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SOME ENHANCED EXISTENCE RESULTS FOR STRONG VECTOR EQUILIBRIUM PROBLEMS

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Abstract. This paper explores some sufficient conditions for the enhanced solvability of strong vector equilibrium problems, which can be established via a variational approach. Enhanced solvability here means existence of solutions, which are strong with respect to the partial ordering, complemented with inequalities estimating the distance from the solution set (namely, error bounds). This kind of estimates plays a crucial role in the tangential (first-order) approximation of the solution set as well as in formulating optimality conditions for mathematical programming with equilibrium constraints (MPEC).

The approach here followed characterizes solutions as zeros (or global minimizers) of some merit functions associated to the original problem. Thus, to achieve the main results the traditional employment of the KKM theory is replaced by proper conditions on the slope of the merit functions. In turn, to make such conditions verifiable, some tools of nonsmooth analysis are exploited. As a result, several conditions for the enhanced solvability of strong equilibrium problems are derived, which are expressed in terms of generalized (Bouligand) derivatives, various normal and (Fenchel and Mordukhovich) subdifferential constructions.

1. Introduction

Since more than two decades vector equilibrium problems are a topic of active investigations, typically conducted by methods of nonlinear and convex analysis. The investigations exposed in the present paper consider vector equilibrium problems, which are defined by a vector valued bifunction taking values in a partially ordered space $\mathbb{Y}$, namely $f : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{Y}$, and a (nonempty) closed constraint set $K \subseteq \mathbb{X}$. Throughout the paper, the relation $\le_C$ partially ordering $\mathbb{Y}$ is supposed to be induced in the standard way by a fixed closed, nontrivial, convex and pointed cone $C \subseteq \mathbb{Y}$ (hence, in particular, $\{\mathbf{0}\} \neq C \neq \mathbb{Y}$).

Similarly as in vector optimization, also for vector equilibrium problems the solution concept is not unambiguously defined a priori. Given the aforementioned data, by strong vector equilibrium the following problem is meant:

$$(\texttt{SVE}) \qquad \text{find } \bar{x} \in K \text{ such that } f(\bar{x}, x) \in C, \quad \forall x \in K.$$

The solution set associated with problem (SVE) will be denoted throughout the paper by $\mathcal{SE}$. Sufficient conditions for its nonemptiness and estimates for the distance from it are the main theme of the present paper.

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With the above data, one may also consider the different problem

$$\text{(VE)} \qquad \text{find } \bar{x} \in K \text{ such that } f(\bar{x}, x) \notin -C\backslash\{\mathbf{0}\}, \quad \forall x \in K,$$

called vector equilibrium problem. Furthermore, if $\operatorname{int} C \neq \varnothing$, it makes sense to consider the so-called weak vector equilibrium problem, meaning

$$\text{(WVE)} \qquad \text{find } \bar{x} \in K \text{ such that } f(\bar{x}, x) \notin -\operatorname{int} C, \quad \forall x \in K.$$

It is clear from the respective definitions that every solution to a problem (SVE) is a fortiori a solution to the problems (VE) and (WVE), defined by the same data (whence the terminology). Of course, elementary examples show that the converse is not true. If, in particular, $\mathbb{Y} = \mathbb{R}$ and $C = [0, +\infty)$, then (SVE), (VE) and (WVE) collapse to the same problem, namely what is called equilibrium problem after Blum and Oettli. By their seminal paper [9], they definitely contribute to popularize this kind of problem, in stressing its unifying feature and undertaking a thorough study of it. In fact, equilibrium problem revealed to be a convenient format to treat in a unified framework various problems which are relevant in operations research and mathematical programming, such as single and multicriteria optimization problems, saddle point problems, complementarity problems, variational inequalities, fixed point problems, Nash equilibrium problems. In a similar manner, vector equilibrium problems provide a format able to subsume vector optimization problems, vector complementarity problems and vector variational inequalities (see [1, 4, 16] and references therein). For this reason, in the last two decades vector equilibrium problems became the subject of many investigations. As it is reasonable, within the fast growing literature in this area, a remarkable amount of research work focussed on solution existence and related issues (see, for instance, [1–4, 7, 16]). One of the main techniques of analysis in this context consists in adapting to the vector case the approach due to Ky Fan, originally proposed for scalar equilibrium problems (see, for instance, [8]). To achieve the nonemptiness of the solution set, regarded as the intersection of a proper family of sets, this approach leads to apply the Knaster-Kuratowski-Mazurkiewicz theorem (a.k.a. Three Polish theorem) or some variant of it (see [7, 11]). Other approaches to solution existence specific for strong vector equilibrium problems rely on different techniques, such as the employment of separation theorems for convex sets (see [16]) or the Kakutani fixed point theorem (see [3]).

The aim of the present paper is to enhance the study of solvability for strong vector equilibrium problems by complementing results about solution existence with inequalities estimating the distance from the solution set. These error bounds for (SVE) provide useful quantitative information on the set of solutions, which may be exploited in various contexts of application. For example, it is well known that error bounds enable to describe the local geometry of the solution set of a problem through its tangential (first-order) approximation. Moreover, error bounds are known to be connected with the metric subregularity and calmness properties of set-valued mappings (see, for instance, [14]). Thus, according to a recognized approach of analysis, error bounds reveal to be an essential tool for establishing optimality conditions via penalization techniques. More precisely, the estimates presented in this paper should be propaedeutic in order for deriving optimality conditions for

MPEC, where equilibrium constraints take the form of strong vector equilibrium problems (see [21, Section 5.2.3]). Furthermore, error bounds turn out to play an important role in the convergence theory of numerical methods.

It is plain to see that a problem (SVE) can be reformulated as a set-valued inclusion. If a set-valued mapping $F_{f,K} : \mathbb{X} \rightrightarrows \mathbb{Y}$ is defined as

$$F_{f,K}(x) = f(x, K) = \{y \in \mathbb{Y} : \ y = f(x, z), \ z \in K\},$$

then problem (SVE) becomes

$$\text{find } \bar{x} \in K \text{ such that } F_{f,K}(\bar{x}) \subseteq C.$$

Various elements for a solution analysis of the latter problem have been recently proposed in a series of papers [25–28], where several theoretical aspects of the solution behaviour (including existence and stability issues) have been investigated. The work presented in this paper can be viewed as an attempt to specialize that line of research to the context of vector equilibria.

Following a variational approach, the first step consists in introducing some functional characterizations of $\mathcal{SE}$. This is done by associating to a problem (SVE) a sort of merit function $\nu : \mathbb{X} \longrightarrow [0, +\infty]$, which is defined as

$$\begin{aligned} \nu(x) &= \operatorname{exc}(F_{f,K}(x), C) = \sup_{y \in F_{f,K}(x)} \operatorname{dist}(y, C) \\ &= \sup_{z \in K} \operatorname{dist}(f(x, z), C) . \end{aligned} \tag{1.1}$$

In order to embed also the constraining set K, it is useful to consider as well the function $\nu_{+K} : \mathbb{X} \longrightarrow [0, +\infty]$, given by

$$\nu_{+K}(x) = \nu(x) + \operatorname{dist}(x, K) . \tag{1.2}$$

Such merit functions, incorporating all problem data, enable to reduce strong vector equilibria to zeros (or global minimizers) of a functional.

Remark 1.1. Since C is closed, one sees that $\bar{x} \in \mathcal{SE}$ iff $\bar{x} \in K$ and $\nu(\bar{x}) = 0$. Equivalently, it holds

$$\mathcal{SE} = \nu^{-1}((-\infty, 0]) \cap K = \nu^{-1}(0) \cap K.$$

Analogously, since K is closed, one sees that $\bar{x} \in \mathcal{SE}$ iff $\nu_{+K}(\bar{x}) = 0$, namely

$$\mathcal{SE} = \nu_{+K}^{-1}((-\infty, 0]) = \nu_{+K}^{-1}(0).$$

The above functional characterizations of $\mathcal{SE}$ enable also to clarify at once some of its structural vector-topological properties. Namely, whenever ν and K are convex, $\mathcal{SE}$ is convex (possibly empty). Whenever ν is l.s.c. on $\mathbb{X}$ (K being closed), $\mathcal{SE}$ is closed (possibly empty). A sufficient condition for the latter property of ν to hold is, for instance, that each function $x \mapsto f(x, z)$ is continuous on $\mathbb{X}$, for every $z \in K$. Indeed, in such an event, as the distance function $y \mapsto \operatorname{dist}(y, C)$ is Lipschitz continuous on $\mathbb{Y}$ and therefore each function $x \mapsto \operatorname{dist}(f(x, z), C)$ is continuous on $\mathbb{X}$, then, according to (1.1), ν can be expressed as an upper envelope of continuous functions on $\mathbb{X}$.

The contents of the paper are organized as follows. Section 2 contains some preliminary technicalities dealing with the basic tools of analysis. Other advanced tools are recalled in the subsequent section, contextually to their use. Section 3 contains the main results of the paper arranged in two subsections: in the first one, conditions for the enhanced existence are presented, which rely on metric increase behaviour of the involved bifunctions, whereas in the second one some conditions are expressed in terms of several subdifferentials and normal cone constructions. Section 4 is reserved for concluding remarks.

Below, let us introduce the basic notations employed in the paper. The acronyms l.s.c., u.s.c. and p.h. stand for lower semicontinuous, upper semicontinuous and positively homogeneous, respectively. In a metric space setting, the closed ball centered at an element x, with radius $r \geq 0$, is denoted by $\mathrm{B}(x, r)$. In particular, in a Banach space, $\mathbb{B} = \mathrm{B}(\mathbf{0}, 1)$, whereas $\mathbb{S}$ stands for the unit sphere. The distance of a point x from S is denoted by $\mathrm{dist}\,(x, S)$, with the convention that $\mathrm{dist}\,(x, \varnothing) = +\infty$. The function $x \mapsto \mathrm{dist}\,(x, S)$ is sometimes indicated by d_S, if convenient. $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|)$ denote real Banach spaces, whose null vector is indicated by $\mathbf{0}$. Given a subset S of a Banach space, $\mathrm{int}\, S$ denotes its interior, $\mathrm{bd}\, S$ its boundary, whereas $\mathrm{cone}\, S$ its conical hull. By $\mathcal{P}(\mathbb{X}, \mathbb{Y})$ the Banach space of all continuous p.h. operators acting between $\mathbb{X}$ and $\mathbb{Y}$ is denoted, equipped with the operator norm $\|h\|_{\mathcal{P}} = \sup_{u \in \mathbb{S}} \|h(u)\|$, $h \in \mathcal{P}(\mathbb{X}, \mathbb{Y})$. $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ denotes its subspace of all bounded linear operators and, if $\Lambda \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$, $\Lambda^* \in \mathcal{L}(\mathbb{Y}^*, \mathbb{X}^*)$ indicates the adjoint operator to Λ. In particular, $\mathbb{X}^* = \mathcal{L}(\mathbb{X}, \mathbb{R})$ stands for the dual space of $\mathbb{X}^*$, in which case $\|\cdot\|_{\mathcal{P}}$ is simply marked by $\|\cdot\|$. The null vector, the unit ball and the unit sphere in a dual space will be marked by $\mathbf{0}^*$, $\mathbb{B}^*$, and $\mathbb{S}^*$, respectively. The duality pairing of a Banach space with its dual will be denoted by $\langle \cdot, \cdot \rangle$. If S is a subset of a dual space, $\overline{\mathrm{conv}}^{\,*} S$ stands for its convex closure with respect to the weak* topology. Whenever $C \subseteq \mathbb{Y}$ is a cone, by $C^{\ominus} = \{y^* \in \mathbb{Y}^* : \ \langle y^*, y \rangle \leq 0, \quad \forall y \in C\}$ its negative dual cone is denoted. Given a function $\varphi : \mathbb{X} \longrightarrow \mathbb{R} \cup \{\pm\infty\}$, by $[\varphi \leq 0] = \varphi^{-1}([-\infty, 0])$ its 0-sublevel set is denoted, whereas $[\varphi > 0] = \varphi^{-1}((0, +\infty])$ denotes the strict 0-superlevel set of φ. The symbol $\mathrm{dom}\, \varphi = \varphi^{-1}(\mathbb{R})$ indicates the domain of the function φ, while $\partial\varphi(x)$ the subdifferential of φ at x in the sense of convex analysis (a.k.a. Fenchel subdifferential), with the convention $\partial\varphi(x) = \varnothing$ if $x \notin \mathrm{dom}\, \varphi$. The normal cone to a set S at x in the sense of convex analysis is denoted by $\mathrm{N}(x; S)$.

As a standing assumption, throughout the paper the set-valued mapping $x \rightsquigarrow F_{f,K}(x) + C$ will be supposed to take closed values.

2. Preliminary tools of analysis

A first group of technical preliminaries relate to semicontinuity and convexity properties of the merit functions ν and ν_{+K} defined as in (1.1) and (1.2). Recall that, according to [19], given a closed, convex cone C, a mapping $g : \mathbb{X} \longrightarrow \mathbb{Y}$ between Banach spaces is said to be C-l.s.c. (resp. C-u.s.c.) at $x_0 \in \mathbb{X}$ if for any neighbourhood V of $g(x_0)$ there exists a neighbourhood U of x_0 in $\mathbb{X}$ such that

$$g(x) \in V + C, \qquad (\text{resp. } g(x) \in V - C) \qquad \forall x \in U.$$

Clearly, continuous mappings are both C-l.s.c. and C-u.s.c., whereas C-semicontinuity does not imply continuity, in general.

Lemma 2.1. *If $g : \mathbb{X} \longrightarrow \mathbb{Y}$ is C-u.s.c. at $x_0 \in \mathbb{X}$, then function $d_C \circ g$ is l.s.c. at x_0.*

Proof. In the current Banach space setting the property of semicontinuity can be proven to hold by a sequential argument. So, fix arbitrarily $\epsilon > 0$ and a sequence $(x_n)_n$, with $x_n \to x_0$ as $n \to \infty$. By the C-upper semicontinuity of g at x_0, corresponding to $V = \mathrm{B}(g(x_0), \epsilon)$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$g(x_n) \in \mathrm{B}(g(x_0), \epsilon) - C, \quad \forall n \geq n_\epsilon.$$

This means that there are $v_n \in \epsilon\mathbb{B}$ and $c_n \in C$ such that it is possible to write

$$g(x_n) = g(x_0) + v_n - c_n, \quad \forall n \geq n_\epsilon. \tag{2.1}$$

Thus, as it is $c_n + C \subseteq C$ and hence $C \subseteq C - c_n$, on account of the representation (2.1) one obtains

$$\begin{aligned}
(d_C \circ g)(x_0) &= \inf_{c \in C} \|g(x_0) - c\| = \inf_{c \in C} \|g(x_n) - v_n + c_n - c\| \\
&\leq \inf_{c \in C} \|g(x_n) - (c - c_n)\| + \|v_n\| \\
&= \inf_{y \in C - c_n} \|g(x_n) - y\| + \|v_n\| \leq \inf_{c \in C} \|g(x_n) - c\| + \epsilon \\
&= (d_C \circ g)(x_n) + \epsilon, \quad \forall n \geq n_\epsilon.
\end{aligned}$$

From these inequalities it follows

$$\liminf_{n \to \infty} (d_C \circ g)(x_n) \geq (d_C \circ g)(x_0) - \epsilon,$$

which by arbitrariness of ϵ and $(x_n)_n$ proves the assertion in the thesis. □

Remark 2.2. For the purposes of the present analysis it is useful to note that, as a straightforward consequence of Lemma 2.1, one can deduce that if each function $x \mapsto f(x, z)$ is C-u.s.c. on K for every $z \in K$, then function $\nu(x) = \sup_{z \in K}(d_C \circ f)(x, z) = \sup_{z \in K} \mathrm{dist}\,(f(x, z), C)$ is l.s.c. on K, as an upper envelope of functions being l.s.c. on K. Since d_K is (Lipschitz) continuous on $\mathbb{X}$, the same is true for ν_{+K}.

In order to formulate the next lemma it is convenient to recall that a subset S of a Banach space partially ordered by a cone C is said to be C-bounded if there exists a constant $m \geq 0$ such that $S \backslash C \subseteq m\mathbb{B}$.

Lemma 2.3. *Let $f : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{Y}$ be a given bifunction, $K \subseteq \mathbb{X}$ and $x_0 \in K$. If the set $f(x_0, K)$ is C-bounded, then $x_0 \in \mathrm{dom}\,\nu$. If, in addition, each function $x \mapsto f(x, z)$ is C-l.s.c. at x_0 uniformly in $z \in K$, then ν is bounded from above in a neighbourhood of x_0 and hence $x_0 \in \mathrm{int}\,\mathrm{dom}\,\nu$.*

Proof. By hypothesis, for some $m > 0$ it holds

$$f(x_0, K) \backslash C \subseteq m\mathbb{B},$$

whence it follows

$$\nu(x_0) = \sup_{y \in f(x_0, K) \backslash C} \mathrm{dist}\,(y, C) \leq \sup_{y \in m\mathbb{B}} \mathrm{dist}\,(y, C) \leq \sup_{y \in m\mathbb{B}} \|y\| = m < +\infty.$$

Thus, $x_0 \in \mathrm{dom}\, \nu$.

According to the additional hypothesis, corresponding to $V = l\mathbb{B}$ there exists a neighbourhood U of x_0 (not depending on $z \in K$) such that

$$f(x,z) \in \mathrm{B}(f(x_0,z),l) + C, \quad \forall x \in U,\ \forall z \in K.$$

This amounts to say that, for any $z \in K$ and $x \in U$, there exist $v \in \mathrm{B}(f(x_0,z),l)$ and $c \in C$ (both depending on $x \in U$ and $z \in K$), such that $f(x,z) = v + c$. Thus, recalling that $C + C \subseteq C$, for any $z \in K$ one finds

$$\begin{aligned} \mathrm{dist}\,(f(x,z),C) &= \mathrm{dist}\,(v+c,C) = \inf_{y\in C} \|v+c-y\| \\ &\le \inf_{c_1\in C}\inf_{c_2\in C} \|v+c-(c_1+c_2)\| \\ &\le \inf_{c_1\in C}\inf_{c_2\in C} [\|v-c_1\| + \|c-c_2\|] \le \inf_{c_1\in C} \|v-c_1\| \\ &= \mathrm{dist}\,(v,C)\,, \quad \forall x \in U. \end{aligned}$$

Therefore, it follows

$$\begin{aligned} \sup_{x\in U} \mathrm{dist}\,(f(x,z),C) &\le \|v - f(x_0,z)\| + \mathrm{dist}\,(f(x_0,z),C) \\ &\le l + \mathrm{dist}\,(f(x_0,z),C)\,, \quad \forall z \in K. \end{aligned}$$

Consequently, one obtains

$$\begin{aligned} \sup_{x\in U} \nu(x) &= \sup_{x\in U}\sup_{z\in K} \mathrm{dist}\,(f(x,z),C) = \sup_{z\in K}\sup_{x\in U} \mathrm{dist}\,(f(x,z),C) \\ &\le \sup_{z\in K} [l + \mathrm{dist}\,(f(x_0,z),C)] = l + \nu(x_0) < +\infty, \end{aligned}$$

which means that ν is bounded from above on U. □

Another key assumption for the present approach is C-concavity for mappings. Following [15], a mapping $g : \mathbb{X} \longrightarrow \mathbb{Y}$ between Banach spaces is said to be C-concave on the convex set $K \subseteq \mathbb{X}$ if for every $x_1,\, x_2 \in K$ and for every $t \in [0,1]$ it is true that

$$tg(x_1) + (1-t)g(x_2) \le_C g(tx_1 + (1-t)x_2).$$

Example 2.4. (i) It is readily seen that if a mapping $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is defined by components $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, each of which is concave on a convex set $K \subseteq \mathbb{R}^n$, then g is $\mathbb{R}^m_+$-concave on K.

(ii) A remarkable class of C-concave mappings is the subclass of $\mathcal{P}(\mathbb{X},\mathbb{Y})$ formed by the superlinear operators taking values in a Kantorovich space $\mathbb{Y}$ (i.e. a Dedekind complete normed vector lattice), partially ordered by a cone C. Following [24], a mapping $h \in \mathcal{P}(\mathbb{X},\mathbb{Y})$ is said to be superlinear if $h(x) + h(z) \le_C h(x+z)$, for every $x,\, z \in \mathbb{X}$. It is well known that, as a consequence of the Hahn-Banach-Kantorovich theorem (see [18]), any superlinear operator admits the following infimal representation

$$h(x) = \min_{\le_C} \{\Lambda x :\ \Lambda \in \overline{\partial} h\},$$

where $\overline{\partial} h = \{\Lambda \in \mathcal{L}(\mathbb{X},\mathbb{Y}) :\ h(x) \le_C \Lambda x,\ \forall x \in \mathbb{X}\}$ and $\min_{\le_C} S$ denotes the smallest element of a set $S \subseteq \mathbb{Y}$ with respect to the partial order $\le_C$.

Lemma 2.5. *If $g : \mathbb{X} \longrightarrow \mathbb{Y}$ is C-concave on the convex set $K \subseteq \mathbb{X}$, then function $d_C \circ g$ is convex on K.*

Proof. Take arbitrary x_1, $x_2 \in K$ and $t \in [0, 1]$. Owing to the C-concavity of g on K, one has

$$c_0 = g(tx_1 + (1-t)x_2) - [tg(x_1) + (1-t)g(x_2)] \in C.$$

Since it is $c_0 + C \subseteq C$, from the above inclusion one obtains

$$\begin{aligned}(d_C \circ g)(tx_1 + (1-t)x_2) &= \inf_{c \in C} \|g(tx_1 + (1-t)x_2) - c\| \\ &\leq \inf_{c \in c_0 + C} \|g(tx_1 + (1-t)x_2) - c\| \\ &\leq \inf_{c \in C} \|g(tx_1 + (1-t)x_2) - \{g(tx_1 + (1-t)x_2) \\ &\quad -[tg(x_1) + (1-t)g(x_2)] + c\}\| \\ &= \mathrm{dist}\,(tg(x_1) + (1-t)g(x_2), C)\,.\end{aligned}$$

By recalling that the function $y \mapsto \mathrm{dist}\,(y, C)$ is sublinear on $\mathbb{Y}$ as C is a convex cone, then from the above inequalities it follows

$$\begin{aligned}(d_C \circ g)(tx_1 + (1-t)x_2) &\leq t\mathrm{dist}\,(g(x_1), C) + (1-t)\mathrm{dist}\,(g(x_2), C) \\ &= t(d_C \circ g)(x_1) + (1-t)(d_C \circ g)(x_2),\end{aligned}$$

which, by arbitrariness of x_1, $x_2 \in K$ and $t \in [0, 1]$ completes the proof. □

The next lemma singles out a sufficient condition on the problem data of (SVE) in order for ν to be a (proper) convex and continuous function.

Lemma 2.6. *Let $f : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{Y}$ be a given bifunction and let $K \subseteq \mathbb{X}$ be a convex set. Suppose that:*

(i) *each function $x \mapsto f(x, z)$ is C-concave on K, for every $z \in K$;*
(ii) *there exists $x_0 \in K$ such that $f(x_0, K)$ is C-bounded and function $x \mapsto f(x, z)$ is C-l.s.c. at x_0, uniformly in $z \in K$.*

Then, function $\nu : K \longrightarrow [0, +\infty]$ is convex and continuous on $\mathrm{int}\,\mathrm{dom}\,\nu \neq \varnothing$.

Proof. According to Lemma 2.3, by virtue of hypothesis (ii) $x_0 \in \mathrm{int}\,\mathrm{dom}\,\nu$ and ν turns out to be bounded from above in a neighbourhood of x_0. According to Lemma 2.5, each function $x \mapsto \mathrm{dist}\,(f(x, z), C)$ is convex on K and therefore so is function ν, which can be regarded as an upper envelope of functions $\mathrm{dist}\,(f(\cdot, z), C)$ over $z \in K$. As a convex function, which is bounded from above in a neighbourhood of x_0, ν must be continuous on the interior of its domain, in the light of a well-known result in convex analysis (see, for instance, [30, Theorem 2.2.9]). □

Remark 2.7. In view of the subsequent analysis, it is convenient to notice that, under the hypotheses of Lemma 2.6, also function ν_{+K} turns out to be convex and continuous, with $\mathrm{dom}\,\nu_{+K} = \mathrm{dom}\,\nu \neq \varnothing$.

A characteristic feature of the main results established in the next section is to provide, along with solution existence, quantitative (metric) information about the solution set to strong vector equilibrium problems. Following a standard technique of variational analysis, estimates for the distance from the solution set to (SVE) will be investigated by means of a metric slope of the merit functions. Given a function $\varphi : \mathbb{X} \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ defined on a Banach space, a closed set $K \subseteq \mathbb{X}$ and $x \in K \cap \operatorname{dom} \varphi$, the nonnegative value

$$|\nabla_K \varphi|(x) = \begin{cases} 0, & \text{if } x \text{ is a local minimizer} \\ & \text{of } \varphi \text{ subject to } x \in K, \\ \limsup\limits_{u \xrightarrow{K} x} \dfrac{\varphi(x) - \varphi(u)}{\|x - u\|}, & \text{otherwise,} \end{cases}$$

where $u \xrightarrow{K} x$ means $u \to x$ while $u \in K$, represents the slope of φ at x restricted to K. It may be regarded as a restricted version of the well-known notion of (strong) slope of a function $\varphi : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ defined on a metric space (X, d), denoted here by $|\nabla\varphi|(x)$, which was introduced in [13] and subsequently employed in various contexts of variational analysis (see, among the others, [5, 6, 17, 22]).

Remark 2.8.

(i) Directly from the above definition, one sees that in general it holds
$$|\nabla_K \varphi|(x) \le |\nabla\varphi|(x),$$
while, whenever it is $x \in \operatorname{int} K$, one has $|\nabla_K \varphi|(x) = |\nabla\varphi|(x)$.

(ii) Remember that, whenever φ is Fréchet differentiable at $x \in \operatorname{dom} \varphi$, it holds $|\nabla\varphi|(x) = \|\mathrm{D}\varphi(x)\|$.

(iii) Whenever φ is a proper, convex and l.s.c. function, one has $|\nabla\varphi|(x) = \operatorname{dist}(\mathbf{0}^*, \partial\varphi(x))$ (see, for instance, [6, Proposition 3.1] or [14, Theorem 5(ii)]).

The seminal condition for the solution existence of (SVE) presented in Section 3 will be formulated in terms of the following crucial value associated through ν with a problem (SVE):

$$|\nabla_K f|^{>} = \inf_{x \in [\nu > 0] \cap K} |\nabla_K \nu|(x).$$

It is well known from variational analysis (see, for instance, [5, 6, 17, 22, 25]) that distances from sublevel sets of a l.s.c. function can be estimated in terms of its slope, while a certain positivity behaviour of the slope allows one to establish solvability of inequalities. All of this leads to investigate metric behaviours of the bifunction f, which ensure these conditions to hold. The property defined below pursues this purpose. It is a uniform variant of a conceptual tool that was already considered in [25], in connection with the analysis of solvability and stability properties of set-valued inclusions (see also [27]).

Definition 2.9. Let $f : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{Y}$ be a bifunction and let $K \subseteq \mathbb{X}$ and $S \subseteq \mathbb{X}$ be given sets. Then, f is said to be *metrically C-increasing* on the set S, *uniformly in $z \in K$*, if there exists $\alpha > 1$ such that for every $x_0 \in S$ there is $\delta_0 > 0$ (not depending on z) such that

$$\begin{aligned} &\forall r \in (0, \delta_0)\ \exists x \in \mathrm{B}(x_0, r) \cap S : \\ &\mathrm{B}(f(x, z), \alpha r) \subseteq \mathrm{B}(f(x_0, z) + C, r), \quad \forall z \in K. \end{aligned} \tag{2.2}$$

The value

$$\operatorname{inc}_C(f; S) = \sup\{\alpha > 1 : \ \text{condition (2.2) holds}\}$$

is called *exact bound of uniform C-metric increase* of f over S.

In view of the next proposition, it is useful to recall some relations involving the behaviour of the excess of sets that will be exploited in its proof. Let $C \subseteq \mathbb{Y}$ be a closed, convex cone and let $S \subseteq \mathbb{Y}$ a set with $S \not\subseteq C$ (hence, nonempty). Then, it holds

(p_1) $\mathrm{exc}(\mathrm{B}(S, r), C) = \mathrm{exc}(S, C) + r, \quad \forall r > 0$ (see [25, Lemma 2.2]);
(p_2) $\mathrm{exc}(S + C, C) = \mathrm{exc}(S, C)$ (see [25, Remark 2.1]).

The next proposition shows that the metric C-increase property is able to capture a behavior of f, which is useful in providing estimates from below of $|\nabla_K f|^>$ that are convenient to the present approach.

Proposition 2.10. *Let $f : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{Y}$ be a given bifunction and let $K \subseteq \mathbb{X}$. Suppose that:*

(i) *the function ν, associated with f and K as in* (1.1)*, is l.s.c. on K;*
(ii) *$f(\cdot, z)$ is metrically C-increasing on $K \cap [\nu > 0]$, uniformly in $z \in K$;*
(iii) *the set-valued mapping $F_{f,K}$ takes C-bounded values on K.*

Then, it holds

$$|\nabla_K f|^> \geq \mathrm{inc}_C(f; K \cap [\nu > 0]) - 1. \tag{2.3}$$

Proof. Take arbitrary $\alpha \in (1, \mathrm{inc}_C(f; K \cap [\nu > 0]))$ and $x_0 \in K \cap [\nu > 0]$. Since ν is l.s.c. at x_0, there exists $\delta > 0$ such that $\mathrm{B}(x_0, \delta) \cap K \subseteq [\nu > 0]$. By virtue of hypothesis (ii), for any $r \in (0, \delta_0)$, where $\delta_0 \in (0, \delta)$ is as in Definition 2.9, there exists $x_r \in \mathrm{B}(x_0, r) \cap K \cap [\nu > 0]$ such that $\mathrm{B}(f(x_r, z), \alpha r) \subseteq \mathrm{B}(f(x_0, z) + C, r)$, for every $z \in K$. This implies

$$\mathrm{B}(f(x_r, z), \alpha r) \subseteq \mathrm{B}(f(x_0, K) + C, r), \quad \forall z \in K,$$

whence it is possible to deduce

$$\mathrm{B}(f(x_r, K), \alpha r) \subseteq \mathrm{B}(f(x_0, K) + C, r), \tag{2.4}$$

because the set $\mathrm{B}(f(x_0, K) + C, r)$ is closed. Notice that it must be $x_r \neq x_0$. Indeed, assume to the contrary that $x_r = x_0$. From inclusion (2.4) it follows

$$\mathrm{B}(F_{f,K}(x_0), \alpha r) \subseteq \mathrm{B}(F_{f,K}(x_0) + C, r). \tag{2.5}$$

Let us show that such an inclusion under the hypotheses taken leads to an absurdum. To this aim, observe that, in turn, inclusion (2.5) implies for any $\tilde{\alpha} \in (1, \alpha)$

$$\mathrm{B}(F_{f,K}(x_0) + C, \tilde{\alpha} r) \subseteq \mathrm{B}(F_{f,K}(x_0) + C, r). \tag{2.6}$$

To check this fact notice that, as $\tilde{\alpha} < \alpha$, it holds

$$\mathrm{B}(F_{f,K}(x_0) + C, \tilde{\alpha} r) \subseteq \mathrm{B}(F_{f,K}(x_0), \alpha r) + C$$

and

$$\mathrm{B}(F_{f,K}(x_0) + C, r) + C \subseteq \mathrm{B}(F_{f,K}(x_0) + C + C, r).$$

Thus, by taking into account inclusion (2.5), from the above relations one gets

$$\begin{aligned}
\mathrm{B}(F_{f,K}(x_0) + C, \tilde{\alpha} r) &\subseteq \mathrm{B}(F_{f,K}(x_0), \alpha r) + C \\
&\subseteq \mathrm{B}(F_{f,K}(x_0) + C, r) + C \\
&\subseteq \mathrm{B}(F_{f,K}(x_0) + C + C, r) \\
&= \mathrm{B}(F_{f,K}(x_0) + C, r),
\end{aligned}$$

so inclusion (2.6) holds true. Now, since $F_{f,K}(x_0)$ is C-bounded according to hypothesis (iii), there exists $m > 0$ such that $F_{f,K}(x_0)\backslash C \subseteq m\mathbb{B}$. Consequently, it is

$$F_{f,K}(x_0) + C \neq \mathbb{Y},$$

because it is

$$\begin{aligned} F_{f,K}(x_0) + C &= [(F_{f,K}(x_0)\backslash C) \cup (F_{f,K}(x_0) \cap C)] + C \\ &\subseteq [m\mathbb{B} \cup C] + C = (m\mathbb{B} + C) \cup (C + C) \\ &= m\mathbb{B} + C. \end{aligned}$$

Of course, it is $m\mathbb{B} + C \neq \mathbb{Y}$. To see this, take $u_c \in \mathbb{S} \cap C$. Since C is pointed and $u_c \neq \mathbf{0}$, it must be $-u_c \notin C$. As C is closed, it is $\delta_c = \operatorname{dist}(-u_c, C) > 0$. So, by recalling that function $y \mapsto \operatorname{dist}(y, C)$ is p.h. as C is a cone, one obtains

$$\begin{aligned} \operatorname{dist}\left(-\frac{2m}{\delta_c}u_c, m\mathbb{B} + C\right) &\geq \operatorname{dist}\left(-\frac{2m}{\delta_c}u_c, C\right) - m \\ &= \frac{2m}{\delta_c}\operatorname{dist}(-u_c, C) - m = m > 0. \end{aligned}$$

The positive distance of $-\frac{2m}{\delta_c}u_c$ from $m\mathbb{B}+C$ allows one to conclude that $-\frac{2m}{\delta_c}u_c \in \mathbb{Y}\backslash(m\mathbb{B} + C) \neq \varnothing$. By recalling that $F_{f,K}(x_0) \not\subseteq C$, and hence $F_{f,K}(x_0) + C \not\subseteq C$, and that C is closed, it is possible to claim the existence of $y_0 \in \mathbb{Y}$ and $\eta \in (0, r)$ such that

$$y_0 \in \operatorname{bd}[F_{f,K}(x_0) + C]$$

and

$$\mathrm{B}(y_0, \eta) \cap C = \varnothing.$$

Then one can pick $y_r \in \mathrm{B}(y_0, \eta)\backslash[(F_{f,K}(x_0) + C) \cup C]$. As $F_{f,K}(x_0) + C$ is closed, these choices imply

$$0 < \operatorname{dist}(y_r, F_{f,K}(x_0) + C) \leq \|y_r - y_0\| \leq \eta < r \tag{2.7}$$

and

$$\operatorname{dist}(y_r, C) > 0. \tag{2.8}$$

From the above constructions, by remembering that $F_{f,K}(x_0) + C \subseteq m\mathbb{B} + C$, one obtains the following estimate for any $t \in [1, +\infty)$

$$\begin{aligned} \operatorname{dist}(ty_r, F_{f,K}(x_0) + C) &\geq \operatorname{dist}(ty_r, m\mathbb{B} + C) \geq \operatorname{dist}(ty_r, C) - m \\ &= t\operatorname{dist}(y_r, C) - m. \end{aligned}$$

This estimate and inequality (2.8) clearly imply

$$\lim_{t\to+\infty} \operatorname{dist}(ty_r, F_{f,K}(x_0) + C) = +\infty.$$

Let $\gamma : [1, +\infty) \longrightarrow [0, +\infty)$ be the continuous function defined by

$$\gamma(t) = \operatorname{dist}(ty_r, F_{f,K}(x_0) + C).$$

According to inequality (2.7), one has

$$\gamma(1) = \operatorname{dist}(y_r, F_{f,K}(x_0) + C) < r. \tag{2.9}$$

On the other hand, it has been shown that it holds

$$\lim_{t\to+\infty} \gamma(t) = +\infty.$$

By virtue of a well-known property of continuous functions of one real variable, one can deduce the existence of $t_{\tilde{\alpha}} \in (1, +\infty)$ such that

$$\gamma(t_{\tilde{\alpha}}) = \mathrm{dist}\,(t_{\tilde{\alpha}} y_r, F_{f,K}(x_0) + C) = \tilde{\alpha} r > r.$$

In the light of the last estimate, one can conclude that

$$t_{\tilde{\alpha}} y_r \in \mathrm{B}(F_{f,K}(x_0) + C, \tilde{\alpha} r) \backslash \mathrm{B}(F_{f,K}(x_0) + C, r),$$

which contradicts inclusion (2.6) and hence (2.5). Therefore, as claimed, it must be $x_r \neq x_0$.

Now, since $F_{f,K}(x_r) \not\subseteq C$, on account of the above recalled relations (p_1) and (p_2), one obtains

$$\begin{aligned} \nu(x_r) &= \mathrm{exc}(\mathrm{B}(F_{f,K}(x_r), \alpha r), C) - \alpha r \\ &\leq \mathrm{exc}(\mathrm{B}(F_{f,K}(x_0) + C, r), C) - \alpha r \\ &\leq \nu(x_0) + r - \alpha r. \end{aligned}$$

As x_r belongs to $\mathrm{B}(x_0, r) \cap K \cap [\nu > 0]$, one finds

$$\nu(x_0) - \nu(x_r) \geq (\alpha - 1) r \geq (\alpha - 1)\|x_r - x_0\|.$$

By arbitrariness of $r \in (0, \delta_0)$, it results in

$$|\nabla_K \nu|(x_0) \geq \alpha - 1.$$

Since the last inequality is true all over $K \cap [\nu > 0]$ by the arbitrariness of x_0, then one can conclude that $|\nabla_K f|^{>} \geq \alpha - 1$. From this inequality one achieves the estimate in (2.3) by taking into account the arbitrariness of $\alpha \in (1, \mathrm{inc}_C(f; K \cap [\nu > 0]))$. □

3. Enhanced solution existence for strong vector equilibrium problems

The present approach to the strong solvability of vector equilibrium problems starts with a general result which relies on the following specialization of [5, Theorem 1.10].

Proposition 3.1. *Let (X, d) be a complete metric space and let $\varphi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. Assume that $[\varphi < +\infty] \neq \varnothing$ and*

$$\tau = \inf_{x \in [0 < \varphi < +\infty]} |\nabla \varphi|(x) > 0.$$

Then, it is $[\varphi \leq 0] \neq \varnothing$ and

$$\mathrm{dist}\,(x, [\varphi \leq 0]) \leq \frac{\max\{\varphi(x), 0\}}{\tau}, \quad \forall x \in [\varphi < +\infty]. \tag{3.1}$$

Proof. The thesis follows directly from the assertion *a)* in [5, Theorem 1.10], with the choice $\alpha = \gamma = 0$ and $\beta = +\infty$ for the parameters appearing in its statement. Notice, in particular, that the nonemptiness of $[\varphi \leq 0]$ comes as a consequence of inequality (3.1), whose left-side term must be a real number for every $x \in [\varphi < +\infty]$. □

A first enhanced existence result for strong equilibrium problems can be established in terms of constructions described in Section 2 as follows.

Theorem 3.2. *With reference to a problem* (`SVE`)*, suppose that:*

(i) *each function* $x \mapsto f(x, z)$ *is* C*-u.s.c. on* K*, for every* $z \in K$*;*
(ii) *there exists* $x_0 \in K$ *such that* $f(x_0, K)$ *is* C*-bounded;*
(iii) *it is* $|\nabla_K f|^{>} > 0$*.*

Then, $\mathcal{SE}$ *is nonempty and closed and the following estimate holds*

$$\mathrm{dist}\,(x, \mathcal{SE}) \leq \frac{\nu(x)}{|\nabla_K f|^{>}}, \quad \forall x \in K. \tag{3.2}$$

Proof. If $K \cap [\nu > 0] = \varnothing$ it means that $\mathcal{SE} = K$, so all the assertions in the thesis become trivially true. Assume henceforth that $K \cap [\nu > 0] \neq \varnothing$. In the light of Remark 2.2, under the made assumptions function $\nu : K \longrightarrow [0, +\infty]$ turns out to be l.s.c. on K and, by virtue of Lemma 2.3, it is $x_0 \in [\nu < +\infty] \neq \varnothing$. As K is closed, the metric space (K, d), where d is the metric induced by the norm of $\mathbb{X}$ on K, is complete. Thus it is possible to invoke Proposition 3.1, in such a way to get, in consideration of the functional characterization of $\mathcal{SE}$ (remember Remark 1.1) $\mathcal{SE} \neq \varnothing$. By taking into account that ν takes nonnegative values only, from inequality (3.1) one can reach the estimate in the thesis for every $x \in K \cap [\nu < +\infty]$. The extension of the validity of inequality (3.2) in such a way to include also $x \in K \cap [\nu = +\infty]$ is obvious. As for the closedness of $\mathcal{SE}$, it comes as an immediate consequence of the lower semicontinuity property of ν on K. This completes the proof. □

As it happens for existence and error bound results related to several problems, which can be achieved by following the present variational approach (see, among the others, [25, 27]), Theorem 3.2 provides a sufficient condition for solvability, which generally fails to be also necessary. The example below aims at illustrating this fact.

Example 3.3. Let $\mathbb{X} = \mathbb{Y} = \mathbb{R}^2$ be equipped with its standard Euclidean space structure, $C = \mathbb{R}^2_+$, $K = -\mathbb{R}^2_+$ and let $f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$f(x, z) = \begin{pmatrix} -x_1^2 + e^{-\|z\|} \\ -x_2^2 + \frac{1}{\|z\|+1} \end{pmatrix}, \qquad x = (x_1, x_2),\ z = (z_1, z_2) \in \mathbb{R}^2. \tag{3.3}$$

It is evident that $\bar{x} = \mathbf{0}$ is a solution to the problem (`SVE`) defined by the given data. More precisely, it is $\mathcal{SE} = \{\mathbf{0}\}$. Indeed, if $\hat{x} \in -\mathbb{R}^2_+ \backslash \{\mathbf{0}\}$ it must be $\min\{\hat{x}_1\ \hat{x}_2\} < 0$. According to the expression in (3.3), if taking $z_k = (-k, 0) \in -\mathbb{R}^2_+$ for every $k \in \mathbb{N}$, one finds

$$f(\hat{x}, z_k) = \begin{pmatrix} -\hat{x}_1^2 + e^{-k} \\ -\hat{x}_2^2 + \frac{1}{k+1} \end{pmatrix} \overset{k\to\infty}{\longrightarrow} \begin{pmatrix} -\hat{x}_1^2 \\ -\hat{x}_2^2 \end{pmatrix} \not\in \mathbb{R}^2_+,$$

and hence, as $\mathbb{R}^2_+$ is closed, for some $z_k \in -\mathbb{R}^2_+$, it must be true that $f(\hat{x}, z_k) \not\in \mathbb{R}^2_+$, so $\hat{x} \not\in \mathcal{SE}$. Since f is continuous on $\mathbb{R}^2$, in particular each function $x \mapsto f(x, z)$ is C-u.s.c. on $-\mathbb{R}^2_+$, for every $z \in -\mathbb{R}^2_+$. From being $e^{-\|z\|}$, $(\|z\| + 1)^{-1} \in (0, 1]$ for every $z \in -\mathbb{R}^2_+$, one deduces that

$$F_{f,K}(x) = f(x, -\mathbb{R}^2_+) \subseteq \begin{pmatrix} -x_1^2 \\ -x_2^2 \end{pmatrix} + ([0,1] \times [0,1]), \quad \forall x \in -\mathbb{R}^2_+,$$

which shows that the set $f(x, -\mathbb{R}^2_+)$ is bounded (and hence, a fortiori, $\mathbb{R}^2_+$-bounded) for every $x \in -\mathbb{R}^2_+$. Whereas hypotheses (i) and (ii) of Theorem 3.2 happen to be satisfied, hypothesis (iii) does not. Indeed, for the problem under consideration the merit function $\nu : \mathbb{R}^2 \longrightarrow [0, +\infty)$ is clearly given by the expression

$$\nu(x) = \left\| \begin{pmatrix} -x_1^2 \\ -x_2^2 \end{pmatrix} \right\| = \sqrt{x_1^4 + x_2^4}, \quad \forall x \in -\mathbb{R}^2_+.$$

By taking into account what noticed in Remark 2.8, one obtains in particular

$$|\nabla_K \nu|(x) = |\nabla \nu|(x) = \|\mathrm{D}\nu(x)\|, \quad \forall x \in \mathrm{int}\,(-\mathbb{R}^2_+) = -\mathrm{int}\,\mathbb{R}^2_+.$$

Then, elementary calculations lead to find

$$\|\mathrm{D}\nu(x)\| = \left\| \begin{pmatrix} \frac{2x_1^3}{\sqrt{x_1^4+x_2^4}} \\ \frac{2x_2^3}{\sqrt{x_1^4+x_2^4}} \end{pmatrix} \right\| = \frac{2}{\sqrt{x_1^4 + x_2^4}} \sqrt{x_1^6 + x_2^6}, \quad \forall x \in -\mathrm{int}\,\mathbb{R}^2_+.$$

Therefore, if setting $x_n = -(1/n, 1/n) \in -\mathrm{int}\,\mathbb{R}^2_+$, one obtains

$$|\nabla_K \nu|(x_n) = 2\sqrt{\frac{\frac{1}{n^6} + \frac{1}{n^6}}{\frac{1}{n^4} + \frac{1}{n^4}}} = \frac{2}{n}, \quad \forall n \in \mathbb{N}\backslash\{0\}.$$

Consequently, it results in

$$|\nabla_K f|^{>} = \inf_{x \in -\mathbb{R}^2_+ \backslash \{\mathbf{0}\}} |\nabla_K \nu|(x) \leq \inf_{n \in \mathbb{N}\backslash\{0\}} |\nabla_K \nu|(x_n) = 0.$$

So, hypothesis (iii) is not fulfilled. In spite of this, it happens that $\mathcal{SE} \neq \varnothing$. Nevertheless, it is worth observing that, while a solution to the problem ($\mathtt{SVE}$) actually exists, an error bound such as inequality (3.2), with $|\nabla_K f|^{>}$ replaced with any positive constant τ, fails to work for the problem at the issue. This because the inequality

$$\mathrm{dist}\,(x, \mathcal{SE}) = \mathrm{dist}\,(x, \{\mathbf{0}\}) = \sqrt{x_1^2 + x_2^2} \leq \frac{\sqrt{x_1^4 + x_2^4}}{\tau}, \quad \forall x \in -\mathbb{R}^2_+,$$

can never be true, no matter how the value of $\tau > 0$ is chosen.

3.1. **Enhanced existence conditions under metric C-increase.** Further enhanced existence results for ($\mathtt{SVE}$) can be derived by exploiting conditions able to guarantee hypothesis (iii) of Theorem 3.2 to hold ceteris paribus. As seen in Section 2, the property of metric C-increase offers the possibility to estimate the merit function's slope in a way which is useful in the present context.

Corollary 3.4 (Existence under metric increase). *With reference to a problem* ($\mathtt{SVE}$)*, suppose that:*

(i) *each function $x \mapsto f(x, z)$ is C-u.s.c. on K, for every $z \in K$;*
(ii) *the set-valued mapping $F_{f,K}$ takes C-bounded values on K;*

(iii) $f(\cdot, z)$ *is metrically* C*-increasing on* $K\backslash\mathcal{SE}$*, uniformly in* $z \in K$.
Then, $\mathcal{SE}$ *is nonempty and closed, and the following estimate holds true*

$$\operatorname{dist}(x, \mathcal{SE}) \leq \frac{\nu(x)}{\operatorname{inc}_C(f; K\backslash\mathcal{SE}) - 1}, \quad \forall x \in K. \tag{3.4}$$

Proof. It suffices to observe that the current hypotheses enable one to apply Proposition 2.10, according to which one has

$$|\nabla_K f|^{>} \geq \operatorname{inc}_C(f; K\backslash\mathcal{SE}) - 1 > 0.$$

Therefore also hypothesis (iii) of Theorem 3.2 is satisfied. So all the assertions in the thesis follow at once from Theorem 3.2. □

The next step in the present investigation is to derive from Corollary 3.4 verifiable conditions for the enhanced existence of solutions to (`SVE`), which rely on differential calculus, with the aim of making the last result more suitable for applications. This can be done to a level of generality large enough to include also certain nonsmooth mappings.

Following [23], let us say that a mapping $g : \mathbb{X} \longrightarrow \mathbb{Y}$ between Banach spaces is Bouligand-differentiable (for short, B-differentiable) at $x_0 \in \mathbb{X}$ if there exists a mapping $\mathrm{D}_B g(x_0) \in \mathcal{P}(\mathbb{X}, \mathbb{Y})$ (henceforth called the B-derivative of g at x_0) such that

$$\lim_{x \to x_0} \frac{g(x) - g(x_0) - \mathrm{D}_B g(x_0; x - x_0)}{\|x - x_0\|} = \mathbf{0}. \tag{3.5}$$

The reader should notice that, since $\mathrm{D}_B g(x_0)$ is required to be continuous by the above notion, then g is continuous at x_0 whenever it is B-differentiable at the same point. Besides, whenever it happens, in particular, that $\mathrm{D}_B g(x_0) \in \mathcal{L}(\mathbb{X}, \mathbb{Y}) \subseteq \mathcal{P}(\mathbb{X}, \mathbb{Y})$, then g turns out to be Fréchet differentiable at x_0, with $\mathrm{D}_B g(x_0; v) = \mathrm{D}g(x_0)v$ for every $v \in \mathbb{X}$. Therefore B-differentiability extends Fréchet differential calculus to a broader class of mappings. A sufficient condition for the metric C-increase of B-differentiable mappings can be formulated in terms of existence of directions, along which the B-derivative is "firmly positive".

Proposition 3.5 (Differential condition for metric increase)**.** *Given a mapping* $g : \mathbb{X} \longrightarrow \mathbb{Y}$*, a convex subset* $S \subseteq \mathbb{X}$ *and a closed convex cone* $C \subseteq \mathbb{Y}$*. If* g *is B-differentiable at each point of* $S\backslash g^{-1}(C)$ *and there exists* $\sigma > 0$ *such that for every* $x_0 \in S\backslash g^{-1}(C)$

$$\exists u_0 \in \mathbb{S} \cap \operatorname{cone}(S - x_0) : \ \mathrm{D}_B g(x_0; u_0) + \sigma\mathbb{B} \subseteq C, \tag{3.6}$$

then g *is metrically* C*-increasing on* $S\backslash g^{-1}(C)$*, with*

$$\operatorname{inc}_C(g; S\backslash g^{-1}(C)) \geq \sigma + 1. \tag{3.7}$$

Proof. Fix arbitrary $\epsilon \in (0, \min\{\sigma, 1\})$ and $x_0 \in S\backslash g^{-1}(C)$ and set $\alpha = \sigma + 1 - \epsilon > 1$. According to (3.5), there exists $\delta > 0$ such that

$$g(x_0 + tv) - g(x_0) - t\mathrm{D}_B g(x_0; v) \in \epsilon t\|v\|\mathbb{B} \subseteq \epsilon t\mathbb{B}, \quad \forall v \in \mathbb{B},\ \forall t \in [0, \delta].$$

In particular, taking $u_0 \in \mathbb{S} \cap \operatorname{cone}(S - x_0)$ as in the assumption (3.6), one finds

$$g(x_0 + tu_0) \in g(x_0) + t\mathrm{D}_B g(x_0; u_0) + \epsilon t\mathbb{B}, \quad \forall t \in [0, \delta]. \tag{3.8}$$

Notice that $u_0 \in \mathbb{S} \cap \mathrm{cone}\,(S - x_0)$ implies the existence of $\lambda_0 > 0$ and $s_0 \in S\backslash\{x_0\}$ (having to be $u_0 \neq \mathbf{0}$) such that $u_0 = \lambda_0(s_0 - x_0)$. Therefore, if one assumes that $\delta \in (0, \lambda_0^{-1})$, for every $t \in [0, \delta]$ it is possible to write

$$x_0 + tu_0 = x_0 + t\lambda_0(s_0 - x_0) = (1 - t\lambda_0)x_0 + t\lambda_0 s_0,$$

where it is $t\lambda_0 \in [0, 1]$ because $0 \leq t\lambda_0 \leq \delta\lambda_0 < 1$. Thus, up a reduction in the value of δ, if needed, as x_0, $s_0 \in S$, by convexity of S it holds $x_0 + tu_0 \in S$ for every $t \in [0, \delta]$. Since g is continuous on $S\backslash g^{-1}(C)$ (as a consequence of its B-differentiability on the same set) and C is closed, without any loss of generality, one can assume that $\mathrm{B}(x_0, \delta) \cap S \subseteq S\backslash g^{-1}(C)$. So, by setting $\delta_0 = \delta$, fixed any $r \in (0, \delta_0)$, let us choose $x_r = x_0 + ru_0$. In this way, one has $x_r \in S\backslash g^{-1}(C)$ and, on account of the inclusions (3.8) and in (3.6), it results in

$$\begin{aligned}
\mathrm{B}(g(x_r), \alpha r) &= g(x_r) + \alpha r\mathbb{B} \subseteq g(x_0) + r\mathrm{D}_B g(x_0; u_0) + \epsilon r\mathbb{B} + \alpha r\mathbb{B} \\
&= g(x_0) + r\mathrm{D}_B g(x_0; u_0) + \epsilon r\mathbb{B} + (\sigma + 1 - \epsilon)r\mathbb{B} \\
&\subseteq g(x_0) + r[\mathrm{D}_B g(x_0; u_0) + \sigma\mathbb{B}] + (\epsilon + 1 - \epsilon)r\mathbb{B} \\
&\subseteq g(x_0) + rC + r\mathbb{B} \subseteq \mathrm{B}(g(x_0) + C, r).
\end{aligned}$$

As x_0 was arbitrarily chosen in $S\backslash g^{-1}(C)$, the above inclusion amounts to say that g is metrically C-increasing on $S\backslash g^{-1}(C)$, with $\mathrm{inc}_C(g; S\backslash g^{-1}(C)) \geq \sigma + 1 - \epsilon$. The estimate in (3.7) then follows for arbitrariness of ϵ. □

When dealing with bifunctions, the above differentiability notion will be employed in its (partial) uniform variant: given a bifunction $f : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{Y}$ and a subset $K \subseteq \mathbb{X}$, f is said to be (partially) B-differentiable at $x_0 \in \mathbb{X}$, uniformly in $z \in K$, if there exist mappings $\mathrm{D}_B f(\cdot, z)(x_0) \in \mathcal{P}(\mathbb{X}, \mathbb{Y})$ with the property that for every $\epsilon > 0$ there exists $\delta > 0$ (depending on x_0 and ϵ, but not on $z \in K$) such that

$$\sup_{z \in K} \frac{\|f(x, z) - f(x_0, z) - \mathrm{D}_B f(\cdot, z)(x_0; x - x_0)\|}{\|x - x_0\|} \leq \epsilon, \quad \forall x \in \mathrm{B}(x_0, \delta). \tag{3.9}$$

Example 3.6. (i) Whenever a bifunction $f : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{Y}$ takes the following additively separable form

$$f(x, z) = g(x) + h(z), \quad (x, z) \in \mathbb{X} \times K,$$

where $g : \mathbb{X} \longrightarrow \mathbb{Y}$ is B-differentiable at x_0 and $h : K \longrightarrow \mathbb{Y}$, then f is B-differentiable at x_0 uniformly in K and it holds $\mathrm{D}_B f(\cdot, z)(x_0; v) = \mathrm{D}_B g(x_0; v)$, for every $z \in K$ and $v \in \mathbb{X}$.

(ii) Whenever a bifunction $f : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{Y}$ takes the following factorable form

$$f(x, z) = \lambda(z)g(x), \quad (x, z) \in \mathbb{X} \times K,$$

where $g : \mathbb{X} \longrightarrow \mathbb{Y}$ is B-differentiable at x_0 and $\lambda : K \longrightarrow \mathbb{R}$ is bounded on K, then f is B-differentiable at x_0 uniformly in K and it holds $\mathrm{D}_B f(\cdot, z)(x_0; v) = \lambda(z)\mathrm{D}_B g(x_0; v)$, for every $z \in K$ and $v \in \mathbb{X}$.

The next technical lemma provides a sufficient condition for the uniform metric C-increase property of uniformly B-differentiable bifunctions, along with an estimate of the exact bound of uniform metric C-increase.

Lemma 3.7. *With reference to problem* (SVE), *let* $f : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{Y}$ *be a bifunction and let* $K \subseteq \mathbb{X}$ *be a convex set. Suppose that* f *is B-differentiable in the first argument at each point of* $K \backslash \mathcal{SE}$*, uniformly in* $z \in K$*, with B-derivatives* $\mathrm{D}_B f(\cdot, z)(x_0) \in \mathcal{P}(\mathbb{X}, \mathbb{Y})$*. If there exists* $\sigma > 0$ *such that for every* $x_0 \in K \backslash \mathcal{SE}$

$$\exists u_0 \in \mathbb{S} \cap \mathrm{cone}\,(K - x_0) : \ \mathrm{D}_B f(\cdot; z)(x_0; u_0) + \sigma \mathbb{B} \subseteq C, \quad \forall z \in K, \tag{3.10}$$

then f *is metrically* C*-increasing on* $K \backslash \mathcal{SE}$*, uniformly in* $z \in K$*, and it holds*

$$\mathrm{inc}_C(f; K \backslash \mathcal{SE}) \geq \sigma + 1. \tag{3.11}$$

Proof. One needs to adapt the proof of Proposition 3.5 to the context of uniform B-differentiability for bifunctions. Fix arbitrary $\epsilon \in (0, \min\{\sigma, 1\})$ and $x_0 \in K \backslash \mathcal{SE}$ and set $\alpha = \sigma + 1 - \epsilon > 1$. By uniform B-differentiability of f at x_0, for some $\delta > 0$, taking $u_0 \in \mathbb{S} \cap \mathrm{cone}\,(K - x_0)$ as in (3.10) (not depending on $z \in K$), one has

$$f(x_0 + tu_0, z) \in f(x_0, z) + t\mathrm{D}_B f(\cdot; z)(x_0; u_0) + \epsilon t \mathbb{B}, \quad \forall t \in [0, \delta],\ \forall z \in K.$$

Observe that, since

$$x_0 \in K \backslash \mathcal{SE} = K \backslash \bigcap_{z \in K} f(\cdot, z)^{-1}(C) = \bigcup_{z \in K} [K \backslash f(\cdot, z)^{-1}(C)],$$

there exists $z_0 \in K$ such that $x_0 \in K \backslash f(\cdot, z_0)^{-1}(C)$. As the mapping $x \mapsto f(x, z_0)$ is continuous at x_0, up to a reduction of the value of δ one finds

$$\mathrm{B}(x_0, \delta) \cap K \subseteq K \backslash f(\cdot, z_0)^{-1}(C) \subseteq K \backslash \mathcal{SE}.$$

So, by setting $\delta_0 = \delta$ and choosing $x_r = x_0 + ru_0$ for any $r \in (0, \delta_0)$, through the same reasoning as in the proof of Proposition 3.5, one obtains

$$\mathrm{B}(f(x_r, z, \alpha r) \subseteq \mathrm{B}(f(x_0, z) + C, r), \quad \forall z \in K.$$

This shows that the conditions in Definition 2.9 are satisfied, along with the related estimate, thereby completing the proof. □

Corollary 3.8. *Under the assumptions (i) and (ii) of Corollary 3.4, suppose in addition that:*

- (iii) K *is convex;*
- (iv) f *is B-differentiable in the first argument on* $K \backslash \mathcal{SE}$*, uniformly in* $z \in K$*, and* K *is convex;*
- (v) *there exists* $\sigma > 0$ *satisfying the condition in* (3.10).

Then, $\mathcal{SE}$ *is nonempty and closed, and the following estimate holds true*

$$\mathrm{dist}\,(x, \mathcal{SE}) \leq \frac{\nu(x)}{\sigma}, \quad \forall x \in K. \tag{3.12}$$

Proof. It suffices to observe that, under the above assumptions, Lemma 3.7 can be applied, so f turns out to be metrically C-increasing on $K \backslash \mathcal{SE}$ uniformly in $z \in K$, with the estimate in (3.11) being valid. In such a circumstance, all its hypotheses being satisfied, it remains to invoke Corollary 3.4 to get all the assertions in the thesis. □

Example 3.9. Let $\mathbb{X} = \mathbb{Y} = \mathbb{R}^2$ be equipped with their standard Euclidean space structure, let $f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$f(x, z) = g(z) - g(x),$$

where $g(x) = x$, let $\mathbb{R}^2$ be partially ordered by the cone $C = \mathbb{R}^2_+$ and let the constraining set be given by

$$K_\theta = \left\{ x = (r \cos t, r \sin t) \in \mathbb{R}^2 : \ r \geq 0, \ \theta \leq t \leq \frac{\pi}{2} - \theta \right\},$$

for any fixed $\theta \in (0, \frac{\pi}{4})$. It is clear that the strong equilibrium problem defined by the above data is equivalent to finding the strong efficient solutions to the vector optimization problem

$$\min_{\leq_C} g(x) \qquad \text{subject to} \qquad x \in K_\theta.$$

So one readily sees that $\mathcal{SE} = \{\mathbf{0}\}$. Since f is continuous over $\mathbb{R}^2 \times \mathbb{R}^2$, assumption (i) of Corollary 3.4 is fulfilled. As it is

$$f(x, K_\theta) = g(K_\theta) - g(x) = K_\theta - x$$

and $\theta \in (0, \frac{\pi}{4})$, one sees that the set-valued mapping $x \rightsquigarrow f(x, K_\theta)$ takes $\mathbb{R}^2_+$-bounded values on K_θ. Moreover, since $K_\theta - x + \mathbb{R}^2_+$ is a translation of $\mathbb{R}^2_+$, it is is a closed set, so the standing assumption on $F_{f,K} + C$ to be closed-valued is satisfied. Since f takes a form such as in Example 3.6(i), with g being differentiable, assumption (iv) of Corollary 3.8 is satisfied with

$$\mathrm{D}_B f(\cdot, z)(x; v) = -v, \quad \forall x \in K_\theta \backslash \{\mathbf{0}\}.$$

Let us check that also assumption (v) of Corollary 3.8 is fulfilled. To this aim, observe that, if taking $x_0 \in \operatorname{int} K_\theta$, then it is $\operatorname{cone}(K_\theta - x_0) = \mathbb{R}^2$. Therefore, by choosing $u_0 = (-1/\sqrt{2}, -1/\sqrt{2}) \in \operatorname{cone}(K_\theta - x_0) \cap \mathbb{S}$, one finds

$$\mathrm{D}_B f(\cdot, z)(x_0; u_0) + \frac{1}{\sqrt{2}}\mathbb{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + \frac{1}{\sqrt{2}}\mathbb{B} \subseteq \mathbb{R}^2_+. \tag{3.13}$$

Now take $x_0 = (r_0 \cos \theta, r_0 \sin \theta) \in K_\theta \backslash \{\mathbf{0}\}$, for any $r_0 > 0$. In this case, one has

$$\operatorname{cone}(K_\theta - x_0) = \{x = (r \cos t, r \sin t) \in \mathbb{R}^2 : \ r \geq 0, \ \theta \leq t \leq \theta + \pi\}.$$

Thus, if taking $u_0 = (\cos(\theta + \pi), \sin(\theta + \pi)) = -(\cos \theta, \sin \theta)$, one finds

$$\mathrm{D}_B f(\cdot, z)(x_0; u_0) + \sin \theta \, \mathbb{B} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} + \sin \theta \, \mathbb{B} \subseteq \mathbb{R}^2_+. \tag{3.14}$$

Notice that the last inclusion is true because it is $\sin \theta < \cos \theta$, as $\theta \in (0, \frac{\pi}{4})$. The remaining case $x_0 = \left(r_0 \cos\left(\frac{\pi}{2} - \theta\right), r_0 \sin\left(\frac{\pi}{2} - \theta\right)\right) \in K_\theta \backslash \{\mathbf{0}\}$, for any $r_0 > 0$, can be discussed in a similar manner by taking into account the symmetry of K_θ with respect to the axe of equation $x_2 = x_1$. Thus, since it is $\sin \theta < \frac{1}{\sqrt{2}}$, by inclusions (3.13) and (3.14) one verifies that condition (3.10) is fulfilled with $\sigma = \sin \theta$. As a consequence, Corollary 3.8 can be applied to the problem under examination.

In order to check the validity of the error bound provided in its thesis, one needs to find the expression of ν. This is easily done, inasmuch as one has

$$\begin{aligned}\nu(x) &= \sup_{z\in K_\theta} \mathrm{dist}\left(z-x,\mathbb{R}^2_+\right) = \sup_{z\in K_\theta} \mathrm{dist}\left(z,x+\mathbb{R}^2_+\right)\\ &\geq \mathrm{dist}\left(\mathbf{0},x+\mathbb{R}^2_+\right) = \|x\|, \quad \forall x\in K_\theta.\end{aligned}$$

On the other hand, as $K_\theta \subseteq \mathbb{R}^2_+$, one gets

$$\nu(x) \leq \sup_{z\in\mathbb{R}^2_+} \mathrm{dist}\left(z,x+\mathbb{R}^2_+\right) = \mathrm{dist}\left(\mathbf{0},x+\mathbb{R}^2_+\right) = \|x\|, \quad \forall x\in K_\theta.$$

From the last inequalities, one obtains

$$\nu(x) = \|x\|, \quad \forall x\in K_\theta.$$

So, in the light of the above computations, it is possible to check that it actually holds

$$\mathrm{dist}\,(x,\mathcal{SE}) = \mathrm{dist}\,(x,\{\mathbf{0}\}) = \|x\| \leq \frac{\|x\|}{\sin\theta} = \frac{\nu(x)}{\sigma}, \quad \forall x\in K_\theta,$$

in accordance with inequality (3.12).

The reader should be warned that the same conclusions can not be drawn in the (critical) case with $\theta = 0$, corresponding to a vector equilibrium problem defined by the same bifunction f, but with $K = \mathbb{R}^2_+$. There is no $\sigma > 0$ for which assumption (v) of Corollary 3.8 happens to be satisfied. Indeed, if taking $x_0 = (r_0, 0) \in K\backslash\{\mathbf{0}\}$, for some $r_0 > 0$, one can not find $u_0 \in \mathrm{cone}\,(\mathbb{R}^2_+ - (r_0,0)) = \{x = (x_1,x_2) \in \mathbb{R}^2_+ : x_2 \geq 0\}$ such that

$$-u_0 + \sigma\mathbb{B} \subseteq \mathbb{R}^2_+.$$

In spite of this, it is still $\nu(x) = \|x\|$ for every $x\in\mathbb{R}^2_+$ and $\mathcal{SE} = \{\mathbf{0}\}$, so the error bound

$$\mathrm{dist}\,(x,\mathcal{SE}) = \|x\| \leq \nu(x), \quad \forall x\in\mathbb{R}^2_+$$

still holds true. This fact shows once more that the conditions for the enhanced existence established by the present approach can be only sufficient.

3.2. **Subdifferential conditions for the enhanced solution existence.** Throughout the current subsection the Banach space $(\mathbb{X}\,\|\cdot\|)$ is assumed to be reflexive.

In order to formulate the next enhanced existence results for problem $(\mathtt{SVE})$, some more technical tools from nonsmooth analysis need to be recalled. Let $K \subseteq \mathbb{X}$ be a nonempty closed subset and let $x\in\mathbb{X}\backslash K$ such that $\Pi(x;K)\neq\varnothing$. In [20, Proposition 1.102] it has been proved that

$$\widehat{\partial}\mathrm{dist}\,(\cdot,K)\,(x) \subseteq \bigcap_{w\in\Pi(x;K)} \widehat{\mathrm{N}}(w;K)\cap\mathbb{S}^*,$$

where

$$\widehat{\partial}\varphi(x) = \left\{x^*\in\mathbb{X}^* : \liminf_{v\to\mathbf{0}} \frac{\varphi(x+v)-\varphi(x)-\langle x^*,v\rangle}{\|v\|} \geq 0\right\}$$

denotes the (regular) Fréchet subdifferential of a function φ at $x \in \operatorname{dom}\varphi$, and

$$\widehat{\mathrm{N}}(w; K) = \left\{ x^* \in \mathbb{X}^* : \limsup_{x \xrightarrow{K} w} \frac{\langle x^*, x - w \rangle}{\|x - w\|} \le 0 \right\}$$

denotes the cone of the Fréchet normals (a.k.a. prenormal cone) to K at w. On the other hand, if $x \in K$ then [20, Corollary 1.96] provides the different representation

$$\widehat{\partial}\mathrm{dist}\,(\cdot, K)\,(x) = \widehat{\mathrm{N}}(x; K) \cap \mathbb{B}^*.$$

Since in what follows both the cases have to be considered, it is convenient to deal with the set-valued mapping $\widehat{B}^*_K : \mathbb{X} \rightrightarrows \mathbb{X}^*$, defined by

$$\widehat{B}^*_K(x) = \begin{cases} \widehat{\mathrm{N}}(x; K) \cap \mathbb{B}^* & \text{if } x \in K, \\ \displaystyle\bigcap_{w \in \Pi(x;K)} \widehat{\mathrm{N}}(w; K) \cap \mathbb{S}^* & \text{if } x \notin K. \end{cases} \tag{3.15}$$

Whenever K is a closed convex set, in a reflexive Banach space setting it is $\Pi(x; K) \neq \varnothing$ for every $x \in \mathbb{X}$. Thus, since in such an event it turns out that $\widehat{\partial}\mathrm{dist}\,(\cdot, K)\,(x) = \partial\mathrm{dist}\,(\cdot, K)\,(x)$ and $\widehat{\mathrm{N}}(w; K) = \mathrm{N}(w; K)$ (see [20, Theorem 1.93] and [20, Proposition 1.3], respectively), the set-valued mapping $\widehat{B}^*_K$ is well-defined and takes the special form

$$\widehat{B}^*_K(x) = \begin{cases} \mathrm{N}(x; K) \cap \mathbb{B}^* & \text{if } x \in K, \\ \displaystyle\bigcap_{w \in \Pi(x;K)} \mathrm{N}(w; K) \cap \mathbb{S}^* & \text{if } x \notin K. \end{cases}$$

Remark 3.10. It is well known that if $x \in \mathbb{X}\backslash K$ and $(\mathbb{X}, \|\cdot\|)$ has a uniformly Gâteaux differentiable norm, then function $\mathrm{dist}\,(\cdot, K)$ is strictly differentiable at x, with $\mathrm{Ddist}\,(\cdot, K)\,(x) \in \mathbb{S}^*$ (see [29, Corollary 4.2]), so in this case one has $\widehat{B}^*_K(x) = \{\mathrm{Ddist}\,(\cdot, K)\,(x)\}$. The class of Banach spaces admitting a uniformly Gâteaux differentiable norm is known to include all separable spaces (see [12, Chapter II, Corollary 6.9(i)]).

Theorem 3.11 (C-concave case)**.** *With reference to a problem* (SVE)*, suppose that:*

(i) *each function $x \mapsto f(x, z)$ is C-concave on the convex set K, for every $z \in K$;*
(ii) *there exists $x_0 \in K$ such that $f(x_0, K)$ is C-bounded;*
(iii) *each function $x \mapsto f(x, z)$ is C-u.s.c. on $\mathbb{X}$, for every $z \in K$;*
(iv) *there exists $\gamma > 0$ such that*

$$\left[\partial\nu(x) + \widehat{B}^*_K(x)\right] \cap \gamma\mathbb{B}^* = \varnothing, \quad \forall x \in \mathbb{X}\backslash\mathcal{SE}. \tag{3.16}$$

Then, $\mathcal{SE}$ is nonempty, closed and convex, and the following estimate holds true

$$\mathrm{dist}\,(x, \mathcal{SE}) \le \frac{\nu_{+K}(x)}{\gamma}, \quad \forall x \in \mathbb{X}. \tag{3.17}$$

Proof. Observe first that, on account of Remark 2.2 and hypothesis (iii), the merit function ν_{+K} is l.s.c. on $\mathbb{X}$. Moreover, as stated in Remark 2.7, under hypotheses (i) and (ii) ν_{+K} is convex on $\mathbb{X}$ and $x_0 \in \operatorname{dom}\nu_{+K}$, so that $[\nu_{+K} < +\infty] \neq \varnothing$. Now, take $x \in [0 < \nu_{+K} < +\infty]$. Since ν_{+K} is proper, convex and l.s.c. on $\mathbb{X}$, then according to Remark 2.8(iii), the following slope estimate holds

$$|\nabla\nu_{+K}|(x) = \operatorname{dist}\left(\mathbf{0}^*, \partial\nu_{+K}(x)\right).$$

Since both ν and $\operatorname{dist}(\cdot, K)$ are convex functions, while ν is l.s.c. and $\operatorname{dist}(\cdot, K)$ is continuous at $x_0 \in \operatorname{dom}\nu \cap \operatorname{dom}\operatorname{dist}(\cdot, K)$, by the Moreau-Rockafellar theorem one obtains

$$\partial\nu_{+K}(x) = \partial\nu(x) + \partial\operatorname{dist}(\cdot, K)(x) \subseteq \partial\nu(x) + \widehat{B}^*_K(x).$$

Thus, the condition in hypothesis (iv) implies

$$|\nabla\nu_{+K}|(x) = \operatorname{dist}\left(\mathbf{0}^*, \partial\nu_{+K}(x)\right) \geq \gamma, \quad \forall x \in [0 < \nu_{+K} < +\infty].$$

This estimate shows that is possible to apply Proposition 3.1 with $X = \mathbb{X}$ and $\varphi = \nu_{+K}$, what leads to obtain the nonemptiness of $\mathcal{SE}$ and the inequality

$$\operatorname{dist}(x, \mathcal{SE}) \leq \frac{\nu_{+K}(x)}{\gamma}, \quad \forall x \in [\nu_{+K} < +\infty].$$

The last inequality readily allows one to achieve the estimate in (3.17). The convexity of $\mathcal{SE}$ is a direct consequence of the convexity of ν_{+K}, as observed in Remark 1.1. Thus the proof is complete. □

Remark 3.12. It should be noticed that inequality (3.17), in particular, entails

$$\operatorname{dist}(x, \mathcal{SE}) \leq \frac{\nu(x)}{\gamma}, \quad \forall x \in K.$$

As a comment to Theorem 3.11, it is to be pointed out that the condition in hypothesis (iv), involving function ν, is not explicitly formulated in terms of problem data. In fact, expressing the subdifferential of ν in terms of generalized derivatives of f may imply nontrivial calculations. Nevertheless, in the special case in which K is compact and each function $x \mapsto f(x, z)$ is smooth, with surjective derivatives, this issue can be faced by means of well-known subdifferential calculus rules and other technical results in nonsmooth analysis. To see this in detail, given $x \in K$ let us set

$$\overline{K}_x = \{z \in K : \ \operatorname{dist}(f(x, z), C) = \nu(x)\}$$

and let us introduce the set-valued mapping $B^*_C : \mathbb{X} \rightrightarrows \mathbb{Y}^*$, defined by

$$B^*_C(x) = \begin{cases} C^{\ominus} \cap \mathbb{B}^* & \text{if } \nu(x) = 0, \\ \\ C^{\ominus} \cap \mathbb{S}^* & \text{if } \nu(x) > 0. \end{cases}$$

Proposition 3.13. *With reference to a problem* (SVE), *suppose that:*

(i) *K is compact and convex and $(\mathbb{Y}, \|\cdot\|)$ is reflexive;*
(ii) *$f : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{Y}$ is continuous on $\mathbb{X} \times \mathbb{X}$;*
(iii) *each function $x \mapsto f(x, z)$ is C-concave on $\mathbb{X}$, for every $z \in K$;*
(iv) *each function $x \mapsto f(x, z)$ is $\mathrm{C}^1(\mathbb{X})$, with $\mathrm{D}f(\cdot, z)(x) \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ being onto, for every $z \in K$;*

(v) *there exists* $\gamma > 0$ *such that*

$$\left[\overline{\mathrm{conv}}^{\,*}\left(\bigcup_{z\in\overline{K}_x} \mathrm{D}f(\cdot,z)(x)^*(B^*_C(x))\right) + \widehat{B}^*_K(x)\right] \cap \gamma\mathbb{B}^* = \varnothing, \quad \forall x \in \mathbb{X}\backslash\mathcal{SE}.$$

Then, $\mathcal{SE}$ *is nonempty, closed and convex, and inequality* (3.17) *holds true.*

Proof. Let us start with observing that, by virtue of hypothesis (ii), the bifunction $d_C \circ f$ is continuous on $\mathbb{X}\times\mathbb{X}$. By the compactness of K, this implies that, for every $x \in K$, the value $\nu(x) = \sup_{z\in K} \mathrm{dist}\,(f(x,z),C)$ is attained at some $z \in K$, so that $\overline{K}_x \neq \varnothing$ for every $x \in K$. Furthermore, the continuity of $d_C \circ f$ entails that each function $x \mapsto \mathrm{dist}\,(f(x,z),C)$ is continuous on $\mathbb{X}$ and each function $z \mapsto \mathrm{dist}\,(f(x,z),C)$ is, in particular, u.s.c. on $\mathbb{X}$. Since by virtue of hypothesis (iii) each function $x \mapsto \mathrm{dist}\,(f(x,z),C)$ is convex on $\mathbb{X}$ (remember Lemma 2.5), these facts enable one to apply the well-known max rule for the subdifferential of a supremum of convex functions (see, for instance, [30, Theorem 2.4.18]), which gives

$$\begin{aligned}\partial\nu(x) &= \partial\left(\sup_{z\in K}\mathrm{dist}\,(f(\cdot,z),C)\right)(x)\\ &= \overline{\mathrm{conv}}^{\,*}\left(\bigcup_{z\in\overline{K}_x}\partial\mathrm{dist}\,(f(\cdot,z),C)\,(x)\right).\end{aligned} \tag{3.18}$$

Since each function $f : x \mapsto (x,z)$ is, in particular, strictly differentiable and its derivative is surjective according to hypothesis (iv), then it is possible to employ the formula in [20, Proposition 1.112(i)], which rules the subdifferential under composition with smooth mappings. In doing so, by recalling that the Mordukhovich (a.k.a. basic or limiting) subdifferential coincides here with the subdifferential in the sense of convex analysis by the convexity of the involved functions, one finds

$$\partial\mathrm{dist}\,(f(\cdot,z),C)\,(x) \subseteq \mathrm{D}f(\cdot,z)(x)^*\partial d_C(f(x,z)).$$

Now, observe that if $x \in \mathbb{X}\backslash\mathcal{SE}$ and $\nu(x) > 0$, then for every $z \in \overline{K}_x$ it must be $f(x,z) \notin C$. Consequently, as also $(\mathbb{Y},\|\cdot\|)$ is reflexive, $\Pi(f(x,z);C) \neq \varnothing$ and therefore

$$\partial d_C(f(x,z)) \subseteq \bigcap_{w\in\Pi(f(x,z);C)} \mathrm{N}(w;C)\cap\mathbb{S}^* \subseteq C^{\ominus}\cap\mathbb{S}^*.$$

If $x \in \mathbb{X}\backslash\mathcal{SE}$ but $\nu(x) = 0$, then it happens that $f(x,z) \in C$ also for $z \in \overline{K}_x$, so one can only say

$$\partial d_C(f(x,z)) \subseteq C^{\ominus}\cap\mathbb{B}^*.$$

Thus, from inclusion (3.18) one obtains in any case

$$\partial\nu(x) \subseteq \overline{\mathrm{conv}}^{\,*}\left(\bigcup_{z\in\overline{K}_x}\mathrm{D}f(\cdot,z)(x)^*(B^*_C(x))\right), \quad \forall x \in [\nu_{+K} > 0].$$

In the light of the last inclusion, it is clear that hypothesis (v) implies the validity of condition (iv) in Theorem 3.11.

It remains to notice that, for any fixed $x \in \mathbb{X}$, by continuity of the function $z \mapsto \mathrm{dist}\,(f(x,z), C)$ and by compactness of K, also the set $f(x,K)$ is compact and hence C-bounded, so hypothesis (ii) of Theorem 3.11 is fulfilled. As for hypothesis (iii) of Theorem 3.11, it comes true as an easy consequence of the current hypothesis (ii). Thus, the thesis can be achieved by applying Theorem 3.11. □

Another worthwhile comment refers to condition (3.16), which, as a requirement for subgradients to be sufficiently away from $\mathbf{0}^*$, can be regarded as a regularity condition. The reader should notice that if $x \in [\nu_{+K} > 0] \cap K$, then $\nu_{+K}(x) = \nu(x)$, so it must $\nu(x) > 0$. If K is such that $\nu(x) > \mathrm{dist}\,(f(x,z), C)$ for at least some $z \in K$, that is $\mathrm{dist}\,(f(x,\cdot), C)$ is not constant over $z \in K$, then x cannot be a minimizer of ν, with the consequence that $\mathbf{0}^* \notin \partial\nu(x)$. Therefore, if $\partial\nu(x)$ lies sufficiently faraway from the origin, it may actually happen that $\left[\partial\nu(x) + \widehat{B}^*_K(x)\right] \cap \gamma\mathbb{B}^* = \varnothing$. On the other hand, if $x \in [\nu_{+K} > 0]\backslash K$, then it may happen that $\nu(x) = 0$, so $\mathbf{0}^* \in \partial\nu(x)$, but in such an event, according to (3.15), it is $\mathbf{0}^* \notin \widehat{B}^*_K(x)$. Thus, again the condition (3.16) may actually take place.

By replacing the Fenchel subdifferential with more involved tools of nonsmooth analysis, the above line of investigation can be expanded in such a way to consider also problems without C-concave bifunctions. Below, the reader will find an attempt to develop the analysis by employing Fréchet and Mordukhovich subgradients. Let $\varphi : \mathbb{X} \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ be function, l.s.c. around a point $x \in \mathrm{dom}\,\varphi$ and let $(\mathbb{X}, \|\cdot\|)$ be an Asplund space. In this setting, an equivalent way to introduce the Mordukhovich subdifferential $\partial_{\mathrm{M}}\varphi(x)$ of φ at $\bar{x}$ by using the Fréchet subdifferential is to define

$$\partial_{\mathrm{M}}\varphi(\bar{x}) = \operatorname*{Limsup}_{x \xrightarrow{\varphi} \bar{x}} \widehat{\partial}\varphi(x),$$

where $\mathrm{Limsup}_{x \xrightarrow{\varphi} \bar{x}}$ denotes the Painlevé-Kuratowski upper limit of the set-valued mapping $\widehat{\partial}\varphi : \mathbb{X} \rightrightarrows \mathbb{X}^*$ as $x \to \bar{x}$ and $\varphi(x) \to \varphi(\bar{x})$, with respect to the norm topology on $\mathbb{X}$ and the weak* topology on $\mathbb{X}^*$ (see, for more details, [20] and Theorem 2.34 therein).

Theorem 3.14 (Mordukhovich subdifferential condition)**.** *With reference to a problem* (`SVE`)*, suppose that:*

(i) *the set K is convex;*
(ii) *each function $x \mapsto f(x,z)$ is C-u.s.c. on $\mathbb{X}$, for every $z \in K$;*
(iii) *there exists $x_0 \in K$ such that $f(x_0, K)$ is C-bounded;*
(iv) *there exists $\gamma > 0$ such that*

$$\left[\partial_{\mathrm{M}}\nu(x) + \widehat{B}^*_K(x)\right] \cap \gamma\mathbb{B}^* = \varnothing, \quad \forall x \in \mathbb{X}\backslash\mathcal{SE}. \tag{3.19}$$

Then, $\mathcal{SE}$ is nonempty and closed, and the the estimate in (3.17) *holds true.*

Proof. By hypothesis (ii) the merit function $\nu_{+K} : \mathbb{X} \longrightarrow [0, +\infty]$ is l.s.c. on $\mathbb{X}$ and therefore the set $[\nu_{+K} > 0]$ is open. By hypothesis (iii) it is $x_0 \in [\nu_{+K} < +\infty] \neq \varnothing$. Moreover, observe that $(\mathbb{X}, \|\cdot\|)$, as a reflexive Banach space, is an Asplund space. According to [17, Theorem 1, Chapter 2], this is equivalent to the fact that $(\mathbb{X}, \|\cdot\|)$ is $\widehat{\partial}$-trustworthy. Such a property allows one to exploit the slope estimate in [17, Proposition 1, Chapter 3] valid for l.s.c. functions on open subsets of $\widehat{\partial}$-trustworthy Banach spaces, according to which

$$\inf_{x\in[\nu_{+K}>0]} |\nabla\nu_{+K}|(x) \geq \inf_{x\in[\nu_{+K}>0]} \mathrm{dist}\left(\mathbf{0}^*, \widehat{\partial}\nu_{+K}(x)\right). \tag{3.20}$$

By recalling that for any $x \in \mathbb{X}$ the following general relation between subdifferentials holds

$$\widehat{\partial}\nu_{+K}(x) \subseteq \partial_{\mathrm{M}}\nu_{+K}(x),$$

from inequality (3.20) one obtains

$$\inf_{x\in[\nu_{+K}>0]} |\nabla\nu_{+K}|(x) \geq \inf_{x\in[\nu_{+K}>0]} \mathrm{dist}\left(\mathbf{0}^*, \partial_{\mathrm{M}}\nu_{+K}(x)\right). \tag{3.21}$$

Now, as ν and $\mathrm{dist}\,(\cdot, K)$ are l.s.c. and Lipschitz continuous on $\mathbb{X}$, respectively, they form a semi-Lipschitzian sum at any $x \in \mathrm{dom}\,\nu_{+K}$ in the sense of [20, Chapter 2.4]. Thus, as $(\mathbb{X}, \|\cdot\|)$ is an Asplund space, according to the sum rule for the Mordukhovich subdifferential (see [20, Theorem 2.33(c)]) one finds

$$\partial_{\mathrm{M}}\nu_{+K}(x) \subseteq \partial_{\mathrm{M}}\nu(x) + \partial_{\mathrm{M}}\mathrm{dist}\,(\cdot, K)\,(x) \subseteq \partial_{\mathrm{M}}\nu(x) + \widehat{B}^*_K(x),$$

where the last inclusion holds because the Mordukhovich subdifferential coincides with the Fenchel subdifferential, in consideration of the convexity of K. By virtue of hypothesis (iv), the last inclusion implies

$$\inf_{x\in[\nu_{+K}>0]} |\nabla\nu_{+K}|(x) \geq \gamma.$$

Such an inequality enables one to apply Proposition 3.1, from which all the assertions in the thesis can be deduced. This completes the proof. □

Remark 3.15. The author is aware of the fact that Theorem 3.14 can be subsumed in a more general scheme of analysis, where the Fréchet and the Mordukhovich subdifferentials are replaced with any subdifferential $\partial_\Box$, axiomatically defined as in [17, Section 1.5, Chapter 2], and $(\mathbb{X}, \|\cdot\|)$ is supposed to be $\partial_\Box$-trustworthy, as meant in [17, Definition 4, Chapter 2]). It is clear that, following such an approach, the subdifferential condition (3.19) should be expected to take a more involved form, because of the need of employing fuzzy sum rules for the $\partial_\Box$ subdifferential.

4. Conclusions

The contents of the present paper describe an attempt to conduct a study of solvability and error bounds for vector equilibrium problems via a variational approach. This attempt leads to obtain, as a main result, conditions which are expressed in terms of B-derivatives, convex normals and subgradients as well as Mordukhovich subdifferential. With respect to results focussing exclusively on the existence issue and obtained by different approaches, the conditions here established clearly demonstrate the crucial role that nonsmooth analysis can play in the development of this area. The achievements here discussed leave open the possibility for subsequent refinements and improvements. Some of them should come from a convenient expressions of subdifferentials of ν in terms of proper generalized derivatives of the bifunction f.

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