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A CHARACTERIZATION OF THE SET OF DE BRANGES MATRICES $\mathfrak{E}$ FOR WHICH THERE EXISTS A J-INNER MATRIX FUNCTION U SUCH THAT THE SPACES $\mathcal{B}(\mathfrak{E})$ AND $\mathcal{H}(U)$ COINCIDE

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Abstract. In his studies of canonical differential systems L de Branges introduced two classes of reproducing kernel Hilbert spaces of $m \times 1$ vector valued functions that (in our terminology) we refer to as $\mathcal{H}(U)$ spaces based on an $m \times m$ J-inner matrix valued function $U(\lambda)$ and $\mathcal{B}(\mathfrak{E})$ spaces based on an $m \times 2m$ matrix valued function $\mathfrak{E}(\lambda)$ that we call a de Branges matrix. In a previous publication we have shown that every $\mathcal{H}(U)$ space is automatically a $\mathcal{B}(\mathfrak{E})$ space.

The converse is not true. A transparent characterization of those de Branges matrices for which this holds is presented.

1. Introduction

If J is an $m \times m$ signature matrix and U is J-inner with respect to the open upper half-plane $\mathbb{C}_+$, then the kernel

$$K_\omega^U(\lambda) = \frac{J - U(\lambda)JU(\omega)^*}{\rho_\omega(\lambda)} \quad \text{if } \lambda, \omega \in \mathfrak{h}_U \text{ and } \lambda \neq \overline{\omega} \tag{1.1}$$

in which $\mathfrak{h}_U$ denotes the domain of holomorphy of U in $\mathbb{C}$ and

$$\rho_\omega(\lambda) = -2\pi i(\lambda - \overline{\omega}),$$

is positive on $\mathfrak{h}_U \times \mathfrak{h}_U$. Therefore, by the matrix version of a Theorem of Aronszjan (see e.g., Theorem 5.2 and Lemma 5.6 in [1]) $K_\omega^U(\lambda)$ is the RK (reproducing kernel) of exactly one RKHS (reproducing kernel Hilbert space) $\mathcal{H}(U)$ of $m \times 1$ vvf's that are holomorphic on $\mathfrak{h}_U$; in fact,

$$\mathfrak{h}_U = \bigcap_{f \in \mathcal{H}(U)} \mathfrak{h}_f \quad \text{(where } \mathfrak{h}_f \text{ denotes the domain of holomorphy of } f).$$

The RKHS $\mathcal{H}(U)$ is invariant with respect to the generalized backward shift operator R_α that is defined by the rule

$$(R_\alpha f)(\lambda) = \begin{cases} \dfrac{f(\lambda) - f(\alpha)}{\lambda - \alpha} & \text{for } \lambda, \alpha \in \mathfrak{h}_f \text{ and } \lambda \neq \alpha \\ f'(\alpha) & \text{for } \lambda, \alpha \in \mathfrak{h}_f \text{ and } \lambda = \alpha. \end{cases}$$

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Moreover, there exists a characterization of the RKHS $\mathcal{H}(U)$ by an identity (see e.g., (4.1) below) that is expressed in terms of R_α that is due to de Branges [6] (with an important technical improvement of this characterization due to Rovnyak [9]).

There is another RKHS $\mathcal{B}(\mathfrak{E})$ that originates with de Branges [7], [8] that plays a significant role in many problems of analysis. The RK of this space

$$(1.2)\qquad K_\omega^{\mathfrak{E}}(\lambda) = \frac{E_+(\lambda)E_+(\omega)^* - E_-(\lambda)E_-(\omega)^*}{\rho_\omega(\lambda)} \quad \text{if } \lambda, \omega \in \mathfrak{h}_{\mathfrak{E}} \text{ and } \lambda \neq \overline{\omega}$$

is based on an $m \times 2m$ mvf (matrix valued function) $\mathfrak{E} = \begin{bmatrix} E_- & E_+ \end{bmatrix}$ with $m \times m$ components that are of bounded type in each of the two half-planes in $\mathbb{C} \setminus \mathbb{R}$ with matching non tangential limits a.e. on $\mathbb{R}$ (i.e., $E_\pm \in \Pi^{m\times m}$, see below (1.7)) and are such that $\det E_+(\lambda) \not\equiv 0$ in $\mathbb{C}_+$ and $\chi = E_+^{-1}E_-$ is an $m \times m$ inner mvf. We shall call each such $m \times 2m$ mvf $\mathfrak{E}(\lambda)$ a **de Branges matrix**.

In previous publications [3], [5] we identified the de Branges space $\mathcal{H}(U)$ with the de Branges space $\mathcal{B}(\mathfrak{E})$ based on the de Branges matrix $\mathfrak{E} = \begin{bmatrix} E_- & E_+ \end{bmatrix}$ with $m \times m$ components

$$(1.3)\qquad E_-(\lambda) = P_- + U(\lambda)P_+ \quad \text{and} \quad E_+(\lambda) = P_+ + U(\lambda)P_-,$$

in which $P_+ = (I_m + J)/2$ and $P_- = (I_m - J)/2$ and explored this connection in detail in [5]. In particular:

Every J-inner matrix U generates a de Branges matrix $\mathfrak{E}$ via the recipe (1.3).

However, the converse implication is false: Not every de Branges space $\mathcal{B}(\mathfrak{E})$ can be identified as an $\mathcal{H}(U)$ space.

The **main objective** of this note is to establish a transparent characterization of those de Branges matrices $\mathfrak{E}$ that may be expressed in terms of J-inner matrices U via (1.3). This characterization will be established in Section 3. Section 2 is devoted to some preliminary analysis that reviews and expands upon the conclusions of [5]. Section 4 presents a sample calculation that illustrates the usefulness of this characterization.

The rest of this section is devoted to notation.

The symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{C}^{p\times q}$, $\mathbb{C}^p$ denote the set of complex numbers, real numbers, $p \times q$ complex valued matrices and $p \times 1$ complex valued vectors, respectively; $\mathbb{C}_+$ (resp., $\mathbb{C}_-$) denotes the open upper (resp., lower) half-plane; $\mathfrak{h}_f$ stands for the domain of holomorphy of a mvf f, and

$$\mathfrak{h}_f^+ = \mathfrak{h}_f \cap \mathbb{C}_+, \quad \mathfrak{h}_f^- = \mathfrak{h}_f \cap \mathbb{C}_-, \quad \mathfrak{h}_f^0 = \mathfrak{h}_f \cap \mathbb{R}.$$

The notation

$$(1.4)\qquad \begin{aligned} &f^\#(\lambda) = f(\overline{\lambda})^* \quad \text{and} \quad A \succ O \quad (A \succeq O) \quad \text{for} \\ &\qquad\text{positive definite (resp., semidefinite) matrices } A, \end{aligned}$$

will be useful.

The following classes of mvf's will play a role:

- $H_2^{p\times q}$, the Hardy space of $p\times q$ mvf's f that are holomorphic in $\mathbb{C}_+$ for which

(1.5) $$\|f\|^2_{H_2^{p\times q}} = \sup_{\nu>0}\int_{-\infty}^{\infty} \operatorname{trace}\{f(\mu+i\nu)^* f(\mu+i\nu)\}d\mu < \infty.$$

  $H_2^{p\times q}$ is a Hilbert space with norm defined as above.
- $H_\infty^{p\times q}$, the Hardy space of $p\times q$ mvf's f that are holomorphic in $\mathbb{C}_+$ for which

(1.6) $$\|f\|_\infty = \sup_{\omega\in\mathbb{C}_+}\|f(\omega)\| < \infty.$$

  $H_\infty^{p\times q}$ is a Banach space with norm defined as above.
- $\mathcal{S}_{in}^{p\times q}$, the class of $p\times q$ inner mvf's, is the set of mvf's f that belong to $H_\infty^{p\times q}$ with $f(\lambda)^*f(\lambda)\preceq I_q$ for $\lambda\in\mathbb{C}_+$ and $f(\mu)^*f(\mu)=I_q$ a.e. on $\mathbb{R}$.
- $\mathcal{N}^{p\times q}$, the Nevanlinna class of $p\times q$ mvf's f that are meromorphic in $\mathbb{C}_+$ and admit a representation of the form

(1.7) $$f = g^{-1}h \quad \text{with } g\in H_\infty^{1\times 1} \text{ and } h\in H_\infty^{p\times q}.$$

- $\Pi^{p\times q}$, the class of meromorphic $p\times q$ mvf's f on $\mathbb{C}\setminus\mathbb{R}$ such that
  (1) the restriction of f to $\mathbb{C}_+$ belongs to $\mathcal{N}^{p\times q}$,
  (2) the restriction of f to $\mathbb{C}_-$ belongs to the class $\{f : f^\# \in \mathcal{N}^{q\times p}\}$ and
  (3) $\lim_{\nu\downarrow 0} f(\mu+i\nu) = \lim_{\nu\downarrow 0} f(\mu-i\nu)$ a.e. on $\mathbb{R}$.

The abbreviation $\mathcal{X}^p = \mathcal{X}^{p\times 1}$ will be used for each of the spaces considered above. Thus, for example, $H_2^p = H_2^{p\times 1}$.

2. Preliminaries

Recall that a matrix $J\in\mathbb{C}^{m\times m}$ is a **signature matrix** if $J=J^*$ and $JJ^*=I_m$. We shall make use of the signature matrices

$$j_{pq} = \begin{bmatrix} I_p & 0\\ 0 & -I_q\end{bmatrix} \quad \text{with } p+q=m \text{ and } j_p = j_{pp} \quad \text{with } 2p=m.$$

An $m\times m$ mvf $U(\lambda)$ that is meromorphic in $\mathbb{C}_+$ is said to be ***J*-contractive** in $\mathbb{C}_+$ if

(1) $U(\lambda)^*JU(\lambda)\preceq J$ for every point $\lambda\in\mathfrak{h}_U^+$.

It is well known that every J-contractive mvf $U(\lambda)$ has nontangential limits $U(\mu)$ at a.e. point $\mu\in\mathbb{R}$ (see e.g. pp. 169–170 in [1]). A J-contractive mvf $U(\lambda)$ is said to be ***J*-inner** if,

(2) $U(\mu)^*JU(\mu)=J$ for a.e point $\mu\in\mathbb{R}$.

If (1) and (2) hold, then $U(\lambda)$ may be extended to $\mathbb{C}_-$ via the formula

$$J = U(\omega)JU(\overline{\omega})^* \quad \text{for } \omega\in\mathbb{C}_- \text{ such that } \overline{\omega}\in\mathfrak{h}_U^+ \text{ and } \det U(\overline{\omega})\neq 0.$$

The J-inner $m\times m$ mvf $U(\lambda)$ extended to $\mathbb{C}_-$ belongs to the class $\Pi^{m\times m}$.

Additional information on the classes of mvf's listed above and some of their principal applications may be found in [1], [2] and [4] (as well as many other places) and the references cited therein.

A $p \times 2p$ mvf $\mathfrak{E}(\lambda) = \begin{bmatrix} E_-(\lambda) & E_+(\lambda) \end{bmatrix}$ with $p \times p$ blocks $E_\pm(\lambda)$ that belong to $\Pi^{p\times p}$ such that

$$\det E_+(\lambda) \not\equiv 0 \quad \text{in } \mathbb{C}_+ \text{ and the mvf } \chi = E_+^{-1}E_- \text{ belongs to } \quad \mathcal{S}_{in}^{p\times p}$$

will be called a **de Branges matrix**.

If $\mathfrak{E}$ is a de Branges matrix, then the space

$$\begin{aligned} \mathcal{B}(\mathfrak{E}) &= \{p \times 1 \text{ vvf's } f : \ E_+^{-1}f \in H_2^p \ominus \chi H_2^p\} \\ &= \{p \times 1 \text{ vvf's } f : \ E_+^{-1}f \in H_2^p \quad \textit{and} \quad E_-^{-1}f \in (H_2^\perp)^p\} \end{aligned} \tag{2.1}$$

(here $H_2^\perp = L_2 \ominus H_2$) with inner product

$$\langle f, g\rangle_{\mathcal{B}(\mathfrak{E})} = \langle E_+^{-1}f, E_+^{-1}g\rangle_{st} = \int_{-\infty}^{\infty} g(\mu)^*\Delta_{\mathfrak{E}}(\mu)f(\mu)d\mu, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle_{st}$ denotes the standard inner product in L_2^p and

$$\Delta_{\mathfrak{E}}(\mu) = \{E_+(\mu)E_+(\mu)^*\}^{-1} \quad \text{a.e. on } \mathbb{R} \tag{2.3}$$

is a RKHS of $p \times 1$ vvf's that are holomorphic in $\mathfrak{h}_{\mathfrak{E}}$ with RK given by (1.2). Moreover, $\mathcal{B}(\mathfrak{E}) \subset \Pi^p$, and $\mathfrak{h}_{\mathfrak{E}} \subseteq \mathfrak{h}_f$ for every $f \in \mathcal{B}(\mathfrak{E})$; see e.g., Theorem 5.65 in [1].

The next theorem recalls the identification of the RKHS $\mathcal{H}(U)$ of $m \times 1$ vvf's based on a J-inner mvf U as a de Branges space $\mathcal{B}(\mathfrak{E})$ based on an $m \times 2m$ de Branges matrix that is defined in terms of U.

Theorem 2.1. *The RKHS $\mathcal{H}(U)$ with RK given by formula* (1.1) *coincides with the de Branges space $\mathcal{B}(\mathfrak{E})$ based on the de Branges matrix $\mathfrak{E} = \begin{bmatrix} E_- & E_+ \end{bmatrix}$ with $m \times m$ components*

$$E_-(\lambda) = P_- + U(\lambda)P_+ \quad \textit{and} \quad E_+(\lambda) = P_+ + U(\lambda)P_-, \tag{2.4}$$

in which

$$P_+ = \frac{I_m + J}{2} \quad \textit{and} \quad P_- = \frac{I_m - J}{2}.$$

Moreover,

$$E_+^{-1} \in H_\infty^{m\times m}, \quad (E_-^\#)^{-1} \in H_\infty^{m\times m} \tag{2.5}$$

and the inner product

$$\langle f, g\rangle_{\mathcal{H}(U)} = \int_{-\infty}^{\infty} g(\mu)^*\Delta_{\mathfrak{E}}(\mu)f(\mu)d\mu \quad \textit{for } f, g \in \mathcal{H}(U), \tag{2.6}$$

where

$$\Delta_{\mathfrak{E}}(\mu) = \{E_+(\mu)E_+(\mu)^*\}^{-1} \quad \textit{a.e. on } \mathbb{R}. \tag{2.7}$$

Furthermore, $U(\lambda)$ and J can be recovered from $\mathfrak{E}(\lambda)$ by the formulas

$$U(\lambda) = E_-(\lambda) + E_+(\lambda) - I_m = \mathfrak{E}(\lambda)\begin{bmatrix} I_m \\ I_m \end{bmatrix} - I_m \tag{2.8}$$

and

$$E_+ - E_- = (I_m - U)J = (2I_m - E_+ - E_-)J, \tag{2.9}$$

respectively.

Proof. This theorem coincides with Theorem 3.1 in [5] except for the addition of (2.9), which is self-evident from (2.4). □

There is a converse that builds upon the observation that the mvf's defined in (2.4) satisfy the condition $E_\pm P_\pm = P_\pm$:

Theorem 2.2. *If* $\mathfrak{E} = \begin{bmatrix} E_- & E_+ \end{bmatrix}$ *is a de Branges matrix of size* $m \times 2m$ *and* $P_\pm \in \mathbb{C}^{m\times m}$ *are complementary orthogonal projections (i.e.,* $P_\pm^2 = P_\pm$*,* $P_\pm^* = P_\pm$ *and* $P_+ + P_- = I_m$*) such that* $E_+P_+ = P_+$ *and* $E_-P_- = P_-$ *then:*

(1) $U = E_+ + E_- - I_m$ *is* J*-inner with respect to the signature matrix* $J = P_+ - P_-$*.*
(2) $\mathcal{H}(U) = \mathcal{B}(\mathfrak{E})$*.*

Proof. It suffices to show that the (numerators of) the reproducing kernels $K_\omega^U(\lambda)$ (defined in (1.1)) and $K_\omega^{\mathfrak{E}}(\lambda)$ (defined in (1.2)) coincide. Under the given assumptions, $UP_+ = E_-P_+$ and $UP_- = E_+P_-$. Consequently, for every pair of points $\lambda, \omega \in \mathfrak{h}_{\mathfrak{E}}$,

$$\begin{aligned} E_+(\lambda)E_+(\omega)^* - E_-(\lambda)E_-(\omega)^* &= E_+(\lambda)P_+E_+(\omega)^* - E_-(\lambda)P_-E_-(\omega)^* \\ &\quad + E_+(\lambda)P_-E_+(\omega)^* - E_-(\lambda)P_+E_-(\omega)^* \\ &= P_+ - P_- + U(\lambda)(P_- - P_+)U(\omega)^* \\ &= J - U(\lambda)JU(\omega)^*. \end{aligned}$$

Therefore, (1) and (2) hold. □

We remark that the **Potapov-Ginzburg transform**

$$S = (P_+ + UP_-)^{-1}(P_- + UP_+) = E_+^{-1}E_- \tag{2.10}$$

which maps the class $\mathcal{U}(J)$ of J-inner mvf's onto the set

$$\{S \in \mathcal{S}_{in}^{m\times m} : \det(P_+ + SP_-) \not\equiv 0\} \tag{2.11}$$

(see e.g., Subsection 2.2 in [1] for the existence of the indicated inverses for transforms of the form (2.10)). The formulas in (2.10) imply that U can be recovered from $\mathfrak{E}$ via the formula

$$U = (P_+ + SP_-)^{-1}(P_- + SP_+) \quad \text{with } S = E_+^{-1}E_-.$$

3. A parametrization of those $\mathfrak{E}$ for which $\mathcal{B}(\mathfrak{E}) = \mathcal{H}(U)$

Theorem 3.1. *If* $\mathfrak{F} = \begin{bmatrix} F_- & F_+ \end{bmatrix}$ *is an* $m \times 2m$ *de Branges matrix such that*

(1) $\mathcal{B}(\mathfrak{F}) = \mathcal{H}(U)$ *for some* $m \times m$ J*-inner mvf* U *with signature matrix* $J \in \mathbb{C}^{m\times m}$ *and*
(2) $K_\omega^{\mathfrak{F}}(\omega) \succ O$ *for at least one (and hence every) point* $\omega \in \mathfrak{h}_{\mathfrak{F}}$*,*

then

$$\begin{bmatrix} F_- & F_+ \end{bmatrix} = \begin{bmatrix} E_- & E_+ \end{bmatrix} V \tag{3.1}$$

for some j_m*-unitary matrix* $V \in \mathbb{C}^{2m\times 2m}$ *and some de Branges matrix* $\mathfrak{E} = \begin{bmatrix} E_- & E_+ \end{bmatrix}$ *with blocks* $E_\pm$ *for which*

$$E_+P_+ = P_+, \quad E_- = E_-P_- \quad \textit{where} \quad P_\pm = (I_m \pm J)/2. \tag{3.2}$$

Proof. If $\mathcal{B}(\mathfrak{F}) = \mathcal{H}(U)$, then, in view of Theorems 5.65 and 5.49 in [1],

$$\mathfrak{h}_{\mathfrak{F}} \subseteq \bigcap_{f\in\mathcal{B}(\mathfrak{F})} \mathfrak{h}_f = \bigcap_{f\in\mathcal{H}(U)} \mathfrak{h}_f = \mathfrak{h}_U$$

and hence, as $\mathcal{H}(U)$ is R_α invariant for every point $\alpha \in \mathfrak{h}_U$, $\mathcal{B}(\mathfrak{F})$ is also R_α invariant for every point $\alpha \in \mathfrak{h}_{\mathfrak{F}}$. Therefore, Lemma 2.4 of [5] guarantees that if $K_\omega^{\mathfrak{F}}(\omega) \succ O$ for at least one point $\omega \in \mathfrak{h}_{\mathfrak{F}}$, then $K_\omega^{\mathfrak{F}}(\omega) \succ O$ for every point $\omega \in \mathfrak{h}_{\mathfrak{F}}$.

In view of Theorem 2.1, $\mathcal{H}(U) = \mathcal{B}(\mathfrak{E})$, where the blocks $E_\pm$ of $\mathfrak{E}$ are specified in terms of U and $P_\pm$ by the formulas

$$E_+ = P_+ + UP_- \quad \text{and} \quad E_- = P_- + UP_+, \tag{3.3}$$

which clearly meet the conditions in (3.2). Thus, $\mathcal{B}(\mathfrak{F}) = \mathcal{H}(U) = \mathcal{B}(\mathfrak{E})$, Theorem 2.3 of [5] ensures that (3.1) holds for some j_m-unitary matrix $V \in \mathbb{C}^{2m\times 2m}$.

□

Remark 3.2. Theorem 3.1 admits a self-evident converse: If $\mathfrak{F} = \begin{bmatrix} F_- & F_+ \end{bmatrix}$ and $\mathfrak{E} = \begin{bmatrix} E_- & E_+ \end{bmatrix}$ are de Branges matrices with blocks $F_\pm$ and $E_\pm$ of size $m \times m$ such that (3.1) holds for some j_m-unitary matrix $V \in \mathbb{C}^{2m\times 2m}$ and there exist a pair of complementary projections $P_\pm \in \mathbb{C}^{m\times m}$ such that (3.2) holds for the blocks of $\mathfrak{E}$, then $\mathcal{B}(\mathfrak{F}) = \mathcal{H}(U)$, where $U = E_+ + E_- - I_m$ is J-inner with respect to the signature matrix $J = P_+ - P_-$.

This follows from the fact that

$$-\mathfrak{F}(\lambda) j_m \mathfrak{F}(\omega)^* = -\mathfrak{E}(\lambda) j_m \mathfrak{E}(\omega)^* = J - U(\lambda) J U(\omega)^*.$$

4. A sample calculation

A de Branges space $\mathcal{B}(\mathfrak{E})$ can be identified as an $\mathcal{H}(U)$ space for some J-inner mvf U if and only if $\mathcal{B}(\mathfrak{E})$ is invariant under R_α for every point $\alpha \in \mathfrak{h}_{\mathfrak{E}}$ and the de Branges identity

$$\langle R_\alpha f, g\rangle_{\mathcal{B}(\mathfrak{E})} - \langle f, R_\beta g\rangle_{\mathcal{B}(\mathfrak{E})} - (\alpha - \overline{\beta})\langle R_\alpha f, R_\beta g\rangle_{\mathcal{B}(\mathfrak{E})} = 2\pi i g(\beta)^* J f(\alpha) \tag{4.1}$$

is in force for every pair of points $\alpha, \beta \in \mathfrak{h}_{\mathfrak{E}}$ and every pair of vvf's $f, g \in \mathcal{B}(\mathfrak{E})$ (see e.g., Theorem 5.21 in [1] for a characterization of $\mathcal{H}(U)$).

In this section we shall indicate how to verify (4.1) for de Branges matrices of the form (2.4) when $J = j_{pq}$ by exploiting the following result, which is Corollary 5.3 in [5]:

Theorem 4.1. *If $\mathfrak{E}$ is a de Branges matrix such that $\mathcal{B}(\mathfrak{E})$ is R_α-invariant for every point $\alpha \in \mathfrak{h}_{\mathfrak{E}}$, then the de Branges identity* (4.1) *holds in $\mathcal{B}(\mathfrak{E})$ if and only if*

$$\int_{-\infty}^{\infty} \frac{g(\overline{\alpha})^* \Delta_{\mathfrak{E}}(\mu) f(\mu) - g(\mu)^* \Delta_{\mathfrak{E}}(\mu) f(\alpha)}{\mu - \alpha} d\mu = 2\pi i g(\overline{\alpha})^* J f(\alpha) \tag{4.2}$$

for all vvf's $f, g \in \mathcal{B}(\mathfrak{E})$ and all points $\alpha \in \mathfrak{h}_{\mathfrak{E}}$ such that $\overline{\alpha} \in \mathfrak{h}_{\mathfrak{E}}$.

Discussion Suppose that (3.2) and (3.3) hold with $J = j_{pq}$, $p + q = m$ and $U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$. Then

$$E_+ = \begin{bmatrix} I_p & u_{12} \\ O & u_{22} \end{bmatrix} \quad \text{and} \quad E_- = \begin{bmatrix} u_{11} & O \\ u_{21} & I_q \end{bmatrix}.$$

Then

$$E_+^{-1}E_- = \begin{bmatrix} u_{11} - u_{12}u_{22}^{-1}u_{21} & -u_{12}u_{22}^{-1} \\ u_{22}^{-1}u_{21} & u_{22}^{-1} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \in \mathcal{S}_{in}^{m\times m}$$

and

$$\Delta_{\mathfrak{E}} = (E_+E_+^*)^{-1} = \begin{bmatrix} I_p & s_{12} \\ s_{12}^* & I_q \end{bmatrix} \quad \text{a.e. on } \mathbb{R}.$$

Therefore, the vvf $f = \mathrm{col}\,(f_1, f_2)$ with components $f_1 \in \Pi^p$ and $f_2 \in \Pi^q$ belongs to $\mathcal{B}(\mathfrak{E})$ if and only if

$$E_+^{-1}f = \begin{bmatrix} I_p & -u_{12}u_{22}^{-1} \\ O & u_{22}^{-1} \end{bmatrix} f = \begin{bmatrix} I_p & s_{12} \\ O & s_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in H_2^m$$

and

$$\langle E_+^{-1}f, (E_+^{-1}E_-)g\rangle_{st} = 0 \quad \text{for every } g \in H_2^m.$$

Thus, if $g = \mathrm{col}\,(g_1, g_2)$ with $g_1 \in H_2^p$ and $g_2 \in H_2^q$, then

$$\left\langle \begin{bmatrix} f_1 + s_{12}f_2 \\ s_{22}f_{22} \end{bmatrix}, \begin{bmatrix} s_{12} \\ s_{22} \end{bmatrix} g_2 \right\rangle_{st} = 0$$

for every $g_2 \in H_2^q$. Thus,

$$f_1 + s_{12}f_2 \in H_2^p \quad \text{and} \quad \langle \left[s_{12}^*(f_1 + s_{12}f_2) + s_{22}^*s_{22}f_2\right], g_2\rangle_{st} = 0$$

for every $g_2 \in H_2^q$. Since $s_{12}(\mu)^*s_{12}(\mu) + s_{22}(\mu)^*s_{22}(\mu) = I_q$ a.e. on $\mathbb{R}$, the second equality implies that $s_{12}^\# f_1 + f_2 \in (H_2^\perp)^q$.

We are now ready to evaluate the integral in (4.2). We assume that $\alpha \in \mathfrak{h}_{\mathfrak{E}}^+$ and $\overline{\alpha} \in \mathfrak{h}_{\mathfrak{E}}^-$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{g(\overline{\alpha})^*\Delta_{\mathfrak{E}}(\mu)f(\mu)}{\mu - \alpha} d\mu &= 2\pi i \left\langle \begin{bmatrix} f_1 + s_{12}f_2 \\ s_{12}^*f_1 + f_2 \end{bmatrix}, \frac{1}{\rho_\alpha} \begin{bmatrix} g_1(\overline{\alpha}) \\ g_2(\overline{\alpha}) \end{bmatrix} \right\rangle_{st} \\ &= 2\pi i g_1(\overline{\alpha})^*[f_1(\alpha) + s_{12}(\alpha)f_2(\alpha)] \end{aligned}$$

since $f_1 + s_{12}f_2 \in H_2^p$ and $s_{12}^\# f_1 + f_2 \in (H_2^\perp)^q$.

The next step is to observe that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{f(\alpha)^*\Delta_{\mathfrak{E}}(\mu)g(\mu)}{\mu - \overline{\alpha}} d\mu &= \int_{-\infty}^{\infty} \frac{f_1(\alpha)^*[g_1(\mu) + s_{12}(\mu)g_2(\mu)]}{\mu - \overline{\alpha}} d\mu \\ &\quad + \int_{-\infty}^{\infty} \frac{f_2(\alpha)^*[s_{12}^\#(\mu)g_1(\mu) + g_2(\mu)]}{\mu - \overline{\alpha}} d\mu \\ &= 0 - 2\pi i f_2(\alpha)^*[s_{12}^\#(\overline{\alpha})g_1(\overline{\alpha}) + g_2(\overline{\alpha})]. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{g(\mu)^*\Delta_{\mathfrak{E}}(\mu)f(\alpha)}{\mu - \alpha} d\mu &= \left(\int_{-\infty}^{\infty} \frac{f(\alpha)^*\Delta_{\mathfrak{E}}(\mu)g(\mu)}{\mu - \overline{\alpha}} d\mu \right)^* \\ &= 2\pi i[g_1(\overline{\alpha})^*s_{12}(\alpha) + g_2(\overline{\alpha})^*]f_2(\alpha) \end{aligned}$$

and hence (4.2) with $J = j_{pq}$ follows by combining these evaluations.

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