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A BREGMAN REGULARIZED PROXIMAL POINT METHOD FOR QUASI-EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we present a Bregman regularized proximal point method for solving quasi-equilibrium problems. Under mild assumptions, we prove that the method finds a solution of the problem and it generalizes some existing works in the literature of equilibrium and quasi-equilibrium problems. A numerical illustration shows the performance of the method in comparison with the classical version of the proximal point method.

1. Introduction

The equilibrium problem (EP) consists of finding $x^* \in K$ such that

$$f(x^*, y) \geq 0 \quad \forall y \in K, \tag{1.1}$$

where K is a closed and convex set and $f : K \times K \to \mathbb{R}$ is an equilibrium bifunction, i.e., $f(x, x) = 0$, for every $x \in K$. This problem will be denoted by $EP(f, K)$ and its solution set by $S_{EP}(f, K)$. A problem related to $EP(f, K)$ is that of finding $y^* \in K$ such that

$$f(x, y^*) \leq 0, \quad \forall x \in K.$$

This problem is called the dual of $EP(f, K)$ and its solution set will be denoted by $S^d_{EP}(f, K)$. Following the paper by Blum and Oettli [1], this problem has attracted much attention of researchers, but the problem itself was studied earlier, in the works of Ky Fan [14] and Nikaidô and Isoda [25]. It has been shown that some important problems such as scalar and vector optimization problems, saddle-point (minimax) problems, variational inequalities, Nash equilibria problems, complementarity problems and fixed point problems, can be formulated in the form of (EP); see for instance Blum and Oettli [1] and Muu and Oettli [24].

One of the most popular strategies for solving EP is the so-called regularization method. Moudafi [22] and Iusem and Sosa [18] proposed an iterative method which

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solves at each iteration the regularized $EP(f_k, K)$, where f_k is given by

$$f_k(x,y) = f(x,y) + \gamma_k \langle x - x^k, y - x \rangle, \tag{1.2}$$

for $\{\gamma_k\}$ a positive bounded auxiliary sequence. In 2012, Burachik and Kassay [5] generalized (1.2) by using Bregman functions on the regularization term as follows:

$$f_k(x,y) = f(x,y) + \gamma_k \langle \nabla\varphi(x) - \nabla\varphi(x^k), y - x \rangle, \tag{1.3}$$

where φ is a Bregman function; see [5] for more details. The notion of Bregman function has its origin in Bregman [2] while the proximal point method became popular after the seminar work of Rockafellar [28]; see also Bruck and Reich [3]. Bregman functions have been extensively used for defining "generalized" versions of the proximal point method (e.g., [6, 8–10, 12, 20]). Note that for $\varphi(x) = \frac{1}{2}||x||^2$ then (1.3) reduces to (1.2).

Recently, Santos and Souza [29] proposed for the first time an extension of the proximal point method to the more general context of quasi-equilibrium problems (QEP). This problem consists of finding $x^* \in C(x^*)$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C(x^*), \tag{1.4}$$

where C is a multivalued mapping from a closed and convex set X into itself such that $C(x) \subset X$ is a non-empty, closed and convex set, for every $x \in X$, and f is an equilibrium bifunction. Since $C(x^*) \subset X$, at a first glance, it seems that solving (1.4) is easier to solve (1.1) with $K = X$. However, in addition to satisfying (1.4), a candidate $x^* \in X$ of solution of the QEP must also to satisfy $x^* \in C(x^*)$. Therefore, the study of existence results or methods for solving QEP is challenging because it requires that an equilibrium problem and a fixed point problem must be solved simultaneously and this is clearly more difficult than only to solve an equilibrium problem. Quasi-equilibrium problems have a large number of important applications, for example, in economics, engineering, and operations research; see for instance [13,31]. In particular, the generalized Nash equilibrium problem, which extends the very important Nash equilibrium problem, can be modeled as a quasi-equilibrium problem and, when the data set is differentiable, as a quasi-variational inequality problem; see [30]. It is well known that quasi-variational inequalities cannot be written as a particular instance of an equilibrium problem; see for instance [21]. This illustrates how important is to study algorithms for solving QEP.

The aim of this paper is to propose a Bregman regularized version of the proximal point method such as in [5] for solving quasi-equilibrium problems. In this sense, our work can be viewed as a generalization from EP to QEP of [5] as well as from a square norm regularization to Bregman regularization in QEP of [29]. In the main convergence result of [5] is assumed that the solution set of the dual equilibrium problem is non-empty while in our approach we replace it by an auxiliary subset of the solution set of the QEP. Furthermore, we use the concept of Mosco continuity of C to deal with the fact that in [5] the constraint set K is fixed while in our context it may change through the mapping C. On the other hand, the assumptions made on the Bregman distance as well as its properties allowed us to extend the results in [29]. A numerical illustration suggests that the Bregman version of the proximal point method outperforms its classical version.

The paper is presented as follows. Section 2 presents some preliminary concepts and results on equilibrium problems and Bregman distances.The algorithm and its convergence analysis is developed in Section 3. Finally, numerical results are reported in Section 4.

2. Preliminary

2.1. **Basic concepts and properties of equilibrium problems.** Let $X \subset \mathbb{R}^n$ be a closed and convex set and $f : X \times X \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(P1) $f(x,x) = 0$, for all $x \in X$;
(P2) $f(\cdot,\cdot) : X \times X \to \mathbb{R}$ is jointly continuous on $X \times X$ (or the graph of f is sequentially closed) in the sense that if $x, y \in X$, $\{x^k\}$ and $\{y^k\}$ are sequences in X converging to x and y, respectively, then $f(x^k, y^k)$ converges to $f(x,y)$;
(P3) $f(x,\cdot) : X \to \mathbb{R}$ is convex, for all $x \in X$;
(P4) f is monotone, i.e., for each pair of points $x, y \in K$, we have that $f(x,y) + f(y,x) \leq 0$.

2.2. **Bregman distances.** Let $A \subseteq \mathbb{R}^n$ be a closed and convex set with $\operatorname{int} A$ nonempty, where $\operatorname{int} A$ denotes the interior of set A. Consider a function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ strictly convex, proper and lower semicontinuous with closed domain $\mathcal{D} := \operatorname{dom}(\varphi)$ and continuously differentiable on $\operatorname{int} A$.

Definition 2.1. The Bregman distance associated to φ with zone A is given by

$$D_\varphi(x,y) = \begin{cases} \varphi(x) - \varphi(y) - \langle \nabla\varphi(y), x - y\rangle, & \forall\, x \in A,\ \forall y \in \operatorname{int} A \\ +\infty, \text{ otherwise.} \end{cases}$$

The following maps are examples of Bregman distances.

Example 2.2. The Bregman function $\varphi(x) = \frac{1}{2}||x||^2$ and its respective Bregman distance

$$D_\varphi(x,y) = \frac{1}{2}||x-y||^2$$

with $A = \mathbb{R}^n$.

Example 2.3. The Bregman function $\varphi(x) = -\sum_{i\in I(x)}^n \log x_i$ and its respective Bregman distance

$$D_\varphi(x,y) = \sum_{i\in I(x)}^n \log(y_i/x_i) + \frac{x_i}{y_i} - 1$$

with $A = \mathbb{R}^n_+$, where $I(x) = \{i : x_i > 0\}$. This function is called Burg entropy.

Example 2.4. The Bregman function $\varphi(x) = \sum_{i=1}^n x_i \log x_i$ called Shannon entropy and its respective Bregman distance

$$D_\varphi(x,y) = \sum_{i=1}^n x_i \log(x_i/y_i) + y_i - x_i$$

with $A = \mathbb{R}^n_+$ known as Kullback-Leibler distance.

Next, we state the well known three point property for Bregman distances. More information on Bregman functions and distances can be found, for example, in the recent paper by Reem et al. [26].

For any $x \in \mathcal{D}$ and $y, z \in \operatorname{int}\mathcal{D}$, it is straightforward to check that

$$\langle \nabla\varphi(y) - \nabla\varphi(z), z - x\rangle = D_\varphi(x, y) - D_\varphi(x, z) - D_\varphi(z, y). \tag{2.1}$$

Following Burachik and Scheimberg [4], we consider throughout this paper the following set of assumptions on φ:

(B1) The right level sets of $D_\varphi(y, \cdot)$:

$$S_{y,\alpha} := \{z \in \operatorname{int}\mathcal{D} \ : \ D_\varphi(y, z) \leq \alpha\}$$

are bounded for all $\alpha \geq 0$ and for all $y \in \mathcal{D}$.

(B2) If $\{x^k\}, \{y^k\} \subset \operatorname{int}\mathcal{D}$ with $\lim_{k\to+\infty} x^k = x$, $\lim_{k\to+\infty} y^k = x$ and

$$\lim_{k\to+\infty} D_\varphi(x^k, y^k) = 0,$$

then

$$\lim_{k\to+\infty} D_\varphi(x, x^k) - D_\varphi(x, y^k) = 0.$$

(B3) If $\{x^k\} \subset \mathcal{D}$ is bounded, $\{y^k\} \subset \operatorname{int}\mathcal{D}$ is such that $\lim_{k\to+\infty} y^k = y$ and $\lim_{k\to+\infty} D_\varphi(x^k, y^k) = 0$, then $\lim_{k\to+\infty} x^k = y$.

(B4) For every $y \in A$, there exists $x \in \operatorname{int}\mathcal{D}$ such that $\nabla\varphi(x) = y$.

Remark 2.5. The Bregman distances in Examples 2.2-2.4 are examples of functions which satisfy assumptions (B1)-(B4); see [6].

3. Bregman regularized PPM

Throughout this paper we consider a quasi-equilibrium problem which consists of finding $x^* \in C(x^*)$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C(x^*). \tag{3.1}$$

We suppose that $X \subset \mathbb{R}^n$ is a closed and convex set such that $X \subset \operatorname{int}\mathcal{D}$, D_φ is a Bregman distance with zone X and $f : X \times X \to \mathbb{R}$ (called bifunction) satisfies the assumptions (P1)–(P4). Additionally, we suppose that $C : X \to \mathcal{P}(X)$ is point-to-set mapping such that, for every $x \in X$, $C(x)$ is a nonempty, closed and convex subset of X. Furthermore, we suppose the M-continuity of the multivalued mapping C. Let us recall that C is said to be M-continuous if:

(i) For $\{x^k\}, \{y^k\} \subset X$ with $y^k \in C(x^k)$, $x^k \to x$ and $y^k \to y$ implies that $y \in C(x)$, which means that the graph of C is sequentially closed.

(ii) For any sequence $\{x^k\} \subset X$ with $x^k \to x$ and for each $y \in C(x)$ there exists a sequence $\{y^k\} \subset X$ with $y^k \in C(x^k)$ such that $y^k \to y$.

We denote the solution set of QEP as $S_{QEP}(f, C)$. Next, we consider a set $S^* \subset S_{QEP}(f, C)$ and we assume that $S^* \neq \emptyset$, where it is given by

$$S^* = \left\{ x \in \bigcap_{z\in X} C(z) \ : \ f(x, y) \geq 0, \quad \forall y \in \bigcup_{z\in X} C(z) \right\}. \tag{3.2}$$

This assumption was considered to study the convergence of extragradient algorithms for solving QEP (see Strodiot et al. [30]), a projection-like method for QVIP (see Zhang et al. [32]) and generalized Nash equilibrium problem (see Han et al. [17]).

Remark 3.1. In equilibrium problems, the hypothesis "the solution set of the EP is nonempty" has been assumed as a mild assumption. In our context, the assumption $S^* \neq \emptyset$ can be viewed as a natural extension to QEP's of the assumption $S_{EP}(f, X) \neq \emptyset$ because if $C(x) \equiv X$, for all $x \in X$, then $\bigcap_{z\in X} C(z) = \bigcup_{z\in X} C(z) = X$ and $S^* = S_{EP}(f, X)$. It is easy to verify that the assumption $S^* \neq \emptyset$ guarantees that $S_{QEP}(f, C) \neq \emptyset$. For QVIP, there are available a great number of results when either X is bounded or the operator C satisfies certain coercivity condition. However, as remarked by Giannessi and Khan [16] many applications deal with QVIP with non-coercive operators defined on unbounded sets.

Next, we present the proximal point method with a Bregman regularization for solving QEP such that the Bregman function φ is strictly convex and satisfies the assumptions (B1)-(B4).

Algorithm 3.2 (Bregman PPM)**.**
Step 0: (Initial data) Take a bounded sequence of positive parameters $\{\gamma_k\}$, choose $x^0 \in X$ and set $k = 0$;
Step 1: (Iterative step) Given x^k, compute $x^{k+1} \in S_{EP}(f_k, C_k)$, where

$$f_k(x, y) = f(x, y) + \gamma_k\langle\nabla\varphi(x) - \nabla\varphi(x^k), y - x\rangle \tag{3.3}$$

and $C_k := C(x^k)$.
Step 2: (Stopping rule) If $x^{k+1} = x^k$, stop and return x^k. Otherwise, set $k = k+1$ and return to Step 1.

Remark 3.3. Note that Algorithm 3.2 solves at each iterate a Bregman regularized equilibrium problem. Thus, the well-definition of the method depends on (3.3) has a solution. In [5, Corollary 3.2], it is proved that (3.3) has a solution if the following assumption holds: given $\bar{x} \in X$ fixed, if for every sequence $\{x^k\} \subset X$ such that $\lim_{k\to\infty} ||x^k|| = \infty$, we have

$$\liminf_{k\to\infty} \Big(f(\bar{x}, x^k) + \gamma\langle\nabla\varphi(\bar{x}) - \nabla\varphi(x^k), \bar{x} - x^k\rangle \Big) > 0. \tag{3.4}$$

Moreover, if φ is strictly convex, then (3.3) has a unique solution. In [5, Remark 3.1], it is shown that the above condition is weaker than to suppose that the Bregman function φ is coercive, i.e.,

$$\lim_{||x||\to\infty} \frac{\varphi(x)}{||x||} = +\infty \tag{3.5}$$

see [27, Lemma 1] and [5, Corollary 3.3]. On the other hand, as mentioned by Censor et al. [7, page 380], if φ is a Bregman function with zone S and $S' \subset S$ is convex and closed, then φ can also be considered as a Bregman function with zone S'. This fact can be applied to Algorithm 3.2 taking into account that $C_k \subset X$ is convex and closed, for all $k \in \mathbb{N}$, together with the assumptions made on the Bregman distance D_φ with zone X and the bifunction f. Therefore, one can ensure that Step 1 of Algorithm 3.2 is well-defined (i.e., (3.3) has a solution) by assuming that (3.4) (or alternatively (3.5)) holds.

Next, we show that if Algorithm 3.2 stops at iterate x^k, then this point is a solution of the QEP (3.1).

Proposition 3.4. *If the stopping rule in Algorithm 3.2 is reached, then it returns a solution of* (3.1).

Proof. Note that $x^{k+1} \in S_{EP}(f_k, C_k)$ implies that $x^{k+1} \in C(x^k)$ and

$$f(x^{k+1}, y) + \gamma_k \langle \nabla\varphi(x^{k+1}) - \nabla\varphi(x^k), y - x^{k+1}\rangle \geq 0, \quad \forall y \in C(x^k).$$

Letting $x^{k+1} = x^k$, then $x^k \in C(x^k)$ and the above inequality implies that

$$f(x^k, y) \geq 0,$$

for all $y \in C(x^k)$. Thus, x^k is a solution of (3.1). □

Remark 3.5. In practice, we use the following stopping rule: given $\epsilon > 0$ a relative error tolerance, we perform the iterative step in Algorithm 3.2 until $||x^{k+1} - x^k|| < \epsilon$. In this case, Algorithm 3.2 returns x^{k+1}. A natural question arises: how good is this output x^{k+1} of Algorithm 3.2? In this paper, we refrain from discussing inexact algorithms as well as inexact computation of the subproblems (3.3), and hence we skip discussions of implementation issues with alternative approaches.

On the other hand, the next proposition provides a partial answer to this question and it will be used in our numerical result in Section 4. It measures the quality of a candidate to solution of QEP and was proved in the context of QEP by Santos and Souza [29]. Given $x \in X$, we define

$$\Upsilon(x) = ||x - \arg\min_{y \in C(x)} \{f(x, y) + \frac{1}{2}||y - x||^2\}||.$$

Proposition 3.6. *A point $x^* \in S_{QEP}(f, C)$ if and only if $\Upsilon(x^*) = 0$.*

Proof. See Santos and Souza [29, Proposition 2.6]. □

Before presenting the convergence analysis of Algorithm 3.2, let us state and prove a Bregman version of [29, Proposition 2.5] that we will use in the next results. To this end, let $\bar{x}$ be fixed define

$$\tilde{f}(x, y) = f(x, y) + \gamma\langle \nabla\varphi(x) - \nabla\varphi(\bar{x}), y - x\rangle, \tag{3.6}$$

for some $\gamma > 0$.

Proposition 3.7. *Let $\bar{x} \in X$ be an arbitrary point, $\tilde{x}, x^* \in X$ such that $\tilde{x} \in S_{EP}(\tilde{f}, C(\bar{x}))$ and $x^* \in S^d_{EP}(f, C(\bar{x}))$. If f satisfies (P1)-(P3), then*

$$D_\varphi(x^*, \tilde{x}) + D_\varphi(\tilde{x}, \bar{x}) \leq D_\varphi(x^*, \bar{x}).$$

Proof. Since that $\tilde{x} \in S_{EP}(\tilde{f}, C(\bar{x}))$, we have that $\tilde{f}(\tilde{x}, y) \geq 0$, for all $y \in C(\bar{x})$. This means that

$$0 \leq f(\tilde{x}, y) + \gamma\langle \nabla\varphi(\tilde{x}) - \nabla\varphi(\bar{x}), y - \tilde{x}\rangle, \quad \forall\, y \in C(\bar{x}). \tag{3.7}$$

Now, as $x^* \in S^d_{EP}(f, C(\bar{x}))$, we have that $x^* \in C(\bar{x})$ and, in addition, $f(x, x^*) \leq 0$, for all $x \in C(\bar{x})$. Making $y = x^*$ in the last inequality and using this fact in (3.7) together with $\gamma > 0$, we obtain

$$0 \leq \langle \nabla\varphi(\tilde{x}) - \nabla\varphi(\bar{x}), x^* - \tilde{x}\rangle.$$

Finally, from (2.1), we have

$$0 \leq \langle \nabla\varphi(\tilde{x}) - \nabla\varphi(\bar{x}), x^* - \tilde{x} \rangle = D_\varphi(x^*, \bar{x}) - D_\varphi(x^*, \tilde{x}) - D_\varphi(\tilde{x}, \bar{x}).$$

Consequently, we get the result. ☐

Next, we prove some classical properties of the proximal point method.

Theorem 3.8. *The following assertions hold:*

i) $\{x^k\}$ *is bounded;*
ii) $\lim_{k\to+\infty} D_\varphi(x^{k+1}, x^k) = 0$.

Proof. Let $x^* \in S^*$ be arbitrary. Since $S^* \subset S_{EP}(f, C(z))$, for all $z \in X$, we have that $x^* \in S_{EP}(f, C(z))$ and hence $f(x^*, y) \geq 0$, for all $y \in C(z)$. Now, since that f is monotone, we have $f(y, x^*) \leq 0$, for all $y \in C(z)$. This implies that $x^* \in S^d_{EP}(f, C(z))$, for all $z \in X$, and, in particular, for $z = x^k$. From the definition of Algorithm 3.2, we have $x^{k+1} \in S_{EP}(f_k, C(x^k))$. Thus, applying Proposition 3.7 with $\tilde{f} = f_k$ in (3.3), $\tilde{x} = x^{k+1}, \bar{x} = x^k$ we have that

$$D_\varphi(x^*, x^{k+1}) + D_\varphi(x^{k+1}, x^k) \leq D_\varphi(x^*, x^k), \quad k \in \mathbb{N}. \tag{3.8}$$

Since $D_\varphi(x^{k+1}, x^k) \geq 0$, we have

$$D_\varphi(x^*, x^{k+1}) \leq D_\varphi(x^*, x^k), \quad \forall k \in \mathbb{N}, \quad x^* \in S^*.$$

It follows from the last inequality that $\{D_\varphi(x^*, x^k)\}$ is non-increasing, and since it is non-negative, it converges. In particular, it is bounded. Thus, the first assertion directly follows from condition (B1).

Now, from (3.8), we have

$$D_\varphi(x^*, x^{k+1}) \leq D_\varphi(x^*, x^{k+1}) + D_\varphi(x^{k+1}, x^k) \leq D_\varphi(x^*, x^k), \quad \forall k \in \mathbb{N}.$$

Letting $k \to \infty$ in the last inequality and taking into account that $\{D_\varphi(x^*, x^k)\}$ is convergent, we obtain the second assertion. ☐

Next, we prove our main convergence result.

Theorem 3.9. *Every cluster point of* $\{x^k\}$ *belongs to* $S_{QEP}(f, C)$.

Proof. Let $\{x^{k_j}\}$ be a subsequence of $\{x^k\}$ that converges to $\widehat{x}$. From the definition of Algorithm 3.2, we have that $x^{k_j+1} \in C(x^{k_j})$. From Theorem 3.8 (ii), we have

$$\lim_{k\to\infty} D_\varphi(x^{k+1}, x^k) = 0,$$

and hence, we can guarantee from condition (B3) that $\lim_{j\to\infty} x^{k_j+1} = \widehat{x}$. Thus, from the M-Continuity of C, we have that $\widehat{x} \in C(\widehat{x})$ and, given $y \in C(\widehat{x})$, there exists a sequence $\{y^{k_j}\}$ such that $y^{k_j} \to y$ and $y^{k_j} \in C(x^{k_j})$. Now, as $x^{k_j+1} \in S_{EP}(f_{k_j}, C_k)$ we have

$$f_{k_j}(x^{k_j+1}, z) \geq 0, \quad \forall z \in C(x^{k_j}),$$

which means in particular for $z = y^{k_j} \in C(x^{k_j})$ that

$$f(x^{k_j+1}, y^{k_j}) + \gamma_{k_j} \langle \nabla\varphi(x^{k_j+1}) - \nabla\varphi(x^{k_j}), y^{k_j} - x^{k_j+1} \rangle \geq 0.$$

Using the fact that $\{\gamma_{k_j}\}$, $\{x^{k_j}\}$ and $\{y^{k_j}\}$ are bounded sequences, φ is continuously differentiable, f satisfies (P2) and taking the limit as $j \to \infty$ in the last inequality, we have

$$f(\widehat{x}, y) \geq 0.$$

Since we consider an arbitrary $y \in C(\widehat{x})$ this means that $f(\widehat{x}, y) \geq 0$, for all $y \in C(\widehat{x})$, and hence, $\widehat{x} \in S_{QEP}(f, C)$. This completes the proof. □

4. Numerical illustration

In this section, we illustrate the performance of the proposed method on one test problem adapted from [29]. We compare the performance of two Bregman regularized versions with the (classical) proximal point method for quasi-equilibrium problems proposed by [29]. We refrain from discussing computational efficiency of other methods, and hence, we skip discussions of comparisons of the proposed methods with other methods for QEP's.

The algorithms are coded in MATLAB R2020b on a 8 GB RAM Intel Core i7 to obtain the numerical results. The stopping rule is $\|x^{k+1} - x^k\| < 10^{-5}$. We take $\gamma_k = \gamma = 3.5$, for all $k \in \mathbb{N}$. We solve the subproblem (3.3) by using the regularized method in Muu and Quoc [23] in the classical version and the Bregman regularized method in Flam and Antipin [15]. They consider the following iterative method for solving an equilibrium problem: for any starting point $x^0 \in X$ and $\gamma > 0$, given $x^k \in X$ define $x^{k+1} \in X$ such that

$$x^{k+1} = \arg\min_{y \in X} \{\gamma f(x^k, y) + \frac{1}{2}\|y - x^k\|^2\} \tag{4.1}$$

and

$$x^{k+1} = \arg\min_{y \in X} \{\gamma f(x^k, y) + D_\varphi(y, x^k)\}, \tag{4.2}$$

respectively. The solutions of the subproblems in (4.1) and (4.2) are computed by the build-in MATLAB solver "fmincon".

Example 4.1. [29, Example 4.1 - Adapted] Consider the 2-dimensional nonsmooth quasi-equilibrium problem with the bifunction $f : X \times X \to \mathbb{R}$ given by

$$f(x, y) =| y_1 | - | x_1 | + y_2^2 - x_2^2$$

and the multivalued mapping C given by

$$C(x) = \left\{ y \in X \,;\, y_1 + y_2 = 1 + \frac{| x_1 |}{1 + | x_1 |},\ i = 1, 2 \right\},$$

where $X \subset \mathbb{R}^2_{++}$ is given by $X = [0.1, +\infty) \times [0.1, +\infty)$. One can check that f is monotone and the solution set is the single point $x^* = (1, \frac{1}{2})$.

We run Algorithm 3.2 with 100 random starting points in the box $[0.1, 20] \times [0.1, 20]$. We compare the performance of the method with the Bregman functions given in Example 2.2 (called PPM), Example 2.3 (called BPPM-1) and Example 2.4 (called BPPM-2). At each running, the methods start from the same initial point and use the same constant γ. In the Tables 1,2 and 3, we show the results of all methods in terms of number of iterates, CPU time and the accuracy $\Upsilon(x^*)$ (cf.

Proposition 3.6) until the stopping rule is satisfied, respectively. In these tables, min. iter. (resp. min. time), max. iter. (resp. max. time) and med. iter. (resp. med. time) denote the minimal, maximum and median of iterates (resp. CPU time) in 100 runs of the methods as well as min. $\Upsilon(x^*)$, max. $\Upsilon(x^*)$ and med. $\Upsilon(x^*)$ stand respectively to the minimum, maximum and median of the values $\Upsilon(x^*)$ in 100 runs, where x^* is the solution found by the methods.

As we can see in Tables 1 and 2, the Bregman regularized methods outperform the classical proximal point method in both number of iterates and CPU time. Furthermore, as shown in Table 3 all the methods find a good approximation of the solution.

Table 1. Running 100 times Algorithm 3.2 for Example 4.1.

Algorithm	min. iter.(k)	max. iter.(k)	med. iter.(k)
PPM	7	14	12.65
BPPM-1	8	17	11.69
BPPM-2	7	15	12.25

Table 2. Running 100 times Algorithm 3.2 for Example 4.1.

Algorithm	min. CPU time	max. CPU time	med. CPU time
PPM	0.0992826	1.1111657	0.199839135
BPPM-1	0.0840165	0.2752377	0.171949707
BPPM-2	0.0973115	0.4178743	0.178682051

Table 3. Running 100 times Algorithm 3.2 for Example 4.1.

Algorithm	min. $\Upsilon(x^*)$	max. $\Upsilon(x^*)$	med. $\Upsilon(x^*)$
PPM	0	2.360499186405201e-06	4.720998372810402e-08
BPPM-1	0	2.880658707330110e-06	5.761317414660219e-08
BPPM-2	0	2.687690141983094e-06	5.375380283966188e-08

In Figures 1,2 and 3, we consider a particular instance of each method (using the same random starting point $x^0 = (8.493, 18.3231)$) to illustrate the assertions in Theorems 3.8 and 3.9. Figure 1 shows that the sequence $\{D_\varphi(x^{k+1}, x^k)\}$ converges to zero faster than $\{||x^{k+1} - x^k||\}$ using both Bregman distances in Example 2.3 and 2.4. In Figures 2 and 3 we can see that the sequence $\{x^k\}$ generated by the methods BPPM-1 and BPPM-2 approach the solution of the QEP faster than the Euclidean regularized PPM.

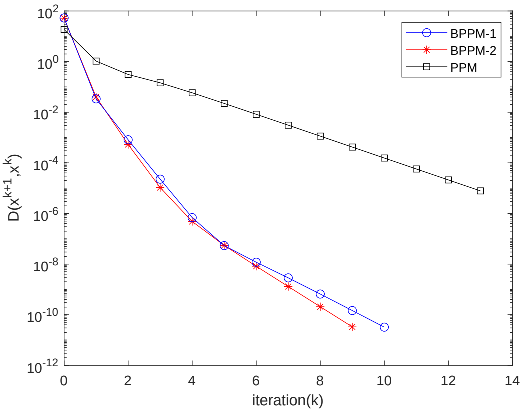

FIGURE 1. Behavior of $\{D_\varphi(x^{k+1}, x^k)\}$ and $\{||x^{k+1} - x^k||\}$.

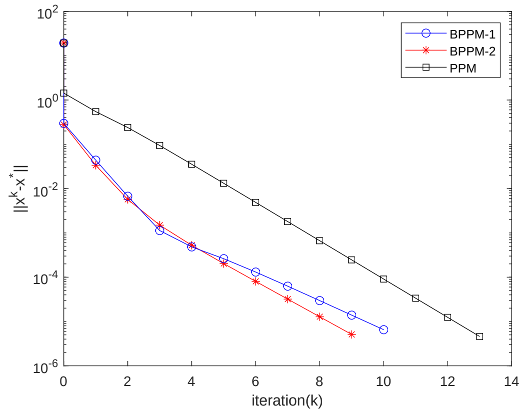

FIGURE 2. Behavior of $\{||x^k - x^*||\}$.

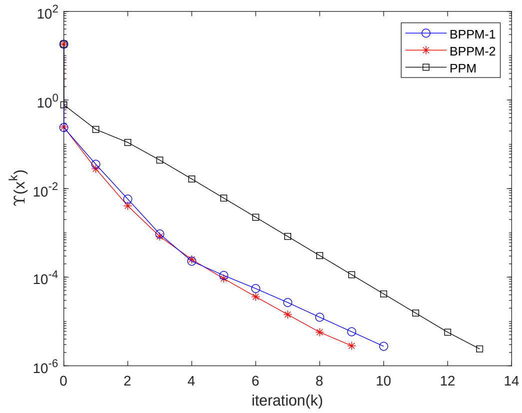

FIGURE 3. Behavior of $\Upsilon(x^k)$.

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