\theta_r) = \limsup_{\theta_r\to\overline{\theta},\ r\geq p} \mathcal{M}(\theta_r) = \mathcal{M}\left(\overline{\theta}\right).$$
- $\mathcal{M}$ is *pseudo-Lipschitz* (pl) at $\left(\overline{\theta}, \overline{x}\right) \in \operatorname{gph} \mathcal{M}$ if there exist neighborhoods V of $\overline{\theta}$ and U of $\overline{x}$, and a scalar $\kappa \geq 0$ such that
$$\operatorname{dist}(x, \mathcal{M}(\theta)) \leq \kappa d\left(\theta, \theta'\right), \forall \theta, \theta' \in V, \forall x \in \mathcal{M}(\theta') \cap U.$$
This property is equivalent to the *metric regularity* of the inverse mapping $\mathcal{M}^{-1}$ at $\left(\overline{x}, \overline{\theta}\right)$.

We denote by $\mathcal{X}_W$ and $\mathcal{X}_E$ the set-valued mappings associating with each $\theta \in \Theta$ its sets of weak efficient solutions and efficient solutions, respectively. Tables 1 and 2 summarize the available information on the type of stability properties of $\mathcal{X}_W$ and $\mathcal{X}_E$ (the unique ones analyzed in the literature), whose rows correspond to the stability properties under study while the columns are the type of multiobjective SIP problem considered by each work: “LSIP” (respectively, “CSIP”, “NCSIP”) when the work deals with multiobjective linear SIP (multiobjective convex SIP, multiobjective nonconvex SIP, resp.). The nonconvexity is usually some kind of extended convexity of the data, e.g., quasiconvexity or continuity (recall that any real-valued convex function on $\mathbb{R}^n$ is continuous).

	LSIP	**CSIP**	**NCSIP**
lsc	[92]	[9]	[103]
usc	[92]	[9]	[103]
pkc	[92]		[51], [76], [77]
pl	[13]		

Table 1: Stability of $\mathcal{X}_W$

	LSIP	**NCSIP**
lsc	[92]	[11], [12], [35], [52], [103]
usc	[92]	[12], [52], [103]
pkc	[13], [92]	

Table 2: Stability of $\mathcal{X}_E$

Each empty cell in Tables 1 and 2 detects an open problem. For instance, regarding the efficient solution set $\mathcal{X}_E$, neither its pseudo-Lipschitz property nor its stability for multiobjective CSIP have been studied up to now (even though most NCSIP sufficient stability conditions apply to CSIP). The same happens with other desirable properties of the parameters, that have only been analyzed for multiobjective LSIP :

- $\overline{\theta}$ is *well-posed* w.r.t. a certain property at $\overline{\theta}$ when this property is satisfied by any perturbed problem provided the perturbation is sufficiently small: [87, LSIP], [88, LSIP], [90, LSIP], [91, LSIP].
- A property is *generic* when it holds on some 'large subset of Θ', e.g., on an open dense subset or on a G_δ subset (i.e., a countable intersection of open sets): [86, LSIP], [88, LSIP]; [23] shows that every vector CSIP problem can be arbitrarily approximated by stable CSIP problems (in the sense that either $\mathcal{X}_W$ is simultaneously lsc and usc or $\mathcal{X}_E$ is usc) are dense in Θ.

As it can be seen from the definition, the well-posedness depends strongly on the topologies chosen in the parameter and image spaces. To avoid this obstacle it would be appropriate to search into two different directions. The first, which was the purpose of the papers [90], [91], is to look for good properties providing information about the continuity without having in mind the topologies in both spaces. We shall mention here the so-called *domination properties*, defined by Bednarczuk [5], and the following stability concept introduced in [90]:

- $\overline{\theta} \in \Theta$ is *nice* if $\mathcal{X}_E\left(\overline{\theta}\right) = \mathcal{X}_W\left(\overline{\theta}\right)$.

It turns out that the property of being nice is closely related to the well-posedness of the problem in some sense. However, there are not topological notions in it. This property guarantees also the closedness of $\mathcal{X}_E\left(\overline{\theta}\right)$, which is not true in general and it could be interpreted as the vector counterpart of uniqueness of the optimal solutions in the definition of well-posedness in scalar optimization.

The second direction, is to find suitable topologies, especially in the image space, giving the possibility to prove easily well-posedness not only of the restricted maps. It is obvious that one can not prove generic well-posedness using the well known Hausdorff topology. Regarding LSIP, [92] uses topologies in the space of closed subsets of a Banach space (like Mosco, or bounded Hausdorff topology) to prove generic well-posedness and Painlevé-Kuratowski convergence of the efficient sets.

The extension of the above mentioned results on stability from multiobjective LSIP to the CSIP setting remains a challenging open problem. Stability results for a very general class of multiobjective CSIP and for multiobjective GSIP can be found in [53] and [24], respectively, and references therein. There exist also stability results on vector IP as [75] and [76].

References

[1] F. Akbari, M. Ghaznavi and E. Khorram, *A revised Pascoletti-Serafini scalarization method for multiobjective optimization problems*, J. Optim. Theory Appl. **178** (2018), 560–590.

[2] E. J. Anderson and P. Nash, *Linear Programming in Infinite-Dimensional Spaces*, Wiley, Chichester, 1987.

[3] A. Auslender, M. A. Goberna and M. A. López, *Penalty and smoothing methods for convex semi-infinite programming*, Math. Oper. Res. **34** (2009), 303–319.

[4] A. Auslender, A. Ferrer, M. A. Goberna and M. A. López, *Comparative study of RPSALG algorithm for convex semi-infinite programming*, Comput. Optim. Appl. **60** (2015), 59–87.

[5] E. Bednarczuk, *Some stability results for vector optimization problems in partially ordered topological vector spaces*, in V. Lakshmikantham (ed.), World Congress of Nonlinear Analysts '92, De Gruyter, Berlin, 1996.

[6] G. Calafiore and M. C. Campi, *Uncertain convex programs: randomized solutions and confidence levels*, Math. Program. **102A** (2005), 25–46.

[7] Y. Censor, *Necessary conditions for Pareto optimality in simultaneous Chebyshev best approximation*, J. Approx. Theory **27** (1979), 127–134.

[8] V. Chankong and Y. Y. Haimes, *Multiobjective decision making,* in: Theory and Methodology, North-Holland Series in System Science and Engineering, 8. North-Holland Publishing Co., New York, 1983.

[9] G. Y. Chen and B. D. Craven, *Existence and continuity of solutions for vector optimization*, J. Optim. Theory Appl. **81** (1994), 459–468.

[10] J. Chen, L. Huang, Y. Lv and Ch.-F. Wen, *Optimality conditions of robust convex multiobjective optimization via ε-constraint scalarization and image space analysis*, Optimization **69** (2020), 1849–1879.

[11] T. D. Chuong, *Lower semi-continuity of the Pareto solution map in quasiconvex semi-infinite vector optimization*, J. Math. Anal. Appl. **388** (2012), 443–450.

[12] T. D. Chuong, N. Q. Huy and J. C. Yao, *Stability of semi-infinite vector optimization problems under functional perturbations*, J. Global Optim., **45** (2009), 583-595.

[13] T. D. Chuong, N. Q. Huy and J. C. Yao, *Pseudo-Lipschitz property of linear semi-infinite vector optimization problems*, European J. Oper. Res. **200** (2010), 639–644.

[14] T. D. Chuong and D. S. Kim, *Nonsmooth semi-infinite multiobjective optimization problems*, J. Optim. Theory Appl. **160** (2014), 748-762.

[15] T. D. Chuong and D. S. Kim, *Normal regularity for the feasible set of semi-infinite multiobjective optimization problems with applications*, Ann. Oper. Res. **267** (2018), 81–99.

[16] T. D. Chuong and D. S. Kim, *Approximate solutions of multiobjective optimization problems*, Positivity **20** (2016), 187–207.

[17] T. D. Chuong and J. C. Yao, *Isolated and proper efficiencies in semi-infinite vector optimization problems*, J. Optim. Theory Appl. **162** (2014), 447–462.

[18] N. Dinh, M. A. Goberna, M. A. López and T. Q. Son, *New Farkas-type constraint qualifications in convex infinite programming*, ESAIM Control Optim. Calc. Var. **13** (2007), 580–597.

[19] N. Dinh, M. A. Goberna, M. A. López and M. Volle, *A unifying approach to robust convex infinite optimization duality*, J. Optim. Theory Appl. **174** (2017), 650–685.

[20] H. Djelassi, A. Mitsos and O. Stein, *Recent advances in nonconvex semi-infinite programming: applications and algorithms*, EURO J. Comput. Optim. **9** (2021):100006

[21] M. Ehrgott, *Multicriteria Optimization (2nd ed)*, Springer, Berlin, 2005.

[22] X. Fan, C. Cheng and H. Wang, *Stability of semi-infinite vector optimization problems without compact constraints*, Nonlinear Anal. **74** (2011), 2087–2093.

[23] X. Fan, C. Cheng and H. Wang, *Density of stable semi-infinite vector optimization problems*, Oper. Res. Letters **40** (2012),140-143.

[24] X. Fan and T. Quin, *Stability analysis for generalized semi-infinite optimization problems under functional perturbations*, J. Ind. Manage. Optim. **16** (2020), 1221–1233

[25] M. J. Gao, K.-F. Yiu and S.-Y. Wu, *A perturbation exchange algorithm for convex semi-infinite programming with applications in sparse beamformer design*, Pac. J. Optim. **14** (2018), 15–30.

[26] M. A. Goberna, F. Guerra-Vazquez and M.I. Todorov, *Constraint qualifications in linear vector semi-infinite optimization*, European J. Oper. Res. **227** (2013), 12–21.

[27] M. A. Goberna, F. Guerra-Vazquez and M. I. Todorov, *Constraint qualifications in convex vector semi-infinite optimization*, European J. Oper. Res. **249** (2016), 32–40.

[28] M. A. Goberna, V. Jeyakumar and M. A. López, *Necessary and sufficient constraint qualifications for solvability of systems of infinite convex inequalities*, Nonlinear Anal. **68** (2008), 1184–1194.

[29] M. A. Goberna and N. Kanzi, *Optimality conditions in convex multiobjective SIP*, Math. Program. **164A** (2017), 167–191.

[30] M. A. Goberna and M. A. López, *Linear Semi-Infinite Optimization*, Wiley, Chichester, 1998.

[31] M. A. Goberna and M. A. López, *Recent contributions to linear semi-infinite optimization: an update*, Ann. Oper. Res. **271** (2018), 237–278.

[32] M. A. Goberna, M. A. López and M. Volle, *New glimpses on convex infinite optimization duality*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **109** (2015), 431–450.

[33] M. A. Goberna, M. A. López and M. Volle, *Modified Lagrangian duality for the supremum of convex functions*, Pac. J. Optim. **13** (2017), 501–514.

[34] M. A. Goberna, M. L. Rodríguez and V.N. Vera de Serio, *The LFM data qualification in convex multiobjective semi-infinite programming*, in: The Mathematics of the Uncertain, 845–854, Stud. Syst. Decis. Control, 142, Springer, Cham, 2018.

[35] X. Gong, *Lower semicontinuity of the efficient solution mapping in semi-infinite vector optimization*, J. Syst. Sci. Complex. **28** (2015), 1312–1325.

[36] J. Guddat, F. Guerra-Vasquez, K. Tammer and K. Wendler, *Multiobjective and Stochastic Optimization Based on Parametric Optimization*, Mathematical Research, 26. Akademie-Verlag, Berlin, 1985.

[37] F. Guerra-Vázquez and J.-J. Rückmann, *On proper efficiency in multiobjective semi-infinite optimization*, in: Xu, H. et al. (eds.), Optimization and Control Techniques and Applications. Springer, Berlin, 2014, pp. 115–135.

[38] F. Guerra-Vázquez and J.-J. Rückmann, *On duality in multiobjective semi-infinite optimization, Optimization* **66** (2017), 1237–1249.

[39] F. Guerra-Vázquez, J.-J. Rückmann, O. Stein and G. Still, *Generalized semi-infinite programming: a tutorial*, J. Comput. Appl. Math. **217** (2008), 394–419.

[40] R. Hartley, *On cone-efficiency, cone-convexity and cone-compactness*, SIAM J. Appl. Math. 34 (1978), 211–222.

[41] S. Helbig, *Relationships between vector optimization and parametric semi-infinite optimization in certain ordered groups*, in: Optimization in mathematical physics, Methoden Verfahren Math. Phys., 34, Peter Lang, Frankfurt am Main, 1987, pp. 61–75.

[42] R. P. Hettich and H. Th. Jongen, *Semi-infinite programming: conditions of optimality and applications. Optimization techniques,* in: Proc. 8th IFIP Conf., Würzburg, 1977, Part 2, Lecture Notes in Control and Information Sci., Vol. 7, Springer, Berlin, 1978, pp. 1–11.

[43] R. Hettich and K. O. Kortanek, *Semi-infinite programming: theory, methods, and applications*, SIAM Rev. **35** (1993), 380–429.

[44] N. Ho-Nguyen and F. Kılınç-Karzan, *Online first-order framework for robust convex optimization*, Oper. Res. **66** (2018), 1670–1692.

[45] J. Jahn, *Duality in vector optimization*, Math. Programming **25** (1983), 343–353.

[46] L. Jiao, D. S. Kim and Y. Zhou, *Quasi $\varepsilon-$solutions in a semi-infinite programming problem with locally Lipschitz data*, Optim. Letters **15** (2021), 1759–1772.

[47] L. Jiao and J. H. Lee, *Finding efficient solutions in robust multiple objective optimization with SOS-convex polynomial data*, Ann. Oper. Res. **296** (2021), 803–820.

[48] H. T. Jongen, F. Twilt and G. W. Weber, *Semi-infinite optimization: structure and stability of feasible set*, J. Optim. Theory Appl. **72** (1992), 529–552.

[49] N. Kanzi, J. S. Ardekani and G. Caristi, *Optimality, scalarization and duality in linear vector semi-infinite programming*, Optimization **67** (2018), 523–536.

[50] R. Kasimbeyli, *A conic scalarization method in multi-objective optimization*, J. Global Optim. **56** (2013), 279–297.

[51] S. Kapoor and C. S. Lalitha, *Stability in unified semi-infinite vector optimization*, J. Global Optim. **74** (2019), 383–399.

[52] S. Kapoor and C. S. Lalitha, *Essential stability in unified vector optimization*, J. Global Optim. **80** (2021), 161–175.

[53] S. Kapoor and C. S. Lalitha, *Continuity and closedness of constraint and solution set mappings in unified parametric semi-infinite vector optimization*, J. Math. Anal. Appl. **506** (2022), to appear.

[54] P. Kesarwani and J. Dutta, *Charnes–Cooper scalarization and convex vector optimization*, Optim. Letters **15** (2021), 833–846.

[55] D. S. Kim and T. Q. Son, *Characterizations of solution sets of a class of nonconvex semi-infinite programming problems*, J. Nonlinear Convex Anal. **12** (2011), 429–440.

[56] D. S. Kim and T. Q. Son, *An approach to duality theorems for nonconvex semi-infinite multiobjective optimization problems*, Taiwan. J. Math. **22** (2018), 1261–1287.

[57] K. O. Kortanek, *Duality, semi-infinite programming, and some aspects of control in business and economic systems*, Thesis (Ph.D.)–Northwestern University. 1964, ProQuest LLC.

[58] K. O. Kortanek and H. No, *A central cutting plane algorithm for convex semi-infinite programming problems*, SIAM J. Optim. **3** (1993), 901–918.

[59] H. Kuhn and A. Tucker, *Nonlinear programming*, in: J. Newman (ed.), Proc. of 2nd Berkeley symposium on mathematical statistics and probability, University of California Press, Berkeley, 1951, pp. 481-492.

[60] C. Li and K. F. Ng, *On constraint qualification for an infinite system of convex inequalities in a Banach space*, SIAM J. Optim. **15** (2005), 48–512.

[61] W. Li, C. Nahak and I. Singer, *Constraint qualifications in semi-infinite systems of convex inequalities*, SIAM J. Optim. **11** (2000), 31–52.

[62] C. Li, K. F. Ng and T. K. Pong, *Constraint qualifications for convex inequality systems with applications in constrained optimization*, SIAM J. Optim. **19** (2008), 163–187.

[63] C. Li, X. Zhao and Y. Hu, *Quasi-Slater and Farkas–Minkowski qualifications for semi-infinite programming with applications*, SIAM J. Optim. **23** (2013), 2208–2230.

[64] Y. Liu, *New constraint qualification and optimality for linear semi-infinite programing*, Pac. J. Optim. **12** (2016), 223–232.

[65] Q. Long, L. Jiang and G. Li, *A nonlinear scalarization method for multi-objective optimization problems*, Pac. J. Optim. **16** (2020), 39–65.

[66] M. A. López and G. Still, *Semi-infinite programming*, European J. Oper. Res. **180** (2007), 491–518.

[67] T. Maeda, *Constraint qualifications in multiobjective optimization problems: differentiable case*, J. Optim. Theory Appl. **80** (1994), 483-500.

[68] S. Mehrotra and D. Papp, *A cutting surface algorithm for semi-infinite convex programming with an application to moment robust optimization*, SIAM J. Optim. **24** (2014), 1670–1697.

[69] S. K. Mishra and M. Jaiswal, *Optimality conditions and duality for nondifferentiable multiobjective semi-infinite programming*, Vietnam J. Math. **40** (2012), 331–343.

[70] S. K. Mishra, M. Jaiswal and L. T. D. An, *Duality for nonsmooth semi-infinite programming problems*, Optim. Letters **6** (2012), 261–271.

[71] B. Mordukhovich and T. T. A. Nghia, *Constraint qualifications and optimality conditions for nonconvex semi-infinite and infinite programs*, Math. Program. **139B** (2013), 271–300.

[72] T. Okuno, S. Hayashi, N. Yamashita and K. Gomoto, *An exchange method with refined subproblems for convex semi-infinite programming problems*, Optim. Methods Softw. **31** (2016), 1305–1324.

[73] L.-P. Pang, J. Lv and J.-H. Wang, *Constrained incremental bundle method with partial inexact oracle for nonsmooth convex semi-infinite programming problems*, Comput. Optim. Appl. **64** (2016), 433–465.

[74] L.-P. Pang, Q. Wu, J.-H. Wang and Q. Wu, *A discretization algorithm for nonsmooth convex semi-infinite programming problems based on bundle methods*, Comput. Optim. Appl. **76** (2020), 125–153.

[75] Z. Y. Peng, X. J. Long, X. F. Wang and Y. B. Zhao, *Generalized Hadamard well-posedness for infinite vector optimization problems*, Optimization **66** (2017), 1563–1575.

[76] Z. Y. Peng, J. W. Peng, X. J. Long and J. C. Yao, *On the stability of solutions for semi-infinite vector optimization problems*, J. Global Optim. **70** (2018), 55–69.

[77] Z.-Y. Peng and J.-T. Zhou, *Stability of efficient solutions for semi-infinite vector optimization problems*, J. Nonlinear Sci. Appl. **9** (2016), 3203–3211.

[78] T.-H. Pham, *Optimality conditions and duality for multiobjective semi-infinite programming with data uncertainty via Mordukhovich subdifferential*, Yugoslav J. Oper. Res. **31** (2021), 495–514.

[79] T.-H. Pham, *Approximate optimality conditions and approximate duality theorems for nonlinear semi-infinite programming problems with uncertainty data*, SeMA Journal **80** (2023), 111–129.

[80] R. Puente and V. N. Vera de Serio, *Locally Farkas-Minkowski linear inequality systems*, TOP **7** (1999), 103–121.

[81] S. Qu, M. Goh, S.-Y. Wu and R. De Souza, *Multiobjective DC programs with infinite convex constraints*, J. Global Optim. **59** (2014), 41–58.

[82] C. Sagastizábal and M. Solodov, *An infeasible bundle method for nonsmooth convex constrained optimization without a penalty function or a filter*, SIAM J. Optim. **16** (2005), 146–169.

[83] T. Q. Son, N. Van Tuyen and C. F. Wen, Optimality conditions for approximate Pareto solutions of a nonsmooth vector optimization problem with an infinite number of constraints, Acta Math Vietnam **45** (2020) 435-448.

[84] T. Q. Son, N. Van Tuyen, and C. F. Wen, Optimality conditions for approximate Pareto solutions of a nonsmooth vector optimization problem with an infinite number of constraints, Acta Math Vietnam **45** (2020) 435–448.

[85] O. Stein, *How to solve a semi-infinite optimization problem*, European J. Oper. Res. **223** (2012), 312–320.

[86] M. I. Todorov, *Generic properties in the linear vector semi-infinite optimization,* C. R. Acad. Bulgare Sci. **42** (1989), 27–29.

[87] M. I. Todorov, *Properties resembling well-posedness in the linear vector semi-infinite optimization*, C. R. Acad. Bulgare Sci. **42** (1989), 23–25.

[88] M. I. Todorov, *Generic uniqueness and well-posedness in linear vector semi-infinite optimization*, in: Mathematics and Mathematical Education (Bulgarian) (Slanchev Bryag, 1990), Publ. House Bulgar. Acad. Sci., Sofia, 1990, 413–418.

[89] M. I. Todorov, *Existence of solutions of linear vector semi-infinite optimization problems*, Math. Balkanica **4** (1990), 390–395.

[90] M. I. Todorov, *Linear vector optimization. Properties of the efficient sets*, Serdica **18** (1992), 179–185.

[91] M. I. Todorov, *Well-posedness in the linear vector semi-infinite optimization*, in G. H. Tzeng (ed.), Multiple Criteria Decision Making, Springer, N.Y., 1994, pp. 141–150.

[92] M. I. Todorov, *Kuratowksi convergence of the efficient sets in the parametric linear vector semi-infinite optimization*, European J. Oper. Res. **94** (1996), 610-617.

[93] L. T. Tung, *Karush-Kuhn-Tucker optimality conditions and duality for convex semi-infinite programming with multiple interval-valued objective functions*, J. Appl. Math. Comput. **62** (2020), 67–91.

[94] L. T. Tung, *Strong Karush-Kuhn-Tucker optimality conditions for Borwein properly efficient solutions of multiobjective semi-infinite programming*, Bull. Braz. Math. Soc. **52** (2021), 1–22.
[95] L. T. Tung, *Karush-Kuhn-Tucker optimality conditions and duality for multiobjective semi-infinite programming with equilibrium constraints*, Yugosl. J. Oper. Res. **31** (2021) 429–453.
[96] L. T. Tung, *Karush–Kuhn–Tucker optimality conditions and duality for multiobjective semi-infinite programming with vanishing constraints*, Ann Oper Res **311** (2022) 1307–1334.
[97] L. T. Tung, *Karush–Kuhn–Tucker optimality conditions and duality for multiobjective semi-infinite programming problems with vanishing constraints*, Appl. Set-Valued Anal. Optim. **4** (2022) 1–26.
[98] L. T. Tung and D. H. Tam, *Optimality conditions and duality for nonsmooth multiobjective semi-infinite programming on Hadamard Manifolds*, Bull. Iran Math. Soc. **48** (2022), 2191–2219.
[99] H. Wang and H. Wang, *Duality theorems for nondifferentiable semi-infinite interval-valued optimization problems with vanishing constraints*, J. Inequal. Appl. **182** (2021).
[100] B. Wei, W. B. Haskell and S. Zhao, *The CoMirror algorithm with random constraint sampling for convex semi-infinite programming*, Ann. Oper. Res. **295** (2020), 809–841.
[101] B. Wei, W. B. Haskell and S. Zhao, A*n inexact primal-dual algorithm for semi-infinite programming,* Math. Methods Oper. Res. **91** (2020), 501–544.
[102] Z. Y. Wu, and C. R. Sun, *An unconstrained convex programming approach to convex semi-infinite programming*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms **17** (2010), 581–598.
[103] S. W. Xiang and Y. H. Zhou, *Continuity properties of solutions of vector optimization*, Nonlinear Anal. **64** (2006) 2496–2506.
[104] T. Yadav and S. K. Gupta, *On duality theory for multiobjective semi-infinite fractional optimization model using higher order convexity*, RAIRO Oper. Res. **55** (2021) 1343–1370.
[105] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, New Jersey, 2002.
[106] M. Zarepisheh, E. Khorram and P. M. Pardalos, *Generating properly efficient points in multiobjective programs by the nonlinear weighted sum scalarization method*, Optimization **63** (2014), 473–486.
[107] L. Zhang, S.-Y. Wu and M. A. López, *A new exchange method for convex semi-infinite programming, SIAM J. Optim.* **20** (2010), 2959–2977.
[108] L. Zhang, S.-C. Fang and S.-Y. Wu, *An entropy based central cutting plane algorithm for convex min-max semi-infinite programming problems*, Sci. China Math. **56** (2013), 201–211.

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